On the Hausdorff Dimension of the Mather Quotient

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Abstract

Under appropriate assumptions on the dimension of the ambient manifold and the regularity of the Hamiltonian, we show that the Mather quotient is small in term of the Hausdorff dimension. Then we present applications in dynamics. © 2008 Wiley Periodicals, Inc.

1 Introduction

Let M be a smooth manifold without boundary. We denote by TM the tangent bundle and by $\pi : TM \to M$ the canonical projection. A point in TM will be denoted by (x, v) with $x \in M$ and $v \in T_xM = \pi^{-1}(x)$. In the same way a point of the cotangent bundle T^*M will be denoted by (x, p) with $x \in M$ and $p \in T_x^*M$ a linear form on the vector space T_xM. We will suppose that g is a complete Riemannian metric on M. For $v \in T_xM$, the norm $\|v\|_x$ is $g_x(v, v)^{1/2}$. We will denote by $\|\cdot\|_x$ the dual norm on T^*M. Moreover, for every pair $x, y \in M$, $d(x, y)$ will denote the Riemannian distance from x to y.

We will assume throughout this paper that $H : T^*M \to \mathbb{R}$ is a Hamiltonian of class $C^{k,\alpha}$, with $k \geq 2$, $\alpha \in [0, 1]$, which satisfies the three following conditions:

(H1) C^2*-strict convexity*: $\forall (x, p) \in T^*M$, the second derivative along the fibers $\partial^2 H/\partial p^2(x, p)$ is strictly positive definite;

(H2) *uniform superlinearity*: for every $K \geq 0$ there exists a finite constant $C(K)$ such that

$$\forall (x, p) \in T^*M, \quad H(x, p) \geq K\|p\|_x + C(K);$$

(H3) *uniform boundedness in the fibers*: for every $R \geq 0$, we have

$$\sup_{x\in M} \{H(x, p) \mid \|p\|_x \leq R\} < +\infty.$$

Communications on Pure and Applied Mathematics, Vol. LXII, 0445–0500 (2009)

By the weak KAM theorem we know that, under the above conditions, there is $c(H) \in \mathbb{R}$ such that the Hamilton-Jacobi equation

$$H(x, d_x u) = c$$

admits a global viscosity solution $u : M \to \mathbb{R}$ for $c = c(H)$ and does not admit such solution for $c < c(H)$; see [7, 13, 15, 17, 24]. In fact, for $c < c(H)$, the Hamilton-Jacobi equation does not admit any viscosity subsolution (for the theory of viscosity solutions, we refer the reader to the monographs [1, 2, 15]). Moreover, if M is assumed to be compact, then $c(H)$ is the only value of c for which the Hamilton-Jacobi equation above admits a viscosity solution. The constant $c(H)$ is called the *critical value* or the *Mañé critical value* of H. In what follows, a viscosity solution $u : M \to \mathbb{R}$ of $H(x, d_x u) = c(H)$ will be called a *critical viscosity solution* or a *weak KAM solution*, while a viscosity subsolution u of $H(x, d_x u) = c(H)$ will be called a *critical viscosity subsolution* (or *critical subsolution* if u is at least C^1).

The Lagrangian $L : TM \to \mathbb{R}$ associated to the Hamiltonian H is defined by

$$\forall (x, v) \in TM, \quad L(x, v) = \max_{p \in T_x^* M} \{p(v) - H(x, p)\}.$$

Since H is of class C^k, with $k \geq 2$, and satisfies the three conditions (H1)–(H3), it is well-known (see, for instance, [15] or [17, lemma 2.1])) that L is finite everywhere of class C^k and is a Tonelli Lagrangian, i.e., satisfies the analogues of conditions (H1)–(H3). Moreover, the Hamiltonian H can be recovered from L by

$$\forall (x, p) \in T_x^* M, \quad H(x, p) = \max_{v \in T_x M} \{p(v) - L(x, v)\}.$$

Therefore the following inequality is always satisfied:

$$p(v) \leq L(x, v) + H(x, p).$$

This inequality is called the Fenchel inequality. Moreover, due to the strict convexity of L, we have equality in the Fenchel inequality if and only if

$$(x, p) = \mathcal{L}(x, v),$$

where $\mathcal{L} : TM \to T^* M$ denotes the Legendre transform defined as

$$\mathcal{L}(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v) \right).$$

Under our assumption $\mathcal{L}$ is a diffeomorphism of class at least C^1. We will denote by ϕ_t^L the Euler-Lagrange flow of L, and by X_L the vector field on TM that generates the flow ϕ_t^L. If we denote by ϕ_t^H the Hamiltonian flow of H on $T^* M$, then, as is well-known (see, e.g., [15]), this flow ϕ_t^H is conjugate to ϕ_t^L by the Legendre transform $\mathcal{L}$. Moreover, thanks to assumptions (H1)–(H3), the flow ϕ_t^H (and so also ϕ_t^L) is complete; see [17].

As done by Mather in [29], it is convenient to introduce for $t > 0$ fixed the function $h_t : M \times M \to \mathbb{R}$ defined by

$$\forall x, y \in M, \quad h_t(x, y) = \inf \int_0^t L(\gamma(s), \dot{\gamma}(s))ds,$$

where the infimum is taken over all the absolutely continuous paths $\gamma : [0, t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$. The *Peierls barrier* is the function $h : M \times M \to \mathbb{R}$ defined by

$$h(x, y) = \liminf_{t \to \infty} \{h_t(x, y) + c(H)t\} .$$

It is clear that this function satisfies for all $t > 0$

$$\begin{aligned} \forall x, y, z \in M, \quad h(x, z) &\le h(x, y) + h_t(y, z) + c(H)t, \\ h(x, z) &\le h_t(x, y) + c(H)t + h(y, z), \end{aligned}$$

and therefore it also satisfies the triangle inequality

$$\forall x, y, z \in M, \quad h(x, z) \le h(x, y) + h(y, z).$$

Moreover, given a weak KAM solution u, we have

$$\forall x, y \in M, \quad u(y) - u(x) \le h(x, y).$$

In particular, we have $h > -\infty$ everywhere. It follows, from the triangle inequality, that the function h is either identically $+\infty$ or is finite everywhere. If M is compact, h is finite everywhere. In addition, if h is finite, then for each $x \in M$ the function $h_x(\cdot) = h(x, \cdot)$ is a critical viscosity solution (see [15] or [18]). The *projected Aubry set* $\mathcal{A}$ is defined by

$$\mathcal{A} = \{x \in M \mid h(x, x) = 0\}.$$

Following Mather, see [29, p. 1370], we symmetrize h to define the function $\delta_M : M \times M \to \mathbb{R}$ by

$$\forall x, y \in M, \quad \delta_M(x, y) = h(x, y) + h(y, x).$$

Since h satisfies the triangle inequality and $h(x, x) \ge 0$ everywhere, the function δ_M is symmetric and everywhere nonnegative and satisfies the triangle inequality. The restriction $\delta_M : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ is a genuine semidistance on the projected Aubry set. We will call this function δ_M the *Mather semidistance* (even when we consider it on M rather than on $\mathcal{A}$). We define the *Mather quotient* $(\mathcal{A}_M, \delta_M)$ to be the metric space obtained by identifying two points $x, y \in \mathcal{A}$ if their semidistance $\delta_M(x, y)$ vanishes (we mention that this set is also called the *quotient Aubry set*). When we consider δ_M on the quotient space $\mathcal{A}_M$, we will call it the *Mather distance*.

In [32], Mather formulated the following problem:

> *Mather's problem*. If L is C^∞, is the set $\mathcal{A}_M$ totally disconnected for the topology of δ_M; i.e., is each connected component of $\mathcal{A}_M$ reduced to a single point?

In [31], Mather brought a positive answer to that problem in low dimension. More precisely, he proved that if M has dimension 2 or if the Lagrangian is the kinetic energy associated to a Riemannian metric on M in dimension ≤ 3, then the Mather quotient is totally disconnected. Notice that one can easily show that for a dense set of Hamiltonians, the set $(\mathcal{A}_M, \delta_M)$ is reduced to one point; see [26]. Mather mentioned in [32, p. 1668] that it would be even more interesting to be able to prove that the Mather quotient has vanishing one-dimensional Hausdorff measure, because this implies the upper semicontinuity of the mapping $H \mapsto \mathcal{A}$. He also stated that for Arnold's diffusion a result generic in the Lagrangian but true for every cohomology class was more relevant. This was obtained recently by Bernard and Contreras [6].

The aim of the present paper is to show that the vanishing of the one-dimensional Hausdorff measure of the Mather quotient is satisfied under various assumptions. Let us state our results.

THEOREM 1.1 *If* $\dim M = 1, 2$ *and* H *of class* C^2 *or* $\dim M = 3$ *and* H *of class* $C^{k,1}$ *with* $k \geq 3$*, then the Mather quotient* $(\mathcal{A}_M, \delta_M)$ *has vanishing one-dimensional Hausdorff measure.*

Above the projected Aubry $\mathcal{A}$, there is a compact subset $\tilde{\mathcal{A}} \subset TM$ called the Aubry set (see Section 2.1). The projection $\pi : TM \to M$ induces a homeomorphism $\pi|_{\tilde{\mathcal{A}}}$ from $\tilde{\mathcal{A}}$ onto $\mathcal{A}$ (whose inverse is Lipschitz by a theorem due to Mather). The Aubry set can be defined as the set of $(x, v) \in TM$ such that $x \in \mathcal{A}$ and v is the unique element in T_xM such that $d_xu = \partial L/\partial v(x, v)$ for any critical viscosity subsolution u. The Aubry set is invariant under the Euler-Lagrange flow $\phi_t^L : TM \to TM$. Therefore, for each $x \in \mathcal{A}$, there is only one orbit of ϕ_t^L in $\tilde{\mathcal{A}}$ whose projection passes through x. We define the *stationary Aubry set* $\tilde{\mathcal{A}}^0 \subset \tilde{\mathcal{A}}$ as the set of points in $\tilde{\mathcal{A}}$ that are fixed points of the Euler-Lagrange flow $\phi_t(x, v)$, i.e.,

$$\tilde{\mathcal{A}}^0 = \{(x, v) \in \tilde{\mathcal{A}} \mid \forall t \in \mathbb{R},\ \phi_t^L(x, v) = (x, v)\}.$$

In fact (see Proposition 3.2), it can be shown that $\tilde{\mathcal{A}}^0$ is the intersection of $\tilde{\mathcal{A}}$ with the zero section of TM,

$$\tilde{\mathcal{A}}^0 = \{(x, 0) \mid (x, 0) \in \tilde{\mathcal{A}}\}.$$

We define the *projected stationary Aubry set* $\mathcal{A}^0$ as the projection onto M of $\tilde{\mathcal{A}}^0$,

$$\mathcal{A}^0 = \{x \mid (x, 0) \in \tilde{\mathcal{A}}\}.$$

At the very end of his paper [31], Mather noticed that the argument he used in the case where L is a kinetic energy in dimension 3 proves the total disconnectedness of the Mather quotient in dimension 3 as long as $\mathcal{A}_M^0$ is empty. In fact, if we consider the restriction of δ_M to $\mathcal{A}^0$, we have the following result on the quotient metric space $(\mathcal{A}_M^0, \delta_M)$:

THEOREM 1.2 *Suppose that L is at least C^2 and that the restriction $x \mapsto L(x,0)$ of L to the zero section of TM is of class $C^{k,1}$. Then $(\mathcal{A}^0_M, \delta_M)$ has vanishing Hausdorff measure in dimension* $2\dim M/(k+3)$. *In particular, if $k \geq 2\dim M - 3$ then $\mathcal{H}^1(\mathcal{A}^0_M, \delta_M) = 0$, and if $x \mapsto L(x,0)$ is C^∞, then $(\mathcal{A}^0_M, \delta_M)$ has zero Hausdorff dimension.*

As a corollary, we have the following result, which was more or less already mentioned by Mather in [32, sec. 19, p. 1722] and proved by Sorrentino [36].

COROLLARY 1.3 *Assume that H is of class C^2 and that its associated Lagrangian L satisfies the following conditions*:

(i) $\forall x \in M$, $\min_{v \in T_x M} L(x,v) = L(x,0)$.
(ii) *The mapping $x \in M \mapsto L(x,0)$ is of class $C^{l,1}(M)$ with $l \geq 1$.*

If $\dim M = 1, 2$ *or* $\dim M \geq 3$ *and* $l \geq 2\dim M - 3$, *then $(\mathcal{A}_M, \delta_M)$ is totally disconnected. In particular, if $L(x,v) = \frac{1}{2}\|v\|_x^2 - V(x)$, with $V \in \mathrm{C}^{l,1}(M)$ and $l \geq 2\dim M - 3$ ($V \in \mathrm{C}^2(M)$ if $\dim M = 1, 2$), then $(\mathcal{A}_M, \delta_M)$ is totally disconnected.*

Since $\mathcal{A}^0$ is the projection of the subset $\tilde{\mathcal{A}}^0 \subset \tilde{\mathcal{A}}$ consisting of points in $\tilde{\mathcal{A}}$ that are fixed under the Euler-Lagrange flow ϕ_t^L, it is natural to consider $\mathcal{A}^p$ the set of $x \in \mathcal{A}$ that are projections of a point $(x,v) \in \tilde{\mathcal{A}}$ whose orbit under the Euler-Lagrange flow ϕ_t^L is periodic with strictly positive period. We call this set the *projected periodic Aubry set*. We have the following result:

THEOREM 1.4 *If* $\dim M \geq 2$ *and H of class $C^{k,1}$ with $k \geq 2$, then $(\mathcal{A}^p_M, \delta_M)$ has vanishing Hausdorff measure in dimension* $8\dim M/(k+8)$. *In particular, if $k \geq 8\dim M - 8$, then $\mathcal{H}^1(\mathcal{A}^p_M, \delta_M) = 0$, and if H is C^∞, then $(\mathcal{A}^p_M, \delta_M)$ has zero Hausdorff dimension.*

In the case of compact surfaces, using the finiteness of exceptional minimal sets of flows, we have:

THEOREM 1.5 *If M is a compact surface of class C^∞ and H is of class C^∞, then $(\mathcal{A}_M, \delta_M)$ has zero Hausdorff dimension.*

In the last section, we present applications to the theory of dynamical systems, of which Theorem 1.6 below is a corollary. If X is a C^k vector field on M, with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is defined by

$$\forall (x,v) \in TM, \quad L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2.$$

We will denote by $\mathcal{A}_X$ the projected Aubry set of the Lagrangian L_X.

The first author has raised the following problem; compare with the list of questions `http://www.aimath.org/WWN/dynpde/articles/html/20a/`.

Problem. Let $L_X : TM \to \mathbb{R}$ be the Mañé Lagrangian associated to the C^k vector field X ($k \geq 2$) on the compact connected manifold M.

(1) Is the set of chain-recurrent points of the flow of X on M equal to the projected Aubry set $\mathcal{A}_X$?
(2) Give a condition on the dynamics of X that insures that the only weak KAM solutions are the constants.

The theorems obtained in the first part of the paper together with the applications in dynamics developed in Section 6 give an answer to this question when $\dim M \leq 3$.

THEOREM 1.6 *Let X be a C^k vector field, with $k \geq 2$, on the compact connected C^∞ manifold M. Assume that one of the following conditions hold*:

(i) *The dimension of M is* 1 *or* 2*.*
(ii) *The dimension of M is* 3*, and the vector field X never vanishes.*
(iii) *The dimension of M is* 3*, and X is of class $C^{3,1}$.*

Then the projected Aubry set $\mathcal{A}_X$ of the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is the set of chain-recurrent points of the flow of X on M. Moreover, the constants are the only weak KAM solutions for L_X if and only if every point of M is chain-recurrent under the flow of X.

The outline of the paper is the following: Sections 2 and 3 are devoted to preparatory results. Section 4 is devoted to the proofs of Theorems 1.1, 1.2, and 1.4. Sections 5 and 6 present applications in dynamics.

2 Preliminary Results

Throughout this section, M is assumed to be a complete Riemannian manifold. As before, $H : T^*M \to \mathbb{R}$ is a Hamiltonian of class at least C^2 satisfying the three usual conditions (H1)–(H3), and L is the Tonelli Lagrangian that is associated to it by Fenchel's duality.

2.1 Some Facts about the Aubry Set

We recall the results of Mather on the Aubry set and also an important complement due to Dias Carneiro.

The following results are due to Mather; see [28, 29] for the proof in the compact case.

THEOREM 2.1 (Mather) *There exists a closed subset $\tilde{\mathcal{A}} \subset TM$ such that*:

(i) *The set $\tilde{\mathcal{A}}$ is invariant under the Euler-Lagrange flow.*
(ii) *The projection $\pi : TM \to M$ is injective on $\tilde{\mathcal{A}}$. Moreover, we have $\pi(\tilde{\mathcal{A}}) = \mathcal{A}$, and the inverse map $(\pi|\tilde{\mathcal{A}})^{-1} : \mathcal{A} \to \tilde{\mathcal{A}}$ is locally Lipschitz.*

(iii) *Let (x,v) be in $\tilde{\mathcal{A}}$, and call $\gamma_{(x,v)}$ the curve that is the projection of the orbit $\phi_t^L(x,v)$ of the Euler-Lagrange flow through (x,v)*

$$\gamma_{(x,v)}(t) = \pi\phi_t(x,v).$$

This curve is entirely contained in $\mathcal{A}$, and it is an L-minimizer. Moreover, we have

$$\forall t, t' \in \mathbb{R}, \quad \delta_M(\gamma(t), \gamma(t')) = 0;$$

therefore the whole curve $\gamma_{(x,v)}$ projects to the same point as x in the Mather quotient.

(iv) *If $x \in \mathcal{A}$ and $\gamma_n : [0, t_n] \to M$ is a sequence of L-minimizers such that $t_n \to +\infty$, $\gamma_n(0) = \gamma_n(t_n) = x$, and $\int_0^{t_n} L(\gamma_n(s), \dot{\gamma}_n(s))ds + c(H)t_n \to 0$, then both sequences $\dot{\gamma}_n(0), \dot{\gamma}_n(t_n)$ converge in T_xM to the unique $v \in T_xM$ such that $(x,v) \in \tilde{\mathcal{A}}$.*

The following theorem of Dias Carneiro [9] is a nice complement to the theorem above:

THEOREM 2.2 *For every $(x,v) \in \tilde{\mathcal{A}}$, we have*

$$H\left(x, \frac{\partial L}{\partial v}(x,v)\right) = c(H).$$

We end this subsection with the following important estimation of the Mather semidistance (due to Mather); see [29, p. 1375].

PROPOSITION 2.3 *For every compact subset $K \subset M$, we can find a finite constant C_K such that*

$$\forall x \in \mathcal{A} \cap K,\ \forall y \in K, \quad \delta_M(x,y) \le C_K d(x,y)^2,$$

where d is the Riemannian distance on M.

Note that one can prove directly this proposition from the fact that h is locally semiconcave on $M \times M$ by using that $\delta_M \ge 0$, together with the fact that $\delta_M(x,x) = 0$ for every $x \in \mathcal{A}$.

2.2 Aubry Set and the Hamilton-Jacobi Equation

In this section we recast the above results in terms of viscosity solutions of the Hamilton-Jacobi as is done in [15, 17, 18].

We first recall the notion of domination. If $c \in \mathbb{R}$, a function $u : M \to \mathbb{R}$ is said to be dominated by $L + c$ (which we denote by $u \prec L + c$), if for every continuous piecewise C^1 curve $\gamma : [a,b] \to M$, $a < b$, we have

$$u(\gamma(b)) - u(\gamma(a)) \le \int_a^b L(\gamma(s), \dot{\gamma}(s))ds + c(b-a).$$

In fact, this is simply a different way to define the notion of viscosity subsolution for H. More precisely, we have (see [15] or [17, prop. 5.1, p. 12]):

THEOREM 2.4 *A $u : M \to \mathbb{R}$ is dominated by $L + c$ if and only if it is a viscosity subsolution of the Hamilton-Jacobi equation $H(x, d_x u) = c$. Moreover, we have $u \prec L + c$ if and only if u is Lipschitz and $H(x, d_x u) \leq c$ almost everywhere.*

Note that Rademacher's theorem states that every Lipschitz function is differentiable almost everywhere. For the proof that dominated functions are Lipschitz, see Lemma B.2. It is not difficult to see that a function $u : M \to \mathbb{R}$ is dominated by $L + c$ if and only if

$$\forall t > 0,\ \forall x, y \in M, \quad u(y) - u(x) \leq h_t(x, y) + ct.$$

With this notation, we observe that a function u is a critical subsolution if and only if $u \prec L + c(H)$.

We now give the definition of calibrated curves. If $u : M \to \mathbb{R}$ and $c \in \mathbb{R}$, we say that the curve $\gamma : [a, b] \to M$ is (u, L, c)-calibrated if we have the equality

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot\gamma(s))ds + c(b - a).$$

If γ is a curve defined on the not necessarily compact interval I, we will say that γ is (u, L, c)-calibrated if its restriction to any compact subinterval of I is (u, L, c)-calibrated.

In fact, this condition of calibration is useful only when $u \prec L + c$. In this case γ is an L-minimizer. Moreover, if $[a', b']$ is a subinterval of $[a, b]$, then the restriction $\gamma|[a', b']$ is also (u, L, c)-calibrated.

As in [15], if $u : M \to \mathbb{R}$ is a critical subsolution, we denote by $\tilde{\mathcal{I}}(u)$ the subset of TM defined as

$$\tilde{\mathcal{I}}(u) = \{(x, v) \in TM \mid \gamma_{(x,v)} \text{ is } (u, L, c(H))\text{-calibrated}\},$$

where $\gamma_{(x,v)}$ is the curve (already introduced in Theorem 2.1) defined on $\mathbb{R}$ by

$$\gamma_{(x,v)}(t) = \pi\phi_t^L(x, v).$$

The following properties of $\tilde{\mathcal{I}}(u)$ are shown in [15]:

THEOREM 2.5 *The set $\tilde{\mathcal{I}}(u)$ is invariant under the Euler-Lagrange flow ϕ_t^L. If $(x, v) \in \tilde{\mathcal{I}}(u)$, then $d_x u$ exists, and we have*

$$d_x u = \frac{\partial L}{\partial v}(x, v) \quad \textit{and} \quad H(x, d_x u) = c(H).$$

It follows that the restriction $\pi|_{\tilde{\mathcal{I}}(u)}$ of the projection is injective; therefore, if we set $\mathcal{I}(u) = \pi(\tilde{\mathcal{I}}(u))$, then $\tilde{\mathcal{I}}(u)$ is a continuous graph over $\mathcal{I}(u)$. Moreover, the map $x \mapsto d_x u$ is locally Lipschitz on $\mathcal{I}(u)$.

Since the inverse of the restriction $\pi|\tilde{\mathcal{I}}(u)$ is given by $x \mapsto \mathcal{L}^{-1}(x, d_x u)$, and the Legendre transform $\mathcal{L}$ is C^1, it follows that the inverse of $\pi|\tilde{\mathcal{I}}(u)$ is also locally Lipschitz on $\mathcal{I}$.

Using the sets $\tilde{\mathcal{I}}(u)$, one can give the following characterization of the Aubry set and its projection:

THEOREM 2.6 *The Aubry set $\tilde{\mathcal{A}}$ is given by*

$$\tilde{\mathcal{A}} = \bigcap_{u \in \mathcal{SS}} \tilde{\mathcal{I}}(u),$$

where $\mathcal{SS}$ is the set of critical viscosity subsolutions. The projected Aubry set $\mathcal{A}$, which is simply the image $\pi(\tilde{\mathcal{A}})$, is also

$$\mathcal{A} = \bigcap_{u \in \mathcal{SS}} \mathcal{I}(u).$$

Note that the fact that the Aubry set is a locally Lipschitz graph (i.e., part (ii) of Theorem 2.1) follows from the above results, since $\tilde{\mathcal{A}} \subset \tilde{\mathcal{I}}(u)$ for any critical subsolution u. Moreover, Theorem 2.2 also follows from the results above.

2.3 Mather Semidistance and Critical Subsolutions

While generalizing Mather's examples given in [32], the first author observed (see [14]) that a representation formula for δ_M in terms of C^1 critical subsolutions is extremely useful. This has also been used more recently by Sorrentino [36].

To explain this representation formula, as in Theorem 2.6, we call $\mathcal{SS}$ the set of critical viscosity subsolutions and $\mathcal{S}_-$ the set of critical viscosity (or weak KAM) solutions. Hence $\mathcal{S}_- \subset \mathcal{SS}$. If $u : M \to \mathbb{R}$ is a critical viscosity subsolution, we recall that

$$\forall x, y \in M, \quad u(y) - u(x) \leq h(x, y).$$

In [18], Fathi and Siconolfi proved that for every critical viscosity subsolution $u : M \to \mathbb{R}$, there exists a C^1 critical subsolution whose restriction to the projected Aubry set is equal to u. Recently Patrick Bernard [4] has even shown that u can be assumed $C^{1,1}$, i.e., differentiable everywhere with a (locally) Lipschitz derivative; see also Appendix B below. In what follows, we denote by $\mathcal{SS}^1$ (respectively, $\mathcal{SS}^{1,1}$) the set of C^1 (respectively, $C^{1,1}$) critical subsolutions. The representation formula is given by the following lemma:

LEMMA 2.7 *For every $x, y \in \mathcal{A}$,*

$$\begin{aligned}
\delta_M(x, y) &= \max_{u_1, u_2 \in \mathcal{S}_-} \{(u_1 - u_2)(y) - (u_1 - u_2)(x)\} \\
&= \max_{u_1, u_2 \in \mathcal{SS}} \{(u_1 - u_2)(y) - (u_1 - u_2)(x)\} \\
&= \max_{u_1, u_2 \in \mathcal{SS}^1} \{(u_1 - u_2)(y) - (u_1 - u_2)(x)\} \\
&= \max_{u_1, u_2 \in \mathcal{SS}^{1,1}} \{(u_1 - u_2)(y) - (u_1 - u_2)(x)\}.
\end{aligned}$$

PROOF: Let $x, y \in \mathcal{A}$ be fixed. First, we notice that if u_1 and u_2 are two critical viscosity subsolutions, then we have

$$\begin{aligned}(u_1 - u_2)(y) - (u_1 - u_2)(x) &= (u_1(y) - u_1(x)) + (u_2(x) - u_2(y)) \\ &\le h(x, y) + h(y, x) = \delta_M(x, y).\end{aligned}$$

On the other hand, if we define $u_1, u_2 : M \to \mathbb{R}$ by $u_1(z) = h(x, z)$ and $u_2(z) = h(y, z)$ for any $z \in M$, by the properties of h the functions u_1 and u_2 are both critical viscosity solutions. Moreover,

$$\begin{aligned}(u_1 - u_2)(y) - (u_1 - u_2)(x) &= (h(x, y) - h(y, y)) - (h(x, x) - h(y, x)) \\ &= h(x, y) + h(y, x) = \delta_M(x, y),\end{aligned}$$

since $h(x, x) = h(y, y) = 0$. Thus we easily obtain the first and second equalities. The last equalities are an immediate consequence of the work of Fathi and Siconolfi and that of Bernard recalled above. □

2.4 Norton's Generalization of the Morse Vanishing Lemma

We will need in a crucial way Norton's elegant generalization of the Morse vanishing lemma; see [33, 35]. This result, like Ferry's lemma (see Lemma A.3) are the two basic pieces that allow us to prove generalizations of the Morse-Sard theorem (see, e.g., the work of Bates).

LEMMA 2.8 (Generalized Morse Vanishing Lemma) *Suppose M is an n-dimensional (separable) manifold endowed with a distance d coming from a Riemannian metric. Let $k \in \mathbb{N}$ and $\alpha \in [0, 1]$. Then for any subset $A \subset M$, we can find a countable family $B_i, i \in \mathbb{N}$, of C^1 embedded compact disks in M of dimension $\le n$ and a countable decomposition of $A = \bigcup_{i\in\mathbb{N}} A_i$, with $A_i \subset B_i$ for every $i \in \mathbb{N}$ such that every $f \in C^{k,\alpha}(M, \mathbb{R})$ vanishing on A satisfies, for each $i \in \mathbb{N}$,*

$$\forall y \in A_i, x \in B_i, \quad |f(x) - f(y)| \le M_i d(x, y)^{k+\alpha}, \tag{2.1}$$

for a certain constant M_i (depending on f).

Let us make some comments. In his statement of the lemma above (see [35]), Norton distinguishes a countable A_0 in his decomposition. In fact, in the statement we give, this corresponds to the (countable numbers of) disks in the family B_i where the dimension of the disk B_i is 0, in which case A_i is also a point. Therefore there is no need to distinguish this countable subset when formulating the generalized Morse vanishing lemma. The second comment is that we have stated this generalized Morse vanishing lemma, Lemma 2.8, directly for (separable) manifolds. This is a routine generalization of the case $M = \mathbb{R}^n$, which is done by Norton (see, for example, the way we deduce Lemma A.3 from Lemma A.1).

3 Proofs of Theorems 1.1, 1.2, 1.4, and 1.5

3.1 Proof of Theorem 1.1

Let us first assume that $\dim M = 1, 2$. The proof is the same as Mather's proof of total disconnectedness given in [31]. It also uses Proposition 2.3, but instead of using the results of Mather contained in [30], it uses the stronger Lemma A.3 due to Ferry and proved in Appendix A below.

We cover M by an increasing countable union K_n of compact subsets. For a given n, by Proposition 2.3 we can find a finite constant C_n such that

$$\forall x, y \in \mathcal{A} \cap K_n, \quad \delta_M(x, y) \le C_n d(x, y)^2.$$

Since $\dim M \le 2$ by Lemma A.3, we obtain that $(\mathcal{A} \cap K_n, \delta_M)$ has vanishing one-dimensional Hausdorff measure. Since $\mathcal{A}$ is the countable union of the $\mathcal{A} \cap K_n$, we also conclude that $(\mathcal{A}, \delta_M)$ has vanishing one-dimensional Hausdorff measure.

Let us now assume that $\dim M = 3$. The fact that $(\mathcal{A}^0_M, \delta_M)$ has vanishing one-dimensional Hausdorff measure will follow from Theorem 1.2. So it suffices to prove that the semimetric space $(\mathcal{A} \setminus \mathcal{A}^0, \delta_M)$ has vanishing one-dimensional Hausdorff measure.

Consider for every $x \in \mathcal{A}$ the unique vector $v_x \in T_x M$ such that $(x, v_x) \in \tilde{\mathcal{A}}$. Call γ_x the curve defined by $\gamma_x(t) = \pi\phi_t^L(x, v_x)$. Since $\tilde{\mathcal{A}}$ is invariant by ϕ_t^L, the projected Aubry set is laminated by the curves γ_x, $x \in \mathcal{A}$. Let us define $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}^0$. Since, by Proposition 3.2, any point of the form $(z, 0) \in \tilde{\mathcal{A}}$ is fixed under ϕ_t^L and $\dot{\gamma}_x(0) = v_x$, we have $\gamma_x(t) \in \mathcal{A}'$ for all $x \in \mathcal{A}'$ and all $t \in \mathbb{R}$. Moreover, the family γ_x, $x \in \mathcal{A}'$, is a genuine one-dimensional Lipschitz lamination on $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}^0$. For each $x \in \mathcal{A}'$, we can find a small C^∞ two-dimensional submanifold S_x of M such that S_x is transverse to γ_x. By transversality and continuity, the union U_x of the curves γ_y, $y \in \mathcal{A}'$, such that $\gamma_y \cap S_x \neq \varnothing$ is a neighborhood of x in $\mathcal{A}'$ (for the topology induced by the manifold topology). Therefore since M is metric separable, we can find a countable subfamily $(S_{x_i})_{i \in \mathbb{N}}$ such that $\gamma_y \cap (\bigcup_{i \in \mathbb{N}} S_{x_i}) \neq \varnothing$ for every $y \in \mathcal{A}'$. By part (iii) of Theorem 2.1 above, for every $z \in \mathcal{A}$ and every $t, t' \in \mathbb{R}$, we have

$$\delta_M(\gamma_z(t), \gamma_z(t')) = 0.$$

It follows that the countable union of the images of $S_{x_i} \cap \mathcal{A}$ in $\mathcal{A}_M$ covers the image of $\mathcal{A}'$ in $\mathcal{A}_M$. Therefore by the countable additivity of the Hausdorff measure, we have to show that $(S_{x_i} \cap \mathcal{A}, \delta_M)$ has one-dimensional Hausdorff measure equal to 0. Since S_{x_i} is two-dimensional, this follows from Proposition 2.3 and Lemma A.3 as above.

3.2 Proof of Theorem 1.2

Before giving the proof we need a better understanding of the sets $\tilde{\mathcal{A}}^0$ and $\mathcal{A}^0$.

LEMMA 3.1 *The function $\tilde{H} : M \to \mathbb{R}$ defined by*

$$\tilde{H}(x) = \inf\{H(x,p) \mid p \in T_x^* M\}$$

satisfies the following properties:

(i) *For every $x \in M$, we have $\tilde{H}(x) \le c(H)$.*

(ii) *We have $H(x,p) = \tilde{H}(x)$ if and only if $p = \partial L/\partial v(x,0)$.*

(iii) *For every $x \in M$, we have*

$$\tilde{H}(x) = H\left(x, \frac{\partial L}{\partial v}(x,0)\right) = -L(x,0).$$

Therefore $\tilde{H}$ is as smooth as $x \mapsto L(x,0)$.

(iv) *The point x is a critical point of $\tilde{H}$ (or of $x \mapsto L(x,0)$) if and only if the point $(x, \partial L/\partial v(x,0))$ is a critical point of H. In particular, the point $(x, \partial L/\partial v(x,0))$ is a critical point of H for every x such that $\tilde{H}(x) = c(H)$.*

PROOF: Since there exists a C^1 critical subsolution $u : M \to \mathbb{R}$ that satisfies

$$\forall x \in M, \quad H(x, d_x u) \le c(H),$$

we must have

$$\tilde{H}(x) = \inf\{H(x,p) \mid p \in T_x^* M\} \le c(H).$$

By strict convexity the infimum $\tilde{H}(x)$ is attained at the unique point $\tilde{p}(x) \in T_x^* M$ that satisfies

$$\frac{\partial H}{\partial p}(x, \tilde{p}(x)) = 0. \tag{3.1}$$

Because $(x,p) \mapsto (x, \partial H/\partial p(x,p))$ is the inverse of the Legendre transform $(x,v) \mapsto \partial L/\partial v(x,v)$, we obtain

$$\tilde{p}(x) = \frac{\partial L}{\partial v}(x,0),$$

and therefore by the Fenchel equality

$$\tilde{H}(x) = H(x, \tilde{p}(x)) = \frac{\partial L}{\partial v}(x,0) \cdot 0 - L(x,0) = -L(x,0).$$

To prove part (iv), we first observe that

$$\frac{\partial H}{\partial p}\left(x, \frac{\partial L}{\partial v}(x,0)\right) = 0.$$

Then we differentiate (in a coordinate chart) the equality obtained in (ii) to obtain

$$\begin{aligned} d_x \tilde{H} &= \frac{\partial H}{\partial x}\left(x, \frac{\partial L}{\partial v}(x,0)\right) + \frac{\partial H}{\partial p}\left(x, \frac{\partial L}{\partial v}(x,0)\right) \circ \frac{\partial^2 L}{\partial v\,\partial x}(x,0) \\ &= \frac{\partial H}{\partial x}\left(x, \frac{\partial L}{\partial v}(x,0)\right). \end{aligned}$$

Therefore, by the two previous equations, the first part of (iv) follows. The last part of (iv) is a consequence of (i), which implies that each x satisfying $\tilde{H}(x) = c(H)$ is a global maximum of $\tilde{H}$. □

We can now give a characterization of the stationary Aubry set $\tilde{\mathcal{A}}^0$.

PROPOSITION 3.2 *The set $\tilde{\mathcal{A}}^0$ of points in $\tilde{\mathcal{A}}$ that are fixed for the Euler-Lagrange flow ϕ_t^L is exactly the intersection of $\tilde{\mathcal{A}}$ with the zero section in TM, i.e.,*

$$\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}} \cap \{(x,0) \mid x \in M\}.$$

Its projection $\mathcal{A}^0 = \pi(\tilde{\mathcal{A}}^0)$ on M is precisely the set of points x in M at which $\tilde{H}$ takes the value $c(H)$, i.e.,

$$\mathcal{A}^0 = \{x \in M \mid \tilde{H}(x) = c(H)\}.$$

PROOF: Let (x, v) be in $\tilde{\mathcal{A}}^0$. Since the Euler-Lagrange flow ϕ_t^L is conjugated to the Hamiltonian flow ϕ_t^H of H by the Legendre transform $\mathcal{L}$, we obtain that $(x, \partial L/\partial v(x, v))$ is fixed under ϕ_t^H, and therefore $(x, \partial L/\partial v(x, v))$ is a critical point of H. In particular, we have

$$\frac{\partial H}{\partial p}\left(x, \frac{\partial L}{\partial v}(x,v)\right) = 0.$$

Since $(x, p) \mapsto (x, \partial H/\partial p(x, p))$ is the inverse of the Legendre transform, we conclude that $v = 0$, yielding the proof of the inclusion $\tilde{\mathcal{A}}^0 \subset \tilde{\mathcal{A}} \cap \{(x,0) \mid x \in M\}$.

Suppose now that that $(x, 0)$ is in $\tilde{\mathcal{A}}$. By Theorem 2.2, the Legendre transform of the Aubry set is contained in the set where H is equal to $c(H)$, i.e.,

$$H\left(x, \frac{\partial L}{\partial v}(x,0)\right) = c(H).$$

We obtain by Lemma 3.1 that x is a critical point of $\tilde{H}$, and therefore we get that $(x, \partial L/\partial v(v, 0))$ is a critical point of H. This implies that this point is invariant under ϕ_t^H; hence $(x, 0)$ is fixed under the Euler-Lagrange flow ϕ_t^L. By this we get the equality $\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}} \cap \{(x,0) \mid x \in M\}$.

Note that we have proved that if $(x, 0)$ in $\tilde{\mathcal{A}}$, then $\tilde{H}(x) = c(H)$. Therefore $\mathcal{A}^0$ is contained in the set $\tilde{H}^{-1}(c(H))$.

It remains to show that any x such that $\tilde{H}(x) = c(H)$ is in $\mathcal{A}^0$. Suppose that x is such that $\tilde{H}(x) = c(H)$. Since $\tilde{H}(x) = -L(x, 0)$, we get $L(x, 0) + c(H) = 0$. If we consider now the constant curve $\gamma :]-\infty, +\infty[\to \{x\}$, we see that

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))ds + c(H)t = \int_0^t L(x,0)ds + c(H)t = 0.$$

Therefore $h_t(x, x) + c(H)t = 0$ for every $t \geq 0$. This implies that $x \in \mathcal{A}$. It remains to show that the point $(x, v) \in \tilde{\mathcal{A}}$ above x is necessarily $(x, 0)$, which will imply that $x \in \mathcal{A}^0$. Note that again by Theorem 2.2 we have

$$H\left(x, \frac{\partial L}{\partial v}(x,v)\right) = c(H).$$

But $\inf_{p \in T_x^* M} H(x,p) = \tilde{H}(x) = c(H)$, and this infimum is only attained at $p = \partial L/\partial v(x,0)$. This implies that $\partial L/\partial v(x,v) = \partial L/\partial v(x,0)$. The invertibility of the Legendre transform yields $v = 0$. $\square$

We now start the proof of Theorem 1.2. Replacing L by $L + c(H)$, we can assume, without loss of generality, that $c(H) = 0$. Notice now that, for every compact subset $K \subset M$, there exists $\alpha_K \geq 0$ such that

$$x \in K, \quad H(x,p) \leq 0 \Longrightarrow \|p - \tilde{p}(x)\|_x \leq \alpha_K \sqrt{-\tilde{H}(x)}. \tag{3.2}$$

In fact, since $\partial^2 H/\partial p^2(x,p)$ is positive definite everywhere and the set

$$S(K) = \{(x,p) \in T^*M \mid x \in K,\ H(x,p) \leq 0\}$$

is compact, Taylor's formula (in integral form) yields a $\beta_K > 0$ such that

$$\begin{aligned} &\forall (x,p),\ (x,p') \in S(K), \\ &H(x,p) \geq H(x,p') + \frac{\partial H}{\partial p}(x,p')(p-p') + \beta_K \|p - p'\|_x^2. \end{aligned}$$

Using the equalities (3.1) and $\tilde{H}(x) = H(x, \tilde{p}(x))$, and that $H(x,p) \leq 0$ on $S(K)$, the inequality above yields

$$\forall (x,p) \in S(K), \quad 0 \geq \tilde{H}(x) + \beta_K \|p - \tilde{p}(x)\|_x^2.$$

This yields (3.2) with $\alpha_K = 1/\sqrt{\beta_K}$. If $u : M \to \mathbb{R}$ is a C^1 critical subsolution, we know that $H(x, d_x u) \leq 0$ for every $x \in M$; therefore we obtain

$$\forall x \in M, \quad \|d_x u - \tilde{p}(x)\|_x \leq \alpha_K \sqrt{-\tilde{H}(x)}.$$

It follows that for every pair u_1, u_2 of critical subsolutions, we have

$$\forall x \in M, \quad \|d_x(u_2 - u_1)\|_x \leq 2\alpha_K \sqrt{-\tilde{H}(x)}. \tag{3.3}$$

We now use Lemma 2.8 for $C^{k,1}$ functions to decompose $\mathcal{A}^0$ as

$$\mathcal{A}^0 = \bigcup_{i \in \mathbb{N}} A_i,$$

with each $A_i \subset B_i$, where $B_i \subset M$ is a C^1 embedded compact disk of dimension $\leq \dim M$. Since $\tilde{H}$ is a $C^{k,1}$ function vanishing on $\mathcal{A}^0$, by (2.1) we know that we can find for each $i \in \mathbb{N}$ a finite constant M_i such that

$$\forall x \in A_i,\ \forall y \in B_i, \quad -\tilde{H}(y) = |\tilde{H}(x) - \tilde{H}(y)| \leq M_i d(x,y)^{k+1}.$$

Since B_i is compact, we can combine this last inequality with (3.3) above to obtain for every pair of critical subsolutions u_1, u_2 and every $i \in \mathbb{N}$

$$\forall x \in A_i,\ \forall y \in B_i, \quad \|d_y(u_2 - u_1)\|_y \leq 2\alpha_{B_i}\sqrt{M_i}\, d(x, y)^{(k+1)/2}.$$

We know that B_i is C^1 diffeomorphic to the unit ball $\mathbb{B}^{n_i}$, with $n_i \in \{0, \dots, \dim M\}$. To avoid heavy notation we will identify in the remainder of the proof B_i with $\mathbb{B}^{n_i}$. Since this identification is C^1, we can replace in the inequality above the Riemannian norm by the Euclidean norm $\|\cdot\|_{\text{euc}}$ on $\mathbb{R}^{n_i}$ to obtain the inequality

$$\forall x \in A_i,\ \forall y \in B_i \approx \mathbb{B}^{n_i}, \quad \|d_y(u_2 - u_1)\|_{\text{euc}} \leq C_i \|y - x\|_{\text{euc}}^{(k+1)/2}.$$

for some suitable finite constant depending on i. If we integrate this inequality along the segment from x to y in $\mathbb{B}^{n_i} \approx B_i$, we obtain

$$\forall x \in A_i,\ \forall y \in B_i \approx \mathbb{B}^{n_i},$$
$$|(u_1 - u_2)(y) - (u_1 - u_2)(x)| \leq \tilde{C}_i \|y - x\|_{\text{euc}}^{\frac{k+1}{2}+1}.$$

By Lemma 2.7 we deduce that

$$\forall x, y \in A_i, \quad \delta_M(x, y) \leq \tilde{C}_i \|y - x\|_{\text{euc}}^{\frac{k+1}{2}+1}.$$

Since $A_i \subset B_i \approx \mathbb{B}^{n_i} \subset \mathbb{R}^{n_i}$, and obviously $1 + \frac{k+1}{2} > 1$, we conclude from Lemma A.1 that the Hausdorff measure $\mathcal{H}^{n_i/(1+(k+1)/2)}(A_i, \delta_M)$ is equal to 0. Therefore, since $n_i \leq \dim M$ and $\mathcal{A}^0$ is the countable union of the A_i, we conclude that

$$\mathcal{H}^{2\dim M/(k+3)}(\mathcal{A}^0, \delta_M) = 0.$$

In particular, if $k + 3 \geq 2\dim M$, that is, $k \geq 2\dim M - 3$, the one-dimensional Hausdorff dimension of $(\mathcal{A}^0_M, \delta_M)$ vanishes.

3.3 Proof of Theorem 1.4

We will give a proof of Theorem 1.4 that does not use conservation of energy (complicating a little bit some of the steps). It will use instead the completeness of the Euler-Lagrange flow, which is automatic for Tonelli Lagrangians independent of time; see [17, cor. 2.2, p. 6]. It can therefore be readily adapted to the case where L depends on time, is 1-periodic in time, and has a complete Euler-Lagrange flow as in the work of Mather [28, 29].

In a flow the period function on the periodic nonfixed points is not necessarily continuous. Therefore when we pick a local Poincaré section for a closed orbit, the nearby periodic points of the flow do not give rise to fixed points of the Poincaré return map. This will cause us some minor difficulties in the proof of Theorem 1.4. We will use the following general lemma to get around these problems easily.

PROPOSITION 3.3 *Let X be a metric space and $(\phi_t)_{t\in\mathbb{R}}$ a continuous flow on X. Call* $\mathrm{Fix}(\phi_t)$ *the set of fixed points of the flow $(\phi_t)_{t\in\mathbb{R}}$, and* $\mathrm{Per}(\phi_t)$ *the set of periodic nonfixed points of $(\phi_t)_{t\in\mathbb{R}}$. Let $T : \mathrm{Per}(\phi_t) \to\ ]0, \infty[$ be the function such that $T(x)$ is the smallest period > 0 of $x \in \mathrm{Per}(\phi_t)$.*

We can write $\mathrm{Per}(\phi_t)$ *as a countable union* $\mathrm{Per}(\phi_t) = \bigcup_{n\in\mathbb{N}} C_n$ *where each* C_n *is a closed subset on which the period map* T *is continuous.*

PROOF: For $t \in \mathbb{R}$, call F_t the set of fixed points of the map ϕ_t. Using the continuity of $(\phi_t)_{t\in\mathbb{R}}$ on the product $\mathbb{R}\times X$, it is not difficult to see that $\bigcup_{t\in[a,b]} F_t$ is a closed subset of X for every compact subinterval $[a,b]$ contained in $\mathbb{R}$. For $n \in \mathbb{Z}$ we set

$$F^n = \bigcup_{2^n\le t\le 2^{n+1}} F_t = \{x \in X \mid \exists t \in [2^n, 2^{n+1}] \text{ with } \phi_t(x) = x\}.$$

Note that F^m is closed. Moreover, since $\phi_t(x) = x$ with $2^{m-1} \le t \le 2^m$ implies $\phi_{2t}(x) = \phi_t \circ \phi_t(x) = \phi_t(x) = x$ and $2^m \le 2t \le 2^{m+1}$, we get $F^{m-1} \subset F^m$ for every $m \in \mathbb{Z}$. Therefore we have that $F^n \setminus F^{n-1} = F^n \setminus \bigcup_{i\le n-1} F^i$ is the set of periodic nonfixed points with $2^n < T(x) \le 2^{n+1}$. In particular, $\mathrm{Per}(\phi_t) = \bigcup_{n\in\mathbb{Z}} F^n \setminus F^{n-1}$. Note also that if $x \in F^n \setminus F^{n-1}$ and $t \in]0, 2^{n+1}]$ are such that $\phi_t(x) = x$, then necessarily $t = T(x)$. In fact, we have $t/T(x) \in \mathbb{N}^*$, but $t/T(x) \le 2^{n+1}/T(x) < 2$, hence $t/T(x) = 1$.

We now show that the period map T is continuous on $F^n \setminus F^{n-1}$. For this we have to show that for a sequence $x_\ell \in F^n \setminus F^{n-1}$ that converges to $x_\infty \in F^n \setminus F^{n-1}$, we necessarily have $T(x_\ell) \to T(x_\infty)$ when $\ell \to \infty$. Since $T(x_\ell) \in [2^n, 2^{n+1}]$, which is compact, it suffices to show that any accumulation point T of $T(x_\ell)$ satisfies $T = T(x_\infty)$. Pick up an increasing subsequence $\ell_k \nearrow \infty$ such that $T(x_{\ell_k}) \to T$ when $k \to \infty$. By continuity of the flow, $T \in [2^n, 2^{n+1}]$ and $\phi_T(x_\infty) = x_\infty$. Since $x_\infty \in F^n \setminus F^{n-1}$, by what we have shown above we have $T = T(x_\infty)$.

Since $\mathrm{Per}(\phi_t)$ is the countable union $\bigcup_{n\in\mathbb{Z}} F^n \setminus F^{n-1}$, to finish the proof of the lemma it remains to show that each $F^n \setminus F^{n-1}$ is itself a countable union of closed subsets of X. This is obvious because $F^n \setminus F^{n-1} = F^n \cap (X \setminus F^{n-1})$ is the intersection of a closed and an open subset in the metric X, but an open subset in a metric space is itself a countable union of closed sets. □

We will also need the following proposition, which relates the size of the derivative of a $C^{1,1}$ critical subsolution at a point to minimal actions of loops at that point. We will need to use Lipschitz functions from a compact subset of M to a compact subset of TM. We therefore need distances on M and TM. On M we already have a distance coming from the Riemannian metric. Since all distances obtained from Riemannian metrics are Lipschitz equivalent on compact subsets, the precise distance we use on TM is not important. We therefore just assume that we have chosen some Riemannian metric on TM (not necessarily related to the one on M), and we will use the distance on TM coming from this Riemannian metric.

PROPOSITION 3.4 *Suppose that* K *is a given compact set and that* $t_0, t_0' \in \mathbb{R}$ *satisfy* $0 < t_0 \le t_0'$. *We can find a compact set* K' *such that, for any finite number* ℓ, *we* *can find a finite number C such that any critical C^1 subsolution $u : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u)$ is Lipschitz on K' with Lipschitz constant $\leq \ell$ satisfies*

$$\forall x \in K,\ \forall t \in [t_0, t_0'], \quad [c(H) - H(x, d_x u)]^2 \leq C[h_t(x, x) + c(H)t].$$

Moreover, for every such ℓ, we can find a constant C' such that any pair of critical C^1 subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that both maps $x \mapsto (x, d_x u_i)$, $i = 1, 2$, are Lipschitz on K' with Lipschitz constant $\leq \ell$ satisfies

$$\forall x \in K,\ \forall t \in [t_0, t_0'], \quad \|d_x u_2 - d_x u_1\|_x^4 \leq C'[h_t(x, x) + c(H)t].$$

When M is compact, we can take $t_0' = +\infty$, and the above proposition becomes:

PROPOSITION 3.5 *Suppose the manifold M is compact and that $t_0 > 0$ is given. For any finite number ℓ, we can find a finite number C such that any critical C^1 subsolution $u : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u)$ is Lipschitz on M with Lipschitz constant $\leq \ell$ satisfies*

$$\forall x \in M,\ \forall t \geq t_0, \quad [c(H) - H(x, d_x u)]^2 \leq C[h_t(x, x) + c(H)t].$$

Moreover, for every such ℓ, we can find a constant C' such that any pair of critical C^1 subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that both maps $x \mapsto (x, d_x u_i)$, $i = 1, 2$, are Lipschitz on M with Lipschitz constant $\leq \ell$ satisfies

$$\forall x \in M,\ \forall t \geq t_0, \quad \|d_x u_2 - d_x u_1\|_x^4 \leq C'[h_t(x, x) + c(H)t].$$

To prove these propositions, we first need to prove some lemmas.

LEMMA 3.6 *Suppose K is a compact subset of M and that $t_0, t_0' \in \mathbb{R}$ satisfy $0 < t_0 \leq t_0'$. We can find a compact subset $K' \subset M$ containing K (and depending on K, t_0, t_0') such that any L-minimizer $\gamma : [a, b] \to M$ with $t_0 \leq b - a \leq t_0'$ and $\gamma(a), \gamma(b) \in K$ is contained in K'.*

Of course, when M is compact we could take $K' = M$ and the lemma is trivial.

PROOF OF LEMMA 3.6: Since M is a complete Riemannian manifold, we can find $g : [a, b] \to M$, a geodesic with $g(a) = \gamma(a)$, $g(b) = \gamma(b)$, and whose length is $d(\gamma(a), \gamma(b))$. Since g is a geodesic, the norm $\|\dot{g}(s)\|_{g(s)}$ of its speed is a constant that we denote by C. Therefore we have

$$d(\gamma(a), \gamma(b)) = \text{length}(g) = \int_a^b \|\dot{g}(s)\|_{g(s)}\, ds = C(b - a).$$

This yields that the norm of speed $\|\dot{g}(s)\|_{g(s)} = C = d(\gamma(a), \gamma(b))/(b - a)$ is bounded by $\text{diam}(K)/t_0$. If we set

$$A = \sup\{L(x, v) \mid (x, v) \in TM, \|v\|_x \leq \text{diam}(K)/t_0\},$$

we know that A is finite by the uniform boundedness of L in the fiber. It follows that we can estimate the action of g by

$$\int_a^b L(g(s), \dot{g}(s))ds \le A(b-a).$$

Since γ is a minimizer with the same endpoints as g, we also get

$$\int_a^b L(\gamma(s), \dot{\gamma}(s))ds \le A(b-a).$$

By the uniform superlinearity of L in the fibers, we can find a constant $C > -\infty$ such that

$$\forall (x, v) \in TM, \quad C + \|v\|_x \le L(x, v).$$

Applying this to $(\gamma(s), \dot{\gamma}(s))$ and integrating, we get

$$C(b-a) + \text{length}(\gamma) \le \int_a^b L(\gamma(s), \dot{\gamma}(s))ds \le A(b-a).$$

Therefore

$$\text{length}(\gamma) \le (A-C)(b-a).$$

Therefore γ is contained in the set K defined by

$$K' = \bar{V}_{(A-C)(b-a)}(K) = \{y \mid \exists x \in K, d(x, y) \le (A-C)(b-a)\}.$$

Notice that K' is contained in a ball of radius (diam $K + (A-C)(b-a)$), which is finite, and balls of finite radius are compact in a complete Riemannian manifold. Therefore K' is compact. □

LEMMA 3.7 *For every compact subset K' of M and every $t_0 > 0$, we can find a constant $C = C(t_0, K')$ such that every L-minimizer $\gamma : [a, b] \to M$, with $b - a \ge t_0$ and $\gamma([a, b]) \subset K'$, satisfies*

$$\forall s \in [a, b], \quad \|\dot{\gamma}(s)\|_{\gamma(s)} \le C.$$

PROOF: Since any $s \in [a, b]$ with $b - a \ge t_0$ is contained in a subinterval of length exactly t_0, and any subcurve of a minimizer is a minimizer, it suffices to prove the lemma under the condition $b - a = t_0$. Using the action of a geodesic from $\gamma(a)$ to $\gamma(b)$, and the uniform boundedness of L in the fibers as in the proof of Lemma 3.6, we can find a constant A (depending on diam(K) and t_0 but not on γ) such that

$$\int_a^b L(\gamma(s), \dot{\gamma}(s))ds \le A(b-a).$$

Therefore, we can find $s_0 \in [a, b]$ such that $L(\gamma(s_0), \dot{\gamma}(s_0)) \le A$. By the uniform superlinearity of L, the subset

$$\mathcal{K} = \{(x, v) \in TM \mid x \in K', L(x, v) \le A\}$$

is compact (and does not depend on γ). Since γ is a minimizer, we have

$$(\gamma(s), \dot{\gamma}(s)) = \phi_{s-s_0}(\gamma(s_0), \dot{\gamma}(s_0))$$

and $|s - s_0| \leq b - a = t_0$; we conclude that the speed curve of the minimizer γ is contained in the set (independent of γ)

$$\mathcal{K}' = \bigcup_{|t| \leq t_0} \phi_t^L(\mathcal{K}),$$

which is compact by the continuity of the Euler-Lagrange flow. $\square$

LEMMA 3.8 *For every K compact subset of M, every $t_0 > 0$, and every $t_0' \in [t_0, +\infty[$ (respectively, $t_0' = +\infty$ when M is compact), we can find $K' \supset K$ a compact subset of M (respectively, $K' = M$ when M is compact) and finite constants C_0 and C_1 such that for every C^1 critical subsolution $u : M \to \mathbb{R}$, if $\omega_{u,K'} : [0, \infty[\to [0, \infty[$ is a continuous, nondecreasing modulus of continuity of $x \mapsto (x, d_x u)$ on K', then for every $x, y \in K$ and every $t \in \mathbb{R}$ with $t_0 \leq t \leq t_0'$ we have*

$$\omega_{u,K'}^{-1}\left(\frac{c(H) - H(x, d_x u)}{2C_1}\right) \frac{c(H) - H(x, d_x u)}{2C_0} \leq \\ h_t(x, y) + c(H)t + u(x) - u(y),$$

where

$$\omega_{u,K'}^{-1}(t) = \begin{cases} \inf\{t' \mid \omega_{u,K'}(t') = t\} & \text{if } t \in \omega_{u,K'}([0, +\infty[), \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, if $\omega_{u,K'}$ is the linear function $t \mapsto Ct$, with $C > 0$, then for every $x, y \in K$ and every $t \in \mathbb{R}$ with $t_0 \leq t \leq t_0'$, we have

$$\frac{[c(H) - H(x, d_x u)]^2}{4CC_0C_1} \leq h_t(x, y) + c(H)t + u(x) - u(y).$$

PROOF: We first choose K'. If M is compact, we set $K' = M$ and we allow $t_0' = +\infty$. If M is not compact, we assume $t_0' < +\infty$. By Lemma 3.6, we can find a compact subset $K' \supset K$ of M such that every L-minimizer $\gamma : [a, b] \to M$ with $t_0 \leq b - a \leq t_0'$ and $\gamma(a), \gamma(b) \in K$ is contained in K'. With this choice of K', we apply Lemma 3.7 to find a finite constant C_0 such that every L-minimizer $\gamma : [a, b] \to M$ contained in K', with $b - a \geq t_0$, has a speed bounded in norm by C_0.

Therefore we conclude that for every L-minimizer $\gamma : [0, t] \to M$, with $t_0 \leq t'$ and $\gamma(0), \gamma(t) \in K$, we have $\gamma([0, t]) \subset K'$, and $\|\dot\gamma(s)\|_{\gamma(t)} \leq C_0$ (this is valid both in the compact and noncompact case). In particular, for such a minimizer γ, we have

$$\forall s, s' \in [0, t], \quad d(\gamma(s), \gamma(s')) \leq C_0 |s - s'|.$$

We call C_1 a Lipschitz constant of H on the compact subset $\{(x, p) \in T^*M \mid x \in K', H(x, p) \leq c(H)\}$.

Suppose now u is a critical subsolution. Given $x, y \in K$ and $t \in [t_0, t_0']$, we pick $\gamma : [0, t] \to M$, a minimizer with $\gamma(0) = x$ and $\gamma(t) = y$. Therefore we have

$$h_t(x, y) = \int_0^t L(\gamma(s), \dot{\gamma}(s))ds.$$

Since $\gamma([0, t]) \subset K'$ and $H(\gamma(s), d_{\gamma(s)}u) \leq c(H)$, we have

$$\begin{aligned} \forall s, s' \in [0, t], \quad & |H(\gamma(s'), d_{\gamma(s')}u) - H(\gamma(s), d_{\gamma(s)}u)| \\ & \leq C_1 d[(\gamma(s'), d_{\gamma(s')}u), (\gamma(s), d_{\gamma(s)}u)] \\ & \leq C_1 \omega_{u,K'}(d(\gamma(s'), \gamma(s)) \leq C_1 \omega_{u,K'}(C_0|s - s'|). \end{aligned} \tag{3.4}$$

Integrating the Fenchel inequality

$$d_{\gamma(s)}u(\dot{\gamma}(s)) \leq L(\gamma(s), \dot{\gamma}(s)) + H(\gamma(s), d_{\gamma(s)}u),$$

we get

$$u(y) - u(x) \leq h_t(x, y) + \int_0^t H(\gamma(s), d_{\gamma(s)}u)ds.$$

Since $H(\gamma(s), d_{\gamma(s)}u) \leq c(H)$, for every $t' \in [0, t]$, from (3.4) above we can write

$$\begin{aligned} & \int_0^t H(\gamma(s), d_{\gamma(s)}u)ds \\ & \quad = \int_0^t c(H) + [H(\gamma(s), d_{\gamma(s)}u) - c(H)]\, ds \\ & \quad \leq c(H)t + \int_0^{t'} H(\gamma(s), d_{\gamma(s)}u) - c(H)ds \\ & \quad \leq c(H)t + \int_0^{t'} H(\gamma(0), d_{\gamma(0)}u) - c(H) + C_1\omega_{u,K'}(C_0 s)ds \\ & \quad = c(H)t + \int_0^{t'} H(x, d_x u) - c(H) + C_1\omega_{u,K'}(C_0 s)ds. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \forall t' \in [0, t], \quad u(y) - u(x) \leq h_t(x, y) + c(H)t + \int_0^{t'} & H(x, d_x u) - c(H) \\ & + C_1\omega_{u,K'}(C_0 s)ds, \end{aligned}$$

which yields

$$\begin{aligned} \forall t' \in [0, t], \quad \int_0^{t'} c(H) - H(x, d_x u) - C_1\omega_{u,K'}(C_0 s)ds \leq & \\ h_t(x, y) + c(H)t + u(x) - u(y). & \end{aligned} \tag{3.5}$$

Since $c(H) - H(x, d_x u) \le c(H) - \inf\{H(x, p) \mid (x, p) \in T^*M\} < +\infty$, up to a choice of C_0 big enough, we can assume

$$t_0 > \frac{1}{C_0}\,\omega_{u,K'}^{-1}\left(\frac{c(H) - H(x, d_x u)}{2C_1}\right).$$

Then, recalling that $t \ge t_0$, if we set

$$t' = \frac{1}{C_0}\,\omega_{u,K'}^{-1}\left(\frac{c(H) - H(x, d_x u)}{2C_1}\right),$$

since $\omega_{u,K'}$ is nondecreasing, we obtain

$$\forall s \in [0, t'], \quad C_1\omega_{u,K'}(C_0 s) \le \frac{c(H) - H(x, d_x u)}{2}.$$

Hence

$$\forall s \in [0, t'], \quad c(H) - H(x, d_x u) - C_1\omega_{u,K'}(C_0 s) \ge \frac{c(H) - H(x, d_x u)}{2}.$$

Combining this with (3.5), we obtain

$$\omega_{u,K'}^{-1}\left(\frac{c(H) - H(x, d_x u)}{2C_1}\right)\frac{c(H) - H(x, d_x u)}{2C_0} \le \\ h_t(x, y) + c(H)t + u(x) - u(y).$$

This finishes the proof. ☐

PROOF OF PROPOSITION 3.4: We apply Lemma 3.8 above to obtain the compact set K'. This lemma also gives for every $\ell \ge 0$ a constant $A = A(\ell)$ such that any $C^{1,1}$ critical subsolution $u : M \to \mathbb{R}$ that is ℓ-Lipschitz on K' satisfies

$$\forall x \in K,\ \forall t \in [t_0, t_0'], \quad \frac{[c(H) - H(x, d_x u)]^2}{A} \le h_t(x, x) + c(H)t.$$

To prove the second part, we will use the strict C^2 convexity of H. Since the set $\{(x, p) \mid x \in K, H(x, p) \le c(H)\}$ is compact, the C^2 strict convexity allows us to find $\beta > 0$ such that for all $x \in K$ and $p_1, p_2 \in T_x^*M$, with $H(x, p_i) \le c(H)$, we have

$$H(x, p_2) - H(x, p_1) \ge \frac{\partial H(x, p_1)}{\partial p}(p_2 - p_1) + \beta\|p_2 - p_1\|_x^2.$$

Since H is convex in p for all $x \in K$ and $p_1, p_2 \in T_x^*M$, with $H(x, p_i) \le c(H)$, we also have $H(x, (p_1 + p_2)/2) \le c(H)$. Therefore we can apply the above inequality to the pairs $((p_1 + p_2)/2, p_1)$ and $((p_1 + p_2)/2, p_2)$ to obtain

$$H(x, p_1) - H\left(x, \frac{p_1 + p_2}{2}\right) \ge \frac{\partial H(x, \frac{p_1+p_2}{2})}{\partial p}\left(\frac{p_1 - p_2}{2}\right) + \beta\left\|\frac{p_1 - p_2}{2}\right\|_x^2,$$
$$H(x, p_2) - H\left(x, \frac{p_1 + p_2}{2}\right) \ge \frac{\partial H(x, \frac{p_1+p_2}{2})}{\partial p}\left(\frac{p_2 - p_1}{2}\right) + \beta\left\|\frac{p_2 - p_1}{2}\right\|_x^2.$$

If we add these two inequalities, using $H(x, p_i) \leq c(H)$ and dividing by 2, we obtain

$$c(H) - H\left(x, \frac{p_1 + p_2}{2}\right) \geq \beta \left\| \frac{p_2 - p_1}{2} \right\|_x^2 .$$

Therefore if $u_1, u_2 : M \to \mathbb{R}$ are two C^1 critical subsolutions, we get

$$c(H) - H\left(x, \frac{d_x u_1 + d_x u_2}{2}\right) \geq \beta \left\| \frac{d_x u_2 - d_x u_1}{2} \right\|_x^2 . \tag{3.6}$$

We denote by $T^*M \oplus T^*M$ the Whitney sum of T^*M with itself (i.e., we consider the vector bundle over M whose fiber at $x \in M$ is $T_x^*M \times T_x^*M$). The maps

$$T^*M \to T^*M \oplus T^*M, \qquad (x, p) \mapsto (x, p, 0),$$
$$T^*M \to T^*M \oplus T^*M, \qquad (x, p) \mapsto (x, 0, p),$$

and

$$T^*M \oplus T^*M \to T^*M, \qquad (x, p_1, p_2) \mapsto \left(x, \frac{p_1 + p_2}{2}\right),$$

are all C^∞. Therefore they are Lipschitz on any compact subset. Since for a critical subsolution $u : M \to \mathbb{R}$ the values $(x, d_x u)$ for $x \in K'$ are all in the compact subset $\{(x, p) \mid x \in K',\ H(x, p) \leq c(H)\}$, we can find a constant $B < \infty$ such that for any two C^1 critical subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u_i)$, $i = 1, 2$, has a Lipschitz constant $\leq \ell$ on K', the map $x \mapsto (x, (d_x u_1 + d_x u_2)/2)$ has a Lipschitz constant $\leq B\ell$. Since $(u_1 + u_2)/2$ is also a critical subsolution, applying the first part of the proposition proved above with Lipschitz constant $\ell_1 = B\ell$, we can find a constant C_1 such that

$$\forall x \in K,\ \forall t \in [t_0, t_0'],$$
$$\left[c(H) - H\left(x, \frac{d_x u_1 + d_x u_2}{2}\right)\right]^2 \leq C_1(h_t(x, x) + c(H)t).$$

Combining this inequality with (3.6) above, we get

$$\forall x \in K,\ \forall t \in [t_0, t_0'], \quad \beta^2 \left\| \frac{d_x u_2 - d_x u_1}{2} \right\|_x^4 \leq C_1(h_t(x, x) + c(H)t).$$

This yields the second part of the proposition with $C' = \beta^{-2} C_1$. □

We now can start the proof of Theorem 1.2. Let $\tilde{\mathcal{A}}^p$ be the set of points in the Aubry set $\tilde{\mathcal{A}}$ that are periodic but not fixed under the Euler-Lagrange flow ϕ_t^L. This set projects on $\mathcal{A}^p$. Denote by $T : \tilde{\mathcal{A}}^p \to]0, +\infty[$ the period map of Euler-Lagrange flow ϕ_t^L; i.e., if $(x, v) \in \tilde{\mathcal{A}}^p$, the number $T(x)$ is the smallest positive number t such that $\phi_t^L(x, v) = (x, v)$. Using Proposition 3.3 above, we can write $\tilde{\mathcal{A}}^p = \bigcup_{n \in \mathbb{N}} \tilde{F}_n$, with each $\tilde{F}_n$ compact and such that the restriction $T|\tilde{F}_n$ is continuous. We denote by F_n the projection of $\tilde{F}_n \subset TM$ on the base M. We have

$\mathcal{A}^p = \bigcup_{n\in\mathbb{N}} F_n$. If we want to show that $\mathcal{H}^d(\mathcal{A}^p, \delta_M) = 0$ for some dimension $d > 0$, by the countable additivity of the Hausdorff measure in dimension d, it suffices to show that $\mathcal{H}^d(F_n, \delta_M) = 0$ for every $n \in \mathbb{N}$.

Therefore from now on we fix some compact subset $\tilde{F} \subset \tilde{\mathcal{A}}^p$ on which the period map is continuous, and we will show that its Hausdorff measure in the appropriate dimension d is 0. We now perform one more reduction. In fact, we claim that it suffices for each $(x, v) \in \tilde{F}$ to find $\tilde{S}_{(x,v)} \subset TM$, a C^∞ codimension 1 transverse section to the Euler-Lagrange flow ϕ_t^L such that $(x, v) \in \tilde{S}_{(x,v)}$ and $\mathcal{H}^d(\pi(\tilde{F} \cap \tilde{S}_{(x,v)}), \delta_M) = 0$, where $\pi : TM \to M$ is the canonical projection. Indeed, if this were the case, since, by transversality of $\tilde{S}$ to the flow ϕ_t^L, the set $\tilde{V}_{(x,v)} = \bigcup_{t\in\mathbb{R}} \phi_t^L(\tilde{S}_{(x,v)})$ would be open in TM, we could cover the compact set $\tilde{F}$ by a finite number of sets $\tilde{V}_{(x_i,v_i)}$, $i = 1, \ldots, \ell$. Note that by part (iii) of Mather's theorem, Theorem 2.1, the sets $\pi(\tilde{F} \cap \tilde{V}_{(x,v)})$ and $\pi(\tilde{F} \cap \tilde{S}_{(x,v)})$ have the same image in the quotient Mather set; therefore we get

$$\mathcal{H}^d(\pi(\tilde{F} \cap \tilde{V}_{(x,v)}), \delta_M) = \mathcal{H}^d(\pi(\tilde{F} \cap \tilde{S}_{(x,v)}), \delta_M) = 0.$$

Hence $F = \pi(\tilde{F})$, which is covered by the finite number of sets $\pi(\tilde{F} \cap \tilde{V}_{(x_i,v_i)})$, also satisfies $\mathcal{H}^d(F, \delta_M) = 0$.

Fix now (x_0, v_0) in $\tilde{F} \subset \tilde{\mathcal{A}}^p$. We proceed to construct the transverse $\tilde{S} = \tilde{S}_{(x_0,v_0)}$. We start with a C^∞ codimension 1 submanifold $\tilde{S}_0 \subset TM$ that is transverse to the flow ϕ_t^L and that intersects the compact periodic orbit $\phi_t^L(x_0, v_0)$ at exactly (x_0, v_0). If L (or H) is $C^{k,1}$, the Poincaré first return time $\tau : \tilde{S}_1 \to]0, \infty[$ on T_0 is defined and $C^{k-1,1}$ on some smaller transverse $\tilde{S}_1 \subset \tilde{S}_0$ containing (x_0, v_0). We set $\theta : \tilde{S}_1 \to \tilde{S}_0, (x, v) \mapsto \phi^L_{\tau(x,v)}(x, v)$. This is the Poincaré return map, and it is also $C^{k-1,1}$, as a composition of $C^{k-1,1}$ maps. Of course, we have $\tau(x_0, v_0) = T(x_0, v_0)$ and $\theta(x_0, v_0) = (x_0, v_0)$. Since T is continuous on F, it is easy to show that $T = \tau$ and θ is the identity on $F \cap \tilde{S}_2$, where $\tilde{S}_2 \subset \tilde{S}_1$ is a smaller section containing (x_0, v_0).

Pick $\epsilon > 0$ small enough so that the radius of injectivity of the Riemannian manifold M is $\geq \epsilon$ for every $x \in B_d(x_0, \epsilon) = \{y \in M \mid d(x_0, y) < \epsilon\}$, where d is the distance obtained from the Riemannian metric on M. This implies that the restriction of the square d^2 of the distance d is of class C^∞ (like the Riemannian metric) on $B_d(x_0, \epsilon/2) \times B_d(x_0, \epsilon/2)$.

We now take a smaller section $\tilde{S}_3 \subset \tilde{S}_2$ around (x_0, v_0) such that for every $(x, v) \in \tilde{S}_3$ both x and $\pi\theta(x, v)$ of M are in the ball $B_d(x_0, \epsilon/2)$. This is possible by continuity since $\theta(x_0, v_0) = (x_0, v_0)$. For $(x, v) \in \tilde{S}_3$, we set

$$\rho(x, v) = \tau(x, v) + d(\pi\theta(x, v), x).$$

We will now give an upper bound for $h_{\rho(x,v)}(x, x)+c(H)\rho(x, v)$ when $(x, v) \in \tilde{S}_3$. For this we choose a loop $\gamma_{(x,v)} : [0, \rho(x, v)] \to M$ at x. This loop $\gamma_{(x,v)}$ is equal to the curve $\gamma_{(x,v),1}(t) = \pi\phi_t(x,v)$ for $t \in [x, \tau(x,v)]$, which joins x to $\pi\theta(x,v)$, followed by the shortest geodesic $\gamma_{(x,v),2} : [\tau(x,v), \rho(x,v)] \to M$ for the Riemannian metric, parametrized by arc length and joining $\pi\theta(x,v)$ to x. Since $\gamma_{(x,v),2}$ is parametrized by arc length and is contained in $B_d(x_0,\epsilon)$, its action is bounded by $Kd(\pi\theta(x,v),x)$, where $K = \sup\{L(x,v) \mid d(x,x_0) \le \epsilon,\ \|v\|_x \le 1\} < \infty$. On the other hand, the action $a(x,v)$ of $\gamma_{(x,v),1}(t)$ is given by

$$a(x,v) = \int_0^{\tau(x,v)} L[\phi_s^L(x,v)]ds.$$

Note that a is also of class $C^{k-1,1}$. It follows that, for $(x,v) \in \tilde{S}_3$, we have

$$h_{\rho(x,v)}(x,x) + c(H)\rho(x,v) \le \\ [a(x,v) + c(H)\tau(x,v)] + [K + c(H)]d(\pi\theta(x,v),x)).$$

Therefore if, for $(x,v) \in \tilde{S}_3$, we define

$$\Psi(x,v) = [a(x,v) + c(H)\tau(x,v)]^2 + d^2(\pi\theta(x,v),x),$$

we obtain

$$\begin{aligned} \forall (x,v) \in \tilde{S}_3, \quad 0 &\le h_{\rho(x,v)}(x,x) + c(H)\rho(x,v) \\ &\le [1 + K + |c(H)|]\sqrt{\Psi(x,v)}. \end{aligned}$$

Notice that Ψ is $C^{k-1,1}$ like a and τ, because $x, \pi\theta(x,v) \in B(x_0,\epsilon/2)$ and d^2 is C^∞ on the ball $B(x_0,\epsilon/2)$. We now observe that Ψ is identically 0 on $\tilde{F} \cap \tilde{S}_3$. Indeed, for $(x,v) \in \tilde{F} \cap \tilde{S}_3$, we have $\theta(x,v) = (x,v)$; therefore $d^2(\pi\theta(x,v),x) = 0$. Moreover, since $(x,v) \in \tilde{F} \subset \tilde{\mathcal{A}}$, the curve $t \mapsto \pi\phi_t^L(x,v)$ calibrates any critical subsolution $u : M \to \mathbb{R}$; in particular,

$$\begin{aligned} u\big(\pi\phi^L_{\tau(x,v)}(x,v)\big) - u(\pi(x,v)) &= \int_0^{\tau(x,v)} L\phi_s^L(x,v)ds + c(H)\tau(x,v) \\ &= a(x,v) + c(H)\tau(x,v). \end{aligned}$$

But $\phi^L_{\tau(x,v)}(x,v) = \theta(x,v) = (x,v)$ for $(x,v) \in \tilde{F} \cap \tilde{S}_3$. Hence $a(x,v) + c(H)\tau(x,v) = 0$ for $(x,v) \in \tilde{F} \cap \tilde{S}_3$. Therefore Ψ is identically 0 on $\tilde{F} \cap \tilde{S}_3$.

To sum up, we have found two functions $\rho, \Psi : \tilde{S}_3 \to [0,+\infty[$ such that

(1) the function ρ is continuous and > 0 everywhere,
(2) the function Ψ is $C^{k-1,1}$ and vanishes identically on $\tilde{F} \cap \tilde{S}_3$, and
(3) there exists a finite constant C such that

$$\forall (x,v) \in \tilde{S}_3, \quad 0 \le h_{\rho(x,v)}(x,x) + c(H)\rho(x,v) \le C\sqrt{\Psi(x,v)}.$$

This is all that we will use in the remainder of the proof.

We now fix a smaller Poincaré section $\tilde{S}_4$ containing (x_0,v_0) whose closure $\mathrm{Cl}(\tilde{S}_4)$ is compact and contained in $\tilde{S}_3$. We now observe that $K = \pi(\mathrm{Cl}(\tilde{S}_4))$ is a compact subset of M and that

$$t_0 = \min\{\tau(x,v) \mid (x,v) \in \mathrm{Cl}(\tilde{S}_4)\}, \quad t_0' = \max\{\tau(x,v) \mid (x,v) \in \mathrm{Cl}(\tilde{S}_4)\},$$

are finite and > 0 since τ is continuous and > 0 on the compact set $\mathrm{Cl}(\tilde{S}_4)$. We can therefore apply Proposition 3.4 to obtain a set K'. We have to choose a constant ℓ needed to apply this Proposition 3.4. For this we invoke Theorem B.1: we can find a constant ℓ such that for any critical subsolution $u : M \to \mathbb{R}$ we can find a $C^{1,1}$ critical subsolution $v : M \to \mathbb{R}$ that is equal to u on the projected Aubry set $\mathcal{A}$ and such that $x \mapsto (x, d_x v)$ has Lipschitz constant $\leq \ell$ on K'. It follows from Lemma 2.7 that

$$\forall x, y \in \mathcal{A}, \quad \delta_M(x,y) = \max\left\{(u_1 - u_2)(y) - (u_1 - u_2)(x)\right\},$$

where the maximum is taken over all the pairs of $C^{1,1}$ critical subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u_i)$, $i = 1, 2$, have a Lipschitz constant $\leq \ell$ on K'. Using this ℓ, we obtain, from Proposition 3.4, a constant C' such that

$$\forall (x,v) \in \mathrm{Cl}(\tilde{S}_4), \quad \|d_x u_2 - d_x u_1\|_x^4 \leq C'[h_{\tau(x,v)}(x,x) + c(H)\tau(x,v)]$$

for every pair of $C^{1,1}$ critical subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u_i)$, $i = 1, 2$, have a Lipschitz constant $\leq \ell$ on K'. Therefore by the properties of τ and Ψ described above, we obtain

$$\forall (x,v) \in \mathrm{Cl}(\tilde{S}_4), \quad \|d_x u_2 - d_x u_1\|_x \leq C\Psi(x,v)^{1/8},$$

again for every pair of $C^{1,1}$ critical subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u_i)$, $i = 1, 2$, have a Lipschitz constant $\leq \ell$ on K'. Since Ψ is of class $C^{k-1,1}$ and is identically 0 on $\tilde{F} \cap \tilde{S}_4$, we can invoke Lemma 2.8 to obtain a decomposition

$$\tilde{F} \cap \tilde{S}_4 = \bigcup_{i \in \mathbb{N}} A_i,$$

with A_i a compact subset, a family $(B_i)_{i \in \mathbb{N}}$ of C^1 compact embedded discs in $\tilde{S}_4$, and constants $(C_i)_{i \in \mathbb{N}}$ such that

$$\begin{aligned}
&\forall (x,v) \in A_i,\ \forall (y,w) \in B_i, \\
&\Psi(y,w) = |\Psi(y,w) - \Psi(x,w)| \leq C_i \tilde{d}[(y,w),(x,v)]^k,
\end{aligned}$$

where $\tilde{d}$ is the distance obtained from a fixed Riemannian metric on $\tilde{S}_4$. Combining with what we obtained above, we find constants C_i' (independent of the pair of functions u_1, u_2) such that

$$\forall (x,v) \in A_i,\ \forall (y,w) \in B_i, \quad \|d_y u_1 - d_y u_2\|_y \leq C_i' \tilde{d}[(y,w),(x,v)]^{k/8}.$$

Since we want to consider u_1 and u_2 as functions on $B_i \subset TM$ composing with $\pi : TM \to M$, we can rewrite this as

$$\begin{aligned}
&\forall (x,v) \in A_i,\ \forall (y,w) \in B_i, \\
&\|d_{(y,w)} u_1 \circ \pi - d_{(y,w)} u_2 \circ \pi\|_y \leq C_i' \tilde{d}[(y,w),(x,v)]^{k/8}.
\end{aligned}$$

Again, as in the proof of the previous theorem, to simplify things we can identify B_I with a Euclidean ball $\mathbb{B}_i$ of some dimension, and since the identification is done by a C^1 diffeomorphism, we can find constants C_i'' (independent of the pair of functions u_1, u_2) such that

$$\begin{aligned}&\forall (x,v) \in A_i,\ \forall (y,w) \in \mathbb{B}_i,\\ &\|d_{(y,w)}u_1 \circ \pi - d_{(y,w)}u_2 \circ \pi\|_{\mathrm{euc}} \le C_i''\|(y,w)-(x,v)\|_{\mathrm{euc}}^{k/8}.\end{aligned}$$

In the Euclidean disc $\mathbb{B}_i$, we can integrate this inequality along the Euclidean segment joining (x,v) to (y,w) to obtain

$$\begin{aligned}&\forall (x,v) \in A_i,\ \forall (y,w) \in \mathbb{B}_i,\\ &|(u_1-u_2)(y)-(u_1-u_2)(x)| \le \frac{C_i''}{1+(k/8)}\|(y,w)-(x,v)\|_{\mathrm{euc}}^{1+(k/8)}.\end{aligned}$$

Of course, since the identification of $B_i \subset TM$ with $\mathbb{B}_i$ is done by a C^1 diffeomorphism changing constants again to some $\tilde{C}_i$ (independent of the pair of functions u_1, u_2), we get

$$\begin{aligned}&\forall (x,v) \in A_i,\ \forall (y,w) \in \mathbb{B}_i,\\ &\qquad\qquad |(u_1-u_2)(y)-(u_1-u_2)(x)| \le \tilde{C}_i\, d[(y,w),(x,v)]^{1+(k/8)},\end{aligned}$$

where d is a distance on TM obtained from a Riemannian metric. Observe now that, by Mather's theorem, the projection $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$ is bijective with an inverse that is locally Lipschitz. Therefore, since A_i is compact and contained in $\tilde{F} \subset \tilde{\mathcal{A}}$, changing again the constants to $\tilde{C}_i'$ (independent of the pair of functions u_1, u_2), we obtain

$$\forall x, y \in \pi(A_i), \quad |(u_1-u_2)(y)-(u_1-u_2)(x)| \le \tilde{C}_i' d[y,x]^{1+(k/8)}.$$

Since this inequality is true now for every pair of $C^{1,1}$ critical subsolutions $u_1, u_2 : M \to \mathbb{R}$ such that $x \mapsto (x, d_x u_i)$, $i = 1, 2$, have a Lipschitz constant $\le \ell$ on K' (with the constant $\tilde{C}_i'$ independent of the pair of functions u_1, u_2), we conclude that

$$\forall x, y \in \pi(A_i), \quad \delta_M(x,y) \le \tilde{C}_i'' d[y,x]^{1+(k/8)}.$$

Therefore by Lemma A.3 we obtain that

$$\mathcal{H}^{8\dim M/(k+8)}(\pi(A_i)) = 0.$$

Again by countable additivity this gives $\mathcal{H}^{8\dim M/(k+8)}(\pi(\tilde{F} \cap \tilde{S}_4)) = 0$. This finishes the proof of the theorem.

Remark 3.9. We observe that, from our proof, for any $\tilde{F} \subset \tilde{\mathcal{A}}$, the semimetric space $(\pi(\tilde{F}), \delta_M)$ has vanishing one-dimensional Hausdorff measure as soon as the following properties are satisfied: there are $r > 0$, $k', l \in \mathbb{N}$, and a function $G : TM \to \mathbb{R}$ of class $C^{k',1}$ such that

(i) $G(x,v) \equiv 0$ on $\tilde{F}$,

(ii) $\{m_r(x)\}^l \leq G(x, v)$ for all $(x, v) \in TM$, and
(iii) $k' \geq 4l(\dim M - 1) - 1$,

where $m_r(x) = \inf_{t \geq r} \{h_t(x, x) + c(H)t\}$.

Remark 3.10. By Proposition 2.3, for every compact subset $K \subset M$ there is a constant $C_K > 0$ such that

$$\forall x \in K, \quad h(x, x) \leq C_K d(x, \mathcal{A})^2,$$

where $d(x, \mathcal{A})$ denotes the Riemannian distance from x to the set $\mathcal{A}$ (which is assumed to be nonempty). Therefore, from the remark above, we deduce that if there are $l \in \mathbb{N}$ and a function $G : M \to \mathbb{R}$ of class $C^{k',1}$ with $k' \geq 2l(\dim M - 1) - 1$ such that

$$\forall x \in M, \quad d(x, \mathcal{A})^l \leq G(x),$$

then $(\mathcal{A}_M, d_M)$ has vanishing one-dimensional Hausdorff measure.

3.4 Proof of Theorem 1.5

By Theorems 1.2 and 1.4, we know that $(\mathcal{A}_M^0 \cup \mathcal{A}_M^p, \delta_M)$ has zero Hausdorff dimension. Thus the result will follow once we will show that $\mathcal{A}_M \setminus (\mathcal{A}_M^0 \cup \mathcal{A}_M^p)$ is a finite set.

We recall that the Aubry set $\tilde{\mathcal{A}} \subset TM$ is given by the set of $(x, v) \in TM$ such that $x \in \mathcal{A}$ and v is the unique $v \in T_x M$ such that $d_x u = \frac{\partial L}{\partial v}(x, v)$ for any critical viscosity subsolution. This set is invariant under the Euler-Lagrange flow ϕ_t^L. For every $x \in \mathcal{A}$, we denote by $\mathcal{O}(x)$ the projection on $\mathcal{A}$ of the orbit of ϕ_t^L that passes through x. We observe that by Theorem 2.1(iii) the following simple fact holds:

LEMMA 3.11 *If $x, y \in \mathcal{A}$ and $\overline{\mathcal{O}(x)} \cap \overline{\mathcal{O}(y)} \neq \varnothing$, then $\delta_M(x, y) = 0$.*

Let us define

$$\mathcal{C}_0 = \{x \in \mathcal{A} \mid \overline{\mathcal{O}(x)} \cap \mathcal{A}_0 \neq \varnothing\}, \quad \mathcal{C}_p = \{x \in \mathcal{A} \mid \overline{\mathcal{O}(x)} \cap \mathcal{A}_p \neq \varnothing\}.$$

Thus, if $x \in \mathcal{C}_0 \cup \mathcal{C}_p$, by Lemma 3.11 the Mather distance between x and $\mathcal{A}^0 \cup \mathcal{A}^p$ is 0, and we are done.

Let us now define $\mathcal{C} = \mathcal{A} \setminus (\mathcal{C}_0 \cup \mathcal{C}_p)$, and let $(\mathcal{C}_M, \delta_M)$ be the quotiented metric space. To conclude the proof, we show that this set consists of a finite number of points.

Let u be a $C^{1,1}$ critical subsolution (whose existence is provided by [4]), and let X be the Lipschitz vector field uniquely defined by the relation

$$\mathcal{L}(x, X(x)) = (x, d_x u),$$

where $\mathcal{L}$ denotes the Legendre transform. Its flow extends on the whole manifold the flow considered above on $\mathcal{A}$. We fix $x \in \mathcal{C}$. Then $\overline{\mathcal{O}(x)}$ is a nonempty, compact, invariant set that contains a nontrivial minimal set for the flow of X (see [34, chap. 1]). By [27], we know that there exists at most a finite number of such nontrivial minimal sets. Therefore, again by Lemma 3.11, $(\mathcal{C}_M, \delta_M)$ consists only in a finite number of points.

4 Applications in Dynamics

Throughout this section, M is assumed to be compact. As before, $H : T^*M \to \mathbb{R}$ is a Hamiltonian of class at least C^2 satisfying the two usual conditions (H1)–(H2) (note that (H3) is automatically satisfied if M is compact), and L is the Tonelli Lagrangian that is associated to it by Fenchel's duality.

As in Section 2.3, we denote by $\mathcal{SS}$ the set of critical viscosity subsolutions and by $\mathcal{S}_-$ the set of critical viscosity (or weak KAM) solutions, so that $\mathcal{S}_- \subset \mathcal{SS}$.

4.1 More about Aubry Sets on Compact Manifolds

From the characterization of the Aubry set given by Theorem 2.6, it is natural to introduce the Mañé set $\tilde{\mathcal{N}}$ given by

$$\tilde{\mathcal{N}} = \bigcup_{u \in \mathcal{SS}} \tilde{\mathcal{I}}(u).$$

As is the case for $\tilde{\mathcal{A}}$, the subset $\tilde{\mathcal{N}}$ of TM is compact and invariant under the Euler-Lagrange flow ϕ_t^L of L.

THEOREM 4.1 (Mañé) *When M is compact, each point of the invariant set $\tilde{\mathcal{A}}$ is chain-recurrent for the restriction $\phi_t^L|_{\tilde{\mathcal{A}}}$. Moreover, the invariant set $\tilde{\mathcal{N}}$ is chain-transitive for the restriction $\phi_t^L|_{\tilde{\mathcal{N}}}$.*

COROLLARY 4.2 *When M is compact, the restriction $\phi_t^L|_{\tilde{\mathcal{A}}}$ to the invariant subset $\tilde{\mathcal{A}}$ is chain-transitive if and only if $\tilde{\mathcal{A}}$ is connected.*

PROOF: This is an easy, well-known result in the theory of dynamical systems: Suppose θ_t, $t \in \mathbb{R}$, is a flow on the compact metric space X. If every point of X is chain-recurrent for θ_t, then θ_t is chain-transitive if and only if X is connected. $\square$

For the following result see [15] or [12, théorème 1].

THEOREM 4.3 *When M is compact, the following properties are satisfied*:

(i) *Two weak KAM solutions that coincide on $\mathcal{A}$ are equal everywhere.*
(ii) *For every $u \in \mathcal{SS}$, there is a unique weak KAM solution $u_- : M \to \mathbb{R}$ such that $u_- = u$ on $\mathcal{A}$; moreover, the two functions u and u_- are also equal on $\mathcal{I}(u)$.*

It follows from the second statement in this theorem that we have

$$\tilde{\mathcal{N}} = \bigcup_{u \in \mathcal{S}_-} \tilde{\mathcal{I}}(u).$$

Moreover, it can be easily shown from the results of [15] that

$$\mathcal{A} = \bigcap_{u \in \mathcal{S}_-} \mathcal{I}(u).$$

We give now the general relationship between uniqueness of weak KAM solutions and the quotient Mather set; see also [7, cor. 4.7, p. 445], [5, cor. 4.7], and [8, prop. 3.4, p. 657].

PROPOSITION 4.4 *Suppose M is compact. The following two statements are equivalent*:

(i) *Any two weak KAM solutions differ by a constant.*
(ii) *The Mather quotient (A_M, δ_M) is trivial, i.e., is reduced to one point.*

Moreover, if any one of these conditions is true, then $\tilde{\mathcal{A}} = \tilde{\mathcal{N}}$, and therefore $\tilde{\mathcal{A}}$ is connected and the restriction of the Euler-Lagrange flow ϕ_t^L to $\tilde{\mathcal{A}}$ is chain-transitive.

PROOF: For every fixed $x \in M$, the function $y \mapsto h(x, y)$ is a weak KAM solution. Therefore if we assume that any two weak KAM solutions differ by a constant, then for $x_1, x_2 \in M$ we can find a constant C_{x_1,x_2} such that

$$\forall y \in M, \quad h(x_1, y) = C_{x_1,x_2} + h(x_2, y).$$

If $x_2 \in \mathcal{A}$, then $h(x_2, x_2) = 0$; therefore, evaluating the equality above for $y = x_2$, we obtain $C_{x_1,x_2} = h(x_1, x_2)$. Substituting in the equality and evaluating, we conclude

$$\forall x_1 \in M,\ \forall x_2 \in \mathcal{A}, \quad h(x_1, x_1) = h(x_1, x_2) + h(x_2, x_1).$$

This implies

$$\forall x_1, x_2 \in \mathcal{A}, \quad h(x_1, x_2) + h(x_2, x_1) = 0,$$

which means that $\delta_M(x_1, x_2) = 0$ for every $x_1, x_2 \in \mathcal{A}$.

To prove the converse, let us recall that for every critical subsolution u, we have

$$\forall x, y \in M, \quad u(y) - u(x) \leq h(x, y).$$

Therefore applying this for a pair $u_1, u_2 \in \mathcal{SS}$, we obtain

$$\forall x, y \in M, \quad u_1(y) - u_1(x) \leq h(x, y), \quad u_2(x) - u_2(y) \leq h(y, x).$$

Adding and rearranging, we obtain

$$\forall x, y \in M, \quad (u_1 - u_2)(y) - (u_1 - u_2)(x) \leq h(x, y) + h(y, x).$$

Since the right-hand side is symmetric in x, y, we obtain

$$\forall x, y \in M, \quad |(u_1 - u_2)(y) - (u_1 - u_2)(x)| \leq h(x, y) + h(y, x).$$

If we assume that (ii) is true, this implies that $u_1 - u_2$ is a constant c on the projected Aubry set $\mathcal{A}$, that is, $u_1 = u_2 + c$ on $\mathcal{A}$. Thus, if u_1, u_2 are weak KAM solutions, then we have $u_1 = u_2 + c$ on M, because any two solutions equal on the Aubry set are equal everywhere by (ii) of Theorem 4.3.

It remains to show the last statement. Notice that if $u_1, u_2 \in \mathcal{SS}$ differ by a constant, then $\tilde{\mathcal{I}}(u_1) = \tilde{\mathcal{I}}(u_2)$. Therefore if any two elements in $\mathcal{S}_-$ differ by a constant, then

$$\tilde{\mathcal{A}} = \tilde{\mathcal{I}}(u) = \tilde{\mathcal{N}},$$

where u is any element in $\mathcal{S}_-$. But, by Mañé's theorem, Theorem 4.1, the invariant set $\tilde{\mathcal{N}}$ is chain-transitive for the flow ϕ_t; hence it is connected by Corollary 4.2. $\square$

We now denote by X_L the Euler-Lagrange vector field of L; that is, the vector field on TM that generates ϕ_t^L. We recall that an important property of X_L is that

$$\forall (x,v) \in TM, \quad T\pi(X_L(x,v)) = v,$$

where $T\pi : T(TM) \to TM$ denotes the canonical projection.

Here is a last ingredient that we will have to use.

PROPOSITION 4.5 (Lyapunov Property) *Suppose $u_1, u_2 \in \mathcal{SS}$. The function $(u_1 - u_2) \circ \pi$ is nondecreasing along any orbit of the Euler Lagrange flow ϕ_t^L contained in $\tilde{\mathcal{I}}(u_2)$. If we assume u_1 is differentiable at $x \in \mathcal{I}(u_2)$ and $(x,v) \in \tilde{\mathcal{I}}(u_2)$, then, using that u_2 is differentiable on $\mathcal{I}(u_2)$, we obtain*

$$X_L \cdot [(u_1 - u_2) \circ \pi](x,v) = d_x u_1(v) - d_x u_2(v) \le 0.$$

Moreover, the inequality above is an equality if and only if $d_x u_1 = d_x u_2$. In that case $H(x, d_x u_1) = H(x, d_x u_2) = c(H)$.

PROOF: If $(x,v) \in \tilde{\mathcal{I}}(u_2)$, then $t \mapsto \pi\phi_t(x,v)$ is $(u_2, L, c(H))$-calibrated, hence

$$\forall t_1 \le t_2, \quad u_2 \circ \pi(\phi_{t_2}(x,v)) - u_2 \circ \pi(\phi_{t_1}(x,v)) = \int_{t_1}^{t_2} L(\phi_s(x,v))ds + c(H)(t_2 - t_1).$$

Since $u_1 \in \mathcal{SS}$, we get

$$\forall t_1 \le t_2, \quad u_1 \circ \pi(\phi_{t_2}(x,v)) - u_1 \circ \pi(\phi_{t_1}(x,v)) \le \int_{t_1}^{t_2} L(\phi_s(x,v))ds + c(H)(t_2 - t_1).$$

Combining these two facts, we conclude

$$\forall t_1 \le t_2, \quad u_1 \circ \pi(\phi_{t_2}(x,v)) - u_1 \circ \pi(\phi_{t_1}(x,v)) \le u_2 \circ \pi(\phi_{t_2}(x,v)) - u_2 \circ \pi(\phi_{t_1}(x,v)).$$

This implies

$$\forall t_1 \le t_2, \quad (u_1 - u_2) \circ \pi(\phi_{t_2}(x,v)) \le (u_1 - u_2) \circ \pi(\phi_{t_1}(x,v)).$$

Recall that u_2 is differentiable at every $x \in \mathcal{I}(u_2)$. Thus, if $d_x u_1$ also exists, if $(x,v) \in \tilde{\mathcal{I}}(u_2)$ we obtain

$$X_L \cdot [(u_1 - u_2) \circ \pi](x,v) \le 0.$$

We remark that $X_L \cdot [(u_1 - u_2) \circ \pi](x,v) = d_x(u_1 - u_2)(T\pi \circ X_L(x,v))$. Since $T\pi \circ X_L(x,v) = v$, we obtain

$$X_L \cdot [(u_1 - u_2) \circ \pi](x,v) = d_x u_1(v) - d_x u_2(v) \le 0.$$

If the last inequality is an equality, we get $d_x u_1(v) = d_x u_2(v)$. Since $(x, v) \in \tilde{\mathcal{I}}(u_2)$, we have $d_x u_2 = \frac{\partial L}{\partial v}(x, v)$ and $H(x, d_x u_2) = c(H)$; therefore the Fenchel inequality yields the equality

$$d_x u_2(v) = L(x, v) + H(x, d_x u_2) = L(x, v) + c(H).$$

Since $u_1 \in \mathcal{SS}$, we know that $H(x, d_x u_1) \leq c(H)$. The previous equality, using the Fenchel inequality $d_x u_1(v) \leq L(x, v) + H(x, d_x u_1)$ and the fact that $d_x u_1(v) = d_x u_2(v)$, implies

$$H(x, d_x u_1) = c(H) \quad \text{and} \quad d_x u_1(v) = L(x, v) + H(x, d_x u_1).$$

This means that we have equality in the Fenchel inequality $d_x u_1(v) \leq L(x, v) + H(x, d_x u_1)$; we therefore conclude that $d_x u_1 = \frac{\partial L}{\partial v}(x, v)$, but the right-hand side of this last equality is $d_x u_2$. $\square$

4.2 Mather Disconnectedness Condition

DEFINITION 4.6 We will say that the Tonelli Lagrangian L on M satisfies the *Mather disconnectedness condition* if for every pair $u_1, u_2 \in \mathcal{S}_-$, the image $(u_1 - u_2)(\mathcal{A}) \subset \mathbb{R}$ is totally disconnected.

Notice that by part (ii) of Theorem 4.3, if L satisfies the Mather disconnectedness condition, then for every pair of critical subsolutions u_1, u_2, the image $(u_1 - u_2)(\mathcal{A}) \subset \mathbb{R}$ is also totally disconnected.

PROPOSITION 4.7 *If $\mathcal{H}^1(\mathcal{A}_M, \delta_M) = 0$, then L satisfies the Mather disconnectedness condition.*

PROOF: If $u_1, u_2 \in \mathcal{SS}$, $u_1 - u_2$ is 1-Lipschitz with respect to δ_M; see the proof of Proposition 4.4. Therefore the one-dimensional Hausdorff measure (i.e., Lebesgue measure) of $(u_1 - u_2)(\mathcal{A})$ is 0 like $\mathcal{H}^1(\mathcal{A}_M, \delta_M)$. The result follows since a subset of $\mathbb{R}$ of Lebesgue measure 0 is totally disconnected. $\square$

By Proposition 4.7, the results obtained above contain the following theorem (assertions (i) and (ii) have already been proved in [31] and assertion (iv) in [36]).

THEOREM 4.8 *Let L be a Tonelli Lagrangian on the compact manifold M; it satisfies the Mather disconnectedness condition in the following five cases*:

(i) *The dimension of M is* 1 *or* 2.
(ii) *The dimension of M is* 3*, and $\tilde{\mathcal{A}}$ contains no fixed point of the Euler-Lagrange flow.*
(iii) *The dimension of M is* 3*, and L is of class $C^{3,1}$.*
(iv) *The Lagrangian is of class $C^{k,1}$, with $k \geq 2 \dim M - 3$, and every point of $\tilde{\mathcal{A}}$ is fixed under the Euler-Lagrange flow ϕ_t^L.*
(v) *The Lagrangian is of class $C^{k,1}$, with $k \geq 8 \dim M - 8$, and either each point of $\tilde{\mathcal{A}}$ is fixed under the Euler-Lagrange flow ϕ_t^L or its orbit in the Aubry set is periodic with* (*strictly*) *positive period.*

LEMMA 4.9 *Suppose that L is a Tonelli Lagrangian L on the compact manifold M that satisfies the Mather disconnectedness condition. For every $u \in \mathcal{SS}$, the set of points in $\tilde{\mathcal{I}}(u)$ that are chain-recurrent for the restriction $\phi_t^L|_{\tilde{\mathcal{I}}(u)}$ of the Euler-Lagrange flow is precisely the Aubry set $\tilde{\mathcal{A}}$.*

PROOF: First of all, we recall that, from Theorem 4.1, each point of $\mathcal{A}$ is chain-recurrent for the restriction $\phi_t^L|_{\tilde{\mathcal{A}}}$. By [18, theorem 1.5], we can find a C^1 critical viscosity subsolution $u_1 : M \to \mathbb{R}$ that is strict outside $\mathcal{A}$, i.e., for every $x \notin \mathcal{A}$ we have $H(x, d_x u_1) < c(H)$. We define θ on TM by $\theta = (u_1 - u) \circ \pi$. By Proposition 2.5, we know that at each point (x, v) of $\tilde{\mathcal{I}}(u)$ the derivative of θ exists and depends continuously on $(x, v) \in \tilde{\mathcal{I}}(u)$. By Proposition 4.5, at each point of (x, v) of $\tilde{\mathcal{I}}(u)$, we have

$$X_L \cdot \theta(x, v) = d_x u_1(v) - d_x u(v) \leq 0,$$

with the last inequality an equality if and only if $d_x u_1 = d_x u$, and this implies $H(x, d_x u_1) = c(H)$. Since u_1 is strict outside $\mathcal{A}$, we conclude that $X_L \cdot \theta < 0$ on $\tilde{\mathcal{I}}(u) \setminus \tilde{\mathcal{A}}$.

Suppose that $(x_0, v_0) \in \tilde{\mathcal{I}}(u) \setminus \tilde{\mathcal{A}}$. By the invariance of both $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{I}}(u)$, every point on the orbit $\phi_t^L(x_0, v_0)$, $t \in \mathbb{R}$, is also contained in $\tilde{\mathcal{I}}(u) \setminus \tilde{\mathcal{A}}$; therefore $t \mapsto c(t) = \theta(\phi_t(x_0, v_0))$ is (strictly) decreasing, and so we have $c(1) < c(0)$. Observe now that $\theta(\tilde{\mathcal{A}}) = (u_1 - u)(\mathcal{A})$ is totally disconnected by the Mather disconnectedness condition. Therefore we can find $c \in \,]c(1), c(0)[\setminus \theta(\tilde{\mathcal{A}})$. By what we have seen, the directional derivative $X_L \cdot \theta$ is < 0 at every point of the level set $L_c = \{(x, v) \in \tilde{\mathcal{I}}(u) \mid \theta(x, v) = c\}$. Since θ is everywhere nonincreasing on the orbits of ϕ_t^L and $X_L \cdot \theta < 0$ on L_c, we get

$$\forall t > 0, \ \forall (x, v) \in L_c, \quad \theta(\phi_t(x, v)) < c.$$

Consider the compact set $K_c = \{(x, v) \in \tilde{\mathcal{I}}(u) \mid \theta(x, v) \leq c\}$. Using again that θ is nonincreasing on the orbits of $\phi_t^L|_{\tilde{\mathcal{I}}(u)}$, we have

$$\forall t \geq 0, \quad \phi_t^L(K_c) \subset K_c \quad \text{and} \quad \phi_t^L(K_c \setminus L_c) \subset K_c \setminus L_c.$$

Using what we obtained above on L_c, we conclude that

$$\forall t > 0, \quad \phi_t^L(K_c) \subset K_c \setminus L_c.$$

We now fix some metric on $\tilde{\mathcal{I}}(u)$ defining its topology. We then consider the compact set $\phi_1^L(K_c)$. It is contained in the open set $K_c \setminus L_c = \{(x, v) \in \tilde{\mathcal{I}}(u) \mid \theta(x, v) < c\}$. We can therefore find $\epsilon > 0$ such that the ϵ-neighborhood $V_\epsilon(\phi_1(K_c))$ of $\phi_1^L(K_c)$ in $\tilde{\mathcal{I}}(u)$ is also contained in K_c. Since for $t \geq 1$ we have $\phi_{t-1}^L(K_c) \subset K_c$, and therefore $\phi_t^L(K_c) \subset \phi_1(K_c)$, it follows that

$$V_\epsilon\Bigg(\bigcup_{t\geq 1} \phi_t^L(K_c)\Bigg) \subset K_c.$$

It is now easy to conclude that every ϵ-pseudo orbit for $\phi_t^L|_{\tilde{\mathcal{I}}(u)}$ that starts in K_c remains in K_c. Since $\theta(\phi_1^L(x_0, v_0)) = c(1) < c < c(0) = \theta(x_0, v_0)$, no α-pseudo orbit starting at (x_0, v_0) can return to (x_0, v_0) for $\alpha \leq \epsilon$ such that the ball of center $\phi_1^L(x_0, v_0)$ and radius α in $\tilde{\mathcal{I}}(u)$ is contained in K_c. Therefore (x_0, v_0) cannot be chain-recurrent. □

THEOREM 4.10 *Let L be a Tonelli Lagrangian on the compact manifold M. If L satisfies the Mather disconnectedness condition, then the following statements are equivalent*:

(i) *The Aubry set $\tilde{\mathcal{A}}$, or its projection $\mathcal{A}$, is connected.*
(ii) *The Aubry set $\tilde{\mathcal{A}}$ is chain-transitive for the restriction of the Euler-Lagrange flow $\phi_t^L|_{\tilde{\mathcal{A}}}$.*
(iii) *Any two weak KAM solutions differ by a constant.*
(iv) *The Aubry set $\tilde{\mathcal{A}}$ is equal to the Mañé set $\tilde{\mathcal{N}}$.*
(v) *There exists $u \in \mathcal{SS}$ such that $\tilde{\mathcal{I}}(u)$ is chain-recurrent for the restriction $\phi_t|_{\tilde{\mathcal{I}}(u)}$ of the Euler-Lagrange flow.*

Remark 4.11. Note that by Proposition 4.4 the above conditions are also equivalent to the triviality of the Mather quotient. In fact, this last condition is equivalent to (iii). Moreover, we observe that, without requiring the Mather disconnectedness condition, one can only prove

$$\text{(i)} \Longleftrightarrow \text{(ii)} \quad \text{and} \quad \text{(iii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(v)}.$$

The assumption that the Mather disconnectedness condition holds allows us to prove that (i) $\Longrightarrow$ (iii) and (v) $\Longrightarrow$ (ii).

PROOF: From Corollary 4.2, we know that (i) and (ii) are equivalent.

If (i) is true, then for $u_1, u_2 \in \mathcal{S}_-$, the image $(u_1 - u_2)(\mathcal{A})$ is a subinterval of $\mathbb{R}$, but by the Mather disconnectedness condition, it is also totally disconnected; therefore $u_1 - u_2$ is constant. Hence (i) implies (iii).

If (iii) is true, then (iv) follows from Proposition 4.4.

Suppose now that (iv) is true. Since for every $u \in \mathcal{SS}$ we have $\tilde{\mathcal{A}} \subset \tilde{\mathcal{I}}(u) \subset \tilde{\mathcal{N}}$, we obtain $\tilde{\mathcal{I}}(u) = \tilde{\mathcal{N}}$. But $\tilde{\mathcal{N}}$ is chain-transitive for the restriction $\phi_t^L|_{\tilde{\mathcal{N}}}$. Hence (iv) implies (v).

If (v) is true for some $u \in \mathcal{SS}$, then every point of $\tilde{\mathcal{I}}(u)$ is chain-recurrent for the restriction $\phi_t^L|_{\tilde{\mathcal{I}}(u)}$. Lemma 4.9 then implies that $\tilde{\mathcal{A}} = \tilde{\mathcal{I}}(u)$, and we therefore satisfy (ii). $\square$

Remark 4.12. For each integer $d > 0$ and each $\epsilon > 0$, John Mather has constructed on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ a Tonelli Lagrangian L of class $C^{2d-3,1-\epsilon}$ such that $\tilde{\mathcal{A}}$ is connected and contained in the fixed points of the Euler-Lagrange flow, and the Mather quotient $(\mathcal{A}_M, \delta_M)$ is isometric to an interval; see [32]. In particular, for such a Lagrangian, Theorem 4.10 cannot be true.

4.3 Mañé Lagrangians

We now give an application to the Mañé example associated to a vector field. Suppose M is a compact Riemannian manifold where the metric g is of class C^∞. If X is a C^k vector field on M with $k \geq 2$, we define the Lagrangian $L_X : TM \to \mathbb{R}$ by

$$L_X(x, v) = \frac{1}{2}\|v - X(x)\|_x^2,$$

where as usual $\|v - X(x)\|_x^2 = g_x(v - X(x), v - X(x))$. We will call L_X the Mañé Lagrangian of X; see the appendix in [25]. The following proposition gives the obvious properties of L_X:

PROPOSITION 4.13 *Let L_X be the Mañé Lagrangian of the C^k vector field X, with $k \geq 2$, on the compact Riemannian manifold M. We have*

$$\frac{\partial L_X}{\partial v}(x, v) = g_x(v - X(x), \cdot).$$

Its associated Hamiltonian $H_X : T^*M \to \mathbb{R}$ is given by

$$H_X(x, p) = \frac{1}{2}\|p\|_x^2 + p(X(x)).$$

The constant functions are solutions of the Hamilton-Jacobi equation

$$H_X(x, d_x u) = 0.$$

Therefore, we obtain $c(H) = 0$. Moreover, we have

$$\tilde{\mathcal{I}}(0) = \mathrm{Graph}(X) = \{(x, X(x)) \mid x \in M\}.$$

If we call ϕ_t the Euler-Lagrange flow of L_X on TM, then for every $x \in M$ and every $t \in \mathbb{R}$, we have $\phi_t(x, X(x)) = (\gamma_x^X(t), \dot{\gamma}_x^X(t))$, where γ_x^X is the solution of the vector field X that is equal to x for $t = 0$. In particular, the restriction $\phi_t|_{\tilde{\mathcal{I}}(0)}$ of the Euler-Lagrange flow to $\tilde{\mathcal{I}}(0) = \mathrm{Graph}(X)$ is conjugated (by $\pi|_{\tilde{\mathcal{I}}(0)}$) to the flow of X on M.

PROOF: Computing $\partial L_X/\partial v$ is easy. For H_X, we recall that $H_X(x,p) = p(v_p) - L(x, v_p)$, where $v_p \in T_xM$ is defined by $p = \partial L_X/\partial v(x, v_p)$. Solving for v_p and substituting yields the result.

If u is a constant function, then $d_x u = 0$ everywhere, and we obviously have $H_X(x, d_x u) = 0$. The fact that $c(H) = 0$ follows, since $c(H)$ is the only value c for which there exists a viscosity solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$.

Define u_0 as the null function on M. Suppose now that $\gamma : (-\infty, +\infty) \to M$ is a solution of X (by compactness of M, solutions of X are defined for all time). We have $d_{\gamma(t)}u_0(\dot\gamma(t)) = 0$ and $H_X(\gamma(t), d_{\gamma(t)}u_0) = 0$; moreover, since $\dot\gamma(t) = X(\gamma(t))$, we also get $L_X(\gamma(t), \dot\gamma(t)) = 0$. It follows that

$$d_{\gamma(t)}u_0(\dot\gamma(t)) = L_X(\gamma(t), \dot\gamma(t)) + H_X(\gamma(t), d_{\gamma(t)}u_0) = L_X(\gamma(t), \dot\gamma(t)).$$

By integration, we see that γ is $(u_0, L_X, 0)$-calibrated; therefore it is an extremal. Hence we get $\phi_t(\gamma(0), \dot\gamma(0)) = (\gamma(t), \dot\gamma(t))$ and $(\gamma(0), \dot\gamma(0)) \in \tilde{\mathcal{I}}(u_0)$. But $\dot\gamma(0) = X(\gamma(0))$, and $\gamma(0)$ can be an arbitrary point of M. This implies $\mathrm{Graph}(X) \subset \tilde{\mathcal{I}}(u_0)$. This finishes the proof because we know that $\tilde{\mathcal{I}}(u_0)$ is a graph on a part of the base M. □

LEMMA 4.14 *Let $L_X : TM \to \mathbb{R}$ be the Mañé Lagrangian associated to the C^k vector field X on the compact connected manifold M with $k \geq 2$. Assume that L_X satisfies the Mather disconnectedness condition. Then we have the following*:

(i) *The projected Aubry set $\mathcal{A}$ is the set of chain-recurrent points of the flow of X on M.*

(ii) *The constants are the only weak KAM solutions if and only if every point of M is chain-recurrent under the flow of X.*

PROOF: To prove (i), we apply Lemma 4.9 to obtain that the Aubry set $\tilde{\mathcal{A}}$ is equal to a set of points in $\tilde{\mathcal{I}}(0) = \mathrm{Graph}(X)$ that are chain-recurrent for the restriction $\phi_t|_{\mathrm{Graph}(X)}$. But from Proposition 4.13 the projection $\pi|_{\mathrm{Graph}(X)}$ conjugates $\phi_t|_{\mathrm{Graph}(X)}$ to the flow of X on M. It now suffices to observe that $\mathcal{A} = \pi(\tilde{\mathcal{A}})$.

We now prove (ii). Suppose that every point of M is chain-recurrent for the flow of X. From what we have just seen, $\mathcal{A} = M$ and thus property (i) of Theorem 4.10 holds. Therefore by property (iii) of that same theorem, we have uniqueness up to constants of weak KAM solutions, but the constants are weak KAM solutions.

To prove the converse, assume that the constants are the only weak KAM solutions. This implies that property (iii) of Theorem 4.10 holds. Therefore by property (iv) of that same theorem $\tilde{\mathcal{A}} = \tilde{\mathcal{N}}$. But $\tilde{\mathcal{I}}(0) = \mathrm{Graph}(X)$ is squeezed between $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{N}}$. Therefore $\tilde{\mathcal{A}} = \mathrm{Graph}(X)$. Taking images by the projection π, we conclude that $\mathcal{A} = M$. By part (i) of the present lemma, every point of M is chain-recurrent for the flow of X on M. □

Combining this last lemma and Theorem 4.8 completes the proof of Theorem 1.6.

4.4 Examples of Gradientlike Vector Fields

We recall the definition of a gradientlike vector field.

DEFINITION 4.15 A vector field X on M is said to be *gradientlike* if we can find a C^1 function $f : M \to \mathbb{R}$ such that

(i) for every $x \in M$, we have $X \cdot f(x) = d_x f(X(x)) \le 0$, and
(ii) for a given $x \in M$, we have $X \cdot f(x) = 0$ if and only if $X(x) = 0$.

As an example of a gradientlike vector field, we can take $X = -\operatorname{grad} f$, where $f : M \to \mathbb{R}$ is C^1 and the gradient is taken with respect to the Riemannian metric on M. In this case

$$X \cdot f(x) = -d_x f(\operatorname{grad} f(x)) = -\frac{1}{2}\|d_x f\|_x^2.$$

Note that if $\varphi : M \to \mathbb{R}$ is a function such that

$$\forall x \in M, \quad \varphi(x) = 0 \iff X(x) = 0,$$

and X is gradientlike, then φX is also gradientlike.

The following fact is easy to prove:

PROPOSITION 4.16 *If X is a C^1 gradientlike vector field, then the nonwandering set $\Omega(\phi_t^X)$ is equal to the zero set $Z(X) = \{x \in M \mid X(x) = 0\}$ of X (or equivalently $\Omega(\phi_t^X) = \operatorname{Fix}(\phi_t^X)$).*

In the case of a Mañé example associated to gradientlike vector field, we have:

PROPOSITION 4.17 *Let X be a gradientlike vector field, and denote by $\mathcal{A}$ the Aubry set of the Mañé Lagrangian L_X. Then the image of $\mathcal{A}^0$ in the Mather quotient $(\mathcal{A}_M, \delta_M)$ is full. Therefore, if X is C^k with $k \ge 2\dim M - 2$, then $\mathcal{H}^1(\mathcal{A}_M, \delta_M) = 0$, and L_X satisfies the Mather disconnectedness condition.*

PROOF: If $x \in \mathcal{A}$, the whole orbit $\phi_t^X(x)$ is contained in $\mathcal{A}$, and any limit point x_∞ of $\phi_t^X(x)$ as $t \to \infty$ is in $\Omega(\phi_t^X)$, and it is therefore fixed. We also know by (iii) of Theorem 2.1 that $\delta_M(x, x_\infty) = 0$. Therefore the image of $\mathcal{A}^0$ in the Mather quotient $(\mathcal{A}_M, \delta_M)$ is full. The rest of the proof follows by Theorem 1.2. □

Let us now give some examples.

We start with a Whitney counterexample to the Sard theorem (see, for example, [20]). Such a counterexample gives a function $f : \mathbb{T}^n \to \mathbb{R}$ that is C^{n-1} and for which we can find a connected set $C \subset \mathbb{T}^n$ such that $d_x f = 0$ for every $x \in C$ and f is not constant on C. Therefore $f(C) = [a, b] \subset \mathbb{R}$ with $a < b$. If we now consider $X = -\operatorname{grad} f$ and $L_X(v) = \frac{1}{2}\|v - X(x)\|_x^2$ on $T\mathbb{T}^n$, then f is a critical C^1 subsolution. In fact,

$$\begin{aligned} H_X(x, d_x f) &= d_x f(X(x)) + \frac{1}{2}\|d_x f\|_x^2 \\ &= -\|d_x f\|_x^2 + \frac{1}{2}\|d_x f\|_x^2 = -\frac{1}{2}\|d_x f\|_x^2. \end{aligned}$$

We see that this critical subsolution is strict outside $Z(X)$; therefore we have $Z(X) = C \supset \mathcal{A}$. Since f and 0 are both critical subsolutions, by the proof of Proposition 4.4 the function f is 1-Lipschitz seen as a map from $(\mathcal{A}_M, \delta_M)$ to $\mathbb{R}$. This implies that

$$\mathcal{H}^1(\mathcal{A}_M, \delta_M) \geq \mathcal{H}^1(f(A)) \geq \mathcal{H}^1(f(C)) = \mathcal{H}^1([a,b]) = b - a > 0.$$

It follows that, for this X, $\mathcal{H}^1(\mathcal{A}_M, \delta_M) > 0$ and L_X does not satisfy the Mather disconnectedness condition.

Note that we can assume that f is C^∞ outside C. Indeed, if this were not the case, we could approximate f in the C^{n-1} topology on $M \setminus C$ with a C^∞ function, so that this approximation would glue back with f on C to a C^{n-1} function.

By a standard result (see, for example, [11]), we can find a C^∞ function $\varphi : M \to [0, +\infty[$, with $\varphi|M \setminus C > 0$ and $\varphi|C = 0$, and such that φX is C^∞. Of course, the vector field φX is still gradientlike, but, since φX is C^∞, the associated Mañé Lagrangian satisfies the Mather disconnectedness condition, and its Aubry set is still $Z(X)$. Note that the orbits of X and φX are the same as $\varphi > 0$ on $M \setminus Z(X)$.

We can also modify a little bit f as suggested by Hurley in [22] to construct a C^{n-1} function $f : \mathbb{T}^n \to \mathbb{R}$ such that its Euclidean gradient $\operatorname{grad} f$ has a chain-recurrent point that is not a critical point of f and for which there exists a connected set $C \subset \mathbb{T}^n$ such that $d_x f = 0$ for every $x \in C$ and f is not constant on C. Although Hurley in [22, pp. 453–454] does it for $n = 2$ or 3, starting from a Whitney counterexample to the Sard theorem, it is clear that one can obtain it for any $n \geq 2$.

Note that again, if we take $X = -\operatorname{grad} f$ and denote by $\mathcal{A}_X$ the Aubry set of L_X as above, we will have $\mathcal{A}_X = Z(X)$, and in that case the chain-recurrent set of X is strictly larger than $\mathcal{A}_X$. Therefore one must have some high differentiability assumption on the vector field X in order to assure that $\mathcal{A}_X$ is equal to the set of chain-recurrent points.

Again taking some care in the construction of Hurley, and applying an approximation theorem, we can assume that f is C^∞ outside C. As above, we can find a C^∞ function $\varphi : \mathbb{T}^n \to [0, +\infty[$, with $\varphi|\mathbb{T}^n \setminus C > 0$ and $\varphi|C = 0$ and such that φX is C^∞. Note that $\mathcal{A}_{\varphi X}$ is equal to the chain-recurrent set of φX (which is the same as the chain-recurrent set of X) because $L_{\varphi X}$ satisfies the Mather disconnectedness condition.

Appendix A: A Lemma of Ferry and a Result of Mather

A.1 Ferry's Lemma

In this appendix, we state and prove a generalization of a lemma due to Ferry in 1976 [19]. This lemma was rediscovered by Bates in 1992 [3] to prove his generalization of Sard's theorem. They proved that if $E \subset \mathbb{R}^n$ is a measurable set,

$f : E \to \mathbb{R}$ is continuous, and $n \geq 2$ is such that f satisfies

$$\forall x, y \in E, \quad |f(x) - f(y)| \leq C\|x - y\|^n,$$

then $f(E)$ has Lebesgue measure zero.

Their proof yields in fact the following generalization:

LEMMA A.1 *Let $\Psi : E \to X$ be a map where E is a subset of $\mathbb{R}^n$ and (X, d_X) is a semimetric space. Suppose that there are p and M such that*

$$\forall x, y \in E, \quad d_X(\Psi(x), \Psi(y)) \leq M\|x - y\|^p.$$

If $p > 1$, then the (n/p)-dimensional Hausdorff measure of $(\Psi(E), d_X)$ is 0.

PROOF: Since all norms on $\mathbb{R}^n$ are equivalent, we can assume

$$\forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad \|x\| = \max_{i=1}^{n} |x_i|.$$

Since it suffices to prove that $\mathcal{H}^{\frac{n}{p}}(\Psi(E \cap K)) = 0$ for each compact set $K \subset \mathbb{R}^n$, we can assume that E is bounded, which in particular implies $\mathcal{L}^n(E) < +\infty$ (we denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$). We now write $E = E_1 \cup E_2$, where E_1 is the set of density points for E and $E_2 = E \setminus E_1$. By the definition of density points

$$\forall x \in E_1, \quad \lim_{r \to 0} \frac{\mathcal{L}^n(E \cap B(x, r))}{\mathcal{L}^n(B(x, r))} = 1.$$

It is a standard result in measure theory that $\mathcal{L}^n(E_2) = 0$. Thus for each $\epsilon > 0$ fixed, there exists a countable family of balls $\{B_i\}_{i \in I}$ such that

$$E_2 \subset \bigcup_{i \in I} B_i \quad \text{and} \quad \sum_{i \in I} (\operatorname{diam} B_i)^n \leq \epsilon.$$

Then we have

$$\begin{aligned}
\mathcal{H}^{\frac{n}{p}}(\Psi(E_2)) &\leq \sum_{i \in I} (\operatorname{diam}_X \Psi(B_i \cap E_2))^{n/p} \\
&\leq M^{\frac{n}{p}} \sum_{i \in I} [(\operatorname{diam} B_i)^p]^{n/p} \leq M^{n/p} \sum_{i \in I} (\operatorname{diam} B_i)^n \leq M^{n/p} \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$, we obtain $\mathcal{H}^{n/p}(\Psi(E_2)) = 0$. Note that in this part of the argument we have not used the condition $p > 1$.

We now want to prove that $\mathcal{H}^{n/p}(\Psi(E_1)) = 0$. Fix $N \in \mathbb{N}$. For every density point $x \in E_1$, there exists $\rho(x) > 0$ such that

$$\forall r \leq \rho(x), \quad \frac{\mathcal{L}^n(E_1 \cap B(x, r))}{\mathcal{L}^n(B(x, r))} = \frac{\mathcal{L}^n(E \cap B(x, r))}{\mathcal{L}^n(B(x, r))} \geq 1 - \frac{1}{2N^n}.$$

Note that, since $\mathcal{L}^n(B(y, s)) = 2^n s^n$, this implies that for such an $x \in E_1$, we have

$$\forall r \leq \rho(x), \ \forall y \in \mathbb{R}^n, \quad \mathcal{L}^n(B(x, r) \setminus E_1) \leq \frac{1}{2} \mathcal{L}^n\left(B\left(y, \frac{r}{N}\right)\right).$$

Therefore, since for $y \in B(x, \frac{N-1}{N} r)$ we have $B(y, \frac{r}{N})) \subset B(x, r)$, we obtain

$$\text{(A.1)} \qquad \forall r \le \rho(x),\ \forall y \in B(x, \frac{N-1}{N} r), \quad E_1 \cap B\left(y, \frac{r}{N}\right) \ne \varnothing.$$

Fix $x \in E_1$. It is now simple to prove that for all $y \in E_1 \cap B(x, r)$, with $r \le \rho(x)$, there exist $N+1$ points $x_0, \dots, x_N \in E_1$, with $x_0 = x$ and $x_N = y$, such that

$$\forall 1 \le i \le N, \quad |x_i - x_{i-1}| \le \frac{3r}{N}.$$

Indeed, first take $y_1, \dots, y_{N-1}$, the $N-1$ points on the line segment $[y, x]$ such that $|y_i - y_{i-1}| = \frac{|y-x|}{N}$. We then observe that, for $i = 1, \dots, N-1$, we have $\|y_i - x\| \le i\,\frac{|y-x|}{N} \le (N-1)\frac{r}{N}$. Hence, by (A.1), the intersection $B(y_i, \frac{r_x}{N}) \cap E_1$ is not empty for each $i = 1, \dots, N-1$, and so it suffices to take a point x_i in that intersection. Then, for all $y \in E_1 \cap B(x, r)$,

$$\begin{aligned} \text{(A.2)} \quad d_X(\Psi(x), \Psi(y)) &\le \sum_{i=1}^{N} d_X(\Psi(x_{i-1}), \Psi(x_i)) \\ &\le M \sum_{i=1}^{N} |x_i - x_{i-1}|^p \le MN \left(\frac{3r}{N}\right)^p = 3^p M N^{1-p} r^p. \end{aligned}$$

It follows that

$$\begin{aligned} &\forall x \in E_1,\ \forall r \le \rho(x), \quad \operatorname{diam}(\Psi(B(x, r) \cap E_1) \\ \text{(A.3)} \qquad &\qquad \le 2(3^p M N^{1-p} r^p) \\ &\qquad = 2^{1-p} 3^p M N^{1-p} [\operatorname{diam}(B(x, r))]^p. \end{aligned}$$

We are now able to prove that $\mathcal{H}^{n/p}(\Psi(E_1)) = 0$. Take an open set $\Omega \supset E_1$ such that $\mathcal{L}^n(\Omega) \le \mathcal{L}^n(E_1) + 1 = \mathcal{L}^n(E) + 1 < +\infty$, and consider the fine covering $\mathcal{F}$ given by $\mathcal{F} = \{B(x, r)\}_{x \in E_1}$ with r such that $B(x, r) \subset \Omega$ and $r \le \frac{\rho(x)}{5}$, where $\rho(x)$ is as defined above. By Vitali's covering theorem (see [10, par. 1.5.1]), there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$E_1 \subset \bigcup_{B \in \mathcal{G}} 5B,$$

where $5B$ denotes the ball concentric to B with radius 5 times that of B. Since the balls in $\mathcal{F}$ are disjoint and contained in Ω, we get

$$\sum_{B \in \mathcal{G}} \mathcal{L}^n(B) \le \mathcal{L}^n(\Omega) \le \mathcal{L}^n(E) + 1 < +\infty.$$

Since the norm on $\mathbb{R}^n$ is the max norm, we have $\mathcal{L}^n(B) = \operatorname{diam}(B)^n$ for every B that is a ball for the norm. Therefore

$$\text{(A.4)} \qquad \sum_{B \in \mathcal{G}} \operatorname{diam}(B)^n \le \mathcal{L}^n(\Omega) \le \mathcal{L}^n(E) + 1 < +\infty.$$

We can thus consider the covering of $\Psi(E_1)$ given by $\bigcup_{B\in\mathcal{G}} \Psi(5B \cap E_1)$. In this way, by (A.3), we get

$$\begin{aligned}
\mathcal{H}^{\frac{n}{p}}(\Psi(E_1)) &\le \sum_{B\in\mathcal{G}} (\operatorname{diam}_X \Psi(5B \cap E_1))^{\frac{n}{p}} \\
&\le \sum_{B\in\mathcal{G}} (2^{1-p} 3^p M N^{1-p} [5 \operatorname{diam}(B)]^p)^{\frac{n}{p}} \\
&= \sum_{B\in\mathcal{G}} 2^{\frac{n(1-p)}{p}} 3^n M^{\frac{n}{p}} N^{\frac{n(1-p)}{p}} 5^n \operatorname{diam}(B)^n \\
&= 2^{\frac{n(1-p)}{p}} 3^n M^{\frac{n}{p}} N^{\frac{n(1-p)}{p}} 5^n \sum_{B\in\mathcal{G}} \operatorname{diam}(B)^n.
\end{aligned}$$

Using (A.4), we obtain

$$\mathcal{H}^{\frac{n}{p}}(\Psi(E_1)) \le 2^{\frac{n(1-p)}{p}} 3^n M^{\frac{n}{p}} N^{\frac{n(1-p)}{p}} 5^n (\mathcal{L}^n(E) + 1).$$

Because $\mathcal{L}^n(E) + 1 < \infty$ and $1 - p < 0$, by letting $N \to \infty$ we obtain $\mathcal{H}^{n/p}(\Psi(E_1)) = 0$. □

Remark A.2. As we said at the beginning of the appendix, the original case of Ferry's lemma plays a crucial role in Steve Bates's [3] version of the Morse-Sard theorem: If $f : M \to \mathbb{R}$ is of class $C^{n-1,1}$, where $n = \dim M \ge 2$, then the set of critical values of f is of Lebesgue measure zero.

In fact, the original case of Ferry's lemma is also a consequence of Bates's [3] version of the Morse-Sard theorem. Indeed, note first that, by uniform continuity, we can extend f to the closure $\bar{E}$ of E in $\mathbb{R}^n$. Of course, by continuity we will also have

$$\forall x, y \in \bar{E}, \quad |f(x) - f(y)| \le C\|x - y\|^n.$$

On the closed set the family $f, Df = 0, \ldots, D^{n-1}f = 0$ satisfy the condition of Whitney's extension theorem with $D^{k-1}f$ Lipschitz (see [37, theorem 4, p. 177]); therefore there exists an extension $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ that is, of class $C^{n-1,1}$. Of course, all points of $\bar{E}$ are critical points of $\bar{f}$, so by Bates's version of the Morse-Sard theorem, $\bar{f}(\bar{E}) = f(\bar{E})$ has measure zero.

It is easy to generalize this result to a finite-dimensional manifolds, since such manifolds are always assumed metric and separable, and therefore second countable.

Before stating this generalization, we recall that on a smooth (in fact, at least C^1) finite-dimensional manifold M the notion of locally Hölder of exponent $p \ge 0$ makes sense. A map $f : A \to X$ where (X, d_X) is a metric space and $A \subset M$ is said to be locally Hölder of exponent p (we allow $p \ge 1$!) if for every $x \in A$, we can find a neighborhood U_x of x and $M_x < \infty$ such that

$$\forall y, y' \in U_x \cap A, \quad d_X(f(y), f(y')) \le M_x d_M(y, y')^p,$$

where d_M is a distance obtained from a Riemannian metric on M. Note that this notion is independent of the choice of d_M, since all distances obtained from Riemannian metrics are locally Lipschitz equivalent. It is not difficult to show that $f : A \to X$ is locally Hölder of exponent p if and only if we can find a family $(U_i, \varphi_i)_{i \in I}$ of smooth (or at least C^1) charts of M, with U_i open subsets of $\mathbb{R}^n$, where $n = \dim M$, and a family $M_i \in I$ of finite numbers such that $A \subset \bigcup_{i \in I} \varphi_i(U_i)$ and

$$\forall i \in I,\ \forall x, x' \in U_i, \quad d_X(f\varphi_i(x), f\varphi_i(x')) \leq M_i \|x - x'\|^p,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. Since M is second countable, we can always assume that I is itself countable, and therefore we can deduce the following generalization of Lemma A.1.

LEMMA A.3 *Let M be a (metric separable) manifold of dimension $n < \infty$ and (X, d_x) be a semimetric space. Suppose $\Psi : A \to X$, where $A \subset M$ is a locally Hölder map of exponent $p > 1$. Then the (n/p)-dimensional Hausdorff measure of $(\Psi(A), d_x)$ is* 0.

A.2 Mather's Result

We would like to show how one can deduce from Ferry's lemma the following result of Mather (compare with [30, prop. 1, p. 1507]):

PROPOSITION A.4 *Let X be a compact, connected subset of $\mathbb{R}^d$, $d \geq 2$. Let $x, y \in X$ and $\epsilon > 0$. Then there exists a sequence $x = x_0, \ldots, x_k = y$ of points in X such that $\sum_{i=0}^{k-1} \|x_{i+1} - x_i\|^d < \epsilon$.*

In fact, if (A, d) is a metric space and $p > 0$, we can introduce a semimetric δ_p on A defined by

$$\delta_p(a, a') = \inf\left\{ \sum_{i=0}^{k-1} d(a_{i+1}, a_i)^p \mid k \geq 1, a_1, \ldots a_{k-1} \in A, a_0 = a, a_k = a' \right\}.$$

It is not difficult to check that δ_p is symmetric and satisfies the triangular inequality, and that $\delta_p(a, a) = 0$ for every $a \in A$. Note that when $p \leq 1$, the function d^p is already a metric. Therefore it follows by the triangular inequality that $\delta_p = d^p$ when $p \leq 1$. However, when $p > 1$, we might have $\delta_p(a, a') = 0$ with $a \neq a'$. This is indeed the case when $A = [0, 1]$ with distance $d(t, t') = |t - t'|$. In fact, if we divide the segment $[t, t']$ by N equally spaced points, we obtain $\delta_p(t, t') \leq N(|t - t'|/N)^p$; hence, letting $N \to \infty$, since $p > 1$ we obtain $\delta_p = 0$. This yields the first of the following remarks.

Remark A.5.

(1) If $p > 1$ and there exists a Lipschitz curve $\gamma : [0, 1] \to A$ with $\gamma(0) = a$ and $\gamma(1) = a'$, then $\delta_p(a, a') = 0$ for every $p > 1$.

(2) We will say that A is Lipschitz-arcwise-connected if for every $a, a' \in A$ there is a Lipschitz curve $\gamma : [0,1] \to A$ with $\gamma(0) = a$ and $\gamma(1) = a'$. It follows from (1) that $\delta_p \equiv 0$ if A is Lipschitz-arcwise-connected and $p > 1$.

(3) If M is a connected, smooth manifold with a distance d coming from a Riemannian metric, then $\delta_p \equiv 0$ for every $p > 1$. This follows from (1) above since any two points in a connected manifold can be joined by a smooth path.

(4) If $A' \subset A$ we can consider the distance δ'_p associated to $(A', d|A)$ and $p > 0$. We always have $\delta_p|A' \leq \delta'_p$ with equality when A' is dense in A.

(5) If $f : A \to B$ is Lipschitz with constant Lipschitz constant $\leq K$, then f is also Lipschitz as a map from (A, δ_p^A) to (B, δ_p^B), with Lipschitz constant $\leq K^p$.

In what follows, we will denote by $(\hat{A}_p, \delta_p)$, or simply $\hat{A}_p$, the metric space obtained by identifying points $a, a' \in A$ such that $\delta_p(a, a') = 0$. We denote by $\hat{\pi}_p : A \to \hat{A}_p$ the canonical projection. It is clear that $\delta_p(a, a') \leq d(a, a')^p$; therefore the projection is Hölder of exponent $p > 0$. It follows that one has the following consequence of Lemma A.3.

PROPOSITION A.6 *Suppose that A is a subset of an n-dimensional manifold M and that d is a distance that is locally Lipschitz equivalent to a restriction to A of a distance on M coming from a Riemannian metric. Then $\mathcal{H}^{n/p}(\hat{A}_p) = 0$ for all $p > 1$. In particular, if $n \geq 2$, we have $\mathcal{H}^1(\hat{A}_n) = 0$, and therefore $\hat{A}_n$ is totally disconnected.*

This proposition follows from Lemma A.3 except for the last statement, which is a general fact: If a metric space X has 0 one-dimensional Hausdorff measure, it is totally disconnected. In fact, if x is fixed, note that the map $d_x : X \to \mathbb{R}$, $y \mapsto d(x, y)$ is Lipschitz; hence the image $d_x(X)$ also has one-dimensional Hausdorff measure, i.e., Lebesgue measure, in $\mathbb{R}$ equal to 0. In particular, we can find a sequence $r_n > 0$, with $r_n \to 0$ and $r_n \notin d_x(X)$. This last condition means that $\{y \in X \mid d(x, y) = r_n\}$ is empty; therefore the boundary of the ball $\bar{B}_d(x, r_n)$ is empty.

It is now easy to obtain Proposition A.4. In fact, if under the hypotheses of Proposition A.6 we also assume that A is connected, then $\hat{A}_n$ is also connected because $\hat{\pi}_p$ is continuous and surjective. But a connected and totally disconnected metric space contains at most one point; therefore $\delta_n(x, y) = 0$ for every pair of points in the connected subset A of $\mathbb{R}^n$ when $n \geq 2$.

Note that we could have obtained Proposition A.4 directly from Bates's [3] version of the Morse-Sard theorem along the lines mentioned in Remark A.2.

Mather gave an extension, Proposition A.4, to Lipschitz laminations; see [30, prop. 2, p. 1510]. In fact, by our method we can give a much more general result. For this we introduce the following definition:

DEFINITION A.7 (Agglutination) A subset A of the finite n-dimensional manifold M is a *Lipschitz agglutination of codimension* k if every $x \in A$ is contained in a subset $B \subset A$ that is Lipschitz-arcwise-connected and of topological dimension $\geq n-k$.

Obviously any subset of the manifold that admits a codimension k Lipschitz lamination, as considered in [30], is a codimension k Lipschitz agglutination. Moreover, any union of Lipschitz agglutination of codimension k is itself a Lipschitz agglutination of codimension k. In particular, any union of codimension k immersed Lipschitz submanifolds is a Lipschitz agglutination of codimension k. We can now state our generalization.

PROPOSITION A.8 *Suppose that A is a codimension-k Lipschitz agglutination of the n-dimensional manifold M, and that d is a distance that is locally Lipschitz-equivalent to the restriction to A of a distance on M coming from a Riemannian metric. Then $\mathcal{H}^{k/p}(\hat{A}_p) = 0$ for all $p > 1$. In particular, if $k \geq 2$, we have $\mathcal{H}^1(\hat{A}_k) = 0$, and therefore $\hat{A}_k$ is totally disconnected.*

We first prove a well-known lemma.

LEMMA A.9 *If M is a finite-dimensional (metric separable) manifold, and d is an integer with $0 \leq d \leq n$, we can find a sequence $(D_i)_{i\in\mathbb{N}}$ of subsets of M, each of which is C^∞ diffeomorphic to a Euclidean disc of dimension $n-d$ such that the topological dimension of $M \setminus \bigcup_{i\in N} D_i$ is $\leq d-1$. In particular, any subset B of M of topological dimension $\geq d$ has to intersect one of the D_i.*

PROOF: We first consider the case $M = \mathbb{R}^n$. Call S_n^d the family of subsets of $\{1,\dots,n\}$ with exactly d elements. For every $I \in S_n^d$ and every $(r_1,\dots,r_n) \in \mathbb{Q}^n$, we define

$$V^I_{(r_1,\dots,r_n)} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_j = r_j \ \forall j \in I\}.$$

Each $V^I_{(r_1,\dots,r_n)}$ is an affine subspace of dimension $n-d$, and this family is countable.

If we denote by $\mathcal{M}_n^{d-1}$ the complement in $\mathbb{R}^n$ of the countable union of the subsets V_r^I, $I \in S_n^d$, $r \in \mathbb{Q}^n$, then the points in $\mathcal{M}_n^{d-1}$ are precisely the points in $\mathbb{R}^n$ that have at most $d-1$ rational coordinates. By [21, example III.6, p. 29] the topological dimension of $\mathcal{M}_n^{d-1}$ is $\leq d-1$ (in fact, it is $d-1$).

We now consider a general (metric separable), n-dimensional smooth manifold M. We can find a countable family of charts $\varphi_j : \mathbb{R}^n \to M$, $j \in \mathbb{N}$, such that $\bigcup_{j\in\mathbb{N}} \varphi_j(\bar{\mathbb{B}}) = M$, where $\bar{\mathbb{B}}$ is the unit closed Euclidean ball in $\mathbb{R}^n$. We consider the countable collection $D_{j,I,r}$, $j \in \mathbb{N}$, $I \in S_n^d$, $r \in \mathbb{Q}^n$, defined by

$$D_{j,I,r} = \varphi_j(V_r^I).$$

Each $D_{j,I,r}$ is C^∞ diffeomorphic to a Euclidean disc of dimension $n-d$. We now show that the topological dimension of the complement

$$\mathscr{C} = M \setminus \bigcup_{\substack{j\in\mathbb{N}\\ I\in S_n^d\\ r\in\mathbb{Q}^n}} D_{j,I,r}$$

is $\leq d-1$. We can write $\mathscr{C} = \bigcup_{j\in\mathbb{N}} \mathscr{C} \cap \varphi_j(\bar{\mathbb{B}})$. Since each $\mathscr{C} \cap \varphi_j(\bar{\mathbb{B}})$ is closed in $\mathscr{C}$, by the countable sum theorem [21, theorem III.2, p. 30], it suffices to show that each $\mathscr{C} \cap \varphi_j(\bar{\mathbb{B}})$ has topological dimension $\leq d-1$. But the map $\varphi_j^{-1} : \varphi_j(\mathbb{R}^n) \to \mathbb{R}^n$ sends $\mathscr{C} \cap \varphi_j(\bar{\mathbb{B}})$ to a subset of $\mathcal{M}_n^{d-1}$ that has topological dimension $\leq d-1$. This implies that the topological dimension of $\mathscr{C} \cap \varphi_j(\bar{\mathbb{B}})$ is $d-1$ by [21, theorem III.1, p. 26]. Note that this last reference also proves the last statement in the lemma. $\square$

PROOF OF PROPOSITION A.8: We apply the lemma above with $d = n-k$ to obtain a countable family D_i, $i \in \mathbb{N}$, of C^∞ discs of dimension $n-d = k$ such that each subset of M whose topological dimension is $\geq d = n-k$ has to intersect one of the D_i. Consider then a Lipschitz agglutination $A \subset M$ of codimension k, and fix $p > 1$.

We first claim that $\hat{A}_p = \bigcup_{i\in\mathbb{N}} \hat{\pi}_p(A \cap D_i)$. In fact, if $x \in A$, by the definition of a Lipschitz agglutination of codimension k, we can find a Lipschitz-arcwise-connected subset $B_x \subset A$ of dimension $\geq n-k$ containing x. By the property of the family D_i, there exists $i_0 \in I$ such that $B_x \cap D_{i_0} \neq \varnothing$. Choose $y \in B_x \cap D_{i_0}$. By (2) of Remark A.5, we have $\delta_p^{B_x}(x,y) = 0$. Since $B_x \subset A$, we conclude that $\delta_p^A(x,y) = 0$. Therefore $\hat{\pi}_p(x) = \hat{\pi}_p(y) \in \hat{\pi}_p(B_x \cap D_i) \subset \hat{\pi}_p(A \cap D_i)$. Since the family D_i is countable, it remains to show that $\mathcal{H}^{k/p}(D_i \cap A, \delta_p^A) = 0$. Note that since D_i is a submanifold of M, the distance d on M induces a distance on D_i that is locally Lipschitz equivalent to a distance coming from a Riemannian metric. Therefore by Proposition A.6, we have $\mathcal{H}^{k/p}(D_i \cap A, \delta_p^{D_i\cap A}) = 0$. But the inclusion $D_i \cap A \hookrightarrow A$ is Lipschitz with Lipschitz constant 1 for the metrics $\delta_p^{D_i\cap A}$ on $D_i \cap A$ and δ_p^A on A. Therefore $\mathcal{H}^{k/p}(D_i \cap A, \delta_p^A) = 0$. $\square$

Appendix B: Existence of $C^{1,1}_{\mathrm{loc}}$ Critical Subsolution on Noncompact Manifolds

In [4], using a kind of Lasry-Lions regularization (see [23]), Bernard proved the existence of $C^{1,1}$ critical subsolutions on compact manifolds. Here, adapting his proof, we show that the same result holds in the noncompact case, and we make clear that the Lipschitz constant of the derivative of the $C^{1,1}_{\mathrm{loc}}$ critical subsolution can be uniformly bounded on compact subsets of M. We consider the two Lax-Oleinik semigroups T_t^- and T_t^+ defined by

$$T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y, x), \qquad T_t^+ u(x) = \sup_{y \in M} u(y) - h_t(x, y),$$

for every $x \in M$.

For any $c \in M$, these two semigroups preserve the set of functions dominated by $L + c$; see, for example, [15] for the compact case or [17] for the noncompact case. It is also well-known that these semigroups have some regularizing effects: namely, for every $t > 0$ and every Lipschitz (or even continuous, when M is compact) function $u : M \to \mathbb{R}$, the function $T_t^+ u$ is finite everywhere and locally semiconvex, while $T_t^- u$ is finite everywhere and locally semiconcave; see, for example, [15] or the explanations below.

In [4], the idea for proving the existence of $C^{1,1}$ critical subsolutions on compact manifolds is the following: it is a known fact that a function is $C^{1,1}$ if and only if it is both locally semiconcave and locally semiconvex. Let now u be a critical viscosity subsolution. If we apply the semigroup T_t^+ to u, we obtain a semiconvex critical viscosity subsolution $T_t^+ u$. Thus, if one proves that, for s small enough, $T_s^- T_t^+ u$ is still semiconvex, since we already know that it is semiconcave, we would have found a $C^{1,1}$ critical subsolution. Since we want to give a uniform bound on the Lipschitz constant of the derivative of the $C^{1,1}_{\mathrm{loc}}$ critical subsolution on compact sets, we will have to bound the constant of semiconvexity of $T_t^+ u$ on compact subsets of M. Let us now prove the result in the noncompact case.

THEOREM B.1 *Assume that H is of class C^2. For every compact subset K of M, there is a constant $\ell = \ell(K) > 0$ such that, if $u : M \to \mathbb{R}$ is a critical viscosity subsolution, then there exists a $C^{1,1}_{\mathrm{loc}}$ critical subsolution $v : M \to \mathbb{R}$ whose restriction to the projected Aubry set is equal to u and such that the mapping $x \mapsto (x, d_x v)$ is ℓ-Lipschitz on K.*

Before proving Theorem B.1, we need a few lemmas.

LEMMA B.2 *There is a constant $A < +\infty$ such that for any $c \in \mathbb{R}$, any function $u : M \to \mathbb{R}$ dominated by $L + c$ is $(A + c)$-Lipschitz on M, that is,*

$$\forall x, y \in M, \quad |u(y) - u(x)| \le (A + c) d(x, y),$$

where d denotes the Riemannian distance associated to the Riemannian metric g on M.

PROOF: Let $u : M \to \mathbb{R}$ be dominated by $L + c$ and $x, y \in M$ be fixed. Let $\gamma_{x,y} : [0, d(x, y)] \to M$ be a minimizing geodesic with constant unit speed joining x to y. By definition of $h_{d(x,y)}(x, y)$, one has

$$h_{d(x,y)}(x, y) \le \int_0^{d(x,y)} L(\gamma_{x,y}(t), \dot{\gamma}_{x,y}(t)) dt \le A d(x, y),$$

where $A = \sup_{x \in M}\{L(x, v) \mid \|v\|_x \leq 1\}$ is finite thanks to the uniform boundedness of L in the fibers. Thus one has

$$u(x) - u(y) \leq h_{d(x,y)}(x, y) + cd(x, y) \leq (A + c)d(x, y).$$

Exchanging x and y, we conclude that u is $(A + c)$-Lipschitz. □

Next we give some estimates on the functions h_t.

LEMMA B.3 *There exists a constant $B < +\infty$ such that*

$$\forall t > 0,\ \forall x \in M, \quad h_t(x, x) \leq Bt.$$

Moreover, for every constant $C < +\infty$, we can find $D(C) > -\infty$ such that

$$\forall t > 0,\ \forall x, y \in M, \quad h_t(x, y) \geq Cd(x, y) + D(C)t.$$

PROOF: Using a constant curve at x, we get

$$h_t(x, x) \leq \int_0^t L(x, 0)ds.$$

Therefore, if we set $B = \sup\{L(x, 0) \mid x \in M\} < +\infty$, we obtain

$$\forall t > 0,\ \forall x \in M, \quad h_t(x, x) \leq Bt.$$

Using the uniform superlinearity of L, for every $C < +\infty$, we can find a constant $D(C) > -\infty$, depending only on C, such that

$$\forall (x, v) \in TM, \quad L(x, v) \geq C\|v\|_x + D(C).$$

Fix now $x, y \in M$. If $\gamma : [0, t] \to M$ is such that $\gamma(0) = x$, $\gamma(t) = y$, we can apply the above equality to $(\gamma(s), \dot{\gamma}(s))$ and integrate to obtain

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))ds \geq C \operatorname{length}(\gamma) + D(C)t \geq Cd(x, y) + D(C)t.$$

To find $h_t(x, y)$, we have to minimize $\int_0^t L(\gamma(s), \dot{\gamma}(s))ds$ over all curves with $\gamma(0) = x$, $\gamma(t) = y$. Therefore, by what we just obtained, we get

$$h_t(x, y) \geq Cd(x, y) + D(C)t. \qquad \square$$

LEMMA B.4 *If $C < +\infty$ is a given constant, we can find $B(C) < +\infty$ such that for every $u : M \to \mathbb{R}$ that is Lipschitz, with Lipschitz constant $\leq C$, we have*

$$\begin{aligned}
&\forall t \geq 0,\ \forall x \in M,\\
&\quad T_t^- u(x) = \inf\{u(y) + h_t(y, x) \mid y \in M, d(x, y) \leq B(C)t\},\\
&\quad T_t^+ u(x) = \sup\{u(y) - h_t(x, y) \mid y \in M, d(x, y) \leq B(C)t\},\\
&\quad |T_t^- u(x) - u(x)| \leq B(C)t,\\
&\quad |T_t^+ u(x) - u(x)| \leq B(C)t.
\end{aligned}$$

PROOF: We will do the proof for T_t^-, as the proof for T_t^+ is analogous. Using the first part of Lemma B.3, we get

$$T_t^- u(x) \leq u(x) + h_t(x, x) \leq u(x) + Bt.$$

By the second part of Lemma B.3, we get

$$T_t^- u(x) \geq \inf_{y \in M} u(y) + Cd(x, y) + D(C)t.$$

Since u is C-Lipschitz, we have $u(x) \leq u(y) + Cd(x, y)$; hence $T_t^- u(x) \geq u(x) + D(c)t$. It follows that

$$|T_t^- u(x) - u(x)| \leq \max\{B, -D(C)\}t.$$

Since $u(x) + h_t(x, x) \leq u(x) + Bt$, we obtain

$$T_t^- u(x) = \inf\{u(y) + h_t(y, x) \mid y \in M, u(y) + h_t(y, x) \leq u(x) + Bt\}.$$

Using again the second part of Lemma B.3 and the fact that u is C-Lipschitz, we know that

$$\begin{aligned} u(y) + h_t(y, x) &\geq u(y) + (C + 1)d(x, y) + D(C + 1)t \\ &\geq u(x) + d(x, y) + D(C + 1)t. \end{aligned}$$

It follows that

$$T_t^- u(x) = \inf\{u(y) + h_t(y, x) \mid y \in M,\ d(x, y) \leq Bt - D(C + 1)t\}.$$

Hence we can take any finite number $\geq \max\{B, -D(C), B - D(C + 1)\}$ as $B(C)$. □

For the next lemmas we need to introduce some notation. We will suppose that (U, φ) is a C^∞ chart on M. Here U is an open subset, and $\varphi : U \to \mathbb{R}^k$ is a C^∞ diffeomorphism on the open subset $\varphi(U)$ of $\mathbb{R}^k$. We will denote by $\|\cdot\|_{\text{euc}}$ the canonical Euclidean norm on $\mathbb{R}^k$. For $r \geq 0$, we set

$$\mathbb{B}(r) = \{v \in \mathbb{R}^k \mid \|v\|_{\text{euc}} \leq r\};$$

i.e., the subset $\mathbb{B}(r)$ is the closed Euclidean ball of radius r and center 0 in $\mathbb{R}^k$.

LEMMA B.5 *Suppose that (U, φ) is a C^∞ chart on M and $\mathbb{B}(r) \subset \varphi(U)$. For any data $r' < r$, $A \geq 1$, $B \geq 1$, and $\epsilon > 0$, there is a $\delta > 0$ such that for any function $u : \mathbb{B}(r) \to \mathbb{R}$ satisfying*

(a) *the function u is $C^{1,1}_{\text{loc}}$ on $\mathbb{B}(r)$,*
(b) *the Lipschitz constant (for the canonical Euclidean metric on $\mathbb{R}^k$) of u on $\mathbb{B}(r)$ is $\leq A$, and*
(c) *the Lipschitz constant (for the canonical Euclidean metric on $\mathbb{R}^k$) of the derivative $x \mapsto d_x(u \circ \varphi^{-1})$ on $\mathbb{B}(r)$ is bounded by B,*

and any $t \le \delta$, the function $T_t^{-,\varphi} u : \mathbb{B}(r) \to \mathbb{R}$ defined by

$$T_t^{-,\varphi} u(x) = \inf_{y \in \mathbb{B}(r)} u(y) + h_t(\varphi^{-1}(y), \varphi^{-1}(x))$$

satisfies

(a′) *the function $T_t^{-,\varphi} u$ is $C^{1,1}$ on a neighborhood of $\mathbb{B}(r')$,*

(b′) *the Lipschitz constant (for the canonical Euclidean metric on $\mathbb{R}^k$) of $T_t^{-,\varphi} u$ is bounded by $A + \epsilon$,*

(c′) *the Lipschitz constant (for the canonical Euclidean metric on $\mathbb{R}^k$) of $x \mapsto d_x(T_t^{-,\varphi} u)$ on $\mathbb{B}(r')$ is bounded by $B + \epsilon$, and*

(d′) *for every $x \in \mathbb{B}(r')$, there is one and only one $y_x \in \mathbb{B}(r)$ such that*

$$\varphi^{-1}(x) = \pi^* \phi_t^H(\varphi^{-1}(y_x), d_{\varphi^{-1}(y_x)}(u \circ \varphi)),$$

where $\pi^* : T^*M \to M$ is the canonical projection, and ϕ_t^H is the Hamiltonian flow of H on T^*M. Moreover, we have

$$(\varphi^{-1}(x), d_{\varphi^{-1}(x)}(u \circ \varphi)) = \phi_t^H(\varphi^{-1}(y_x), d_{\varphi^{-1}(y_x)}(u \circ \varphi)).$$

PROOF: We can assume that $r < +\infty$. To simplify notation, we will suppose that φ is the "identity"; i.e., we will write things in the coordinate system given by φ. Let us choose r'' and R such that $r' < r'' < r < R$ and $\varphi(U) \supset \mathbb{B}(R)$. If we set $A_1 = \sup\{L(x, v) \mid x \in \mathbb{B}(R), \|v\|_{\text{euc}} \le 1\}$, any function $u \prec L + c$ has, on $\mathbb{B}(R)$, a Lipschitz constant $\le A = A_1 + c$. In particular, $\|d_x u\|_{\text{euc}} \le A$ at every point $x \in \mathbb{B}(R)$ where $d_x u$ exists.

By continuity and compactness we can find $\delta_1 > 0$ such that

$$\forall x \in \mathbb{B}(r),\ \forall p \in (\mathbb{R}^k)^* \text{ with } \|p\|_{\text{euc}} \le A,\ \forall t \in [-\delta_1, \delta_1],\qquad \phi_t^H(x, p) \in \mathbb{B}(R) \times (\mathbb{R}^k)^*.$$

By Lemma B.4 we can find $\delta_2 > 0$, with $\delta_2 \le \delta_1$ depending only c, such that for any function $u : M \to \mathbb{R}$ with $u \prec L + c$ and any $t \le \delta_2$, we have

$$\forall x \in \mathbb{B}(r''), \quad T_t^- u(x) = \inf_{y \in \mathbb{B}(r)} u(y) + h_t(y, x).$$

Fix a function u satisfying (a), (b), and (c) of the lemma. We will show that $T_t^- u$ is $C^{1,1}$ on $\mathbb{B}(r'')$ for t small enough (depending on A and B and not on u), and we will compute the Lipschitz constant of the derivative of this function. Classically one shows that $T_t^- u$ is $C^{1,1}$ by using the inverse function theorem for a Lipschitz perturbation of the identity. For a change, we will do it in a (very slightly) different way using that $T_t^- u$ is Lipschitz.

Suppose $t \le \delta_1$. For $x \in \mathbb{B}(r'')$ choose a point $y_x \in \mathbb{B}(r)$ such that $T_t^- u(x) = u(y_x) + h_t(y_x, x)$. If we choose a minimizer $\gamma : [0, t] \to M$ with $\gamma(0) = y_x$, $\gamma(t) = x$, and whose action is $h_t(x, y)$, we know that $\partial L/\partial v(x, \dot\gamma(t))$ is in the upper gradient of $T_t^- u$ at x, and $\partial L/\partial v(x, \dot\gamma(0))$ is in the lower gradient of u at y_x. Since u is differentiable at y_x, we necessarily have $\partial L/\partial v(x, \dot\gamma(0)) =$

$d_{y_x}u$. Moreover, at each point $x \in \mathbb{B}(r'')$ where the Lipschitz function $T_t^- u$ is differentiable, we must have $d_x T_t^- u = \partial L/\partial v(x, \dot\gamma(t))$. Since γ is a minimizer, its speed curve $s \mapsto (\gamma(s), \dot\gamma(s))$ is an orbit of the Euler-Lagrange flow ϕ_s^L associated to L. Since the conjugate of ϕ_s^L is the Hamiltonian flow ϕ_s^H of the Hamiltonian $H : T^*M \to \mathbb{R}$ associated by Fenchel duality to L, we obtain that at each x where $T_t^- u$ is differentiable

$$(x, d_x T_t^- u) = \phi_t^H(y_x, d_{y_x}u). \tag{B.1}$$

Therefore $x = \pi^* \phi_t^H(y_x, d_{y_x}u)$, where π^* is the canonical projection from T^*M to M. In the local coordinates that we are using, $\pi^* : \mathbb{B}(r) \times (\mathbb{R}^k)^* \to \mathbb{B}(r)$ is the projection on the first factor.

To simplify computations we use the norm $\|(x, p)\| = \max(\|x\|_{\text{euc}}, \|p\|_{\text{euc}})$ on $\mathbb{B}(R) \times (\mathbb{R}^k)^* \subset \mathbb{R}^k \times (\mathbb{R}^k)^*$. Let us set $\psi(s, y, p) = \pi^*(\phi_s^H(y, p)) - y$. This map is C^1 and is identically 0 when $s = 0$; therefore on the compact set $\{(y, p) \in \mathbb{B}(r) \times (\mathbb{R}^k)^* \mid \|p\| \le A\}$ the Lipschitz constant $\ell(s)$ of $(y, p) \mapsto \psi(s, y, p)$ tends to 0 as $s \to 0$. Since $y \mapsto d_y u$ has a Lipschitz constant bounded by $B \ge 1$ on $\mathbb{B}(r)$, the map $y \mapsto (y, d_y u)$ has also a Lipschitz constant bounded by B on $\mathbb{B}(r)$. Moreover, since $\|d_y u\|_{\text{euc}}$ is bounded by A on $\mathbb{B}(r)$, we see that on $\mathbb{B}(r)$ we have

$$\pi^* \phi_t^H(y, d_y u) = y + \theta_{t,u}(y),$$

where the map $\theta_{t,u}$ has Lipschitz constant $\le B\ell(t)$. Note that this $\ell(t)$ depends only on A and not on u. Let us set $\Theta_{t,u}(y) = y + \theta_{t,u}(y)$. Note that

$$\begin{aligned}
\|\Theta_{t,u}(y') - \Theta_{t,u}(y)\| &= \|[y' + \theta_{t,u}(y')] - [y + \theta_{t,u}(y)]\| \\
&\ge \|y' - y\| - \|\theta_{t,u}(y') - \theta_{t,u}(y)\| \\
&\ge \|y' - y\| - B\ell(t)\|y' - y\| \\
&= (1 - B\ell(t))\|y' - y\|.
\end{aligned}$$

Therefore, for t small enough to have $1 - B\ell(t) > 0$, the map $\Theta_{t,u} : \mathbb{B}(r) \to \Theta_{t,u}(\mathbb{B}(r))$ is invertible and its inverse $\Theta_{t,u}^{-1} : \Theta_{t,u}(\mathbb{B}(r)) \to \mathbb{B}(r)$ has a Lipschitz constant $\le (1 - B\ell(t))^{-1}$. Note that equation (B.1) above shows that, for every $x \in \mathbb{B}(r'')$ at which $T_t^- u$ is differentiable, we can find $y_x \in \mathbb{B}(r)$ such that $x = \Theta_{t,u}(y_x)$. Since $T_t^- u$ is Lipschitz, it is differentiable a.e., and so the image $\Theta_{t,u}(\mathbb{B}(r))$ contains a set of full Lebesgue measure in $\mathbb{B}(r'')$.

The compactness of $\Theta_{t,u}(\mathbb{B}(r))$ implies that this image has to contain $\mathbb{B}(r'')$. Equation (B.1) tells us now that at each point $x \in \mathbb{B}(r'')$ where $T_t^- u$ is differentiable we have

$$(x, d_x T_t^- u) = \phi_t^H(\Theta_{t,u}^{-1}(x), d_{\Theta_{t,u}^{-1}(x)} u). \tag{B.2}$$

But the right-hand side above is a continuous function defined at least on $\mathbb{B}(r'')$. This implies that the Lipschitz function $T_t^- u$ is differentiable on $\mathbb{B}(r'')$ and its derivative satisfies equation (B.2) above. Therefore on $\mathbb{B}(r'')$ the derivative $x \mapsto d_x T_t^- u$ has a Lipschitz constant bounded by $L(t)B(1 - B\ell(t))^{-1}$, with $L(t)$ the

Lipschitz constant of $(y, p) \mapsto \pi_2\phi_t^H(y, p)$ on the set $\{(y, p) \in \mathbb{B}(r) \times (\mathbb{R}^k)^* \mid \|p\| \leq A\}$, where $\pi_2 : \mathbb{B}(r) \times (\mathbb{R}^k)^* \to (\mathbb{R}^k)^*$ is the projection on the second factor. Since ϕ_t^H is a C^1 flow and ϕ_0^H is the identity, we have $L(t) \to 1$ as $t \to 0$. This finishes the proof since $\ell(t) \to 0$, and $\ell(t)$ and $L(t)$ depend only on A and not on u. □

Recall that a function $f : C \to \mathbb{R}$ defined on the convex subset C of $\mathbb{R}^k$ is said to be K-semiconvex if $x \mapsto f(x) + K\|x\|^2_{\mathrm{euc}}$ is convex on C. If $K \geq 0$ is fixed, for an *open* convex subset C of $\mathbb{R}^k$, the following conditions are equivalent:

- the function $f : C \to \mathbb{R}$ is K-semiconvex;
- for every $x \in C$ we can find $p_x \in (\mathbb{R}^k)^*$ such that
$$\forall y \in C, \quad f(y) \geq p_x(y - x) - K\|x\|^2_{\mathrm{euc}};$$
- for every $x \in C$, if $\tilde{p}_x \in (\mathbb{R}^k)^*$ is a subdifferential of f at x, we have
$$\forall y \in C, \quad f(y) \geq \tilde{p}_x(y - x) - K\|x\|^2_{\mathrm{euc}}.$$

It is not difficult to see that, if C is an open convex subset of $\mathbb{R}^k$, a C^1 function $f : C \to \mathbb{R}$ whose derivative has on C a (global) Lipschitz constant $\leq B$ is $\frac{B}{2}$-semiconvex.

We now state the regularization property of the semigroups T_t^- and T_t^+. These properties are well-known. They have been extensively exploited for viscosity solutions; see [1, 2]. For a proof in the compact case, see [15]. We will sketch a proof relying on the semiconcavity of h_t.

THEOREM B.6 *Suppose that $t_0 > 0$, that c is a finite constant, and that (U, φ) is a C^∞ chart with $\mathbb{B}(r) \subset \varphi(U)$. We can find a constant K such that for every function $u \prec L + c$ and any $t \geq t_0$, the restriction $T_t^- u \circ \varphi^{-1}|\mathbb{B}(r)$ (respectively, $T_t^+ u \circ \varphi^{-1}|\mathbb{B}(r)$) is K-semiconcave (respectively, K-semiconvex).*

PROOF: We do the proof for T_t^-. By Lemma B.2, there exists a constant A such that all functions dominated by $L + c$ have Lipschitz constant on M, which is $\leq A + c$. It follows from Lemma B.4 that we can find a finite constant B such that for any $u \prec L + c$ and any $x \in M$
$$T_t^- u(x) = \inf\{u(y) + h_t(y, x) \mid y \in M, d(x, y) \leq Bt\}.$$
In particular, if C_t is the compact set $\{y \in M \mid d(y, \varphi^{-1}(\mathbb{B}(r)) \leq Bt\}$, we get
$$\forall x \in \varphi^{-1}(\mathbb{B}(r)), \quad T_t^- u(x) = \inf_{y \in C_t} u(y) + h_t(y, x).$$
Since h_t is locally semiconcave on $M \times M$ (see, for example, [16, theorem B.19]) and C_t is a compact subset, using standard arguments for the theory of locally semiconcave functions (again see, for example, [16, app. A]) we can find a constant K_t such that $T_t^- u \circ \varphi^{-1}|\mathbb{B}(r)$ is K_t-semiconcave for every $u \prec L + c$.

It remains to show that we can take K_t independently of $t \geq t_0 > 0$. In fact, since T_t^- preserves the set of functions dominated by $L + c$, we have $T_{t-t_0}^- u \prec$

$L + c$ for any $u \prec L + c$. Therefore, we conclude that $T_t^- u = T_{t_0}^-[T_{t-t_0}^- u]$ also satisfies that $T_t^- u \circ \varphi^{-1}|\mathbb{B}(r)$ is K_{t_0}-semiconcave. $\square$

Next we show that T_t^- preserves semiconvexity for small time t.

LEMMA B.7 *Suppose that (U, φ) is a C^∞ chart on M and $\mathbb{B}(r) \subset \varphi(U)$. For any $r' < r$, any finite number $A \geq 0$, any finite number $K \geq \frac{1}{2}$, and finite $\epsilon > 0$, we can find $\delta > 0$ such that for any function $u : \mathbb{B}(r) \to \mathbb{R}$ satisfying*

(i) *the function u has a Lipschitz constant $\leq A$ on $\mathbb{B}(r)$ and*
(ii) *the function u is K-semiconvex,*

and any $t \leq \delta$, the function $T_t^{-,\varphi} u : \mathbb{B}(r') \to \mathbb{R}$ defined by

$$T_t^{-,\varphi} u(x) = \inf_{y \in \mathbb{B}(r)} u(y) + h_t(\varphi^{-1}(y), \varphi^{-1}(x))$$

is $(K + \epsilon)$-semiconvex in $\mathbb{B}(r')$.

PROOF: As in the previous proof we will assume that φ is the "identity." We also choose r'' and r''' such that $r' < r'' < r''' < r$. Consider the family of functions $v_{\alpha,x,p} : \mathbb{B}(r) \to \mathbb{R}$, where $\alpha \in \mathbb{R}$, $x \in \mathbb{B}(r)$, and $p \in (\mathbb{R}^k)^*$, with $\|p\|_{\text{euc}} \leq A$ defined by

$$v_{\alpha,x,p}(y) = \alpha + p(y - x) - K\|y - x\|^2_{\text{euc}}.$$

It is not difficult to see that the derivative of $v_{\alpha,x,p}$ has, on $\mathbb{B}(r)$, a Lipschitz constant $\leq 2K$, and that this derivative is bounded in norm by $A + 4Kr$. Since $2K \geq 1$, we can apply Lemma B.5 and find $\delta > 0$ such that $T_t^{-,\varphi} v_{\alpha,x,p}$ is $C^{1,1}$ on $\mathbb{B}(r'')$ with a Lipschitz constant for its derivative $\leq 2K + 2\epsilon$ for any $t \leq \delta$. In particular, any such function $T_t^{-,\varphi} v_{\alpha,x,p}$ is $(K + \epsilon)$-semiconvex on $\mathbb{B}(r'')$.

Taking $\delta > 0$ smaller if necessary, we can assume that, for every u satisfying condition (a) of the lemma, every $t \leq \delta$, and every $x \in \mathbb{B}(r'')$, we can find $y_x \in \mathbb{B}(r''')$ such that

$$T_t^{-,\varphi} u(x) = u(y_x) + h_t(y_x, x).$$

If we pick up a minimizer $\gamma : [0, t] \to M$ with $\gamma(0) = y_x$ and $\gamma(t) = x$, we know that $\tilde{p}_x = \partial L / \partial v(y_x, \dot{\gamma}(0))$ will be a subdifferential of u at y_x, and also

$$\pi^* \phi_t^H(y_x, \tilde{p}_x) = x. \tag{B.3}$$

Since u is K-semiconvex on $\mathbb{B}(r)$ and $\tilde{p}_x$ is in the subdifferential of u at y_x, we have

$$\forall y \in \mathbb{B}(r), \quad u(y) \geq u(y_x) + \tilde{p}_x(y - y_x) - K\|y - y_x\|^2_{\text{euc}} = v_{u(y_x),y_x,\tilde{p}_x}(y).$$

Set $v = v_{u(y_x),y_x,\tilde{p}_x}$ to simplify notation. From the inequality above we get

$$T_t^{-,\varphi} u \geq T_t^{-,\varphi} v. \tag{B.4}$$

We also know that $T_t^{-,\varphi} v$ is $C^{1,1}$ and $(K + \epsilon)$-semiconvex. We now show that $T_t^{-,\varphi} u$ and $T_t^{-,\varphi} v$ take the same value at x. By the proof of the previous lemma we know that $T_t^{-,\varphi} v(x) = v(y'_x) + h_t(y'_x, x)$, where y'_x is the only point $y \in \mathbb{B}(r)$

such that $\pi^*\phi_t^H(y, d_y v) = x$. But $d_{y_x} v = \tilde{p}_x$; therefore by equation (B.3) we obtain $y'_x = y_x$. Since we also have $v(y_x) = u(y_x)$, we conclude that $T_t^{-,\varphi} v(x) = v(y_x) + h_t(y_x, x) = u(y_x) + h_t(y_x, x) = T_t^{-,\varphi} u(x)$. Since v is $(K+\epsilon)$-semiconvex on $\mathbb{B}(r'')$, for every $y \in \mathbb{B}(r'')$ we have

$$T_t^{-,\varphi} v(y) \geq T_t^{-,\varphi} v(x) + p_x(y-x) - (K+\epsilon)\|y-x\|_{\mathrm{euc}}^2,$$

where p_x is the derivative at x of the $C^{1,1}$ function $T_t^{-,\varphi} v$. Therefore by equation (B.4) we obtain

$$\forall y \in \mathbb{B}(r''), \quad T_t^{-,\varphi} u(y) \geq T_t^{-,\varphi} u(x) + p_x(y-x) - (K+\epsilon)\|y-x\|_{\mathrm{euc}}^2.$$

Since x was an arbitrary point in $\mathbb{B}(r'')$, this finishes the proof. □

Before giving the proof of Theorem B.1, we also notice that since L is uniformly superlinear in the fibers, there exists a finite constant $C(K')$ such that

$$\forall (x, v) \in TM, \quad L(x, v) \geq 2K'\|v\|_x + C(K').$$

From that, we deduce that for every $t > 0$,

$$\forall x, y \in M, \quad h_t(x, y), h_t(y, x) \geq 2K'd(x, y) + C(K')t. \tag{B.5}$$

The previous two lemmas are also true if we replace $T_t^{-,\varphi} u$ by

$$T_t^{+,\varphi} u(x) = \sup_{y \in \mathbb{B}(r)} u(y) - h_t(x, y),$$

and also replace semiconvexity in the second lemma by semiconcavity.

Proof of Theorem B.1: First we choose a countable family of C^∞ charts $(U_n, \varphi_n)_{n \geq 1}$ on M such that $\varphi_n(U_n) = \mathbb{R}^k$ and $M = \bigcup_{n \geq 0} \varphi_n^{-1}(\mathbb{B}(1))$.

Fix a $c \in \mathbb{R}$. We know that any function $u : M \to \mathbb{R}$ dominated by $L + c$ is Lipschitz with Lipschitz constant $\leq A + c$, where A is the constant provided by Lemma B.2. Therefore, for each integer $n \geq 1$, we can find a finite constant A_n such that, for every $u : M \to \mathbb{R}$ dominated by $L + c$, the function $u \circ \varphi_n^{-1}$ has on $\mathbb{B}(2)$ a Lipschitz constant $\leq A_n$ for the canonical Euclidean norm on $\mathbb{R}^k$. We will construct by induction a sequence $B_n \in [1, +\infty[$ and two sequences of > 0 numbers t_n^- and t_n^+ such that if we define, for $u : M \to \mathbb{R}$, the function $S_n(u)$ on m by

$$S_n(u) = T^-_{t_n^-} T^+_{t_n^+} T^-_{t_{n-1}^-} T^+_{t_{n-1}^+} \cdots T^-_{t_1^-} T^+_{t_1^+}(u),$$

with S_0 the identity, then for every $u \prec L + c$ defined on the whole M and every $k = 1, \dots, n$, we have

(i) the supremum $\sup_{x \in M} |S_n(u)(x) - S_{n-1}(u)(x)|$ is less than $1/2^n$,
(ii) the function $S_n(u) \circ \varphi_k^{-1}$ is $C^{1,1}$ on $\mathbb{B}(1 + 2^{-n})$,
(iii) the function $S_n(u) \circ \varphi_k^{-1}$ has on $\mathbb{B}(2)$ Lipschitz constant $\leq A_k$, and
(iv) the derivative of $S_n(u) \circ \varphi_k^{-1}$ on $\mathbb{B}(1 + 2^{-n})$ has Lipschitz constant $\leq B_k + 1 - 2^{-n}$.

Note that, since T_t^- and T_t^+ preserve functions dominated by $L+c$ on M, we will have $S_n(u) \prec L+c$, and condition (iii) above will be satisfied for any choice of t_n^+, t_n^-.

Suppose that S_n has been constructed. We first pick t_{n+1}^+. It follows from Lemma B.4 that there exists δ_1 such that for every $u \prec L+c$ and $t \in [0, \delta_1]$, we have

$$\sup_{x \in M} |T_t^+(u)(x) - u(x)| \le \frac{1}{2^{n+2}}.$$

Given that (i), (ii), (iii), and (iv) are verified, we can apply the version of Lemma B.5 for T_t^+ to the finite set of charts (U_k, φ_k), $k = 1, \dots, n$, with ball $\mathbb{B}(1 + 2^{-n})$ and constants A_k and $B_k + 1 - 2^{-n}$, to find $\delta_2 > 0$ such that for every $t \in [0, \delta_2]$ and every $u \prec L+c$, the function $T_t^+ S_n(u) \circ \varphi_k^{-1}$ is $C^{1,1}$ on $\mathbb{B}(1 + 2^{-n} - 2^{-(n+2)})$ with Lipschitz constant of its derivative $\le B_k + 1 - 2^{-n} + 2^{-(n+2)}$. Let us now fix $t_{n+1}^+ > 0$ with $t_{n+1}^+ \le \min(\delta_1, \delta_2)$. Since $t_{n+1}^+ > 0$, we know by Theorem B.6 that there exists a finite constant $\tilde{B}_{n+1}$ such that for every $u \prec L+c$, the function $T_{t_{n+1}^+}^+ S(u) \circ \varphi_{n+1}^{-1}$ is $\tilde{B}_{n+1}$-semiconvex on the ball $\mathbb{B}(2)$. Therefore, for every $u \prec L+c$, we have

(a) the supremum $\sup_{x \in M} |T_{t_{n+1}^+}^+ S_n(u)(x) - S_n(u)(x)|$ is less than $1/2^{n+2}$,

(b) the function $T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_k^{-1}$ is $C^{1,1}$ on $\mathbb{B}(1 + 2^{-n} - 2^{-(n+2)})$ for $k = 1, \dots, n$,

(c) the function $T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_k^{-1}$ has on $\mathbb{B}(2)$ Lipschitz constant $\le A_k$ for $k = 1, \dots, n+1$,

(d) the derivative of $T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_k^{-1}$ on $\mathbb{B}(1 + 2^{-n})$ has Lipschitz constant $\le B_k + 1 - 2^{-n} + 2^{-(n+2)}$, and

(e) the function $T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_{n+1}^{-1}$ is $\tilde{B}_{n+1}$-semiconvex on the ball $\mathbb{B}(2)$.

We first pick t_{n+1}^-. It follows from Lemma B.4 that there exists δ_1' such that for every $u \prec L+c$ and $t \in [0, \delta_1']$, we have

$$\sup_{x \in M} |T_t^-(u)(x) - u(x)| \le \frac{1}{2^{n+2}}.$$

Given that (b), (c), and (d) are verified, we can apply Lemma B.5 to the finite set of charts (U_k, φ_k), $k = 1, \dots, n$, with ball $\mathbb{B}(1 + 2^{-n} - 2^{-(n+2)})$ and constants A_k and $B_k + 1 - 2^{-n} + 2^{-(n+2)}$ to find $\delta_2' > 0$ such that for every $t \in [0, \delta_2']$ and every $u \prec L+c$, the function

$$T_t^- T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_k^{-1}$$

is $C^{1,1}$ on $\mathbb{B}(1 + 2^{-(n+1)}) = \mathbb{B}(1 + 2^{-n} - 2^{-(n+2)} - 2^{-(n+2)})$ with Lipschitz constant of its derivative $\le B_k + 1 - 2^{-n} + 2^{-(n+2)} + 2^{-(n+2)} = B_k + 1 - 2^{-(n+1)}$.

By condition (e) above, we can also apply Lemma B.7 to find δ_3' such that for every $t \in [0, \delta_3']$ and each $u \prec L + c$, the function

$$T_t^- T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_{n+1}^{-1}$$

is $(\tilde{B}_{n+1} + 1)$-semiconvex on $\mathbb{B}(1 + 2^{-(n+1)})$. Let us now fix $t_{n+1}^- > 0$ with $t_{n+1}^- \leq \min(\delta_1', \delta_2', \delta_3')$. Since $t_{n+1}^- > 0$, we know by Theorem B.6 that there exists a finite constant $\hat{B}_{n+1}$ such that, for every $u \prec L + c$, the function

$$T_{t_{n+1}^-}^- T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_{n+1}^{-1}$$

is $\hat{B}_{n+1}$-semiconcave on the ball $\mathbb{B}(1 + 2^{-(n+1)})$. Hence, if we set $B_{n+1} = 2\max\{\tilde{B}_{n+1} + 1, \hat{B}_{n+1}\} \geq 1$, for every $u \prec L + c$ the function

$$T_{t_{n+1}^-}^+ T_{t_{n+1}^+}^+ S_n(u) \circ \varphi_{n+1}^{-1}$$

is both $B_{n+1}/2$-semiconvex and $B_{n+1}/2$-semiconcave on $\mathbb{B}(1 + 2^{-(n+1)})$. It is therefore $C^{1,1}$ on $\mathbb{B}(1 + 2^{-(n+1)})$, with a derivative with Lipschitz constant $\leq B_{n+1}$. It is not difficult now to verify that with this choice of t_{n+1}^+, t_{n+1}^-, the operator S_{n+1} satisfies the required conditions (i), (ii), (iii), and (iv). □

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Received November 2007.