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Weak KAM from a PDE point of view: viscosity solutions of the Hamilton–Jacobi equation and Aubry set

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We introduce the notion of a viscosity solution for the first-order Hamilton–Jacobi equation, in the more general setting of manifolds, to obtain a weak KAM theory using only tools from partial differential equations. This work should be accessible to people with no prior knowledge of the subject.

We introduce the notion of viscosity solutions for the Hamilton–Jacobi equation (HJE), which is a first-order partial differential equation (PDE). There is also an extensive literature on viscosity solutions of second-order PDEs, but we do not cover this topic at all (see, for example, [8]).

The notion of a viscosity solution is due to Crandall and Lions (see [6]). There are two excellent books on the subject: one by Barles [3] and another by Bardi and Capuzzo-Dolceta [2]. An introduction to viscosity solutions can be found in Evans [9]. Our treatment has been significantly influenced by the content of these three books. Although many things are standard, we will cover the theory on general manifolds since this is the right setting for weak KAM theory. This is probably the first time that a general introduction on viscosity solutions on manifolds has appeared in print. Anything that cannot be found in the standard references is the result of joint work with Antonio Siconolfi (see [11, 12]). Of course, our treatment follows some unpublished notes [10]. We hardly touch the dynamical implications of the theory, and refer the interested reader to Bernard's companion notes [4].

We would like to apologize for the small number of references. Nowadays, for a work of this size, giving a full set of references on this subject is an impossible task. A look at the references in [2] shows that doing so 15 years ago would have been very difficult. However, a larger set of references can be easily found on the Internet.

We denote by M a connected, paracompact $\mathcal{C}^\infty$ manifold without boundary. For any $x \in M$, the tangent and cotangent spaces of M at x are T_xM and T_x^*M, respectively. The tangent and cotangent bundle are TM and T^*M, respectively.

*This paper is a late addition to the papers surveying active areas in partial differential equations, published in issue 141.2, which were based on a series of mini-courses held in the International Centre for Mathematical Sciences (ICMS) in Edinburgh during 2010.

A point in TM (respectively, T^*M) will be denoted by (x,v) (respectively, (x,p)) where $x \in M$, and $v \in T_xM$ (respectively, $p \in T_x^*M$). With this notation the canonical projection $\pi\colon TM \to M$ (respectively, $\pi^*\colon T^*M \to M$) is nothing but $(x,v) \mapsto x$ (respectively, $(x,p) \mapsto x$).

We will assume in what follows that M is endowed with a $\mathcal{C}^\infty$ Riemannian metric g. For $v \in T_xM$, we will set $\|v\|_x = (g_x(v,v))^{1/2}$. We will also denote by $\|\cdot\|_x$ the norm on T_x^*M dual to $\|\cdot\|_x$ on T_xM.

1. The different forms of the Hamilton–Jacobi equation

We will suppose that M is a fixed manifold, and that $H\colon T^*M \to \mathbb{R}$ is a continuous function, which we will call the Hamiltonian.

Definition 1.1 (stationary HJE). The HJE associated to H is the equation

$$H(x, d_xu) = c,$$

where c is some constant.

A first good example to keep in mind is

$$H(x,p) = \tfrac{1}{2}\|p\|_x^2 + V(x),$$

where the norm comes from the Riemannian metric on the manifold M, and where $V\colon M \to \mathbb{R}$ is a continuous (even $\mathcal{C}^\infty$-function). An even better example is to modify H in the following way. Consider a continuous vector field $X\colon M \to TM$, and define H as

$$H(x,p) = \tfrac{1}{2}\|p\|_x^2 + V(x) + p(X(x)).$$

A classical solution of the HJE $H(x, d_xu) = c$ on the open subset U of M is a $\mathcal{C}^1$ map $u\colon U \to \mathbb{R}$ such that $H(x, d_xu) = c$ for each $x \in U$.

We will usually deal only with the case $H(x, d_xu) = 0$, since we can reduce the general case to that particular case by replacing the Hamiltonian H by H_c defined as $H_c(x,p) = H(x,p) - c$.

Definition 1.2 (evolutionary HJE). The evolutionary HJE associated to the Hamiltonian H is the equation

$$\frac{\partial u}{\partial t}(t,x) + H\left(x, \frac{\partial u}{\partial x}(t,x)\right) = 0.$$

A classical solution to this evolutionary HJE on the open subset W of $\mathbb{R} \times T^*M$ is a $\mathcal{C}^1$ map $u\colon W \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t}(t,x) + H\left(x, \frac{\partial u}{\partial x}(t,x)\right) = 0$$

for each $(t,x) \in W$.

The evolutionary form can be reduced to the stationary form by introducing the Hamiltonian $\hat{H}\colon T^*(\mathbb{R} \times M)$ defined as

$$\hat{H}(t,x,s,p) = s + H(x,p),$$

where $(t,x) \in \mathbb{R} \times M$, and $(s,p) \in T^*_{(t,x)}(\mathbb{R} \times M) = \mathbb{R} \times T_x^*M$.

It is also possible to consider a time-dependent Hamiltonian defined on an open subset of $\mathbb{R} \times M$. Consider, for example, a Hamiltonian $H\colon \mathbb{R} \times TM^* \to \mathbb{R}$. The evolutionary form of the HJE for that Hamiltonian is

$$\frac{\partial u}{\partial t}(t,x) + H\left(t, x, \frac{\partial u}{\partial x}(t,x)\right) = 0.$$

A classical solution of that equation on the open subset W of $\mathbb{R} \times M$ is, of course, a $\mathcal{C}^1$ map $u\colon W \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t}(t,x) + H\left(t, x, \frac{\partial u}{\partial x}(t,x)\right) = 0$$

for each $(t,x) \in W$. This form of the HJE can also be reduced to the stationary form by introducing the Hamiltonian $\tilde{H}\colon T^*(\mathbb{R} \times M) \to \mathbb{R}$ defined as

$$\tilde{H}(t,x,s,p) = s + H(t,x,p).$$

It is usually impossible to find global $\mathcal{C}^1$ solutions of the HJE $H(x, d_x u) = c$. For example, if the Hamiltonian is of the form

$$H(x,p) = \tfrac{1}{2}\|p\|_x^2 + V(x),$$

and u is a classical solution of $H(x, d_x u) = c$, we get $c = \frac{1}{2}\|d_x u\|_x^2 + V(x) \geqslant V(x)$, hence $c \geqslant \sup_M V$. If we assume that M is compact, then u has at least two distinct critical points (minimum and maximum) x_1, x_2. At these critical points we get $c = H(x, d_{x_i} u) = V(x_i)$, since $d_{x_i} u = 0$. Therefore, on the compact manifold M, a classical solution of $H(x, d_u) = c$ for such a Hamiltonian can only occur at $c = \max V$. Moreover, if this equation has a classical solution, then V must necessarily achieve its maximum at two distinct points. In particular, if we choose V such that its maximum on the compact manifold M is achieved at a single point, then the HJE does not have classical solutions.

2. Viscosity solutions

We will suppose in this section that M is a manifold and $H\colon T^*M \to M$ is a Hamiltonian.

Since it is generally impossible to find $\mathcal{C}^1$-solutions to the HJE, one has to admit more general functions. A first attempt is to consider Lipschitz functions.

Definition 2.1 (very weak solution). We say that $u\colon M \to \mathbb{R}$ is a very weak solution of $H(x, d_x u) = c$ if it is Lipschitz, and it satisfies $H(x, d_x u) = c$ almost everywhere (this makes sense since the derivative of u exists almost everywhere by Rademacher's theorem).

This is too general because it gives too many solutions. The notion of a weak solution is useful if it gives a unique or, at least, a small number of solutions. This is not satisfied by this notion of a very weak solution, as can be seen in the following example.

Example 2.2. We suppose that $M = \mathbb{R}$, so $T^*M = \mathbb{R} \times \mathbb{R}$, and we take $H(x,p) = p^2 - 1$. Then any continuous piecewise $\mathcal{C}^1$-function u with derivative taking only the values ± 1 is a very weak solution of $H(x, d_x u) = 0$. This is already too large, but there are even weaker solutions. In fact, if A is any measurable subset of $\mathbb{R}$, then the function

$$f_A(x) = \int_0^x 2\chi_A(t) - 1\,\mathrm{d}t,$$

where χ_A is the characteristic function of A, is Lipschitz with derivative ± 1 almost everywhere.

Therefore, we have to define a more stringent notion of solutions. Crandall and Lions introduced the notion of viscosity solutions (see [6, 7]).

Definition 2.3 (viscosity solution). A function $u\colon V \to \mathbb{R}$ is a viscosity *subsolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$ if, for every $\mathcal{C}^1$-function $\phi\colon V \to \mathbb{R}$ and every point $x_0 \in V$ such that $u - \phi$ has a *maximum* at x_0, we have $H(x_0, d_{x_0}\phi) \leqslant c$.

A function $u\colon V \to \mathbb{R}$ is a viscosity *supersolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$ if, for every $\mathcal{C}^1$-function $\psi\colon V \to \mathbb{R}$ and every point $y_0 \in V$ such that $u - \psi$ has a *minimum* at y_0, we have $H(y_0, d_{y_0}\psi) \geqslant c$.

A function $u\colon V \to \mathbb{R}$ is a viscosity *solution* of $H(x, d_x u) = c$ on the open subset $V \subset M$ if it is *both* a subsolution and a supersolution.

This definition is reminiscent of the definition of distributions: since we cannot restrict to differentiable functions, we use *test* functions (namely, ϕ or ψ) which are smooth and on which we can test the condition. We first see that this is indeed a generalization of classical solutions.

Theorem 2.4. *A $\mathcal{C}^1$-function $u\colon V \to \mathbb{R}$ is a viscosity solution of $H(x, d_x u) = c$ on V if and only if it is a classical solution.*

In fact, the $\mathcal{C}^1$-function u is a viscosity subsolution (respectively, supersolution) of $H(x, d_x u) = c$ on V if and only if $H(x, d_x u) \leqslant c$ (respectively, $H(x, d_x u) \geqslant c$) for each $x \in V$.

Proof. We will prove the statement about the subsolution case. Suppose that the $\mathcal{C}^1$-function u is a viscosity subsolution. Since u is $\mathcal{C}^1$, we can use it as a test function. But $u - u = 0$, therefore every $x \in V$ is a maximum, hence $H(x, d_x u) \leqslant c$ for each $x \in V$.

Conversely, suppose $H(x, d_x u) \leqslant c$ for each $x \in V$. If $\phi\colon V \to \mathbb{R}$ is $\mathcal{C}^1$ and $u - \phi$ has a maximum at x_0, then the differentiable function $u - \phi$ must have derivative 0 at the maximum x_0. Therefore, $d_{x_0}\phi = d_{x_0}u$, and $H(x, d_{x_0}\phi) = H(x, d_{x_0}u) \leqslant c$. $\square$

To get a feeling for these viscosity notions, it is better to restate the definitions slightly. We first note that the condition imposed on the test functions (ϕ or ψ) in the definition above is on the derivative. Therefore, to check the condition, we can change our test function by a constant. Suppose now that ϕ (respectively, ψ) is $\mathcal{C}^1$ and $u - \phi$ (respectively, $u - \psi$) has a maximum (respectively, minimum) at x_0 (respectively, y_0). This means that $u(x_0) - \phi(x_0) \geqslant u(x) - \phi(x)$ (respectively, $u(y_0) - \psi(y_0) \leqslant u(x) - \psi(x)$). As we said, since we can add to ϕ (respectively, ψ)

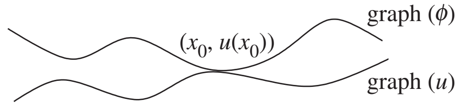

Figure 1. Subsolution: $\phi \geqslant u, u(x_0) = \phi(x_0) \Rightarrow H(x_0, d_{x_0}\phi) \leqslant c$.

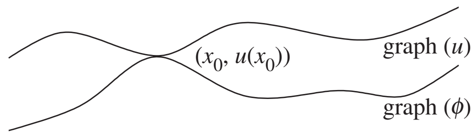

Figure 2. Supersolution: $\psi \leqslant u, u(x_0) = \psi(x_0) \Rightarrow H(x_0, d_{x_0}\psi) \geqslant c$.

the constant $u(x_0) - \phi(x_0)$ (respectively, $u(y_0) - \psi(y_0)$), these conditions can be replaced by $\phi \geqslant u$ (respectively, $\psi \leqslant u$) and $u(x_0) = \phi(x_0)$ (respectively, $u(y_0) = \psi(y_0)$). Therefore, we obtain an equivalent definition for the subsolution and the supersolution.

Definition 2.5 (viscosity solution). A function $u\colon V \to \mathbb{R}$ is a viscosity *subsolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$ if, for every $\mathcal{C}^1$-function $\phi\colon V \to \mathbb{R}$, with $\phi \geqslant u$ everywhere, at every point $x_0 \in V$ where $u(x_0) = \phi(x_0)$, we have $H(x_0, d_{x_0}\phi) \leqslant c$ (see figure 1).

A function $u\colon V \to \mathbb{R}$ is a viscosity *supersolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$ if, for every $\mathcal{C}^1$-function $\psi\colon V \to \mathbb{R}$, with $u \geqslant \psi$ everywhere, at every point $y_0 \in V$ where $u(y_0) = \psi(y_0)$, we have $H(y_0, d_{y_0}\psi) \geqslant c$ (see figure 2).

To see what the viscosity conditions mean, we test them on example 2.2.

Example 2.6. We suppose $M = \mathbb{R}$, so $T^*M = \mathbb{R}\times\mathbb{R}$, and we take $H(x, p) = p^2 - 1$. Any Lipschitz function $u\colon \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $\leqslant 1$ is in fact a viscosity subsolution of $H(x, d_x u) = 0$. To check this, consider ϕ a $\mathcal{C}^1$-function and $x_0 \in \mathbb{R}$ such that $\phi(x_0) = u(x_0)$ and $\phi(x) \geqslant u(x)$, for $x \in \mathbb{R}$. We can write

$$\phi(x) - \phi(x_0) \geqslant u(x) - u(x_0) \geqslant -|x - x_0|.$$

For $x > x_0$, this gives

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \geqslant -1,$$

thereby passing to the limit $\phi'(x_0) \geqslant -1$. On the other hand, if $x < x_0$, we obtain

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \leqslant 1,$$

hence $\phi'(x_0) \leqslant 1$. This yields $|\phi'(x_0)| \leqslant 1$ and, therefore,

$$H(x_0, \phi'(x_0)) = |\phi'(x_0)|^2 - 1 \leqslant 0.$$

So, in fact, any very weak subsolution (that is, a Lipschitz function u such that $H(x, d_x, u) \leqslant 0$ almost everywhere) is a viscosity subsolution. This is due to the fact that, in this example, the Hamiltonian is convex in p (see corollary 10.5).

Of course, the two smooth functions $x \mapsto x$ and $x \mapsto -x$ are the only two classical solutions in that example. It is easy to check that the absolute value

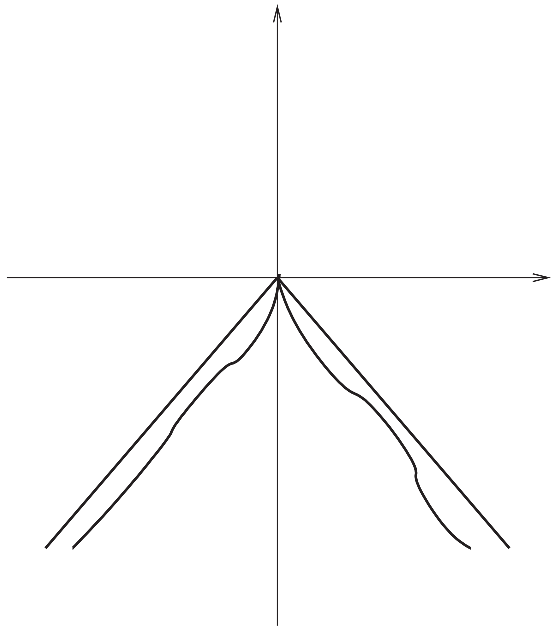

Figure 3. Graphs of $\psi(x) \leqslant -|x|$ with $\psi(0) = 0$.

function $x \mapsto |x|$, which is a subsolution and even a solution on $\mathbb{R} \setminus \{0\}$ (where it is smooth and a classical solution), is not a viscosity solution on the whole of $\mathbb{R}$. In fact the constant function equal to 0 is less than the absolute value everywhere with equality at 0, but we have $H(0, 0) = -1 < 0$, and this violates the supersolution condition.

The function $x \mapsto -|x|$ is a viscosity solution. It is smooth and a classical solution on $\mathbb{R} \setminus \{0\}$. It is a subsolution everywhere. Moreover, any function ϕ with $\phi(0) = 0$ and $\phi(x) \leqslant -|x|$ everywhere cannot be differentiable at 0. This is obvious from seeing the graphs (see figure 3).

Formally, it results from the fact that both $\phi(x)-x$ and $\phi(x)+x$ have a maximum at 0.

Exercise 1. Suppose that $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian on M. For $c \in \mathbb{R}$, define the Hamiltonian $H_c\colon M \to \mathbb{R}$ by

$$H_c(x, p) = H(x, p) - c.$$

Show that $u\colon M \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution, solution) of

$$H(x, d_x u) = c$$

if and only if it is a viscosity subsolution (respectively, supersolution, solution) of

$$H_c(x, d_x u) = 0.$$

Exercise 2. If we consider an open interval $I \subset \mathbb{R}$, then its cotangent space is canonically identified to $I \times \mathbb{R}$. We consider the Hamiltonian $H\colon I \times \mathbb{R} \to \mathbb{R}$ defined by $H(t,p) = p$. In this case, for $c \in \mathbb{R}$, the HJE $H(t, d_t u) = c$ can be written as

$$u'(t) = c.$$

(i) Show that $u\colon \mathbb{R} \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of $u'(t) = c$ if and only if $v(t) = u(t) - ct$ is a viscosity subsolution (respectively, supersolution) of $v'(t) = 0$.

(ii) Show that any non-increasing (respectively, non-decreasing) function $u\colon I \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of $u'(t) = 0$.

(iii) More generally, for $c \in \mathbb{R}$, show that any continuous function $\rho\colon \mathbb{R} \to \mathbb{R}$ such that $t \mapsto \rho(t) - ct$ is non-increasing is a subsolution of $u'(t) = c$.

(iv) Find the classical subsolutions, supersolutions and solutions of $u'(t) = c$.

Exercise 3. Suppose $H\colon T^*M \to \mathbb{R}$ is a Hamiltonian, and $\phi\colon M \to \mathbb{R}$ is a $\mathcal{C}^1$-function. Define the Hamiltonian $H_\phi\colon M \to \mathbb{R}$ by

$$H_\phi(x,p) = H(x, p + d_x\phi).$$

Next, show that v is a subsolution (respectively, supersolution, or solution) of $H_\phi(x, d_x v) = c$ if and only if $u = v + \phi$ is a subsolution (respectively, supersolution, or solution) of $H(x, d_x u) = c$.

Exercise 4. Suppose $H\colon T^*M \to \mathbb{R}$ is a Hamiltonian. Let $u\colon M \to \mathbb{R}$ be a continuous function, and let $c \in \mathbb{R}$ be a constant. We define $U\colon \mathbb{R} \times M \to \mathbb{R}$ by

$$U(x,t) = u(x) - ct.$$

(i) Show that if u is a subsolution (respectively, supersolution or solution) of the HJE

$$H(x, d_x u) = c, \tag{HJ}$$

then U is a viscosity subsolution (respectively, supersolution or solution) of the evolutionary HJE

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0 \tag{EHJ}$$

on $\mathbb{R} \times M$.

(ii) Conversely, if $a, b \in \mathbb{R}$ with $a < b$, and U is a viscosity subsolution (respectively, supersolution, or solution) of (EHJ) on $]a,b[\,\times M$, then u is a subsolution (respectively, supersolution or solution) of (HJ) on M.

3. Lower and upper differentials

We need to introduce the notion of lower and upper differentials.

Definition 3.1. If $u\colon M \to \mathbb{R}$ is a map defined on the manifold M, we say that the linear form $p \in T^*_{x_0}M$ is a lower (respectively, upper) differential of u at $x_0 \in M$ if we can find a neighbourhood V of x_0 and a function $\phi\colon V \to \mathbb{R}$, differentiable at x_0, with $\phi(x_0) = u(x_0)$ and $d_{x_0}\phi = p$, and such that $\phi(x) \leqslant u(x)$ (respectively, $\phi(x) \geqslant u(x)$), for every $x \in V$.

We denote by $D^- u(x_0)$ (respectively, $D^+ u(x_0)$) the set of lower (respectively, upper) differentials of u at x_0.

Exercise 5. Consider the function $u\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. For each $x \in \mathbb{R}$, find $D^- u(x)$ and $D^+ u(x)$. Do the same for $u(x) = -|x|$.

Definition 3.1 is not the one usually given for M an open set of a Euclidean space (see [2,3,5]). It is nevertheless equivalent to the usual definition, as we now show.

Proposition 3.2. *Let $u\colon U \to \mathbb{R}$ be a function defined on the open subset U of $\mathbb{R}^n$. Then the linear form p is in $D^- u(x_0)$ if and only if*

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \geqslant 0.$$

In the same way, $p \in D^+ u(x_0)$ if and only if

$$\limsup_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \leqslant 0.$$

Proof. Supposing $p \in D^- u(x_0)$, we can find a neighbourhood V of x_0 and a function $\phi\colon V \to \mathbb{R}$, differentiable at x_0, with $\phi(x_0) = u(x_0)$ and $d_{x_0}\phi = p$, and such that $\phi(x) \leqslant u(x)$ for every $x \in V$. Therefore, for $x \in V$, we can write

$$\frac{\phi(x) - \phi(x_0) - p(x - x_0)}{\|x - x_0\|} \leqslant \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|}.$$

Since $p = d_{x_0}\phi$, the left-hand side tends to 0 when $x \to x_0$. Therefore,

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \geqslant 0.$$

Conversely, suppose that $p \in \mathbb{R}^{n*}$ satisfies

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \geqslant 0.$$

We pick $r > 0$ such that the ball $\mathring{B}(x_0, r) \subset U$ and, for $h \in \mathbb{R}^n$ such that $0 < \|h\| < r$, we set

$$\epsilon(h) = \min\left(0, \frac{u(x_0 + h) - u(x_0) - p(h)}{\|h\|}\right).$$

It is easy to see that $\lim_{h \to 0} \epsilon(h) = 0$. We can therefore set $\epsilon(0) = 0$. The function $\phi\colon \mathring{B}(x_0, r) \to \mathbb{R}$ defined by $\phi(x) = u(x_0) + p(x - x_0) + \|x - x_0\|\epsilon(x - x_0)$ is differentiable at x_0, with derivative p, it is equal to u at x_0 and satisfies $\phi(x) \leqslant u(x)$ for every $x \in \mathring{B}(x_0, r)$. $\square$

Proposition 3.3. *Let $u\colon M \to \mathbb{R}$ be a function defined on the manifold M.*

(i) *For each x in M, we have $D^+ u(x) = -D^-(-u)(x) = \{-p \mid p \in D^-(-u)(x)\}$ and $D^- u(x) = -D^+(-u)(x)$.*

(ii) *For each x in M, both sets $D^+u(x)$, $D^-u(x)$ are closed convex subsets of T_x^*M.*

(iii) *If u is differentiable at x, then $D^+u(x) = D^-u(x) = \{d_x u\}$.*

(iv) *If both sets $D^+u(x)$, $D^-u(x)$ are non-empty, then u is differentiable at x.*

(v) *If $v\colon M \to \mathbb{R}$ is a function with $v \leqslant u$ and $v(x) = u(x)$, then $D^-v(x) \subset D^-u(x)$ and $D^+v(x) \supset D^+u(x)$.*

(vi) *If U is an open convex subset of a Euclidean space and $u\colon U \to \mathbb{R}$ is convex, then $D^-u(x)$ is the set of subdifferentials of u at $x \in U$. In particular, $D^+u(x) \neq \emptyset$ if and only if u is differentiable at x.*

(vii) *Suppose that d is the distance obtained from the Riemannian metric g on M. If $u\colon M \to \mathbb{R}$ is Lipschitz for d with Lipschitz constant* $\mathrm{Lip}(u)$*, then, for any $p \in D^\pm u(x)$, we have $\|p\|_x \leqslant \mathrm{Lip}(u)$.*

In particular, if M is compact, then the sets

$$D^\pm u = \{(x,p) \mid p \in D^\pm u(x), x \in M\}$$

are relatively compact in T^*M.

Proof. Part (i) and the convexity claim in part (ii) are obvious from definition 3.1.

To prove the fact that $D^+u(x_0)$ is closed for a given $x_0 \in M$, we can assume that M is an open subset of $\mathbb{R}^k$. We will apply proposition 3.2. If $p_n \in D^+u(x_0)$ converges to $p \in \mathbb{R}^{k*}$, we can write

$$\frac{u(x) - u(x_0) - p(x-x_0)}{\|x-x_0\|} \leqslant \frac{u(x) - u(x_0) - p_n(x-x_0)}{\|x-x_0\|} + \|p_n - p\|.$$

Fixing n, and letting $x \to x_0$, we obtain

$$\limsup_{x\to x_0} \frac{u(x) - u(x_0) - p(x-x_0)}{\|x-x_0\|} \leqslant \|p_n - p\|.$$

If we let $n \to \infty$, we see that $p \in D^+u(x_0)$.

We now prove (iii) and (iv) together. If u is differentiable at $x_0 \in M$, then obviously $d_{x_0}u \in D^+u(x_0) \cap D^-u(x_0)$. Supposing now that both $D^+u(x_0)$ and $D^-u(x_0)$ are not empty, pick $p_+ \in D^+u(x_0)$ and $p_- \in D^-u(x_0)$. For small h, we have

$$p_-(h) + \|h\|\epsilon_-(h) \leqslant u(x_0+h) - u(x_0) \leqslant p_+(h) + \|h\|\epsilon_+(h), \qquad (*)$$

where both $\epsilon_-(h)$ and $\epsilon_+(h)$ tend to 0, a $h \to 0$. If $v \in \mathbb{R}^n$, for sufficiently small $t > 0$, we can replace h by tv in the inequalities $(*)$ above. Forgetting the middle term and dividing by t, we obtain

$$p_-(v) + \|v\|\epsilon_-(tv) \leqslant p_+(v) + \|v\|\epsilon_+(tv).$$

Letting t tend to 0, we see that $p_-(v) \leqslant p_+(v)$ for every $v \in \mathbb{R}^n$. Replacing v by $-v$ gives the reverse inequality $p_+(v) \leqslant p_-(v)$, and therefore $p_- = p_+$. This implies that both $D^+u(x_0)$ and $D^-u(x_0)$ are reduced to the same singleton $\{p\}$. The inequality $(*)$ above now gives

$$p(h) + \|h\|\epsilon_-(h) \leqslant u(x_0+h) - u(x_0) \leqslant p(h) + \|h\|\epsilon_+(h),$$

which clearly implies that p is the derivative of u at x_0.

Part (v) follows routinely from the definition.

To prove (vi), we note that, by convexity, $u(x_0+th) \leqslant (1-t)u(x_0) + tu(x_0+h)$. Therefore, for $t>0$, we have

$$u(x_0+h) - u(x_0) \geqslant \frac{u(x_0+th) - u(x_0)}{t}.$$

If p is a linear form, for $t>0$ we obtain

$$\frac{u(x_0+h) - u(x_0) - p(h)}{\|h\|} \geqslant \frac{u(x_0+th) - u(x_0) - p(th)}{\|th\|}.$$

If $p \in D^-u(x_0)$, then the $\liminf$ as $t \to 0$ of the right-hand side is $\geqslant 0$. Therefore, $u(x_0+h) - u(x_0) - p(h) \geqslant 0$, which shows that p is a subdifferential. Conversely, a subdifferential is clearly a lower differential.

It remains to prove (vii). Suppose, for example, that $\phi\colon V \to \mathbb{R}$ is defined on some neighbourhood V of a given $x_0 \in M$, that it is differentiable at x_0, and that $\phi \geqslant u$ on V, with equality at x_0. If $v \in T_{x_0}M$ is given, we pick a $\mathcal{C}^1$ path $\gamma\colon [0,\delta] \to V$, with $\delta > 0, \gamma(0) = x_0$, and $\dot\gamma(0) = v$. We have, for all $t \in [0,\delta]$,

$$\begin{aligned}|u(\gamma(t)) - u(x_0)| &\leqslant \mathrm{Lip}(u) d(\gamma(t), x_0)\\ &\leqslant \mathrm{Lip}(u)\int_0^t \|\dot\gamma(s)\|\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$u(\gamma(t)) - u(x_0) \geqslant -\mathrm{Lip}(u)\int_0^t \|\dot\gamma(s)\|\,\mathrm{d}s.$$

Since $\phi \geqslant u$ on V, with equality at x_0, it follows that

$$\phi(\gamma(t)) - \phi(x_0) \geqslant -\mathrm{Lip}(u)\int_0^t \|\dot\gamma(s)\|\,\mathrm{d}s.$$

Dividing by $t>0$, and letting $t \to 0$, we get

$$d_{x_0}\phi(v) \geqslant -\mathrm{Lip}(u)\|v\|.$$

Since $v \in T_{x_0}M$ is arbitrary, we can change v into $-v$ in the inequality above to conclude that we also have

$$d_{x_0}\phi(v) \leqslant \mathrm{Lip}(u)\|v\|.$$

It then follows that $\|d_{x_0}\phi\| \leqslant \mathrm{Lip}(u)$. □

EXERCISE 6. Suppose V is an open subset of M, and $u\colon V \to \mathbb{R}$ is a continuous function.

(i) Show that we can find a $\mathcal{C}^\infty$-function $\phi\colon V \to \mathbb{R}$ such that $\phi \geqslant u$ (respectively, $\phi \leqslant u$) everywhere. (For an indication, pick a $\mathcal{C}^\infty$ partition of unity $(\varphi_i)_{i\in I}$ such that the support of each φ_i is compact, and consider c_i the maximum of u on the compact support of φ_i.)

(ii) Moreover, supposing that $\epsilon\colon V \to]0,+\infty[$ is a continuous function, show that one can find a $\mathcal{C}^\infty$-function $\phi\colon V \to \mathbb{R}$ such that $u \leqslant \phi \leqslant u + \epsilon$.

LEMMA 3.4. *If $u\colon M \to \mathbb{R}$ is continuous and $p \in D^+u(x_0)$ (respectively, $p \in D^-u(x_0)$), there exists a $\mathcal{C}^1$-function $\phi\colon M \to \mathbb{R}$, such that $\phi(x_0) = u(x_0)$, $d_{x_0}\phi = p$ and $\phi(x) > u(x)$ (respectively, $\phi(x) < u(x)$) for $x \neq x_0$.*

Moreover, if W is any neighbourhood of x_0 and $C > 0$, we can choose ϕ such that $\phi(x) \geqslant u(x) + C$, for $x \notin W$ (respectively, $\phi(x) \leqslant u(x) - C$).

Proof. Assume first that $M = \mathbb{R}^k$. To simplify notation, we can assume $x_0 = 0$. Moreover, subtracting from u the affine function $x \mapsto u(0) + p(x)$. We can assume $u(0) = 0$ and $p = 0$. The fact that $0 \in D^+u(0)$ gives

$$\limsup_{x\to 0} \frac{u(x)}{\|x\|} \leqslant 0.$$

If we take the non-negative part $u^+(x) = \max(u(x), 0)$ of u, this gives

$$\lim_{x\to 0} \frac{u^+(x)}{\|x\|} = 0. \qquad (\spadesuit)$$

If we set

$$c_n = \sup\{u^+(x) \mid 2^{-(n+1)} \leqslant \|x\| \leqslant 2^{-n}\},$$

then c_n is finite and $\geqslant 0$, because $u^+ \geqslant 0$ is continuous. Moreover, using that $2^n u_+(x) \leqslant u^+(x)/\|x\|$ for $\|x\| \leqslant 2^{-n}$, and the limit in ($\spadesuit$) above, we obtain

$$\lim_{n\to\infty} \Big[\sup_{m\geqslant n} 2^m c_m\Big] = 0. \qquad (\heartsuit)$$

We now consider $\theta\colon \mathbb{R}^k \to \mathbb{R}$ a $\mathcal{C}^\infty$ bump function with $\theta = 1$ on the set $\{x \in \mathbb{R}^k \mid \frac{1}{2} \leqslant \|x\| \leqslant 1\}$, and whose support is contained in $\{x \in \mathbb{R}^k \mid \frac{1}{4} \leqslant \|x\| \leqslant 2\}$. We define the function $\psi\colon \mathbb{R}^k \to \mathbb{R}$ by

$$\psi(x) = \sum_{n\in\mathbb{Z}} (c_n + 2^{-2n})\theta(2^n x).$$

This function is well defined at 0 because every term is then 0. For $x \neq 0$ we have $\theta(2^n x) \neq 0$ only if $\frac{1}{4} < \|2^n x\| < 2$. Taking the logarithm in base 2, we see that this can only happen if $-2 - \log_2 \|x\| < n < 1 - \log_2 \|x\|$. Therefore, this can happen for at most three consecutive integers n, hence the sum is also well defined for $x \neq 0$. Moreover, if $x \neq 0$, the set

$$V_x = \{y \neq 0 \mid -1 - \log_2 \|x\| < -\log_2 \|y\| < 1 - \log_2 \|x\|\}$$

is a neighbourhood of x and, for all $y \in V_y$,

$$\psi(y) = \sum_{-3-\log_2 \|x\| < n < 2-\log_2 \|x\|} (c_n + 2^{-2n})\theta(2^n y). \qquad (*)$$

This sum is finite with at most five terms. Therefore, θ is C^∞ on $\mathbb{R}^k \setminus \{0\}$.

We now check that ψ is continuous at 0. Using equation $(*)$ and the limit $(\heartsuit)$, we see that

$$\begin{aligned} 0 \leqslant \psi(x) &\leqslant \sum_{-3-\log_2 \|x\| < n < 2-\log_2 \|x\|} (c_n + 2^{-2n}) \\ &\leqslant 5 \sup_{n > -3-\log_2 \|x\|} (c_n + 2^{-2n}) \to 0 \quad \text{as } x \to 0. \end{aligned}$$

To show that ψ is $\mathcal{C}^1$ on the whole of $\mathbb{R}^k$ with derivative 0 at 0, it suffices to show that $d_x\psi$ tends to 0 as $\|x\| \to 0$. Differentiating equation $(*)$, we see that

$$d_x\psi = \sum_{-3-\log_2 \|x\| < n < 2-\log_2 \|x\|} (c_n + 2^{-2n})2^n d_{2^n x}\theta.$$

Since θ has compact support, $K = \sup_{x \in \mathbb{R}^n} \|d_x\theta\|$ is finite. The equality above and the limit in $(\heartsuit)$ give

$$\|d_x\psi\| \leqslant 5K \sup\{2^n c_n + 2^{-n} \mid n > -3 - \log_2 \|x\|\},$$

but the right-hand side goes to 0 when $\|x\| \to 0$.

We now show $\psi(x) > u(x)$ for $x \neq 0$. There is an integer n_0 such that $\|x\| \in [2^{-n_0+1}, 2^{-n_0}]$, hence $\theta(2^{n_0}x) = 1$ and $\psi(x) \geqslant \theta(2^{n_0}x)(c_{n_0} + 2^{-2n_0}) \geqslant c_{n_0} + 2^{-2n_0}$. Since

$$c_{n_0} = \sup\{u^+(y) \mid \|y\| \in [2^{(-n_0+1)}, 2^{-n_0}]\},$$

we obtain $c_{n_0} \geqslant u^+(x)$ and therefore $\psi(x) > u^+(x) \geqslant u(x)$.

It remains to show that we can get rid of the assumption $M = \mathbb{R}^k$, and to show how to obtain the desired inequality on the complement of W. We pick a small open neighbourhood $U \subset W$ of x_0 which is diffeomorphic to a Euclidean space. Following what we have already done, we can find a $\mathcal{C}^1$-function $\psi\colon U \to \mathbb{R}$ with $\psi(x_0) = u(x_0)$, $d_{x_0}\psi = p$, and $\psi(x) > u(x)$, for $x \in U \setminus \{x_0\}$. We then take a $\mathcal{C}^\infty$ bump function $\varphi\colon M \to [0,1]$ which is equal to 1 on a neighbourhood of x_0 and has compact support contained in $U \subset W$. By exercise 6, we can find a $\mathcal{C}^\infty$-function $\tilde{\psi}\colon M \to \mathbb{R}$ such that $\tilde{\psi} \geqslant u + C$. It is easy to check that the function $\phi\colon M \to \mathbb{R}$ defined by $\phi(x) = (1 - \varphi(x))\tilde{\psi}(x) + \varphi(x)\psi(x)$ has the required property. □

The following simple lemma is very useful.

Lemma 3.5. *Suppose $\psi\colon M \to \mathbb{R}$ is $\mathcal{C}^r$, with $r \geqslant 0$. If $x_0 \in M$, $C \geqslant 0$, and W is a neighbourhood of x_0, there exist two $\mathcal{C}^r$-functions $\psi_+, \psi_-\colon M \to \mathbb{R}$, such that $\psi_+(x_0) = \psi_-(x_0) = \psi(x_0)$ and $\psi_+(x) > \psi(x) > \psi_-(x)$ for $x \neq x_0$. Moreover, $\psi_+(x) - C > \psi(x) > \psi_-(x) + C$ for $x \notin W$. If $r \geqslant 1$, then necessarily $d_{x_0}\psi_+ = d_{x_0}\psi_- = d_{x_0}\tilde{\psi}$.*

Proof. The last fact is clear since $\psi_+ - \psi$ (respectively, $\psi_- - \psi$) achieves a minimum (respectively, maximum) at x_0.

Using the same arguments as at the end of the proof in the previous lemma to obtain the general case, it suffices to assume $C = 0$ and $M = \mathbb{R}^n$. In that case, we can take $\psi_\pm(x) = \psi(x) \pm \|x - x_0\|^2$. □

4. Criteria for viscosity solutions

In this section we fix a continuous function $H\colon T^*M \to \mathbb{R}$.

THEOREM 4.1. *Let $u\colon M \to \mathbb{R}$ be a continuous function.*

(i) *u is a viscosity subsolution of $H(x, d_x u) = 0$ if and only if, for each $x \in M$, and each $p \in D^+u(x)$, we have $H(x,p) \leqslant 0$.*

(ii) *u is a viscosity supersolution of $H(x, d_x u) = 0$ if and only if, for each $x \in M$, and each $p \in D^-u(x)$, we have $H(x,p) \geqslant 0$.*

Proof. Suppose that u is a viscosity subsolution. If $p \in D^+u(x)$, since u is continuous, it follows from lemma 3.4 that there exists a $\mathcal{C}^1$-function $\phi\colon M \to \mathbb{R}$, with $\phi \geqslant u$ on M, $u(x) = \phi(x)$ and $d_x\phi = p$. By the viscosity subsolution condition $H(x,p) = H(x, d_x\phi) \leqslant 0$.

Suppose conversely that, for each $x \in M$ and each $p \in D^+u(x_0)$, we have $H(x,p) \leqslant 0$. If $\phi\colon M \to \mathbb{R}$ is $\mathcal{C}^1$ with $u \leqslant \phi$, then at each point x where $u(x) = \phi(x)$, we have $d_x\phi \in D^+u(x)$ and therefore $H(x, d_x\phi) \leqslant 0$. □

Since $D^\pm u(x)$ depends only on the values of u in a neighbourhood of x, the following corollary is now obvious. It shows the local nature of the viscosity conditions.

COROLLARY 4.2. *Let $u\colon M \to \mathbb{R}$ be a continuous function. Then, if u is a viscosity subsolution (respectively, supersolution, solution) of $H(x, d_x u) = 0$ on M, then any restriction $u|_U$ to an open subset $U \subset M$ is itself a viscosity subsolution (respectively, supersolution, solution) of $H(x, d_x u) = 0$ on U.*

Conversely, if there exists an open cover $(U_i)_{i\in I}$ of M such that every restriction $u|_{U_i}$ is a viscosity subsolution (respectively, supersolution, solution) of $H(x, d_x u) = 0$ on U_i, then u itself is a viscosity subsolution (respectively, supersolution, solution) of $H(x, d_x u) = 0$ on M.

Since, by Rademacher's theorem, a Lipschitz function is differentiable almost everywhere, here is another straightforward consequence of theorem 4.1.

COROLLARY 4.3. *Let $u\colon M \to \mathbb{R}$ be a locally Lipschitz function. If u is a viscosity subsolution (respectively, supersolution, solution) of $H(x, d_x u) = 0$, then $H(x, d_x u) \leqslant 0$ (respectively, $H(x, d_x u) \geqslant 0, H(x, d_x u) = 0$) for almost every $x \in M$.*

In particular, a locally Lipschitz viscosity solution is always a very weak solution.

EXERCISE 7. Let $I \subset \mathbb{R}$, and consider $u\colon I \to \mathbb{R}$ a viscosity subsolution of

$$u'(t) = 0.$$

We want to show that u is non-increasing.

Fix $a < b$ with $a, b \in I$. For every $\epsilon > 0$, consider the function $\theta_\epsilon \colon [a, b[\to \mathbb{R}$ defined by

$$\theta_\epsilon(t) = \frac{\epsilon}{b - t}.$$

(i) Show that $u - \theta_\epsilon$ cannot have a local maximum in the open interval $]a, b[$.

(ii) Show that $u(t) \leqslant u(a) + \theta_\epsilon(t) - \theta_\epsilon(a)$, for every $t \in [a, b[$. Conclude that u is non-increasing.

(iii) What are the supersolutions (respectively, solutions) of $u'(t) = 0$?

(iv) For $c \in \mathbb{R}$, characterize the viscosity subsolutions, supersolutions and solutions of $u'(t) = c$.

We end this section with one more characterization of viscosity solutions.

Proposition 4.4 (criterion for viscosity solutions). *Suppose that $u \colon M \to \mathbb{R}$ is continuous. To check that u is a viscosity subsolution (respectively, supersolution) of $H(x, d_x u) = 0$, it suffices to show that, for each $\mathcal{C}^\infty$-function $\phi \colon M \to \mathbb{R}$ such that $u - \phi$ has a unique strict global maximum (respectively, minimum) attained at x_0, we have $H(x_0, d_{x_0}\phi) \leqslant 0$ (respectively, $H(x_0, d_{x_0}\phi) \geqslant 0$).*

Proof. We treat the subsolution case. We first show that if $\phi \colon M \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function such that $u - \phi$ achieves a (not necessarily strict) maximum at x_0, then we have $H(x_0, d_{x_0}\phi) \leqslant 0$. In fact, applying lemma 3.5, we can find a $\mathcal{C}^\infty$-function $\phi_+ \colon M \to \mathbb{R}$ such that $\phi_+(x_0) = \phi(x_0)$, $d_{x_0}\phi_+ = d_{x_0}\phi$, $\phi_+(x) > \phi(x)$, for $x \neq x_0$. The function $u - \phi_+$ has a unique strict global maximum achieved at x_0, and therefore $H(x_0, d_{x_0}\phi_+) \leqslant 0$. Since $d_{x_0}\phi_+ = d_{x_0}\phi$, this finishes our claim.

Supposing now that $\psi \colon M \to \mathbb{R}$ is $\mathcal{C}^1$ and that $u - \psi$ has a global maximum at x_0, we must show that $H(x_0, d_{x_0}\psi) \leqslant 0$. We fix a *relatively compact* open neighbourhood W of x_0. By lemma 3.5, applied to the continuous function ψ, there exists a $\mathcal{C}^1$-function $\psi_+ \colon M \to \mathbb{R}$ such that $\psi_+(x_0) = \psi(x_0), d_{x_0}\psi_+ = d_{x_0}\psi$, $\psi_+(x) > \psi(x)$ for $x \neq x_0$, and even $\psi_+(x) > \psi(x) + 3$, for $x \notin W$. It is easy to see that $u - \psi_+$ has a strict global maximum at x_0, and that $u(x) - \psi_+(x) < u(x_0) - \psi_+(x_0) - 3$, for $x \notin W$. By smooth approximations, we can find a sequence of $\mathcal{C}^\infty$-functions $\phi_n \colon M \to \mathbb{R}$ such that ϕ_n converges to ψ_+ in the $\mathcal{C}^1$ topology uniformly on compact subsets, and $\sup_{x \in M} |\phi_n(x) - \psi_+(x)| < 1$. This last condition, together with $u(x) - \psi_+(x) < u(x_0) - \psi_+(x_0) - 3$, for $x \notin W$, gives

$$u(x) - \phi_n(x) < u(x_0) - \phi_n(x_0) - 1 \quad \text{for } x \notin W.$$

This implies that the maximum of $u - \phi_n$ on the compact set $\bar{W}$ is a global maximum of $u - \phi_n$. Choose $y_n \in \bar{W}$ where $u - \phi_n$ attains its global maximum. Since ϕ_n is $\mathcal{C}^\infty$, from the first part of the proof we must have $H(y_n, d_{y_n}\phi_n) \leqslant 0$. Extracting a subsequence if necessary, we can assume that y_n converges to $y_\infty \in \bar{W}$. Since ϕ_n converges to ψ_+ uniformly on the compact set $\bar{W}$, $u - \psi_+$ necessarily achieves its maximum on $\bar{W}$ at y_∞. This implies that $y_\infty = x_0$, because the strict global maximum of $u - \tilde{\psi}$ is precisely attained at $x_0 \in W$. Since the convergence of ϕ_n to ψ_+ is in the $\mathcal{C}^1$ topology, we have $(y_n, d_{y_n}\phi_n) \to (x_0, d_{x_0}\psi_+)$. Hence, $H(y_n, d_{y_n}\phi_n) \to H(x_0, d_{x_0}\psi_+)$, by continuity of H. But using $H(y_n, d_{y_n}\phi_n) \leqslant 0$ and $d_{x_0}\psi = d_{x_0}\psi_+$, we get $H(x_0, d_{x_0}\psi) \leqslant 0$. □

5. Coercive Hamiltonians

DEFINITION 5.1 (coercive). A continuous function $H\colon T^*M \to \mathbb{R}$ is said to be coercive above every compact subset if, for each compact subset $K \subset M$ and each $c \in \mathbb{R}$, the set $\{(x,p) \in T^*M \mid x \in K, H(x,p) \leqslant c\}$ is compact.

Choosing any Riemannian metric on M, it is not difficult to see that H is coercive if and only if, for each compact subset $K \subset M$, we have $\lim_{\|p\|_x \to \infty} H(x,p) = +\infty$, the limit being uniform in $x \in K$.

We recall the definition of a locally Lipschitz function on a manifold. A map $f\colon X \to Y$ between metric spaces is locally Lipschitz if each point of X has a neighbourhood on which the function f is Lipschitz. If X is locally compact, it is equivalent to saying that f is Lipschitz on each compact subset of X. If either X or Y are open subsets of a Euclidean space, we will always assume that they are endowed with the Euclidean distance. It is then not difficult to show that $\mathcal{C}^1$ maps between open subsets of Euclidean spaces are locally Lipschitz. Also, a composition of locally Lipschitz maps is locally Lipschitz. If $f\colon M \to N$ is a map between the smooth manifolds, and we assume that the distances on M and N come from Riemannian metrics, then f is locally Lipschitz if and only if f is locally Lipschitz in local coordinates.

THEOREM 5.2. *Suppose that $H\colon T^*M \to \mathbb{R}$ is coercive above every compact subset, and $c \in \mathbb{R}$. Then a viscosity subsolution of $H(x, d_x u) = c$ is necessarily locally Lipschitz, and therefore satisfies $H(x, d_x u) \leqslant c$ almost everywhere.*

Proof. Since this is a local result we can assume $M = \mathbb{R}^k$, and prove only that u is Lipschitz on a neighbourhood of the origin 0. We will consider the usual distance d given by $d(x,y) = \|y - x\|$, where we have chosen the usual Euclidean norm on $\mathbb{R}^k$. We set

$$\ell_0 = \sup\{\|p\| \mid p \in \mathbb{R}^{k*},\ \exists x \in \mathbb{R}^k, \|x\| \leqslant 3, H(x,p) \leqslant c\}.$$

We have $\ell_0 < +\infty$ by the coercivity condition. Suppose $u\colon \mathbb{R}^k \to \mathbb{R}$ is a subsolution of $H(x, d_x u) = c$. Choose $\ell \geqslant \ell_0 + 1$ such that

$$2\ell > \sup\{|u(y) - u(x)| \mid x, y \in \mathbb{R}^k, \|x\| \leqslant 3, \|y\| \leqslant 3\}.$$

Fix $x \in \mathbb{R}^k$ with $\|x\| \leqslant 1$, and define $\phi\colon \mathbb{R}^k \to \mathbb{R}$ by $\phi(y) = \ell\|y - x\|$. Pick $y_0 \in \bar{B}(x,2)$ where the function $y \mapsto u(y) - \phi(y)$ attains its maximum for $y \in \bar{B}(x,2)$. We first observe that y_0 is not on the boundary of $\bar{B}(x,2)$. In fact, if $\|y - x\| = 2$, we have

$$u(y) - \phi(y) = u(y) - 2\ell < u(x) = u(x) - \phi(x).$$

In particular, the point y_0 is a local maximum of $u - \phi$. If y_0 is not equal to x, then $d_{y_0}\phi$ exists, with $d_{y_0}\phi(v) = \ell\langle y_0 - x, v\rangle / \|y_0 - x\|$, and we obtain $\|d_{y_0}\phi\| = \ell$. On the other hand, since $u(y) \leqslant u(y_0) - \phi(y_0) + \phi(y)$, for y in a neighbourhood of y_0, we get $d_{y_0}\phi \in D^+u(y_0)$, and therefore have $H(y_0, d_{y_0}\phi) \leqslant c$. By the choice of ℓ_0, this gives $\|d_{y_0}\phi\| \leqslant \ell_0 < \ell_0 + 1 \leqslant \ell$. This contradiction shows that $y_0 = x$, hence $u(y) - \ell\|y - x\| \leqslant u(x)$, for every x of norm $\leqslant 1$, and every $y \in \bar{B}(x,2)$. This implies that u has Lipschitz constant $\leqslant \ell$ on the unit ball of $\mathbb{R}^k$. $\square$

Let us recall now the locally equi-Lipschitzian concept for a family $\mathcal{F}$ of functions of real-valued functions defined on a manifold M. We assume that M is endowed with a distance d coming from a Riemannian metric on M. The family $\mathcal{F}$ is locally equi-Lipschitzian if, for every $x \in M$, we can find a neighbourhood V of x in M and a constant K such that, for every $y, z \in V$, and every $u \in \mathcal{F}$, we have $|u(y) - u(z)| \leqslant K d(y, z)$. It is not difficult to check that this notion is independent of the choice of Riemannian metric on M. Therefore, $\mathcal{F}$ is locally equi-Lipschitzian if and only if, in local coordinates, $\mathcal{F}$ is equi-Lipschitzian in the usual sense.

We will also need the following characterization of the Lipschitz constant for a function. We leave the proof as an exercise.

Exercise 8. Suppose that $v\colon U \to \mathbb{R}$ is a locally Lipschitz function defined on the open convex subset U of a Euclidean space. If $K < +\infty$, then the following two statements are equivalent:

(i) the Lipschitz constant of v is $\leqslant K$ on U, i.e. for every $x, y \in U$, we have $|v(y) - v(x)| \leqslant \|y - x\|$;

(ii) for almost every $x \in U$, we have $\|d_x u\| \leqslant K$.

(If (i) is true, then (ii) is true at every x where $d_x v$ exists. To prove that (ii) implies (i), prove it first for v is $\mathcal{C}^1$, then use an approximation argument as in lemma 10.3 to conclude in the general Lipschitz case.)

Corollary 5.3. *Let $H\colon T^*M \to \mathbb{R}$ be coercive above every compact subset. For every $c \in \mathbb{R}$, the set $\mathcal{S}(H, c)$ of global viscosity subsolutions*

$$u\colon M \to \mathbb{R} \quad \text{of } H(x, d_x u) = c$$

is locally equi-Lipschitzian.

Proof. The result is essentially local. Using a chart, we can assume $M = \mathbb{R}^n$, and $x = 0$. Denoting, as usual, by $\mathbb{B}$ the Euclidean unit ball, by coercivity of H, the set

$$S = \{(y, p) \in \mathbb{B} \times \mathbb{R}^n \mid H(y, p) \leqslant c\}$$

is compact. Therefore, $K = \sup\{\|p\| \mid (y, p) \in S\}$ is finite. By theorem 5.2 we know that any viscosity $u\colon \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and satisfies $H(y, d_y u) \leqslant c$, for almost every y. By the choice of K, we have $\|d_y u\| \leqslant K$, for almost every $y \in \mathbb{B}$. It now suffices to apply exercise 8 to conclude that u is Lipschitz on $\mathring{\mathbb{B}}$ with Lipschitz constant $\leqslant K$. □

It is important to note that, for the evolutionary Hamilton–Jacobi equation, there are subsolutions which are not locally Lipschitz even if the coercive Hamiltonian is very simple.

Exercise 9. We consider the coercive Hamiltonian $H\colon T^*M \to \mathbb{R}$ defined by

$$H(x, p) = \tfrac{1}{2}\|p\|_x^2.$$

If $\rho\colon \mathbb{R} \to \mathbb{R}$ is a non-increasing function, show that $u(x, t) = \rho(t)$ is a viscosity subsolution of

$$\frac{\partial u}{\partial t}(x, t) + H\left(x, \frac{\partial u}{\partial x}(x, t)\right) = 0.$$

Give an example of such a ρ which is not locally Lipschitz.

6. Stability

THEOREM 6.1 (stability). *Suppose first that the sequence of continuous functions $H_n\colon T^*M \to \mathbb{R}$ converges uniformly on compact subsets to $H\colon T^*M \to \mathbb{R}$. Suppose also that $u_n\colon M \to \mathbb{R}$ is a sequence of continuous functions converging uniformly on compact subsets to $u\colon M \to \mathbb{R}$. If, for each n, the function u_n is a viscosity subsolution (respectively, supersolution, solution) of $H_n(x, d_x u_n) = 0$, then u is a viscosity subsolution (respectively, supersolution, solution) of $H(x, d_x u) = 0$.*

Proof. We show the subsolution case. We use the criterion 4.4. Suppose first that $\phi\colon M \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function such that $u - \phi$ has a unique strict global maximum, achieved at x_0, we have to show $H(x_0, d_{x_0}\phi) \leqslant 0$. We pick a relatively compact open neighbourhood W of x_0. For each n, choose $y_n \in \bar{W}$ where $u_n - \phi$ attains its maximum on the compact subset $\bar{W}$. Extracting a subsequence, if necessary, we can assume that y_n converges to $y_\infty \in \bar{W}$. Since u_n converges to u uniformly on the compact set $\bar{W}$, necessarily $u - \phi$ achieves its maximum on $\bar{W}$ at y_∞. But $u - \phi$ has a strict global maximum at $x_0 \in W$, and therefore $y_\infty = x_0$. By continuity of the derivative of ϕ, we obtain $(y_n, d_{y_n}\phi) \to (x_0, d_{x_0}\phi)$. Since W is an open neighbourhood of x_0, dropping the first terms if necessary, we can assume $y_n \in W$, this implies that y_n is a local maximum of $u_n - \phi$, and therefore $d_{y_n}\phi \in D^+u_n(y)$. Since u_n is a viscosity subsolution of $H_n(x, d_x u_n) = 0$, we get $H_n(y_n, d_{y_n}\phi) \leqslant 0$. The uniform convergence of H_n on compact subsets now implies

$$H(x_0, d_{x_0}\phi) = \lim_{n\to\infty} H_n(y_n, d_{y_n}\phi) \leqslant 0.$$

□

EXERCISE 10. We consider the Hamiltonian $H\colon T^*M \to \mathbb{R}$ on the manifold M. Suppose $U\colon [0, +\infty[\, \times M$ is a viscosity subsolution of the evolutionary HJE

$$\frac{\partial U}{\partial t}(t, x) + H\left(x, \frac{\partial U}{\partial x}(t, x)\right) = 0 \tag{EHJ}$$

on $]0, +\infty[\, \times M$.

(i) If $\rho\colon [0, +\infty[\, \to \mathbb{R}$ is a non-increasing $\mathcal{C}^1$-function, show that $U_\rho\colon [0, +\infty[\, \times M \to \mathbb{R}$ defined by

$$U_\rho(t, x) = U(x, t) + \rho(t)$$

is also a viscosity subsolution of (EHJ) on $]0, +\infty[\, \times M$.

(ii) If $\rho\colon [0, +\infty[\, \to \mathbb{R}$ is an arbitrary non-increasing continuous function, show that it can be uniformly approximated on compact subsets by $\mathcal{C}^\infty$ non-increasing functions. (Hint: use a convolution argument.)

(iii) Show that (i) remains true for arbitrary non-increasing continuous function $\rho\colon [0, +\infty[\, \to \mathbb{R}$.

(iv) Show that U can be uniformly approximated on compact subsets by viscosity subsolutions of (EHJ) which are not locally Lipschitz.

7. Uniqueness

Our goal here is to obtain some uniqueness results especially for the evolutionary HJE. These kind of results are usually obtained through a maximum principle. One of the difficulties is the fact that viscosity solutions are not smooth. There is an efficient tool that has been developed to deal with this problem, namely, the doubling (of variables) argument. It has been extensively used since the beginning of the subject (see, for example, [2, ch. 2, § 3] and [3, ch. 2, §§ 2.4, 2.5]). In our treatment, we found it convenient to use the doubling argument in the next theorem, and to deduce the maximum principles and the uniqueness theorems from this result.

THEOREM 7.1. *Let $H\colon T^*M \to \mathbb{R}$ be a Hamiltonian on the manifold M. Suppose that $u\colon M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_x u) = c_1$, and that $v\colon M \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_x v) = c_2$. Assume further that either u or v is locally Lipschitz on M. If $u - v$ has a local maximum, then necessarily $c_2 \leqslant c_1$.*

Proof. Call $x_0 \in M$ a point where $u-v$ achieves a local maximum. Changing u (or v) by adding an appropriate constant, we can assume that this local maximum of $u-v$ is 0. This means that $u \leqslant v$ in a neighbourhood of x_0, with equality at x_0. If both u and v were differentiable at x_0, we would have $d_{x_0}(u-v) = 0$. Therefore, $d_{x_0}u = d_{x_0}v$, and $c_2 \leqslant H(x_0, d_{x_0}v) = H(x_0, d_{x_0}u) \leqslant c_1$. Since we do not know that these derivatives exist, we must get around this difficulty. The following argument is known in viscosity theory as the doubling argument. The problem is essentially local around x_0. Hence, choosing a chart, we can assume $x_0 = 0$ and $M = \mathbb{R}^n$.

Call $\|\cdot\|$ the usual Euclidean norm in $\mathbb{R}^n$, and denote by $\mathbb{B}^n$ the usual unit ball in $\mathbb{R}^n$. We will also use the canonical identification $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. In this identification, the differential of a function is nothing but its gradient.

Since either u or v are locally Lipschitz on $M = \mathbb{R}^n$, and since $\mathbb{B}^n$ is a compact subset, we can assume that there exists a constant $K < +\infty$ such that either u or v is Lipschitz on $\mathbb{B}^n$ with Lipschitz constant K.

We know that $u \leqslant v$ with equality at 0. Also, $u\colon \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_x u) = c_1$, and $v\colon \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_x u) = c_2$. We want to show that $c_2 \leqslant c_1$. For $\ell \geqslant 1$, we set

$$m_\ell = \sup_{x,y\in\mathbb{B}^n} u(x) - v(y) - \|x\|^2 - \ell\|x-y\|^2. \tag{7.1 a}$$

Note that $m_\ell \geqslant 0$, since $u(0) = v(0)$. By the compactness of $\mathbb{B}^n$, we can find $x_\ell, y_\ell \in \mathbb{B}^n$ such that

$$0 \leqslant m_\ell = u(x_\ell) - v(y_\ell) - \|x_\ell\|^2 - \ell\|x_\ell - y_\ell\|^2. \tag{7.1 b}$$

By the compactness of $\mathbb{B}^n$, we have $A = \sup_{x,y\in\mathbb{B}^n} u(x) - v(y) < +\infty$. It follows that

$$0 \leqslant m_\ell \leqslant A - \ell\|x_\ell - y_\ell\|^2.$$

This implies that $\|x_\ell - y_\ell\|^2 \leqslant A/\ell$, hence $x_\ell - y_\ell \to 0$ when $\ell \to +\infty$. Again, by the compactness of $\mathbb{B}^n$, we can find an extracted subsequence such that x_{ℓ_i} converges to x_∞. Necessarily we also have $y_{\ell_i} \to x_\infty$. By inequality (7.1 b) above,

$u(x_\ell) - v(y_\ell) - \|x_\ell\|^2 \geqslant 0$. Passing to the limit we get $u(x_\infty) - v(y_\infty) - \|x_\infty\|^2 \geqslant 0$. Since $u \leqslant v$, we find that $x_\infty = 0$. Therefore, both x_{ℓ_i} and x_∞ converge to 0. In particular, for sufficiently large i, x_{ℓ_i} and y_{ℓ_i} are in $\mathring{\mathbb{B}}^n$. We can therefore drop some of the first ℓ_i and assume $x_{\ell_i}, y_{\ell_i} \in \mathring{\mathbb{B}}^n$ for all i.

It follows from (7.1 a) and (7.1 b) above that $u(x) - [v(y_{\ell_i}) + \|x\|^2 + \ell_i\|x - y_{\ell_i}\|^2]$ has a local maximum at x_{ℓ_i}. But the function $\varphi(x) = v(y_{\ell_i}) + \|x\|^2 + \ell_i\|x - y_{\ell_i}\|^2$ is $\mathcal{C}^\infty$ with gradient $2x + 2\ell_i(x - y_{\ell_i})$. Therefore, $2x_{\ell_i} + 2\ell_i(x_{\ell_i} - y_{\ell_i}) \in D^+u(x_{\ell_i})$, and using the fact that $u\colon \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_xu) = c_1$, we obtain

$$H(x_{\ell_i}, 2x_{\ell_i} + 2\ell_i(x_{\ell_i} - y_{\ell_i})) \leqslant c_1. \tag{7.1 c}$$

In the same way, we get that $v(y) - [u(x_{\ell_i}) - \|x_{\ell_i}\|^2 - \ell_i\|x_{\ell_i} - y\|^2]$ has a local minimum at y_{ℓ_i}. Therefore, $2\ell_i(x_{\ell_i} - y_{\ell_i}) \in D^-v(y_{\ell_i})$, and using that $v\colon \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_xu) = c_2$, we obtain

$$H(y_{\ell_i}, 2\ell_i(x_{\ell_i} - y_{\ell_i})) \geqslant c_2. \tag{7.1 d}$$

Since x_{ℓ_i}, y_{ℓ_i} are in $\mathbb{B}^n$, and either u or v has Lipschitz constant $\leqslant K$ on $\mathbb{B}^n$, using $2x_{\ell_i} + 2\ell_i(x_{\ell_i} - y_{\ell_i}) \in D^+u(x_{\ell_i}), 2\ell_i(x_{\ell_i} - y_{\ell_i}) \in D^-v(y_{\ell_i})$, from part (vi) of proposition 3.3 we obtain that either

$$\|2x_{\ell_i} + 2\ell_i(x_{\ell_i} - y_{\ell_i})\| \leqslant K \quad \text{or} \quad \|2\ell_i(x_{\ell_i} - y_{\ell_i})\| \leqslant K.$$

Since $x_{\ell_i} \in \mathbb{B}^n$, we conclude that $\|2\ell_i(x_{\ell_i} - y_{\ell_i})\| \leqslant K + 2$ for all i. Therefore, up to extraction, we assume that $2\ell_i(x_{\ell_i} - y_{\ell_i})$ converges to $p \in \mathbb{R}^n$. Since both x_{ℓ_i} and y_{ℓ_i} converge to 0, passing to the limit in (7.1 c) and (7.1 d), we get $c_2 \leqslant H(0, p) \leqslant c_1$. □

Corollary 7.2. *Let $H\colon T^*M \to \mathbb{R}$ be a Hamiltonian coercive above every compact subset of the manifold M. Suppose that $u\colon M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_xu) = c_1$, and $v\colon M \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_xv) = c_2$. If $u - v$ has a local maximum, then necessarily $c_2 \leqslant c_1$.*

Proof. In that case, theorem 5.2 implies that u is locally Lipschitz. Therefore, we can apply theorem 7.1. □

Corollary 7.3. *Suppose $H\colon T^*M \to \mathbb{R}$ is a coercive Hamiltonian on the compact manifold M. If there exists a viscosity subsolution of $H(x, d_xu) = c_1$ and a viscosity supersolution of $H(x, d_xu) = c_2$, then necessarily $c_2 \leqslant c_1$.*

In particular, there exists at most one c for which the HJE $H(x, d_xu) = c$ has a global viscosity solution $u\colon M \to \mathbb{R}$. This only possible value is the smallest c for which $H(x, d_xu) = c$ admits a global viscosity subsolution $u\colon M \to \mathbb{R}$.

Proof. Call $u\colon M \to \mathbb{R}$ a viscosity subsolution of $H(x, d_xu) = c_1$, and call $v\colon M \to \mathbb{R}$ a viscosity supersolution of $H(x, d_xv) = c_2$. By the compactness of M, we can find a point $x_0 \in M$ where $u - v$ achieves its maximum. Therefore, by corollary 7.2, we have $c_2 \leqslant c_1$. □

Theorem 7.4. *Let $H\colon M \to \mathbb{R}$ be a continuous Hamiltonian on the compact manifold M. Next, suppose $U, V\colon [0, +\infty[\, \times M \to \mathbb{R}$ are two continuous functions with*

$U(x,0) \leqslant V(x,0)$, for all $x \in M$. Assume that U (respectively, V) is a viscosity subsolution (respectively, supersolution) of the evolutionary HJE

$$\frac{\partial u}{\partial t}(t,x) + H\left(x, \frac{\partial u}{\partial x}(t,x)\right) = 0$$

on $]0,+\infty[\times M$. If either U or V is locally Lipschitz on $]0,+\infty[\times M$, then $U \leqslant V$ on the whole of $[0,+\infty[\times M$.

Proof. We introduce the Hamiltonian $\hat{H}$ on $\mathbb{R} \times M$ defined by

$$\hat{H}(t,x,s,p) = s + H(x,p),$$

where $(t,x) \in \mathbb{R} \times M$ and $(s,p) \in T^*_{(t,x)}(\mathbb{R} \times M) = \mathbb{R} \times T^*_x M$. With this notation, U (respectively, V) becomes a viscosity subsolution (respectively, supersolution) of the HJE

$$\hat{H}((t,x), d_{(t,x)}u) = 0.$$

Fixing $a, \epsilon > 0$, we will show that, for all $t \in [0,a[$ and for all $x \in M$,

$$U(t,x) + \frac{\epsilon}{t-a} \leqslant V. \tag{$\diamond$}$$

The theorem follows because we can let $\epsilon \to 0$, and $a > 0$ is arbitrary. To simplify notation, define $\rho\colon [0,a] \to \mathbb{R}$ by

$$\rho(t) = \frac{\epsilon}{t-a}.$$

Since $\rho'(t) = -\epsilon/(t-a)^2 \leqslant -\epsilon/a^2$, it is not difficult to see that the continuous function $\hat{U}\colon [0,a[\times M \to$ defined by

$$\hat{U}(t,x) = U(t,x) + \rho(t)$$

is a viscosity subsolution of the HJE

$$\hat{H}((t,x), d_{(t,x)}u) = -\frac{\epsilon}{a^2}.$$

Since ρ is $\mathcal{C}^\infty$, it follows from the hypothesis that either $\hat{U}$ or V is locally Lipschitz on $]0,a[\times M$. Since $-\epsilon/a^2 < 0$, we can apply theorem 7.1 to conclude that $\hat{U} - V$ has no local maximum on $]0,a[\times M$. But $\rho(t) \to -\infty$, as $t \to a$, hence, by the compactness of M, the continuous function $\hat{U} - V = U - V + \rho$ must attain its maximum in $[0,a[\times M$. This maximum can only be in $\{0\} \times M$. But $\hat{U} - V = U - V + \rho$, the function ρ is equal to $\rho(0) = -\epsilon/a$ on $\{0\} \times M$, and $U - V \leqslant 0$ on $\{0\} \times M$. Therefore, we obtain that $\hat{U} - V \leqslant -\epsilon/a \leqslant 0$ on $[0,a[\times M$. This is precisely the inequality ($\diamond$) that we are seeking. □

Corollary 7.5. *Let $H\colon M \to \mathbb{R}$ be a continuous Hamiltonian on the compact manifold M. Suppose that the continuous function $U : [0,+\infty[\times M \to \mathbb{R}$ is locally Lipschitz on $]0,+\infty[\times M$, and is a viscosity solution of the evolutionary HJE*

$$\frac{\partial u}{\partial t}(t,x) + H\left(x, \frac{\partial u}{\partial x}(t,x)\right) = 0, \tag{EHJ}$$

on $]0,+\infty[\times M$.

Any other continuous function $V\colon [0,+\infty[\times M \to \mathbb{R}$, which is a viscosity solution of (EHJ) on $]0,+\infty[\times M$ and coincides with U on $\{0\}\times M$, coincides with U on the whole of $[0,+\infty[\times M$.

8. Construction of viscosity solutions

In this section we will introduce the Perron method for constructing viscosity solutions.

PROPOSITION 8.1. *Let $H\colon T^*M \to \mathbb{R}$ be a continuous function. Suppose $(u_i)_{i\in I}$ is a family of continuous functions $u_i\colon M \to \mathbb{R}$ such that each u_i is a subsolution (respectively, supersolution) of $H(x, d_x u) = 0$. If $\sup_{i\in I} u_i$ (respectively, $\inf_{i\in u} u_i$) is finite and continuous everywhere, then it is also a subsolution (respectively, supersolution) of $H(x, d_x u) = 0$.*

Proof. Set $u = \sup_{i\in I} u_i$. Suppose $\phi\colon M \to \mathbb{R}$ is $\mathcal{C}^1$, with $\phi(x_0) = u(x_0)$ and $\phi(x) > u(x)$ for every $x \in M \setminus \{x_0\}$. We have to show $H(x_0, d_{x_0}\phi) \leqslant 0$. Fix some distance d on M. By continuity of the derivative of ϕ, it suffices to show that, for each sufficiently small $\epsilon > 0$, there exists $x \in \mathring{B}(x_0, \epsilon)$, with $H(x, d_x\phi) \leqslant 0$.

For $\epsilon > 0$ small enough, the closed ball $\bar{B}(x_0, \epsilon)$ is compact. Fix such an $\epsilon > 0$. There is a $\delta > 0$ such that $\phi(y) - \delta \geqslant u(y) = \sup_{i\in I} u_i(y)$ for each $y \in \partial B(x_0, \epsilon)$.

Since $\phi(x_0) = u(x_0)$, we can find $i_\epsilon \in I$ such that $\phi(x_0) - \delta < u_{i_\epsilon}(x_0)$. It follows that the maximum of the continuous function $u_{i_\epsilon} - \phi$ on the compact set $\bar{B}(x_0, \epsilon)$ is not attained on the boundary. Therefore, $u_{i_\epsilon} - \phi$ has a local maximum at some $x_\epsilon \in \mathring{B}(x_0, \epsilon)$. Since the function u_{i_ϵ} is a viscosity subsolution of $H(x, d_x u) = 0$, we have $H(x_\epsilon, d_{x_\epsilon}\phi) \leqslant 0$. □

THEOREM 8.2 (Perron method). *Suppose the Hamiltonian $H\colon TM \to \mathbb{R}$ is coercive above every compact subset. Assume that M is connected and that there exists a viscosity subsolution $u\colon M \to \mathbb{R}$ of $H(x, d_x u) = 0$. Then, for every $x_0 \in M$, the function $S_{x_0}\colon M \to \mathbb{R}$ defined by $S_{x_0}(x) = \sup_v v(x)$, where the supremum is taken over all viscosity subsolutions v satisfying $v(x_0) = 0$, has finite values and is a viscosity subsolution of $H(x, d_x u) = 0$ on M.*

Moreover, it is a viscosity solution of $H(x, d_x u) = 0$ on $M \setminus \{x_0\}$.

Proof. Call $\mathcal{SS}_{x_0}$ the family of viscosity subsolutions $v\colon M \to \mathbb{R}$ of $H(x, d_x v) = 0$ satisfying $v(x_0) = 0$.

Since H is coercive above every compact subset of M, by corollary 5.3, the family of restrictions $v|_K, v \in \mathcal{SS}_{x_0}$ is locally equi-Lipschitzian. We now show that S_{x_0} is finite everywhere. Since M is connected, given $x \in M$, there exists a compact connected set K_{x,x_0} containing both x and x_0. By the local equicontinuity of the family of restrictions $\{v|_{K_{x,x_0}} \mid v \in \mathcal{SS}_{x_0}\}$, and the compactness of K, we can find $\delta > 0$ such that, for each $y, z \in K_{x,x_0}$ with $d(y, z) \leqslant \delta$, we have $|v(y) - v(z)| \leqslant 1$ for each $v \in \mathcal{SS}_{x_0}$.

Since the set K_{x,x_0} is connected, we can find a sequence $x_0, x_1, \ldots, x_n = x$ with$d(x_i, x_{i+1}) \leqslant \delta$. It follows that

$$|v(x)| = |v(x) - v(x_0)| \leqslant \sum_{i=0}^{n-1} |v(x_{i+1}) - v(x_i)| \leqslant n$$

for each $v \in \mathcal{SS}_{x_0}$. Therefore, $\sup_{v\in\mathcal{SS}_{x_0}} v(x)$ is finite everywhere. Moreover, as a finite-valued supremum of a family of locally equicontinuous functions, it is continuous.

By proposition 8.1, the function S_{x_0} is a viscosity subsolution on M itself. It remains to show that it is a viscosity solution of $H(x, d_x u)$ on $M \setminus \{x_0\}$.

Suppose $\psi\colon M \to \mathbb{R}$ is $\mathcal{C}^1$ with $\psi(x_1) = S_{x_0}(x_1)$, where $x_1 \neq x_0$, and $\psi(x) < S_{x_0}(x)$ for every $x \neq x_1$. We want to show that necessarily $H(x_1, d_{x_1}\psi) \geqslant 0$. We argue by contradiction. We therefore suppose that $H(x_1, d_{x_1}\psi) < 0$. By continuity of the derivative of ψ, we have $H(y, d_y\psi) < 0$ for y in a neighbourhood V of x_1. Endowing M with a distance defining its topology, we choose $\epsilon > 0$ such that $\bar{B}(x_1, \epsilon) \subset V$. Since $H(y, d_y\psi) < 0$, for each $y \in \bar{B}(x_1, \epsilon)$, the function ψ is a viscosity subsolution of $H(x, d_x u) = 0$ on $\mathring{B}(x_1, \epsilon)$. Taking sufficiently small $\epsilon > 0$, we assume that $\bar{B}(x_1, \epsilon)$ is a compact subset of M, and $x_0 \notin \bar{B}(x_1, \epsilon)$. Since $\psi < S_{x_0}$ on the boundary $\partial B(x_1, \epsilon)$ of $\bar{B}(x_1, \epsilon)$, we can pick $\delta > 0$ such that $\psi(y) + \delta \leqslant S_{x_0}(y)$ for every $y \in \partial B(x_1, \epsilon)$. We now define $\tilde{S}_{x_0}\colon M \to \mathbb{R}$ by

$$\tilde{S}_{x_0}(x) = \begin{cases} S_{x_0}(x) & \text{on } M \setminus \bar{B}(x_1, \epsilon), \\ \max(\psi(x) + \frac{1}{2}\delta, S_{x_0}(x)) & \text{on } \bar{B}(x_1, \epsilon). \end{cases}$$

The function $\tilde{S}_{x_0}$ is a viscosity subsolution on $\mathring{B}(x_1, \epsilon)$ as the maximum of the two viscosity subsolutions $\psi + \frac{1}{2}\delta$ and S_{x_0}. Moreover, this function $\tilde{S}_{x_0}$ coincides with S_{x_0} outside $K = \{x \in \bar{B}(x_1, \epsilon) \mid \psi(x) + \frac{1}{2}\delta \geqslant S_{x_0}(x))\}$ which is a compact subset of $\mathring{B}(x_1, \epsilon)$. Therefore it is a viscosity subsolution on $M \setminus K$. It follows that $\tilde{S}_{x_0}$ is a viscosity subsolution of $H(x, d_x u)$ on M itself, since its restrictions to both open subsets $M \setminus K$ and $\mathring{B}(x_1, \epsilon)$ are viscosity subsolutions, and $M = \mathring{B}(x_1, \epsilon) \cup (M \setminus K)$.

But $\tilde{S}_{x_0}(x_0) = S_{x_0}(x_0) = 0$ because $x_0 \notin \bar{B}(x_1, \epsilon)$. Moreover,

$$\begin{aligned} \tilde{S}_{x_0}(x_1) &= \max(\psi(x_1) + \tfrac{1}{2}\delta, S_{x_0}(x_1)) \\ &= \max(S_{x_0}(x_1) + \tfrac{1}{2}\delta, S_{x_0}(x_1)) \\ &= S_{x_0}(x_1) + \tfrac{1}{2}\delta \\ &> S_{x_0}(x_1). \end{aligned}$$

This contradicts the definition of S_{x_0}. □

The next argument is inspired by the construction of Busemann functions in Riemannian geometry (see [1]).

Corollary 8.3. *Suppose that $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected* non-compact *manifold M. If there exists a viscosity subsolution of $H(x, d_x u) = 0$ on M, then there exists a viscosity* solution *on M.*

Proof. Fix $\hat{x} \in M$, and pick a sequence $x_n \to \infty$ (this means that each compact subset of M contains only a finite number of points in the sequence).

By arguments analogous to the ones used in the previous proof, the sequence S_{x_n} is locally equicontinuous. Moreover, for each $x \in M$, the sequence $S_{x_n}(x) - S_{x_n}(\hat{x})$ is bounded. Therefore, by Ascoli's theorem, extracting a subsequence if necessary, we can assume that $S_{x_n} - S_{x_n}(\hat{x})$ converges uniformly to a continuous function

$u\colon M \to \mathbb{R}$. It now suffices to show that the restriction of u to an arbitrary, open, relatively compact subset V of M is a viscosity solution of $H(x, d_x u) = 0$ on V. Since $\{n \mid x_n \in \bar{V}\}$ is finite, for sufficiently large n the restriction of $S_{x_n} - S_{x_n}(\hat{x})$ to V is a viscosity solution; therefore, by the stability theorem 6.1, the restriction of the limit u to V is also a viscosity solution. □

The situation is different for compact manifolds, as can be seen from corollary 7.3.

9. Strict subsolutions

Definition 9.1 (strict subsolution). Let $H\colon T^*M \to \mathbb{R}$ be a continuous function. We say that a viscosity subsolution $u\colon M \to \mathbb{R}$ of $H(x, d_x u) = c$ is strict at $x_0 \in M$ if there exists an open neighbourhood V_{x_0} of x_0, and $c_{x_0} < c$ such that $u \mid V_{x_0}$ is a viscosity subsolution of $H(x, d_x u) = c_{x_0}$ on V_{x_0}.

Here is a way to construct viscosity subsolutions that are strict at some point.

Proposition 9.2. *Let $H\colon T^*M \to \mathbb{R}$ be a continuous function. Now, suppose that $u\colon M \to \mathbb{R}$ is a viscosity subsolution of $H(y, d_y u) = c$ on M, which is also a viscosity solution on $M \setminus \{x\}$. If u is not a viscosity solution of $H(y, d_y u) = c$ on M itself, then there exists a viscosity subsolution of $H(y, d_y u) = c$ on M which is strict at x.*

Proof. The argument of the proof is very similar to the end of the proof of theorem 8.2. Assume that u is not a viscosity solution on the whole of M. Since it is a subsolution on M, it is the supersolution condition that is violated. Moreover, since u is a supersolution on $M \setminus \{x\}$, the only possibility is that there exists $\psi\colon M \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\psi(x) = u(x)$, $\psi(y) < u(y)$, for $y \neq x$, and $H(x, d_x\psi) < c$. By continuity of the derivative of ψ, we can find a compact ball $\bar{B}(x, r)$, with $r > 0$, and a $c_x < c$ such that $H(y, d_y\psi) < c_x$, for every $y \in \bar{B}(0, r)$. In particular, the $\mathcal{C}^1$-function ψ is a subsolution of $H(z, d_z v) = c_x$ on $\mathring{B}(x, r)$, and therefore also of $H(z, d_z v) = c$ on the same set since $c_x < c$.

We choose $\delta > 0$ such that, for every $y \in \partial B(x, r)$, we have $u(y) > \psi(y) + \delta$. This is possible since $\partial B(x, r)$ is a compact subset of $M \setminus \{x\}$, where we have the strict inequality $\psi < u$.

First, if we define $\tilde{u}\colon M \to \mathbb{R}$ by $\tilde{u}(y) = u(y)$ if $y \notin \bar{B}(x, r)$ and $\tilde{u}(y) = \max(u(y), \psi(y) + \delta)$, we obtain the desired viscosity subsolution of $H(y, d_y u) \leqslant c$ which is strict at x. In fact, by the choice of $\delta > 0$, the subset

$$K = \{y \in \bar{B}(x, r) \mid \psi(y) + \delta \geqslant u(y)\}$$

is compact and contained in the open ball $\mathring{B}(x, r)$. Therefore, M is covered by the two open subsets $M \setminus K$ and $\mathring{B}(x, r)$. On the first open subset $\tilde{u}$ is equal to u. It is therefore a subsolution of $H(y, d_y u) = c$ on that subset. On the second open subset $\mathring{B}(x, r)$, the function $\tilde{u}$ is the maximum of u and $\psi + \delta$ which are both subsolutions of $H(y, d_y u) = c$ on $\mathring{B}(x, r)$, by proposition 8.1, it is therefore a subsolution of $H(y, d_y u) = c$ on that second open subset. Since $u(x) = \psi(x)$; we have $\tilde{u}(x) = \psi(x) + \delta > u(x)$. Therefore, by continuity, $\tilde{u} = \psi + \delta$ on a neighbourhood $N \subset \mathring{B}(x, r)$ of x. On that neighbourhood, $H(y, d_y\psi) < c_x$, hence $\tilde{u}$ is strict at x. □

Here is another useful result on strict subsolutions.

PROPOSITION 9.3. *Let $H: T^*M \to \mathbb{R}$ be a continuous function. Now suppose that $u: M \to \mathbb{R}$ (respectively, $v: M \to \mathbb{R}$) is a viscosity subsolution (respectively, supersolution) of $H(y, d_y u) = c$ on M. Assume further that either u or v is locally Lipschitz. Then u cannot be strict at any local maximum of $u - v$.*

Proof. We argue by contradiction. Assume x_0 is a local maximum of $u - v$. If u was strict at x_0, we could find an open set V containing x_0, and a $c' < c$ such that $u \mid V$ is a viscosity subsolution of $H(x, d_x u) = c' < c$. But if we apply theorem 7.1 to the restrictions $u \mid V$ and $v \mid V$, we see that we must have $c \leqslant c'$, which contradicts the choice of c'. □

10. Quasi-convexity and viscosity subsolutions

We first recall the definition of a quasi-convex function.

DEFINITION 10.1. The function $f: C \to \mathbb{R}$, defined on the convex subset C of the real vector space E, is said to be quasi-convex if, for every $t \in \mathbb{R}$, the sublevel $\{x \in C \mid f(x) \leqslant t\}$ is convex.

EXERCISE 11. Suppose $f: C \to \mathbb{R}$ is defined on the convex subset C of the real vector space E.

(i) Show that f is quasi-convex if and only if, for every sequence $\alpha_1 \dots, \alpha_\ell \in [0,1]$ with $\sum_{i=1}^{\ell} \alpha_i = 1$, and every sequence $x_1, \dots, x_\ell \in C$, we have

$$f\left(\sum_{i=1}^{\ell} \alpha_i x_i\right) \leqslant \max_{i=1}^{\ell} f(x_i).$$

(ii) Suppose, moreover, that E is a topological vector space, and that f is continuous and quasi-convex. Show that, for any sequence $(\alpha_i)_{i \in \mathbb{N}}$ with $\alpha_i \in [0,1]$ such that $\sum_{i=0}^{\infty} \alpha_i = 1$, and every sequence $(x_i)_{i \in \mathbb{N}}$ such that $\sum_{i=0}^{\infty} \alpha_i x_i$ exists and is in C, we have

$$f\left(\sum_{i \in \mathbb{N}} \alpha_i x_i\right) \leqslant \sup_{i \in \mathbb{N}} f(x_i).$$

(iii) (Difficult.) Suppose further that E is a finite-dimensional vector space, and that the convex set C is Borel measurable. If μ is a Borel probability measure on E with $\mu(C) = 1$, show that $\int_E x \, \mathrm{d}\mu(x) \in C$. (Hint: one can assume that this is true for a vector space whose dimension is strictly lower than that of E, then argue by contradiction. If $x_0 = \int_E x \, \mathrm{d}\mu(x) \notin C$, by the Hahn–Banach theorem and the finite dimensionality of E, find a linear map $\theta: E \to \mathbb{R}$ such that $\theta(x) \leqslant \theta(x_0)$ for every $x \in \mathbb{C}$.)

(iv) If E is finite dimensional, show that (ii) remains true even when f is only assumed Borel measurable on the Borel measurable convex set C.

In this section we are mainly interested in Hamiltonians $H: T^*M \to \mathbb{R}$ that are quasi-convex in the fibres, i.e. for each $x \in M$, the function $p \mapsto H(x,p)$ is quasi-convex on the vector space T_x^*M.

Our first goal in this section is to prove the following theorem.

THEOREM 10.2. *Suppose that the continuous Hamiltonian $H: T^*M \to \mathbb{R}$ is quasi-convex in the fibres. If $u: M \to \mathbb{R}$ is locally Lipschitz and $H(x, d_x u) \leqslant c$ almost everywhere, for some fixed $c \in \mathbb{R}$, then u is a viscosity subsolution of $H(x, d_x u) = c$.*

Before giving the proof of the theorem we need some preliminary material.

If $u: U \to \mathbb{R}$ is a locally Lipschitz function defined on the open subset U of M, it is convenient to introduce the Hamiltonian constant $\mathbb{H}_U(u)$ as the essential supremum on U of $H(x, d_x u)$, i.e. the constant $\mathbb{H}_U(u)$ by

$$\mathbb{H}_U(u) = \inf\{c \in \mathbb{R} \cup \{+\infty\} \mid H(x, d_x u) \leqslant c \text{ for almost every } x \in U\}.$$

Using a sequence $c_n \searrow c[0]$ such that the set $S_n = \{x \in M \mid H(x, d_x u) > c_n\}$ is Lebesgue negligible in M, we obtain that $H(x, d_x u) \leqslant \mathbb{H}_U(u)$ outside of $S = \bigcup_n S_n$. Since S is also negligible – as a countable union of negligible sets – it follows that $H(x, d_x u) \leqslant \mathbb{H}_U(u)$ almost everywhere. Since H takes only finite values, we have $\mathbb{H}_U(u) > -\infty$.

We will use some classical facts about convolution. Let $(\rho_\delta)_{\delta>0}$ be a family of functions $\rho_\delta: \mathbb{R}^k \to [0, \infty[$ of class $\mathcal{C}^\infty$, with $\rho_\delta(x) = 0$, if $\|x\| \geqslant \delta$, and $\int_{\mathbb{R}^k} \rho_\delta(x)\,\mathrm{d}x = 1$. Suppose that V, U are open subsets of $\mathbb{R}^k$, with $\bar{V}$ compact and contained in U. Calling $2\delta_0$ the Euclidean distance of the compact set $\bar{V}$ to the boundary of U, we have $\delta_0 > 0$, and therefore the closed δ_0-neighbourhood

$$\bar{N}_{\delta_0}(\bar{V}) = \{y \in \mathbb{R}^k \mid \exists x \in \bar{V}, \|y - x\| \leqslant \delta_0\}$$

of $\bar{V}$ is compact and contained in U.

If $u: U \to \mathbb{R}$ is a continuous function, then, for $\delta < \delta_0$, the convolution

$$u_\delta(x) = \rho_\delta * u(x) = \int_{\mathbb{R}^k} \rho_\delta(y) u(x-y)\,\mathrm{d}y$$

makes sense and is of class $\mathcal{C}^\infty$ on a neighbourhood of $\bar{V}$. Moreover, the family u_δ converges uniformly on $\bar{V}$ to u, as $t \to 0$.

LEMMA 10.3. *Under the hypothesis above, suppose that $u: U \to \mathbb{R}$ is a locally Lipschitz function. Given any Hamiltonian $H: T^*U \to \mathbb{R}$ that is quasi-convex in the fibres and any $\epsilon > 0$, for every sufficiently small $\delta > 0$, we have*

$$\sup_{x \in V} |u_\delta(x) - u(x)| \leqslant \epsilon \quad \textit{and} \quad \mathbb{H}_V(u_\delta) \leqslant \mathbb{H}_U(u) + \epsilon.$$

Proof. Since u_δ converges uniformly to u on the compact subset $\bar{V}$, we only have to prove that $\mathbb{H}_V(u_\delta) \leqslant \mathbb{H}_U(u) + \epsilon$ for sufficiently small δ. If $\mathbb{H}_U(u) = +\infty$, this is clear. We can therefore assume that $\mathbb{H}_U(u) < +\infty$. We first show that, for $\delta < \delta_0$, we must have

$$d_x u_\delta = \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u\,\mathrm{d}y, \quad \forall x \in V. \qquad (*)$$

Note that the right-hand side makes sense because $d_z u$ exists for almost every $z \in U$. Since we know that u_δ is $\mathcal{C}^\infty$, it suffices to check that

$$\lim_{t \to 0} \frac{u_\delta(x+th) - u_\delta(x)}{t} = \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u(h)\, \mathrm{d}y \qquad (**)$$

for $x \in V$, $\delta < \delta_0$ and $h \in \mathbb{R}^k$. We write

$$\frac{u_\delta(x+th) - u_\delta(x)}{t} = \int_{\mathbb{R}^k} \rho_\delta(y) \frac{u(x+th-y) - u(x-y)}{t}\, \mathrm{d}y.$$

We see that we can obtain $(**)$ from Lebesgue's dominated convergence theorem, since ρ_δ has a compact support contained in $\{y \in \mathbb{R}^k \mid \|y\| < \delta\}$, and for $y \in \mathbb{R}^k$, $t \in \mathbb{R}$ such that $\|y\| < \delta, \|th\| < \delta_0 - \delta$, the two points $x+th-y$, $x-y$ are contained in the compact set $\bar{N}_{\delta_0}(\bar{V})$ on which u is Lipschitz. Equation $(*)$ yields

$$H(x, d_x u_\delta) = H\biggl(x, \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u\, \mathrm{d}y\biggr). \qquad (***)$$

Since $\bar{N}_{\delta_0}(\bar{V})$ is compact and contained in U, and u is locally Lipschitz, we can find $K < \infty$ such that $\|d_z u\| \leqslant K$, for each $z \in \bar{N}_{\delta_0}(\bar{V})$ for which $d_z u$ exists. Since H is continuous, by a compactness argument, we can find $\delta_\epsilon \in]0, \delta_0[$ such that, for $z, z' \in \bar{N}_{\delta_0}(\bar{V})$, with $\|z - z'\| \leqslant \delta_\epsilon$ and $\|p\| \leqslant K$, we have $|H(z', p) - H(z, p)| \leqslant \epsilon$. We deduce that, for all x in V and almost every y with $\|y\| \leqslant \delta_\epsilon$, we have

$$H(x, d_{x-y} u) \leqslant H(x-y, d_{x-y} u) + \epsilon \leqslant \mathbb{H}_U(u) + \epsilon.$$

The quasi-convexity of H in the fibres implies that the set

$$C = \{p \in T_x^* M \mid H(x, p) \leqslant \mathbb{H}_U(u) + \epsilon\}$$

is convex and closed. Since $\rho_\delta dy$ is a probability measure whose support is contained in $\bar{B}(0, \delta) = \{y \in \mathbb{R}^k \mid \|y\| \leqslant \delta\}$, and $d_{x-y} u \in C$, for every $y \in \bar{B}(0, \delta)$, we obtain that the average $\int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u\, \mathrm{d}y$ is also in C. Hence, we obtain, for all $\delta \leqslant \delta_\epsilon$,

$$H\biggl(x, \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u\, \mathrm{d}y\biggr) \leqslant \mathbb{H}_U(u) + \epsilon.$$

It follows from inequality $(***)$ above that $H(x, d_x u_\delta) \leqslant \mathbb{H}_U(u) + \epsilon$ for $\delta \leqslant \delta_\epsilon$ and $x \in V$. This gives $\mathbb{H}_V(u_\delta) \leqslant \mathbb{H}_U(u) + \epsilon$, for $\delta \leqslant \delta_\epsilon$. The inequality

$$\sup_{x \in V} |u_\delta(x) - u(x)| < \epsilon$$

also holds for every sufficiently small δ, since u_δ converges uniformly on $\bar{V}$ to u, as $\delta \to 0$. □

Proof of theorem 10.2. We have to prove that, for each $x_0 \in M$, there exists an open neighbourhood V of x_0 such that $u|_V$ is a viscosity subsolution of $H(x, d_x u) = c$ on V. In fact, if we take V as any open neighbourhood such that $\bar{V}$ is contained in a domain of a coordinate chart, we can apply lemma 10.3 to obtain a sequence $u_n \colon V \to \mathbb{R}$, $n \geqslant 1$, of $\mathcal{C}^\infty$-functions such that u_n converges uniformly to $u|_V$ on V and $H(x, d_x u_n) \leqslant c + 1/n$. If we define $H_n(x, p) = H(x, p) - c - 1/n$, we see that

u_n is a smooth classical, and hence viscosity, subsolution of $H_n(x, d_x w) = 0$ on V. Since H_n converges uniformly to $H - c$, the stability theorem 6.1 implies that $u|_V$ is a viscosity subsolution of $H(x, d_x u) - c = 0$ on V. □

COROLLARY 10.4. *Suppose that the Hamiltonian $H\colon T^*M \to \mathbb{R}$ is continuous and quasi-convex in the fibres. For every $c \in \mathbb{R}$, the set of Lipschitz functions $u\colon M \to \mathbb{R}$ which are viscosity subsolutions of $H(x, d_x u) = c$ is convex.*

Proof. If $u_1, \ldots, u_n$ are such viscosity subsolutions. By corollary 4.3, we know that, at every x where $d_x u_j$ exists, we must have $H(x, d_x u_j) \leqslant c$. If we call A the set of points x where $d_x u_j$ exists for each $j = 1, \ldots, n$, then A has full Lebesgue measure in M. If $a_1, \ldots, a_n \geqslant 0$, and $a_1 + \cdots + a_n = 1$, then $u = a_1 u_1 + \cdots + a_n u_n$ is differentiable at each point of $x \in A$ with $d_x u = a_1 d_x u_1 + \cdots + a_n d_x u_n$. Therefore, by the quasi-convexity of $H(x, p)$ in the variable p, for every $x \in A$, we obtain

$$H(x, d_x u) = H(x, a_1 d_x u_1 + \cdots + a_n d_x u_n) \leqslant \max_{i=1}^{n} H(x, d_x u_i) \leqslant c.$$

Since A is of full measure, by theorem 10.2 we conclude that u is also a viscosity subsolution of $H(x, d_x u) = c$. □

The next corollary shows that the viscosity subsolutions are the same as the very weak subsolutions, at least in the geometric cases we have in mind. This corollary is clearly a consequence of theorems 5.2 and 10.2.

COROLLARY 10.5. *Suppose that the Hamiltonian $H\colon T^*M \to \mathbb{R}$ is continuous, coercive and quasi-convex in the fibres. A continuous function $u\colon M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_x u) = c$ for some $c \in \mathbb{R}$ if and only if u is locally Lipschitz and $H(x, d_x u) \leqslant c$ for almost every $x \in M$.*

We now give a global version of lemma 10.3.

THEOREM 10.6. *Suppose that $H\colon T^*M \to \mathbb{R}$ is a Hamiltonian, which is quasi-convex in the fibres. Let $u\colon M \to \mathbb{R}$ be a locally Lipschitz viscosity subsolution of $H(x, d_x u) = c$ on M. For every pair of continuous functions $\delta, \epsilon : M \to]0, +\infty[$, we can find a $\mathcal{C}^\infty$-function $v\colon M \to \mathbb{R}$ such that $|u(x) - v(x)| \leqslant \delta(x)$ and $H(x, d_x v) \leqslant c + \epsilon(x)$ for each $x \in M$.*

Proof. We pick up a locally finite countable open cover $(V_i)_{i\in\mathbb{N}}$ of M such that each closure $\bar{V}_i$ is compact and contained in the domain U_i of a chart which has a compact closure $\bar{U}_i$ in M. For every $i \in \mathbb{N}$, we set $J(i) = \{j \in \mathbb{N} \mid V_i \cap V_j \neq \emptyset\}$. Note that $j \in J(i)$ if and only if $i \in J(j)$. The local finiteness of the cover $(V_i)_{i\in\mathbb{N}}$ and the compactness of $\bar{V}_i$ imply that, for each $i \in \mathbb{N}$, the set $J(i)$ is finite. Therefore, denoting by $\#A$ the number of elements in a set A, we obtain

$$\begin{aligned} j(i) &= \#J(i) = \#\{j \in \mathbb{N} \mid V_i \cap V_j \neq \emptyset\} < +\infty, \\ \tilde{j}(i) &= \max_{\ell \in J(i)} j(\ell) < +\infty. \end{aligned}$$

We define $R_i = \sup_{x\in\bar{U}_i} \|d_x u\|_x < +\infty$, where the supremum is in fact taken over the subset of full measure of $x \in U_i$ where the locally Lipschitz function u has a derivative. It is finite because $\bar{U}_i$ is compact. Since $J(i)$ is finite, the following quantity $\tilde{R}_i$ is also finite:

$$\tilde{R}_i = \max_{\ell \in J(i)} R_\ell < +\infty.$$

We now choose $(\theta_i)_{i\in\mathbb{N}}$ a $\mathcal{C}^\infty$ partition of unity subordinated to the open cover $(V_i)_{i\in\mathbb{N}}$. We also define

$$K_i = \sup_{x\in M} \|d_x\theta_i\|_x < +\infty,$$

which is finite since θ_i is $\mathcal{C}^\infty$ with support in V_i, which is relatively compact.

Again, by compactness, continuity and finiteness routine arguments, the following numbers are greater than 0:

$$\delta_i = \inf_{x\in\bar{V}_i} \delta(x) > 0, \qquad \tilde{\delta}_i = \min_{\ell\in J(i)} \delta_\ell > 0,$$
$$\epsilon_i = \inf_{x\in\bar{V}_i} \epsilon(x) > 0, \qquad \tilde{\epsilon}_i = \min_{\ell\in J(i)} \epsilon_\ell > 0.$$

Since $\bar{V}_i$ is compact, the subset $\{(x,p)\in T^*M \mid x\in\bar{V}_i, \|p\|_x \leqslant \tilde{R}_i + 1\}$ is also compact. Therefore, by the continuity of H, we can find $\eta_i > 0$ such that, for all $x\in\bar{V}_i$ and for all $p, p' \in T_x^*M$,

$$\|p\|_x \leqslant \tilde{R}_i + 1, \qquad \|p'\|_x \leqslant \eta_i, \qquad H(x,p) \leqslant c + \tfrac{1}{2}\epsilon_i$$
$$\implies H(x, p+p') \leqslant c + \epsilon_i.$$

We can now choose $\tilde{\eta}_i > 0$ such that $\tilde{j}(i)K_i\tilde{\eta}_i < \min_{\ell\in J(i)} \eta_\ell$. Noting that $H(x,p)$ and $\|p\|_x$ are both quasi-convex in p, and that $\bar{V}_i$ is compact and contained in the domain U_i of a chart, by lemma 10.3, for each $i\in\mathbb{N}$, we can find a $\mathcal{C}^\infty$-function $u_i\colon V_i \to \mathbb{R}$ such that, for all $x\in V_i$,

$$|u(x) - u_i(x)| \leqslant \min(\tilde{\delta}_i, \tilde{\eta}_i),$$
$$H(x, d_x u_i) \leqslant \sup_{z\in V_i} H(z, d_z u) + \tfrac{1}{2}\tilde{\epsilon}_i \leqslant c + \tfrac{1}{2}\tilde{\epsilon}_i,$$
$$\|d_x u_i\|_x \leqslant \sup_{z\in V_i} \|d_z u\|_z + 1 = R_i + 1,$$

where the supremum in the last two lines is taken over the set of points $z \in V_i$, where $d_z u$ exists.

We now define $v = \sum_{i\in\mathbb{N}} \theta_i u_i$. It is obvious that v is $\mathcal{C}^\infty$. We fix $x\in M$, and choose $i_0\in\mathbb{N}$ such that $x\in V_{i_0}$. If $\theta_i(x) \neq 0$, then necessarily $V_i \cap V_{i_0} \neq \emptyset$ and therefore $i \in J(i_0)$. Hence,

$$\sum_{i\in J(i_0)} \theta_i(x) = 1 \quad \text{and} \quad v(x) = \sum_{i\in J(i_0)} \theta_i(x) u_i(x).$$

We can now write

$$|u(x) - v(x)| \leqslant \sum_{i\in J(i_0)} \theta_i(x)|u(x) - u_i(x)| \leqslant \sum_{i\in J(i_0)} \theta_i(x)\tilde{\delta}_i$$
$$\leqslant \sum_{i\in J(i_0)} \theta_i(x)\delta_{i_0} = \delta_{i_0} \leqslant \delta(x).$$

We now estimate $H(x, d_x v)$. First we observe that $\sum_{i\in J(i_0)} \theta_i(y) = 1$ and $v(y) = \sum_{i\in J(i_0)}\theta_i(y)u_i(y)$ for every $y \in V_{i_0}$. Since V_{i_0} is a neighbourhood of x, we can differentiate to obtain $\sum_{i\in J(i_0)} d_x\theta_i = 0$, and

$$d_x v = \underbrace{\sum_{i\in J(i_0)} \theta_i(x) d_x u_i}_{p(x)} + \underbrace{\sum_{i\in J(i_0)} u_i(x) d_x\theta_i}_{p'(x)} .$$

Using the quasi-convexity of H in p, we get

$$H(x, p(x)) \leqslant \max_{i\in J(i_0)} H(x, d_x u_i) \leqslant \max_{i\in J(i_0)} c + \tfrac{1}{2}\tilde{\epsilon}_i \leqslant c + \tfrac{1}{2}\epsilon_{i_0}, \qquad (*)$$

where, for the last inequality, we have used that $i \in J(i_0)$ means $V_i \cap V_{i_0} \neq \emptyset$, and therefore $i_0 \in J(i)$, which implies $\tilde{\epsilon}_i \leqslant \epsilon_{i_0}$, by the definition of $\tilde{\epsilon}_i$.

In the same way, we have

$$\|p(x)\|_x \leqslant \max_{i\in J(i_0)} \|d_x u_i\|_x \leqslant \max_{i\in J(i_0)} R_i + 1 \leqslant \tilde{R}_{i_0} + 1. \qquad (**)$$

We now estimate $\|p'(x)\|_x$. Using $\sum_{i\in J(i_0)} d_x\theta_i = 0$, we get

$$\begin{aligned} p'(x) &= \sum_{i\in J(i_0)} u_i(x) d_x\theta_i \\ &= \sum_{i\in J(i_0)} (u_i(x) - u(x)) d_x\theta_i. \end{aligned}$$

Therefore,

$$\begin{aligned} \|p'(x)\|_x &= \Big\| \sum_{i\in J(i_0)} (u_i(x) - u(x)) d_x\theta_i \Big\|_x \\ &\leqslant \sum_{i\in J(i_0)} |u_i(x) - u(x)| \|d_x\theta_i\|_x \\ &\leqslant \sum_{i\in J(i_0)} \tilde{\eta}_i K_i. \qquad (***) \end{aligned}$$

From the definition of $\tilde{\eta}_i$, we get $K_i\tilde{\eta}_i \leqslant \eta_{i_0}/j(i_0)$ for all $i \in J(i_0)$. Hence,

$$\|p'(x)\|_x \leqslant \sum_{i\in J(i_0)} \frac{\eta_{i_0}}{j(i_0)} = \eta_{i_0}.$$

The definition of η_{i_0}, together with inequalities $(*)$–$(***)$ above, implies

$$H(x, d_x v) = H(x, p(x) + p'(x)) \leqslant c + \epsilon_{i_0} \leqslant c + \epsilon(x).$$

□

THEOREM 10.7. *Suppose $H\colon T^*M \to \mathbb{R}$ is a Hamiltonian quasi-convex in the fibres. Let $u\colon M \to \mathbb{R}$ be a locally Lipschitz viscosity subsolution of $H(x, d_x u) = c$ which is strict at every point of an open subset $U \subset M$. For every continuous function $\epsilon\colon U \to]0, +\infty[$, we can find a viscosity subsolution $u_\epsilon\colon M \to \mathbb{R}$ of $H(x, d_x u) = c$*

such that $u = u_\epsilon$ on $M \setminus U, |u(x) - u_\epsilon(x)| \leqslant \epsilon(x)$, for every $x \in M$, and the restriction $u_{\epsilon|U}$ is $\mathcal{C}^\infty$ with $H(x, d_x u) < c$ for each $x \in U$.

Proof. We first define $\tilde\epsilon\colon M \to [0, +\infty[$. If $M = U$, we set $\tilde\epsilon = \epsilon$. If $M \setminus U \neq \emptyset$, we set $\tilde\epsilon(x) = \min(\epsilon(x), d(x, M \setminus U)^2)$ for $x \in U$, and $\tilde\epsilon(x) = 0$ for $x \notin U$. It is clear that $\tilde\epsilon$ is continuous on M and $\tilde\epsilon > 0$ is strictly positive on U.

For each $x \in U$, we can find $c_x < c$, and $V_x \subset V$ an open neighbourhood of x such that $H(y, d_y u) \leqslant c_x$, for almost every $y \in V_x$. The family $(V_x)_{x\in U}$ is an open cover of U. Therefore, we can find a locally finite partition of unity $(\varphi_x)_{x\in U}$ on U subordinated to the open cover $(V_x)_{x\in U}$. We define $\delta\colon U \to]0, +\infty[$ by

$$\delta(y) = \sum_{x\in U} \varphi_x(y)(c - c_x) \quad \text{for } y \in U.$$

It is not difficult to check that $H(y, d_y u) \leqslant c - \delta(y)$ for almost every $y \in U$.

We apply theorem 10.6 to the Hamiltonian $\tilde H\colon T^*U \to \mathbb{R}$ defined by $\tilde H(y, p) = H(y, p) + \delta(y)$ and $u \mid U$ which satisfies $\tilde H(y, d_y u) \leqslant c$ for almost every $y \in U$. We can therefore find a $\mathcal{C}^\infty$-function $u_\epsilon\colon U \to \mathbb{R}$, with $|u_\epsilon(y) - u(y)| \leqslant \tilde\epsilon(y)$, and $\tilde H(y, d_y u_\epsilon) \leqslant c + \frac12\delta(y)$, for each $y \in U$. Therefore, we obtain

$$|u_\epsilon(y) - u(y)| \leqslant \epsilon(y) \quad \text{and} \quad H(y, d_y u_\epsilon) \leqslant c - \tfrac12\delta(y) < c$$

for each $y \in U$. Moreover, since $\tilde\epsilon(y) \leqslant d(y, M \setminus U)^2$, it is clear that we can extend continuously u_ϵ by u on $M \setminus U$. This extension satisfies

$$|u_\epsilon(x) - u(x)| \leqslant d(x, M \setminus U)^2 \quad \text{for every } x \in M.$$

We must verify that u_ϵ is a viscosity subsolution of $H(x, d_x u_\epsilon) = c$. This is clear on U, since u_ϵ is $\mathcal{C}^\infty$ on U, and $H(y, d_y u_\epsilon) < c$, for $y \in U$. It remains to check that if $\phi\colon M \to \mathbb{R}$ is $\mathcal{C}^1$, and such that $\phi \geqslant u_\epsilon$ with equality at $x_0 \notin U$, then $H(x_0, d_{x_0}\phi) \leqslant c$. For this, we note that

$$u_\epsilon(x_0) = u(x_0) \quad \text{and} \quad u(x) - u_\epsilon(x) \leqslant d(x, M \setminus U)^2 \leqslant d(x, x_0)^2.$$

Hence, $u(x) \leqslant \phi(x) + d(x, x_0)^2$ with equality at x_0. The function $x \to \phi(x) + d(x, x_0)^2$ has a derivative at x_0 equal to $d_{x_0}\phi$. Therefore, $H(x_0, d_{x_0}\phi) \leqslant c$, since u is a viscosity solution of $H(x, d_x u) \leqslant c$. □

11. The viscosity semi-distance

We will suppose that $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected manifold M.

Definition 11.1 (Mañé critical value). Let $H\colon T^*M \to \mathbb{R}$ be a continuous Hamiltonian. The Mañé critical value $c[0]$ of H is defined in the following way.

- If there is no $c \in \mathbb{R}$ such that $H(x, d_x u) = c$ admits a global viscosity subsolution $u\colon M \to \mathbb{R}$, we set $c[0] = +\infty$.
- If there is a $c \in \mathbb{R}$ such that $H(x, d_x u) = c$ admits a global viscosity subsolution $u\colon M \to \mathbb{R}$, we define $c[0]$ as the infimum of all $c \in \mathbb{R}$ such that $H(x, d_x u) = c$ admits a global subsolution $u\colon M \to \mathbb{R}$.

Note that when M is compact we are always in the second case above. In fact, if $u\colon M \to \mathbb{R}$ is $\mathcal{C}^1$, then $\mathbb{H}_M(u) = \sup_{x\in M} H(s, d_xu)$ is finite, and u is a viscosity subsolution of $H(x, d_xu) = c$ for any $c \geqslant \mathbb{H}_M(u)$.

PROPOSITION 11.2. *Suppose that $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected manifold M. Then*

$$c[0] \geqslant \sup_{x\in M} \inf_{p\in T_x^*M} H(x,p) > -\infty.$$

Moreover, if the Hamiltonian H on M is defined by $H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x)$, where $V\colon M \to \mathbb{R}$ is continuous, and $\|\cdot\|_x$ is the norm associated to a Riemannian on M, then $c[0] = \sup_M V$.

Proof. We first prove that the function $x \mapsto \inf_{p\in T_x^*M} H(x,p)$ is finite and continuous on M. This is obviously a local result. We can therefore assume that $M = \mathbb{R}^n$. Fix K a compact subset of $\mathbb{R}^n$. By continuity of H, the constant $C_K = \sup_{x\in K} H(x,0)$ is finite. By coercivity of H, the set

$$S_K = \{(x,p) \in K \times \mathbb{R}^n \mid x \in K,\ H(x,p) \leqslant C_K + 1\}$$

is compact. Hence, $R_K = \sup\{\|p\| \mid H(x,p) \leqslant C_K + 1\}$ is finite. Obviously, for $x \in K$, we have

$$\inf_{p\in T_x^*M} H(x,p) = \inf_{p\in \bar{B}(0,R_K)} H(x,p).$$

The continuity of H and the compactness of $K \times \bar{B}(0,R_K)$ imply that

$$\inf_{p\in T_x^*M} H(x,p) = \inf_{p\in \bar{B}(0,R_K)} H(x,p)$$

is a finite and continuous function of $x \in K$. Since $\inf_{p\in T_x^*M} H(x,p)$ is finite, we get

$$\sup_{x\in M} \inf_{p\in T_x^*M} H(x,p) > -\infty.$$

To finish the proof of the first part, we can assume $c[0] < +\infty$. Suppose now that $u\colon M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_xu) = c$, where $c \in \mathbb{R}$. Since H is coercive, we know by theorem 5.2 that u is locally Lipschitz and satisfies $H(x, d_xu) \leqslant c$ almost everywhere in M. This implies that $c \geqslant \inf_{p\in T_x^*M} H(x,p)$ for almost every $x \in M$. But $x \mapsto \inf_{p\in T_x^*M} H(x,p)$ is continuous on M. It follows that

$$c \geqslant \sup_{x\in M} \inf_{p\in T_x^*M} H(x,p).$$

Taking the infimum over all $c \in \mathbb{R}$ such that $H(x, d_xu) = c$ admits a viscosity subsolution, we conclude that

$$c[0] \geqslant \sup_{x\in M} \inf_{p\in T_x^*M} H(x,p).$$

To prove the second part we observe that $\inf_{p\in T_x^*M} H(x,p) = V(x)$. Therefore, $c[0] \geqslant \sup_M V$. Of course, if $\sup_M V = +\infty$, the proof is finished. If $\sup_M V < +\infty$, let $u\colon M \to \mathbb{R}$ be a *constant* function, and then $H(x, d_xu) = H(x,0) = V(x)$ for every $x \in M$. Hence any constant function is a viscosity subsolution of $H(x, d_xu) = \sup_M V$. By definition of $c[0]$, we obtain the reverse inequality $c[0] \leqslant \sup_M V$. $\square$

We denote by $\mathcal{SS}^c$ the set of viscosity subsolutions of $H(x, d_x u) = c$, and by $\mathcal{SS}^c_{\hat{x}} \subset \mathcal{SS}^c$ the subset of subsolutions vanishing at a given $\hat{x} \in M$. Of course, since we can always add a constant to a viscosity subsolution and still obtain a subsolution, we have $\mathcal{SS}^c_{\hat{x}} \neq \emptyset$ if and only if $\mathcal{SS}^c \neq \emptyset$, and in that case $\mathcal{SS}^c = \mathbb{R} + \mathcal{SS}^c_{\hat{x}}$.

Proposition 11.3. *Suppose that $H: T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected manifold M. Assume that there is a $c \in \mathbb{R}$ such that $H(x, d_x u) = c$ has a viscosity subsolution on the whole of M (in particular, the Mañé critical value $c[0]$ is finite). Then there exists a global $u: M \to \mathbb{R}$ viscosity subsolution of $H(x, d_x u) = c[0]$.*

Proof. Fix a point $\hat{x} \in M$. Subtracting $u(\hat{x})$ if necessary, we will assume that all the viscosity subsolutions of $H(x, d_u) = c$ we consider vanish at $\hat{x}$. Since H is coercive above every compact subset of M, by corollary 5.3, for each c the family of functions in $\mathcal{SS}^c_{\hat{x}}$ is locally equi-Lipschitzian. Therefore, by the beginning of the proof of theorem 8.2, using that M is connected and the fact that every $v \in \mathcal{SS}^c_{\hat{x}}$ vanish at $\hat{x}$, we obtain, for all $x \in M$,

$$\sup_{v \in \mathcal{SS}^c_{\hat{x}}} |v(x)| < +\infty.$$

We pick a sequence $c_n \searrow c[0]$, with $c_n \leqslant c$, and a sequence $u_n \in \mathcal{SS}^{c_n}_{\hat{x}}$. Since, by Ascoli's theorem, the family $\mathcal{SS}^c_{\hat{x}}$ is relatively compact in the topology of uniform convergence on each compact subset, extracting a sequence if necessary, we can assume that u_n converges uniformly to u on each compact subset of M. By the stability theorem 6.1, since u_n is a viscosity subsolution of $H(x, d_x u_n) = c_n$, the limit u is a viscosity subsolution of $H(x, d_x u) = c[0]$. □

For $c \geqslant c[0]$, we define

$$S^c(x, y) = \sup_{u \in \mathcal{SS}^c} u(y) - u(x) = \sup_{v \in \mathcal{SS}^c_x} u(y).$$

It follows from theorem 8.2 that, for each $x \in M$, the function $S^c(x, \cdot)$ is finite, and is a viscosity subsolution of $H(y, d_y u) = c$ on M itself, and a viscosity solution on $M \setminus \{x\}$.

Theorem 11.4. *For each $c \geqslant c[0]$, the function S^c is a semi-distance, i.e. it satisfies*

(i) $S^c(x, x) = 0$ *for each* $x \in M$,

(ii) $S^c(x, z) \leqslant S^c(x, y) + S^c(y, z)$ *for each* $x, y, z \in M$.

The function S^c is locally Lipschitz on $M \times M$. If d is a distance coming from a Riemannian metric on M, for every compact subset $K \subset M$, we can find a constant $L_K < +\infty$ such that $|S^c(x, y)| \leqslant L_K d(x, y)$ for every $x, y \in K$.

Moreover, for $c > c[0]$, the symmetric semi-distance

$$\hat{S}^c(x, y) = S^c(x, y) + S^c(y, x)$$

is a distance which is locally Lipschitz-equivalent to any distance coming from a Riemannian metric.

Proof. The fact that S^c is a semi-distance follows easily from the definition

$$S^c(x,y) = \sup_{u \in \mathcal{SS}^c} u(y) - u(x).$$

We now prove that S^c is locally Lipschitz on $M \times M$. By corollary 5.3, the family of functions $\mathcal{SS}^c$ is locally equi-Lipschitzian. Therefore, if $x_0, y_0 \in M$ are given, we can find neighbourhood V_{x_0}, V_{y_0} of x_0 and y_0, respectively, and finite constants L_{x_0}, L_{y_0} such that

$$\begin{aligned} |u(x') - u(x)| &\leqslant L_{x_0} d(x,x'), \quad \forall u \in \mathcal{SS}^c,\ \forall x, x' \in V_{x_0}, \\ |u(y') - u(y)| &\leqslant L_{y_0} d(y,y'), \quad \forall u \in \mathcal{SS}^c,\ \forall y, y' \in V_{y_0}. \end{aligned}$$

This of course implies

$$\begin{aligned} |S^c(x,x')| &\leqslant L_{x_0} d(x,x'), \quad \forall x, x' \in V_{x_0}, \\ |S^c(y,y')| &\leqslant L_{y_0} d(y,y'), \quad \forall y, y' \in V_{y_0}. \end{aligned}$$

If $(x,y),(x'y') \in V_{x_0} \times V_{y_0}$, by the triangular inequality (ii), we obtain

$$S^c(x',y') \leqslant S^c(x',x) + S^c(x,y) + S^c(y,y').$$

Since S^c is finite valued, this yields

$$S^c(x',y') - S^c(x,y) \leqslant S^c(x',x) + S^c(y,y').$$

With the estimation above, we deduce that

$$S^c(x',y') - S^c(x,y) \leqslant L_{x_0} d(x,x') + L_{y_0} d(y',y).$$

Therefore, by symmetry, we obtain

$$|S^c(x',y') - S^c(x,y)| \leqslant L_{x_0} d(x,x') + L_{y_0} d(y',y).$$

Hence, S^c is Lipschitz on the neighbourhood $V_{x_0} \times V_{y_0}$ of (x_0,y_0) in $M \times M$.

Suppose now that $K \subset M$ is compact. Since S^c is locally Lipschitz on the compact set $K \times K$, we can find $\alpha > 0$ and $L < +\infty$ such that, for $x, x', y, y' \in K$, with $d(x,x') + d(y,y') < \alpha$, we have $|S^c(x',y') - S^c(x,y)| \leqslant L[d(x,x') + d(y,y')]$. In particular, since $S^c(z,z) = 0$, if $x, y \in K$ are such that $d(x,y) < \alpha$, we have $|S^c(x,y)| \leqslant L_K d(x,y)$. Since the continuous function S^c is bounded on the compact set $K \times K$, the constant $L_1 = \max_K |S^c|$ is finite, and for $x, y \in K$ with $d(x,y) \geqslant \alpha$, we have

$$|S^c(x,y)| \leqslant L_1 \leqslant \frac{L_1}{\alpha} d(x,y).$$

If we set $L_K = \max(L, L_1/\alpha)$, we obtain that $|S^c(x,y)| \leqslant L_K d(x,y)$, for all $x, y \in K$.

It remains to show a reverse inequality for $c > c[0]$. Fix such a c, and a compact set $K \subset M$. Choose $\delta > 0$ such that $\bar{N}_\delta(K) = \{x \in M \mid d(x,K) \leqslant \delta\}$ is also compact. By the compactness of the set

$$\{(x,p) \mid x \in \bar{N}_\delta(K), H(x,p) \leqslant c[0]\},$$

and the continuity of H, we can find $\epsilon > 0$ such that

$$\begin{aligned} & H(x,p) \leqslant c[0], \quad \|p'\|_x \leqslant \epsilon, \quad \text{for every } x \in \bar{N}_\delta(K) \text{ and every } p, p' \in T_x M, \\ & \quad \Longrightarrow H(x, p+p') \leqslant c. \end{aligned} \tag{$*$}$$

We can find $\delta_1 > 0$ such that the radius of injectivity of the exponential map, associated to the Riemannian metric, is at least δ_1 at every point x in the compact subset $\bar{N}_\delta(K)$. In particular, the distance function $x \mapsto d(x, x_0)$ is $\mathcal{C}^\infty$ on $\mathring{B}(x_0, \delta_1) \setminus \{x_0\}$ for every $x_0 \in \bar{N}_\delta(K)$. The derivative of $x \mapsto d(x, x_0)$ at each point where it exists has norm 1, since this map has (local) Lipschitz constant equal to 1. We can assume $\delta_1 < \delta$. We now pick $\phi\colon \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^\infty$-function, with support in $]\frac{1}{2}, 2[$, and such that $\phi(1) = 1$. If $x_0 \in K$ and $0 < d(y, x_0) \leqslant \frac{1}{2}\delta_1$, the function

$$\phi_y(x) = \phi\left(\frac{d(x, x_0)}{d(y, x_0)}\right)$$

is $\mathcal{C}^\infty$. In fact, if $d(x, x_0) \geqslant \delta_1$, then ϕ_y is zero in a neighbourhood of x, since

$$\frac{d(x, x_0)}{d(y, x_0)} \geqslant \frac{\delta_1}{\frac{1}{2}\delta_1} = 2;$$

if $0 < d(x, x_0) < \delta_1 < \delta$, then it is $\mathcal{C}^\infty$ on a neighbourhood of x; finally $\phi_y(x) = 0$ for x such that $d(x, x_0) \leqslant \frac{1}{2} d(y, x_0)$. In particular, we obtained that $d_x\phi_y = 0$, unless $0 < d(x, x_0) < \delta$, but at each such x, the derivative of $z \mapsto d(z, x_0)$ exists and has norm 1. It is then not difficult to see that $\sup_{x \in M} \|d_x\phi_y\|_x \leqslant A/d(y, x_0)$, where $A = \sup_{t \in \mathbb{R}} |\phi'(t)|$.

Therefore, if we set $\lambda = \epsilon d(y, x_0)/A$, we see that $\|\lambda d_x\phi_y\|_x \leqslant \epsilon$, for $x \in M$. Since ϕ_y is 0 outside the ball $B(x_0, \delta_1) \subset N_{\delta_1}(K)$, it follows from the property $(*)$ characterizing ϵ that we have, for all $(x, p) \in T^*M$,

$$H(x, p) \leqslant c[0] \Rightarrow H(x, p + \lambda d_x\phi_y) \leqslant c.$$

Since $S^{c[0]}(x_0, \cdot)$ is a viscosity subsolution of $H(x, d_x u) = c[0]$, and ϕ_y is $\mathcal{C}^\infty$, we conclude that the function $u(\cdot) = S^{c[0]}(x_0, \cdot) + \lambda\phi_y(\cdot)$ is a viscosity subsolution of $H(x, d_x u) = c$. But the value of u at x_0 is 0, and its value at y is

$$S^{c[0]}(x_0, y) + \lambda\phi_y(y) = S^{c[0]}(x_0, y) + \epsilon d(y, x_0)/A,$$

since $\phi_y(y) = \phi(1) = 1$. Therefore, $S^c(x_0, y) \geqslant S^{c[0]}(x_0, y) + \epsilon d(y, x_0)/A$. Since $x_0 \in K$, and $y \in \bar{B}(x_0, \frac{1}{2}\delta_1)$ were arbitrary, we obtain, for all $x, y \in K$,

$$d(x, y) \leqslant \tfrac{1}{2}\delta_1 \Rightarrow S^c(x, y) \geqslant S^{c[0]}(x, y) + \epsilon A^{-1} d(x, y).$$

Adding up and using $S^{c[0]}(x, y) + S^{c[0]}(y, x) \geqslant S^{c[0]}(x, x) = 0$, we get, for all $x, y \in K$,

$$d(x, y) \leqslant \tfrac{1}{2}\delta_1 \Rightarrow S^c(x, y) + S^c(y, x) \geqslant \frac{2\epsilon}{A} d(x, y).$$

$\square$

12. The projected Aubry set

THEOREM 12.1. *Assume that $H\colon T^*M \to \mathbb{R}$ is a Hamiltonian coercive above every compact subset of the connected manifold M, with $c[0] < +\infty$. For each $c \geqslant c[0]$ and each $x \in M$, the following two conditions are equivalent:*

(i) *the function $S^c(x,\cdot)$ is a viscosity solution of $H(z, d_z u) = c$ on the whole of M;*

(ii) *there is no viscosity subsolution of $H(z, d_z u) = c$ on the whole of M which is strict at x.*

In particular, for every $c > c[0]$, the function $S^c(x,\cdot)$ is not a viscosity solution of $H(z, d_z u) = c$.

Proof. The implication (ii) $\Longrightarrow$ (i) follows from proposition 9.2.

To prove (i) $\Longrightarrow$ (ii), fix $x \in M$ such that $S^c_x(\cdot) = S^c(x,\cdot)$ is a viscosity solution on the whole of M, and suppose that $u\colon M \to \mathbb{R}$ is a viscosity subsolution of $H(y, d_y u) = c$ which is strict at x. Therefore, we can find an open neighbourhood V_x of x, and a $c_x < c$ such that $u|_{V_x}$ is a viscosity subsolution of $H(y, d_y u) = c_x$ on V_x. By definition of S, we have $u(y) - u(x) \leqslant S^c_x(y)$ with equality at $y = x$. This implies that $u - S^c_x$ has a global maximum at x. Applying theorem 7.2 to the restrictions of u and S^c_x to V_x, we see that we must have $c \leqslant c_x < c$, which is a contradiction.

Since a viscosity subsolution of $H(x, d_x u) = c[0]$ is a strict viscosity subsolution of $H(x, d_x u) = c$ for any $c > c[0]$, we obtain the last part of the theorem. □

The above theorem yields the following definition.

DEFINITION 12.2 (projected Aubry set). If $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian, coercive above every compact subset of the connected manifold M. We define the projected Aubry set $\mathcal{A}$ as the set of $x \in M$ such that $S^{c[0]}(x,\cdot)$ is a viscosity solution of $H(z, d_z u) = c[0]$.

PROPOSITION 12.3. *The projected Aubry set $\mathcal{A}$ is closed.*

Proof. We show that $M \setminus \mathcal{A}$ is open. In fact, by the equivalence in theorem 12.1, if $x_0 \notin \mathcal{A}$, there is a viscosity subsolution u which is strict at x_0. By the definition of a strict subsolution there is a whole open neighbourhood U_{x_0} of x_0 on which u is a subsolution of $H(x, d_x u) = c$, for some $c < c[0]$. Obviously, this subsolution u is strict at every $x \in U_{x_0}$. Again, by the equivalence in theorem 12.1, we obtain that $x \notin \mathcal{A}$, for every $x \in U_{x_0}$. □

In fact, the projected Aubry set can be empty even if H is quasi-convex in the fibres (see example 12.7). To be able to proceed in our discussion we will need to restrict to Hamiltonians convex in the fibres.

PROPOSITION 12.4. *Assume that $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian, convex in the fibres, and coercive above every compact subset of the connected manifold M. There exists a viscosity subsolution $v\colon M \to \mathbb{R}$ of $H(x, d_x v) = c[0]$, which is strict at every $x \in M \setminus \mathcal{A}$.*

Proof. We fix some base point $\hat{x} \in M$. For each $x \notin \mathcal{A}$, we can find $u_x \colon M \to \mathbb{R}$, an open subset V_x containing x, and $c_x < c[0]$, such that u_x is a viscosity subsolution of $H(y, d_y u_x) = c[0]$ on M, and $u_x \mid V_x$ is a viscosity subsolution of $H(y, d_y u_x) \leqslant c_x$, on V_x. Subtracting $u_x(\hat{x})$ if necessary, we will assume that $u_x(\hat{x}) = 0$. Since $U = M \setminus \mathcal{A}$ is covered by the family of open sets V_x, $x \notin \mathcal{A}$, we can extract a countable subfamily $(V_{x_i})_{i\in\mathbb{N}}$ covering U. Since H is coercive above every compact set, the sequence $(u_{x_i})_{i\in\mathbb{N}}$ is locally equi-Lipschitzian. Therefore, since M is connected, and all the u_{x_i} vanish at $\hat{x}$, the sequence $(u_{x_i})_{i\in\mathbb{N}}$ is uniformly bounded on every compact subset of M. It follows that the sum

$$v = \sum_{i\in\mathbb{N}} \frac{1}{2^{i+1}} u_{x_i}$$

is uniformly convergent on each compact subset. If we set

$$u_n = (1 - 2^{-(n+1)})^{-1} \sum_{0\leqslant i\leqslant n} \frac{1}{2^{i+1}} u_{x_i},$$

then u_n is a viscosity subsolution of $H(x, d_x u_n) = c[0]$ as a convex combination of viscosity subsolutions (see proposition 10.4). Since u_n converges uniformly on compact subsets to u, the stability theorem 6.1 implies that v is also a viscosity subsolution of $H(x, d_x v) = c[0]$.

On the set $V_{x_{n_0}}$, we have $H(x, d_x u_{x_{n_0}}) \leqslant c_{x_{n_0}}$ for almost every $x \in V_{x_{n_0}}$. Therefore, if we fix $n \geqslant n_0$, we see that, for almost every $x \in V_{x_{n_0}}$, we have

$$\begin{aligned} H(x, d_x u_n) &\leqslant (1 - 2^{-(n+1)})^{-1} \sum_{i=0}^{n} \frac{1}{2^{i+1}} H(x, d_x u_{x_i}) \\ &\leqslant (1 - 2^{-(n+1)})^{-1} \left[\frac{c_{x_{n_0}} - c[0]}{2^{n_0+1}} + \sum_{i=0}^{n} \frac{1}{2^{i+1}} c[0] \right] \\ &= c[0] + \frac{c_{x_{n_0}} - c[0]}{2^{n_0+1} - 2^{n_0-n}}. \end{aligned}$$

Therefore, $u_n | V_{x_{n_0}}$ is a viscosity subsolution of

$$H(x, d_x u_n) = c[0] + \frac{c_{x_{n_0}} - c[0]}{2^{n_0+1} - 2^{n_0-n}}.$$

By the stability theorem 6.1, we obtain that $v | V_{x_{n_0}}$ is a viscosity subsolution of $H(x, d_x v) = c[0] + (c_{x_{n_0}} - c[0])/2^{n_0+1}$. Since $c_{x_{n_0}} - c[0] < 0$, we conclude that $u \mid V_{x_{n_0}}$ is a strict subsolution of $H(x, d_x v) = c[0]$, for each $x \in V_{x_{n_0}}$, and therefore at each $x \in U \subset \bigcup_{n\in\mathbb{N}} V_{x_n}$. □

Corollary 12.5. *Assume that $H \colon T^*M \to \mathbb{R}$ is a Hamiltonian convex in the fibres and coercive, where M is a compact connected manifold. Its projected Aubry set $\mathcal{A}$ is not empty.*

Proof. We argue by contradiction. If $\mathcal{A} = \emptyset$, then by proposition 12.4, we can find a viscosity subsolution u of $H(x, d_x u) = c[0]$ which is strict everywhere. In particular, for every $x \in M$, we can find an open neighbourhood V_x of x and

$c_x < c[0]$ such that $u \mid V_x$ is a viscosity solution of $H(y, d_y v) = c_x$. By compactness of M, we can find a finite number of points $x_1, \ldots, x_\ell$ in M such that $M = V_{x_1} \cup \cdots \cup V_{x_\ell}$. It follows from corollary 4.2 that u is a viscosity subsolution of $H(x, d_x u) = \max(c_{x_1}, \ldots, c_{x_\ell})$ on the whole of M. This is in contradiction of the definition of $c[0]$ since $\max(c_{x_1}, \ldots, c_{x_\ell}) < c[0]$. □

THEOREM 12.6. *Assume that $H\colon T^*M \to \mathbb{R}$ is a Hamiltonian convex in the fibres and coercive, where M is a compact connected manifold. Suppose $u_1, u_2\colon M \to \mathbb{R}$ are a viscosity subsolution and a viscosity supersolution, respectively, of $H(x, d_x u) = c[0]$. If $u_1 \leqslant u_2$ on the projected Aubry set $\mathcal{A}$, then $u_1 \leqslant u_2$ everywhere on M.*

In particular, if two viscosity solutions of $H(x, d_x u) = c[0]$ coincide on $\mathcal{A}$, they coincide on M.

Proof. By proposition 12.4, we can find a viscosity subsolution u_0 of $H(x, d_x u) = c[0]$ which is strict at every point of $M \setminus \mathcal{A}$. We interpolate between u_0 and u_1 by defining $u_t = (1-t)u_0 + tu_1$. As in the proof of proposition 12.4, we can show that u_t is a viscosity subsolution of $H(x, d_x u) = c[0]$ for any $t \in [0,1]$. Moreover, for $t < 1$, the viscosity subsolution u_t is strict at each point of $M \setminus \mathcal{A}$. By the coercivity condition, all subsolutions are locally Lipschitz. Since M is compact, $u_t - u_2$ achieves a maximum on M. By proposition 9.3, for $t < 1$, this maximum is achieved at a point of the compact subset $\mathcal{A}$. Since u_t converges uniformly to u_1, it follows that $u_1 - u_2$ also achieves its maximum on M in the same compact subset $\mathcal{A}$. But $u_1 - u_2 \leqslant 0$ on $\mathcal{A}$. Therefore, $u_1 - u_2 \leqslant 0$ everywhere on M. □

EXAMPLE 12.7. We give an example of a Hamiltonian quasi-convex in the fibres with the Aubry set empty, hence, corollary 12.5 does not hold for quasi-convex Hamiltonians. We will also show that proposition 12.4 does not necessarily hold for this Hamiltonian; in fact, the argument shows more generally that when the Aubry set is empty for a Hamiltonian on a compact manifold, proposition 12.4 cannot hold.

Define the quasi-convex function $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} -t-1 & \text{for } t \leqslant -1, \\ t+1 & \text{for } -1 \leqslant t \leqslant 0, \\ 1 & \text{for } 0 \leqslant t \leqslant 1, \\ t & \text{for } t \geqslant 1. \end{cases}$$

We define a Hamiltonian H on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We use the usual identification of the cotangent space $T^*\mathbb{T}$ with $\mathbb{T} \times \mathbb{R}$. In this usual identification, the derivative du of a function $u\colon \mathbb{T} \to \mathbb{R}$, as a section, is exactly $t \mapsto (t, u'(t))$. The Hamiltonian $H\colon \mathbb{T}\times\mathbb{R}$ is defined by $H(t,s) = h(s)$. Obviously, the constant function $u_0 \equiv 0$ obviously satisfies $H(t, u_0'(t)) = H(t, 0) = h(0) = 1$, therefore $c[0] = 1$, by corollary 7.3. For any $t_0 \in \mathbb{T}$, the function $v_{t_0}(t) = (2\pi)^{-1}\sin(2\pi t + \pi - 2\pi t_0)$ has a derivative $v_{t_0}'(t) = \cos(2\pi t + \pi - 2\pi t_0)$ which is between -1 and 1 everywhere. Therefore, v_{t_0} is a subsolution of $H(t, v'(t)) = 1$. Moreover, its derivative at t_0 is $\cos(\pi) = -1$, Hence, $H(t_0, v_{t_0}'(t_0)) = h(-1) = 0 < 1$. By continuity of the derivative of v_{t_0}, it follows that v_{t_0} is strict at t_0. Since t_0 is arbitrary in $\mathbb{T}$, it follows from theorem 12.1 that the Aubry set of H is empty. This shows that corollary 12.5 cannot be true for general quasi-convex Hamiltonian. We now show that proposition 12.4 cannot be true for H. In fact, if it were true we would obtain a viscosity subsolution which is strict at every point of $\mathbb{T}$. Using the compactness of $\mathbb{T}$ as in the proof of corollary 12.5, we see that this yields a viscosity subsolution of $H(t, v'(t)) = c$, for some $c < 1$. This is impossible since $c[0] = 1$.

13. The representation formula

We still assume that M is compact, and that $H\colon T^*M \to \mathbb{R}$ is a coercive Hamiltonian convex in the fibres.

THEOREM 13.1. *Any viscosity solution $u\colon M \to \mathbb{R}$ for $H(x, d_x u) = c[0]$ satisfies*

$$u(x) = \inf_{x_0 \in \mathcal{A}} u(x_0) + S^{c[0]}(x_0, x), \quad \forall x \in M.$$

This follows easily from the uniqueness theorem 12.6 and the following theorem.

THEOREM 13.2. *For any function $v\colon \mathcal{A} \to \mathbb{R}$ bounded from below, the function*

$$\tilde{v}(x) = \inf_{x_0 \in \mathcal{A}} v(x_0) + S^{c[0]}(x_0, x)$$

is a viscosity solution of $H(x, d_x v) = c[0]$. Moreover, we have $\tilde{v}|_{\mathcal{A}} = v$ if and only if, for all $x, y \in \mathcal{A}$,

$$v(y) - v(x) \leqslant S^{c[0]}(x, y).$$

We start with a lemma.

LEMMA 13.3. *Suppose $H\colon T^*M \to \mathbb{R}$ is a continuous Hamiltonian quasi-convex in the fibres, and coercive above each compact subset of the connected manifold M. Let $u_i\colon M \to \mathbb{R}$, $i \in I$, be a family of viscosity subsolutions of $H(x, d_x u) = c$. If $\inf_{i \in I} u_i(x_0)$ is finite for some $x_0 \in M$, then $\inf_{i \in I} u_i$ is finite everywhere. In that case, the function $u = \inf_{i \in I} u_i$ is a viscosity subsolution of $H(x, d_x u) = c$.*

In particular, if each u_i is a viscosity solution, so is $u = \inf_{i \in I} u_i$.

Proof. We fix an auxiliary Riemannian metric on M, and we use as a distance on M its associated distance.

By the coercivity condition, the family $(u_i)_{i \in I}$ is locally equi-Lipschitzian. Therefore, if K is a compact connected subset of M, there exists a constant $C(K)$ such that, for all $x, y \in K$ and for all $i \in I$,

$$|u_i(x) - u_i(y)| \leqslant C(K).$$

If $x \in M$ is given, we can find a compact connected subset K_x containing x_0 and x, it follows that

$$\inf_{i \in I} u_i(x_0) \leqslant \inf_{i \in I} u_i(x) + C(K_x).$$

Therefore, $\inf_{i \in I} u_i$ is finite everywhere. It now suffices to show that, for a given $\tilde{x} \in M$, we can find an open neighbourhood V of $\tilde{x}$ such that $\inf_{i \in I} u_i \mid V$ is a viscosity subsolution of $H(x, d_x u) = c$ on V. We choose an open neighbourhood V of $\tilde{x}$ such that its closure $\bar{V}$ is compact. Since $\mathcal{C}^0(\bar{V}, \mathbb{R})$ is metric and separable in the topology of uniform convergence, we can find a countable subset $I_0 \subset I$ such that $u_{i|_{\bar V}}$, $i \in I_0$, is dense in $\{u_{i|_{\bar V}} \mid i \in I\}$, for the topology of uniform convergence. Therefore, $\inf_{i\in I} u_i = \inf_{i\in I_0} u_i$ on $\bar V$. Since I_0 is countable, we have reduced the proof to the cases where $I_0 = \{0,\dots,N\}$ or $I_0 = \mathbb{N}$.

Let us start with the first case. Since $u_0,\dots,u_N$, and $u = \inf_{i=0}^N u_i$ are all locally Lipschitzian on V, we can find $E \subset V$ of full Lebesgue measure such that $d_xu, d_xu_0,\dots,d_xu_N$ exists, for each $x \in E$. At each such $x \in E$, we necessarily have $d_xu \in \{d_xu_0,\dots,d_xu_N\}$. In fact, if n is such that $u(x) = u_n(x)$, since $u \leqslant u_n$ with equality at x and both derivatives d_xu_n, d_xu at x exist, they must be equal. Since each u_i is a viscosity subsolution of $H(x, d_xv) = c$, we obtain $H(x, d_xu) \leqslant c$ for every x in the subset E of full measure in V. By the quasi-convexity of H in the fibres, corollary 10.5 implies that u is a viscosity subsolution of $H(x, d_xu) = c$ in V. It remains to consider the case $I_0 = \mathbb{N}$. Define $u^N(x) = \inf_{0\leqslant i\leqslant N} u_i(x)$. By the previous case, u^N is a viscosity subsolution of $H(x, d_xu^N) = c$ on V.

Now $u^N(x) \to \inf_{i\in I_0} u_i(x)$, for each $x \in \bar V$, the convergence is, in fact, uniform on $\bar V$ since $(u_i)_{i\in I_0}$ is equi-Lipschitzian on the compact set $\bar V$. It remains to apply the stability theorem 6.1.

To prove the last part of the lemma, it suffices to recall from proposition 8.1 that an infimum of a family of supersolutions is itself a supersolution. □

Proof of theorem 13.2. By definition of the projected Aubry set, for every $x_0 \in \mathcal{A}$, the function $v(x_0) + S^{c[0]}(x_0,\cdot)$ is a viscosity solution. It follows from lemma 13.3 that $\tilde v$ is a viscosity solution.

Since $\tilde v$ is in particular a subsolution, it satisfies $\tilde v(y) - \tilde v(x) \leqslant S^{c[0]}(x,y)$ everywhere. Therefore, if $v = \tilde v$ on $\mathcal{A}$, we must have that

$$v(y) - v(x) \leqslant S^{c[0]}(x,y), \quad \forall x,y \in \mathcal{A}.$$

Conversely, if v satisfies the property above, from the definition of $\tilde v$ it is obvious that $v = \tilde v$ on $\mathcal{A}$. □

14. Tonelli Hamiltonians and Lagrangians

We now establish part of the relationship between viscosity solutions, weak KAM solutions, and the Lax–Oleinik semi-group for a Tonelli Hamiltonian. A reference for this part is [4]. Another reference is [10].

In this section we will always suppose that the manifold is compact. We first recall the definition of a Tonelli Hamiltonian.

Definition 14.1. Let M be a compact manifold. A Hamiltonian $H\colon T^*M \to \mathbb{R}$ is said to be Tonelli if it is at least $\mathcal{C}^2$, and satisfies the following two conditions.

- **(i) Superlinearity:** for every $K \geqslant 0$, there exists $\mathcal{C}^*(K) < \infty$ such that, for all $(x,p) \in T^*M$,
$$H(x,p) \geqslant K\|p\|_x - C^*(K).$$
- **(ii) $\mathcal{C}^2$ strict convexity in the fibres:** for every $(x,p) \in T^*M$, the second derivative along the fibres $\partial^2 H/\partial p^2(x,p)$ is (strictly) positive definite.

Note that condition (i) is independent of the choice of a Riemannian metric on M. In fact, all Riemannian metrics on the compact manifold M are equivalent. Moreover, condition (i) implies that H is coercive.

To such a Hamiltonian we associate a Lagrangian $L\colon TM \to \mathbb{R}$ defined by

$$L(x,v) = \max_{p \in T_x^*M} \langle p, v\rangle_x - H(x,p), \quad \forall (x,v) \in TM.$$

Since H is of class $\mathcal{C}^2$, finite everywhere, superlinear and $\mathcal{C}^2$ strictly convex in each fibre T_x^*M, it is well known that L is finite everywhere of class $\mathcal{C}^2$, superlinear and $\mathcal{C}^2$ strictly convex in each fibre T_xM, and satisfies, for all $(x,p) \in T^*M$,

$$H(x,p) = \max_{v \in T_xM} \langle p, v\rangle_x - L(x,v).$$

DEFINITION 14.2 (evolution dominated function). A function $U\colon [0,+\infty[\,\times M \to \mathbb{R}$ is said to be evolution dominated by the Tonelli Lagrangian L associated to the Tonelli Hamiltonian H if, for every continuous piecewise $\mathcal{C}^1$ curve $\gamma\colon [a,b] \to M$, with $0 \leqslant a \leqslant b$, we have

$$U(b,\gamma(b)) - U(a,\gamma(a)) \leqslant \int_a^b L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.$$

Note that an evolution dominated function is not necessarily continuous. In fact, since L is superlinear, we have $c = \inf L > -\infty$. If $\rho\colon [0,+\infty[\, \to \mathbb{R}$ is any non-increasing (not necessarily continuous) function, then $U(t,x) = ct + \rho(t)$ is evolution dominated by L.

EXERCISE 12.

(i) Show that a function $U\colon [0,+\infty[\,\times M \to \mathbb{R}$ is evolution dominated by L if and only if, for every continuous piecewise $\mathcal{C}^1$ curve $\gamma\colon [\alpha,\beta] \to M$, with $\alpha,\beta \in \mathbb{R}$, $\alpha \leqslant \beta$, and every $a \geqslant 0$, we have

$$U(a+\beta-\alpha,\gamma(\beta)) - U(a,\gamma(\alpha)) \leqslant \int_\alpha^\beta L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.$$

(Hint: reparametrize the curve γ by a shift in time.)

(ii) Suppose that $U\colon [0,+\infty[\,\times M \to \mathbb{R}$ is evolution dominated by L. If $a \geqslant 0$ show that $V(t,x) = U(t+a,x)$ is also evolution dominated by L.

PROPOSITION 14.3. *If a continuous function $U\colon [0,+\infty[\,\times M \to \mathbb{R}$ is evolution dominated by the Tonelli Lagrangian L associated to the Tonelli Hamiltonian H, then U is a viscosity subsolution of*

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0$$

on the open set $]0,+\infty[\,\times M$.

Proof. Suppose $\phi \geqslant U$, with ϕ of class $\mathcal{C}^1$ and $\phi(t_0,x_0) = U(t_0,x_0)$, where $t_0 > 0$. Fix $v \in T_{x_0}M$, and pick a $\mathcal{C}^1$ curve $\gamma\colon [0,t_0] \to M$ such that $(\gamma(t_0),\dot\gamma(t_0)) = (x_0,v)$.

If $0 \leqslant t \leqslant t_0$, we have

$$U(t_0, \gamma(t_0)) - U(t, \gamma(t)) \leqslant \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s. \tag{$*$}$$

Since $\phi \geqslant U$, with equality at (t_0, x_0), noting that $\gamma(t_0) = x_0$, we obtain from $(*)$ that, for all $t \in]0, t_0[$,

$$\phi(t_0, \gamma(t_0)) - \phi(t, \gamma(t)) \leqslant \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s.$$

Dividing by $t_0 - t > 0$, and letting $t \to t_0$, we get, for all $v \in T_{x_0}M$,

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + \frac{\partial \phi}{\partial x}(t_0, x_0)(v) \leqslant L(x_0, v).$$

Since

$$H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) = \sup_{v \in T_{x_0}M} \frac{\partial \phi}{\partial x}(t_0, x_0)(v) - L(x_0, v),$$

we obtain

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) \leqslant 0.$$

This finishes the proof. □

An important object of the theory is the Lax–Oleinik semi-group. We recall its definition and some of its properties, and send the reader to the last section of [4], or to [10].

If $u\colon M \to \mathbb{R}$ is a continuous function, and $t > 0$, we define $T_t^- u\colon M \to \mathbb{R}$ by

$$T_t^- u(x) = \inf_\gamma \left\{ u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \right\},$$

where the infimum is taken over all the continuous piecewise $\mathcal{C}^1$ curves $\gamma\colon [0, t] \to M$ such that $\gamma(t) = x$.

In fact, for each $t > 0$ the function $T_t^- u$ is continuous (and even Lipschitz). Moreover, setting $T_0^- u = u$, the function $(t, x) \mapsto T_t^- u(x)$ is continuous on $[0, +\infty[\times M$, and is locally Lipschitz on $]0, +\infty[\times M$.

Moreover, the family T_t^-, $t \geqslant 0$, is a semi-group, i.e., for all $t, t' \geqslant 0$ and for all $u \in C^0(M, \mathbb{R})$,

$$T_{t+t'}^- u = T_t^- T_{t'}^- u.$$

Exercise 13.

(i) Suppose that $U\colon [0, +\infty[\times M \to \mathbb{R}$ is a continuous function. For $a \geqslant 0$, set $U_a(x) = U(a, x)$. Show that U is evolution dominated by L if and only if, for every $t, a \geqslant 0$, we have $U_{t+a} \leqslant T_t^- U_a$.

(ii) If $u \in C^0(M, \mathbb{R})$ and $U(t, x) = T_t^- u(x)$, show that U is evolution dominated by L.

THEOREM 14.4. *If* $u \in C^0(M,\mathbb{R})$ *and* $U(t,x) = T_t^- u(x)$, *then* U *is a viscosity solution of*

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0, \tag{EHJ}$$

on the open subset $]0,+\infty[\times M$.

Proof. By proposition 14.3 and part (ii) of exercise 13, the function U is a viscosity subsolution of (EHJ) on $]0,+\infty[\times M$.

To prove that U is a supersolution, we consider $\psi \leqslant U$, with ψ of class $\mathcal{C}^1$. Suppose $U(t_0,x_0) = \psi(t_0,x_0)$ with $t_0 > 0$.

As is well known, by Tonelli's theorem, the infimum in the definition of $T_t^- u(x)$ is attained by a curve which is a minimizer, hence, at least $\mathcal{C}^2$. Therefore, we can pick a $\mathcal{C}^2$ curve $\gamma\colon [0,t_0] \to M$ such that $\gamma(t_0) = x_0$ and

$$U(t_0,x_0) = T_{t_0}^- u(x_0) = u(\gamma(0)) + \int_0^{t_0} L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.$$

Since $U(0,\gamma(0)) = u(\gamma(0))$, this can be rewritten as

$$U(t_0,x_0) - U(0,\gamma(0)) = \int_0^{t_0} L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s. \tag{$**$}$$

Applying the fact that U is evolution dominated twice, for every $t \in [0,t_0]$ we obtain

$$U(t_0,x_0) - U(t,\gamma(t)) \leqslant \int_t^{t_0} L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s,$$
$$U(t,\gamma(t)) - U(0,\gamma(0)) \leqslant \int_0^{t} L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.$$

Adding these two inequalities, by $(**)$, we in fact obtain an equality. Hence, we must have, for all $t \in [0,t_0]$,

$$U(t_0,\gamma(t_0)) - U(t,\gamma(t)) = \int_t^{t_0} L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.$$

Since $\psi \leqslant U$, with equality at (t_0,x_0), for every $t \in [0,t_0]$ we obtain

$$\psi(t_0,\gamma(t_0)) - \psi(t,\gamma(t)) \geqslant \int_t^{t_0} L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.$$

Dividing by $t_0 - t > 0$ and letting $t \to t_0$, we get

$$\frac{\partial\psi}{\partial t}(t_0,x_0) + \frac{\partial\psi}{\partial x}(t_0,x_0)(\dot\gamma(t_0)) \geqslant L(x_0,\dot\gamma(t_0)).$$

By definition of L, we have

$$L(x_0,\dot\gamma(t_0)) \geqslant \frac{\partial\psi}{\partial x}(t_0,x_0)(\dot\gamma(t_0)) - H\left(x_0, \frac{\partial\psi}{\partial x}(t_0,x_0)\right).$$

It follows that

$$\frac{\partial\psi}{\partial t}(t_0,x_0) + \frac{\partial\psi}{\partial x}(t_0,x_0)(\dot\gamma(t_0)) \geqslant \frac{\partial\psi}{\partial x}(t_0,x_0)(\dot\gamma(t_0)) - H\left(x_0, \frac{\partial\psi}{\partial x}(t_0,x_0)\right).$$

Therefore,

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)\right) \geqslant 0.$$

□

Since the continuous function $(t,x) \mapsto T_t^- u(x), (t,x) \in [0,+\infty[\times M$, is locally Lipschitz on $]0,+\infty[\times M$, and is a viscosity solution of

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0$$

on $]0,+\infty[\times M$, we can apply the uniqueness statement of corollary 7.5 to obtain the following theorem.

THEOREM 14.5. *Let $H\colon T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian on the compact manifold M. Suppose that the continuous function $U\colon [0,+\infty[\times M \to \mathbb{R}$ is a viscosity solution of*

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0$$

on the open set $]0,+\infty[\times M$. Then $U(t,x) = T_t^- u(x)$ for every $(t,x) \in [0,+\infty[\times M$, where $u\colon M \to \mathbb{R}$ is defined by $u(x) = U(x,0)$.

We now conclude with the characterization of the solutions of the HJE by the Lax–Oleinik semi-group.

THEOREM 14.6. *Let $H\colon T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian on the compact manifold M. A continuous function $u\colon M \to \mathbb{R}$ is a viscosity solution of $H(x, d_x u) = c$ if and only if $u = T_t^- u + ct$ for all $t \geqslant 0$.*

Proof. We set $U(t,x) = u(x) - ct$. By exercise 4, the function u is a viscosity solution of

$$H(x, d_x u) = c$$

on M if and only if U is a viscosity solution of

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0$$

on $]0,+\infty[\times M$. It now follows from theorem 14.5 that u is a viscosity solution of $H(x, d_x u) = c$ if and only if $u - ct = U(t,x) = T_t^- u(x)$, for all $x \in M$ and $t \geqslant 0$. □

EXERCISE 14. Suppose that $H\colon T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian on the compact manifold M. Assume that the continuous functions $U, V\colon [0,+\infty[\times M \to \mathbb{R}$ are, respectively, a viscosity subsolution and a viscosity supersolution of the evolutionary HJE

$$\frac{\partial U}{\partial t}(t,x) + H\left(x, \frac{\partial U}{\partial x}(t,x)\right) = 0 \tag{EHJ}$$

on the open set $]0,+\infty[\times M$. For $a \geqslant 0$, define $U_a, V_a\colon M \to \mathbb{R}$ by $U_a(x) = U(a,x)$ and $V_a(x) = V(a,x)$.

(i) Show that, for all $t \geqslant 0$, we have

$$U_{t+a} \leqslant T_t^- U_a \quad \text{and} \quad T_t^- V_a \leqslant V_{t+a}.$$

(ii) Show that U is evolution dominated by L.

(iii) Conclude that a continuous function on $[0, +\infty[\, \times M$ is a viscosity subsolution of (EHJ) on $]0, +\infty[\, \times M$ if and only if it is evolution dominated by L.

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