Lecture 1. Viscosity solutions of the Hamilton-Jacobi equation on a non-compact manifold: Preliminaries:

Albert Fathi

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Introduction

The auxiliary Riemannian Metric

We consider a **connected** manifold M endowed with a Riemannian metric.
A point of the tangent bundle TM will be denoted by (x, v), where $x \in M$ and $v \in T_xM$ is a tangent vector at x.
A point of the cotangent bundle T^*M will be denoted by (x, p), where $x \in M$ and $p \in T_x^*M$ is a tangent covector at x. Therefore p is a linear form on T_xM. The canonical projections from the tangent and cotangent bundle are $\pi : TM \to M, (x, v) \mapsto x$ and $\pi^* : T^*M \to M, (x, p) \mapsto x$. Hence $T_xM = \pi^{-1}(x)$ and $T_x^*M = (\pi^*)^{-1}(x)$ are respectively the fibers of the tangent and cotangent bundle at x.
We will denote by $\|\cdot\|_x$ the norm induced by the Riemannian metric on either T_xM or T_x^*M on the fibers above x of the tangent TM or cotangent T^*M bundle of M.
We endow $\mathbb{R} \times M, \mathbb{R} \times M \times M$, and $M \times M$ with the product Riemannian metrics, where the Riemannian metric on $\mathbb{R}$ is the usual one.

If $\gamma : [a, b] \to M$ is a piecewise C^1 (or even an absolutely continuous) curve, its Riemannian length $\ell_g(\gamma)$ is

$$\ell_g(\gamma) = \int_a^b \|\dot\gamma(s)\|_{\gamma(s)}\, ds.$$

We will denote by d the Riemannian distance on M obtained from the Riemannian metric, namely

$$d(x, y) = \inf_\gamma \ell_g(\gamma),$$

where the $\inf$ is taken over all piecewise C^1 curves $\gamma : [a, b] \to M$, with $\gamma(a) = x, \gamma(b) = y$.

As is well-known d is a distance on M that defines its topology. Moreover, this distance is complete (i.e. Cauchy sequences converge) if and only if the geodesic flow is complete (i.e. geodesics are defined for all time) if and only if the closed subsets bounded for d are the compact subsets.

For the rest of the lectures, we will assume that the *Riemannian metric on M is complete*.

Of course, when M is compact all Riemannian metrics are complete. However our main focus in these lectures are non-compact manifolds.

Hamilton-Jacobi Equation

Stationary Hamilton-Jacobi equation

Throughout the lecture $H : T^*M \to \mathbb{R}$ will denote a continuous function which we will call the Hamiltonian.

A good example to keep in mind is the Hamiltonian $H_V : T^*M \to \mathbb{R}$ is defined by

$$H_V(x,p) = \frac{1}{2}\|p\|_x^2 + V(x),$$

where $V : M \to M$ is a continuous function.

The (stationary) Hamilton-Jacobi equation associated to H is the equation

$$H(x, d_x u) = c,$$

where $c \in \mathbb{R}$ is some constant.

A classical solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$ on the open subset U of M is a C^1 map $u : U \to \mathbb{R}$ such that $H(x, d_x u) = c$, for each $x \in U$.

Usually, one deals only with the case $H(x, d_x u) = 0$, since it is possible to reduce the general case to that case by replacing the Hamiltonian H by H_c, defined by $H_c(x,p) = H(x,p) - c$.

Evolutionary Hamilton-Jacobi equation

The evolutionary Hamilton-Jacobi equation associated to the Hamiltonian H is the equation

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

A classical solution to this evolutionary Hamilton-Jacobi equation on the open subset W of $\mathbb{R} \times T^*M$ is a C^1 map $u : W \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0,$$

for each $(t,x) \in W$.
The evolutionary form can be reduced to the stationary form by introducing the Hamiltonian $\hat{H} : T^*(\mathbb{R} \times M)$ defined by

$$\hat{H}(t,x,s,p) = s + H(x,p),$$

where $(t,x) \in \mathbb{R} \times M$, and $(s,p) \in T^*_{(t,x)}(\mathbb{R} \times M) = \mathbb{R} \times T^*_x M$. .

There usually do not exist smooth global subsolutions, supersolutions or solutions to these PDE's.
Therefore, we have to define a weaker notion of solution.
The well-adapted concept for us is the concept of viscosity subsolutions, supersolutions or solutions, that we presently introduce.
We start by recalling the parts of

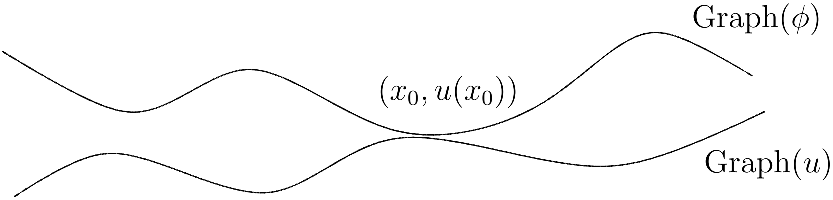
Albert Fathi, *Weak KAM from a PDE point of view: viscosity solutions of the Hamilton-Jacobi equation and Aubry set*, Proc. Roy. Soc. Edinburgh Sect. A, 120 **(2012)** 1193–1236

that are relevant here.

Crash Course on Viscosity

Viscosity Subsolution

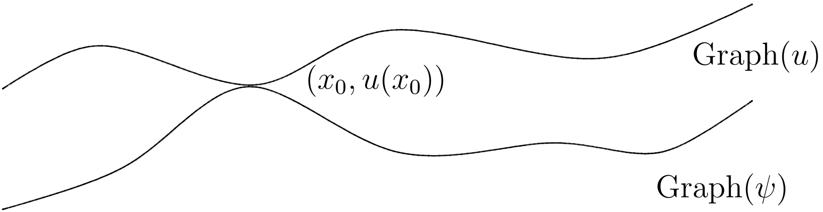
A function $u : V \to \mathbb{R}$ is a *viscosity subsolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if for every C^1 function $\phi : V \to \mathbb{R}$, with $\phi \geq u$ everywhere, at every point $x_0 \in V$ where $u(x_0) = \phi(x_0)$ we have $H(x_0, d_{x_0}\phi) \leq c$.



Subsolution: $\phi \geq u, u(x_0) = \phi(x_0) \Rightarrow H(x_0, d_{x_0}\phi) \leq c$.

Viscosity Supersolution

A function $u : V \to \mathbb{R}$ is a *viscosity supersolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if for every C^1 function $\psi : V \to \mathbb{R}$, with $u \geq \psi$ everywhere, at every point $x_0 \in V$ where $u(x_0) = \psi(x_0)$ we have $H(x_0, d_{x_0}\psi) \geq c$.



Supersolution: $\psi \leq u, u(x_0) = \psi(x_0) \Rightarrow H(x_0, d_{x_0}\psi) \geq c$.

Viscosity Solution

A function $u : V \to \mathbb{R}$ is a *viscosity solution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if it is both a subsolution and a supersolution.

In the sequel of this lecture, we will concentrate on viscosity solutions of the evolutionary Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

We will mainly address the problem of uniqueness of the solution on $[0, T[\times M$ for a given initial condition on $\{0\} \times M$ and its companion the Lax-Oleinik formula in the case of Tonelli Hamiltonians.

Some facts about viscosity solutions

We enumerate some facts about viscosity subsolutions, supersolutons, and solutions.

- A C^1 function is a viscosity solution of the Hamilton-Jacobi equation if and only if it is a classical solution.
- If the viscosity subsolution u (resp. supersolution, solution) of the Hamilton-Jacobi equation $H(x, d_x u) = c$ is differentiable at x_0, then $H(x_0, d_{x_0} u) \leq c$ (resp. $H(x_0, d_{x_0} u) \geq c$, $H(x_0, d_{x_0} u) = c$).
- (Stability) Suppose that $v_n : M \to \mathbb{R}$ is a sequence of continuous functions converging uniformly on compact subsets to $v : M \to \mathbb{R}$. If, for each n, the function v_n is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$, then v is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$.

- If $H(x,p)$ in convex in the momentum variable p, then a locally Lipschitz function u is a viscosity subsolution of $H(x, d_x u) = c$ if and only if $H(x, d_x u) \leq c$ almost everywhere.
- If $H(x,p)$ in convex in the momentum variable$p$ and the two locally Lipschitz function $u_1, u_2 : O \to \mathbb{R}$ are viscosity subsolutions of $H(x, d_x u) = c$, on the open subset $O \subset M$, then so is $\min(u_1, u_2)$.
- If $H(x,p)$ in convex in the momentum variable, and $u : O \to \mathbb{R}$ is a locally Lipschitz viscosity subsolution of $H(x, d_x u) = c$, defined on the open subset $O \subset M$, then for any $\epsilon > 0$ we can find a C^∞ function $v : O \to \mathbb{R}$ which is a viscosity subsolution of $H(x, d_x v) = c + \epsilon$ on O and such that $\sup_{x \in O} |v(x) - u(x)| \leq \epsilon$.

To give further properties we need to introduce:

Definition 1 (Coercive)

A continuous function $H : T^*M \to \mathbb{R}$ is said to be coercive above every compact subset, if for each compact subset $K \subset M$ and each $c \in \mathbb{R}$ the set $\{(x,p) \in T^*M \mid x \in K, H(x,p) \leq c\}$ is compact.

It is not difficult to see that H is coercive if and only if for each compact subset $K \subset M$, we have $\lim_{\|p\|_x \to \infty} H(x,p) = +\infty$, the limit being uniform in $x \in K$.

Theorem 2

Suppose that $H : T^*M \to \mathbb{R}$ is coercive above every compact subset, and $c \in \mathbb{R}$. Then a viscosity subsolution of $H(x, d_x u) = c$ is necessarily locally Lipschitz, and therefore satisfies $H(x, d_x u) \leq c$ almost everywhere.

Note however that the Hamiltonian

$$\hat{H}(t,x,s,p) = s + H(x,p),$$

which give rise to the evolutionary Hamilton-Jacobi equation is never coercive even if H is coercive, since s can $\to -\infty$.
Therefore, it is difficult to assume (or obtain) a priori that a viscosity subsolution of the evolutionary Hamilton-Jacobi equation is locally Lipschitz.
In fact, if U is a viscosity subsolution of

$$\frac{\partial U}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0,$$

and $\rho : [0,+\infty[\to \mathbb{R}$ which is continuous and non-increasing, then $V(x,s) = U(x,s) + \rho(s)$ is a viscosity subsolution of the same equation.
At this point, it is useful to note that the Hamiltonian

$$\tilde{H}(t,x,s,p) = |s| + H(x,p),$$

is coercive above compact subsets, if H is.

The main ingredient to prove uniqueness properties for viscosity solutions is the following one:

Theorem 3

Let $H : T^*M \to \mathbb{R}$ be any continuous Hamiltonian on the manifold M. Suppose that $u : M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_x u) = c_1$, and $v : M \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_x v) = c_2$. Assume further that either u or v is locally Lipschitz on M. If $u - v$ has a local maximum, then necessarily $c_2 \leq c_1$.

Note that, if at x_0 the difference $u - v$ vanishes, then x_0 is a local maximum of $u - v$ if and only if $v \geq u$ in a neighborhood of x_0.

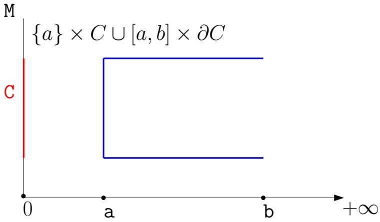
Because this Theorem 3 needs at least one of the functions to be locally Lipschitz, to apply it to the evolution case, we will need to approximate subsolutions by subsolutions which are locally Lipschitz. More on that later.

This previous Theorem 3 implies a maximum principle in the evolutionary case.

Theorem 4 (Maximum Principle)

Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian on the manifold M. Assume $u, v : [a, b] \times C \to \mathbb{R}$ are continuous, where $C \subset M$ is a compact subset, with u a viscosity subsolution and v a viscosity supersolution of $\partial_t U + H(x, \partial_x U) = 0$. If either u or v is locally Lipschitz on $]a, b[\times \mathring{C}$, then

$$\max_{[a,b]\times C} u - v = \max_{\{a\}\times C\cup[a,b]\times\partial C} u - v.$$



Proof.

As usual in proofs of that form of the maximum principle, for $\epsilon, \delta > 0$, we introduce the function $u_{\epsilon,\delta} : [a, b[\times C \to \mathbb{R}$ defined by

$$u_{\epsilon,\delta}(t, x) = u(t, x) - \epsilon(t - a) - \frac{\delta}{b - t}.$$

Note that $u_{\epsilon,\delta}(t, x) \to -\infty$, as $t \to b$, and $u_{\epsilon,\delta} \leq u$.

Moreover, since $t \mapsto -\epsilon(t - a) - \delta/(b - t)$ is C^1, with derivative $t \mapsto -\epsilon - \delta/(b - t)^2 \leq -\epsilon$, the function $u_{\epsilon,\delta}$ is, on $]a, b[\times \mathring{C}$, a viscosity subsolution of

$$\partial_t u_{\epsilon,\delta} + H(x, \partial_x u_{\epsilon,\delta}) = -\epsilon.$$

Introducing as above the Hamiltonian $\hat{H} : T^*(\mathbb{R} \times M)$ defined by

$$\hat{H}\big((t, x), (s, p)\big) = s + H(x, p),$$

we obtain that $u_{\epsilon,\delta}$ is a (stationary) viscosity subsolution of

$$\hat{H}\big((t, x), d_{t,x}u_{\epsilon,\delta}\big) = -\epsilon \text{ on }]a, b[\times \mathring{C}.$$

In the same way, the function v is a viscosity **supersolution** of $\hat{H}\big((t,x), d_{t,x}v\big) = 0$ on $]a,b[\times\mathring{C}$.
Since, either u or v is locally Lipschitz on $]a,b[\times\mathring{C}$ and $u_{\epsilon,\delta} - u$ is C^1, either $u_{\epsilon,\delta}$ or v is locally Lipschitz on $]a,b[\times\mathring{C}$. Therefore, we can apply Theorem 3 with Hamiltonian $\hat{H}$ to the viscosity subsolution $u_{\epsilon,\delta}$ of

$$\hat{H}\big((t,x), d_{t,x}u_{\epsilon,\delta}\big) = -\epsilon$$

on $]a,b[\times\mathring{C}$ and the viscosity supersolution v of

$$\hat{H}\big((t,x), d_{t,x}v\big) = 0$$

on $]a,b[\times\mathring{C}$. Since $-\epsilon < 0$, by this Theorem 3, we conclude that $u_{\epsilon,\delta} - v$ cannot have a local maximum in $]a,b[\times\mathring{C}$.
Since $u_{\epsilon,\delta}(t,x) \to -\infty$, as $t \to b$ and $[a,b] \times C$ is compact, the function $u_{\epsilon,\delta} - v$ achieves a maximum on $[a,b] \times C$. this maximum cannot be attained in in $]a,b[\times\mathring{C}$. Therefore $u_{\epsilon,\delta} - v$ attains its maximum at a point in $[a,b[\times\partial C \cup \{a\} \times C$.

Using that $u_{\epsilon,\delta} \leq u$, we obtain

$$u_{\epsilon,\delta} - v \leq \max_{[a,b[\times\partial C \cup \{a\}\times C} u_{\epsilon,\delta} - v \leq \max_{[a,b]\times\partial C \cup \{a\}\times C} u - v,$$

everywhere on $[a,b[\times C$. Letting $\delta, \epsilon \to 0$, we obtain

$$u - v \leq \max_{[a,b]\times\partial C \cup \{a\}\times C} u - v,$$

on $[a,b[\times C$. Continuity of both u and v yields

$$\max_{[a,b]\times C} u - v \leq \max_{[a,b]\times\partial C \cup \{a\}\times C} u - v. \ \square$$

Remark 5

When H is coercive above compact subsets, as we will show later, this last Theorem remains valid without the assumptions that either u or v is locally Lipschitz.

Enter Tonelli

Tonelli Hamiltonian

Our results will be valid for Tonelli Hamiltonians that we now introduce.

We will assume that $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian (with respect to the Riemannian metric), i.e. H is at least C^2 and satisfies:

(a$*$) (Uniform superlinearity) For every $K \geq 0$, we have
$$C^*(K) = \sup_{(x,p)\in T^*M} K\|p\|_x - H(x,p) < \infty.$$

(b$*$) (Uniform boundedness in the fibers) For every $R \geq 0$, we have
$$A^*(R) = \sup\{H(x,p) \mid \|p\|_x \leq R\} < +\infty \;;$$

(c$*$) (C^2 strict convexity in the fibers) for every $(x,p) \in T^*M$, the second derivative $\partial^2 H/\partial p^2(x,p)$ is positive (strictly) definite.

$A^*(R)$ and $C^*(R)$ are both non-decreasing as functions of $R \in [0 + \infty[$.

Note also that (a$*$) and (b$*$) imply
$$\forall (x,p) \in T^*M, H(x,p) \geq K\|p\|_x - C^*(K).$$
$$\forall (x,p) \in T^*M \in TM, H(x,p) \leq A^*(\|p\|_x).$$

Example

1) The easiest example of a Tonelli Hamiltonian is $H_0 : T^*M \to \mathbb{R}$ defined by

$$H_0(x,p) = \frac{1}{2}\|p\|_x^2.$$

In fact, in this case

$$A_0^*(R) = \sup\{H_0(x,v) \mid \|p\|_x \leq R\} = \frac{R^2}{2},$$

$$C_0^*(K) = \sup_{(x,p)\in T^*M} K\|p\|_x - H_0(x,p) = \sup_{(x,p)\in T^*M} K\|p\|_x - \frac{1}{2}\|p\|_x^2 = \frac{K^2}{2}.$$

2) Let $V : M \to \mathbb{R}$ be a C^r function, with $r \geq 2$, the Hamiltonian $H_V : T^*M \to \mathbb{R}$ defined by

$$H_V(x,v) = \frac{1}{2}\|p\|_x^2 + V(x)$$

is a Tonelli Hamiltonian if and only if V is bounded.

Why Tonelli? The Lagrangian!

The important feature of Tonelli Hamiltonians is that they allow to define an action for curves, using the associated Lagrangian which is convex in the speed. This in turn allows to apply Calculus of Variations.

The Lagrangian $L : TM \to \mathbb{R}, (x, v) \mapsto L(x, v)$, associated to the Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$, is defined by

$$L(x, v) = \sup_{p \in T_x^* M} p(v) - H(x, p).$$

Note that $L(x, v)$ is everywhere finite, since

$$L(x, v) \geq 0(v) - H(x, 0) = -H(x, 0)$$

and

$$p(v) - H(x, p) \leq \|v\|_x \|p\|_x - H(x, p) \leq C^*(\|v\|_x),$$

which implies

$$L(x, v) \leq C^*(\|v\|_x).$$

Tonelli Lagrangian

The Lagrangian $L : TM \to \mathbb{R}$ associated to the Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ is also Tonelli (with respect to the Riemannian metric).

This means that $L : TM \to \mathbb{R}$ is at least C^2 and satisfies:

(a) (Uniform superlinearity) For every $K \geq 0$, we have

$$C(K) = \sup_{(x,v)\in TM} K\|v\|_x - L(x,v) < \infty.$$

(b) (Uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A(R) = \sup\{L(x,v) \mid \|v\|_x \leq R\} < +\infty \,;$$

(c) (C^2 strict convexity in the fibers) for every $(x,v) \in TM$, the second derivative $\partial^2 L/\partial v^2(x,v)$ is positive strictly definite.

$A(R)$ and $C(R)$ are both non-decreasing as functions of $R \in [0 + \infty[$.

Note again that (a) and (b) imply

$$\forall (x,v) \in TM, L(x,v) \geq K\|v\|_x - C(K). \tag{1}$$

$$\forall (x,v) \in TM, L(x,v) \leq A(\|v\|_x). \tag{2}$$

Example

1) The Tonelli Lagrangian $L_0 : TM \to \mathbb{R}$ associated to the Tonelli Hamiltonian $H_0(x,p) = \frac{1}{2}\|p\|_x^2$ is

$$L_0(x,v) = \frac{1}{2}\|v\|_x^2,$$

for which

$$A_0(R) = \sup\{L_0(x,v) \mid \|v\|_x \leq R\} = \frac{R^2}{2},$$

$$C_0(K) = \sup_{(x,v)\in TM} K\|v\|_x - L_0(x,v) = \sup_{(x,v)\in TM} K\|v\|_x - \frac{1}{2}\|v\|_x^2 = \frac{K^2}{2}.$$

2) If $V : M \to \mathbb{R}$ be a bounded C^r function, with $r \geq 2$, the Tonelli Lagrangian $L_V : TM \to \mathbb{R}$ associated to the Tonelli Hamiltonian $H_V(x,p) = \frac{1}{2}\|p\|_x^2 + V(x)$ is

$$L_V(x,v) = \frac{1}{2}\|v\|_x^2 - V(x)$$

Action and Minimizers

Although we assume familiarity with action, minimizers, extremals and Euler-Lagrange Equation for the Lagrangian L, we now sketch some of the definition and properties.
We recall that the action $\mathbb{L}(\gamma)$ of a piecewise C^1 curve $\gamma : [a, b] \to M$ is defined by

$$\mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds.$$

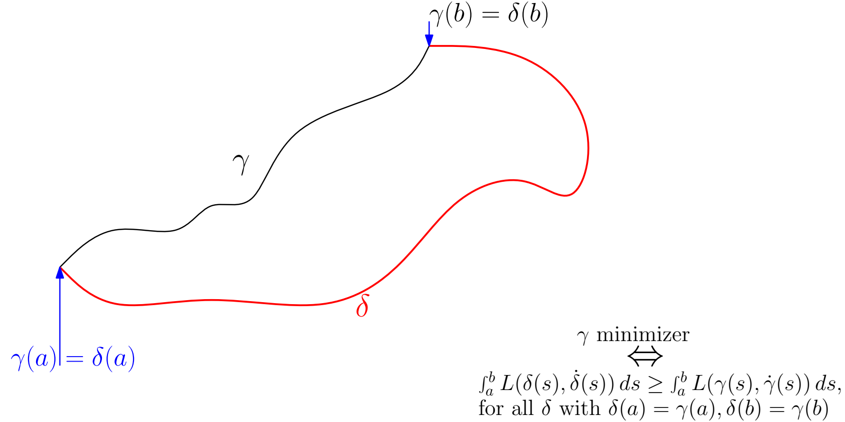
By the superlinearity of L, the action is always bounded below by $-C(0)(b-a)$.

Definition 6 (Minimizer)

A minimizer (for L) is a curve $\gamma : [a, b] \to M$ such that

$$\mathbb{L}(\delta) = \int_a^b L(\delta(s), \dot{\delta}(s))\, ds \geq \mathbb{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

for every curve $\delta : [a, b] \to M$, with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$.


$\gamma(b) = \delta(b)$
γ
δ
$\gamma(a) = \delta(a)$
γ minimizer
$\Longleftrightarrow$
$\int_a^b L(\delta(s), \dot{\delta}(s))\, ds \geq \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$
for all δ with $\delta(a) = \gamma(a), \delta(b) = \gamma(b)$

- Minimizers play a crucial role in Aubry-Mather theory.
- Minimizers (like all minimums of a function) must be critical points for the action functional $\mathbb{L}$.

These critical points are called extremals.

- More precisely, an extremal (for L) is a curve $\gamma : [a, b] \to M$ such that the derivative $D_\gamma \mathbb{L} | \mathcal{E}_\gamma$ at γ vanishes, with

$$\mathcal{E}_\gamma = \{\delta : [a, b] \to M \mid \delta(a) = \gamma(a), \delta(b) = \gamma(b)\}.$$

- By classical theory of Calculus of Variations, the curve γ is an extremal if and only if it satisfies Euler-Lagrange equation, given in local coordinates by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))\right] = \frac{\partial L}{\partial x}(\gamma(t), \dot\gamma(t)). \tag{3}$$

This last first order ODE (3) on TM defines a second order ODE on M.

Since the extremals satisfy a second order ODE on M, if two extremals coincide at some time t_0 in position and speed they have to be equal on their common interval of definition.
Moreover, there exists a flow φ_t on TM, defined for all time by conservation of energy, called the Euler-Lagrange flow, such that $\gamma : [a, b] \to M$ is an extremal if and only if its speed curve $s \mapsto (\gamma(s), \dot{\gamma}(s))$ is an orbit of φ_t.
Moreover, for any $(x, v) \in TM$, the projected curve $\gamma_{x,v}(t) = \pi\varphi_t(x, v)$, where $\pi : TM \to M$ is the canonical projection, is an extremal with $(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) = \varphi_t(x, v)$.
Since the Lagrangian L does not depend on time, it is important to note that for every $t \in \mathbb{R}$ and every curve $\gamma : [a, b] \to M$ the action $\mathbb{L}(\gamma)$ is the same as the action $\mathbb{L}(\gamma_t)$ of the curve $\gamma_t : [a - t, b - t] \to M$, defined by

$$\gamma_t(s) = \gamma(t + s).$$

Therefore $\gamma : [a, b] \to M$ is a minimizer if and only if $\gamma_t : [a - t, b - t] \to M$ is a minimizer

Tonelli's theorem and minimal action

We know recall Tonelli's theorem.

Theorem 7 (Tonelli)

For every $a, b \in \mathbb{R}$, with $a < b$, and every $x, y \in M$, there exists a minimizer $\gamma : [a, b] \to M$, with $\gamma(a) = x, \gamma(b) = y$. Any such minimizer γ is as smooth as L and is a solution of the Euler-Lagrange equation.

Definition 8 (Minimal action h_t)

For $x, y \in M$, and $t > 0$, we define the minimal action $h_t(x, y)$ to join x to y in time t by

$$h_t(x, y) = \inf_{\gamma} \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds,$$

where the infimum is taken over all piecewise C^1 (or even absolutely continuous) curves $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$.

By Tonelli's theorem, the infimum

$$h_t(x,y) = \inf_{\gamma} \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds$$

in the definition of $h_t(x,y)$ is always attained by a minimizer which is as smooth as the Lagrangian.
Since L does not depend on time, it is also useful to note that for $x, y \in M$ and $a, b \in \mathbb{R}$, with $a > b$, we have

$$h_{b-a}(x,y) = \inf_{\gamma} \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

where the infimum is taken over all piecewise C^1 (or even absolutely continuous) curves $\gamma : [a,b] \to M$, with $\gamma(a) = x$ and $\gamma(b) = y$.
We also note that $\gamma : [a,b] \to M$ is a minimizer if and only if $h_{b-a}(\gamma(a), \gamma(b)) = \mathbb{L}(\gamma)$, i.e.

$$h_{b-a}(\gamma(a), \gamma(b)) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Example

1) For the Tonelli Lagrangian $L_0 : TM \to \mathbb{R}$ defined by $L_0(x,v) = \frac{1}{2}\|v\|_x^2$, we have

$$h_t^0(x,y) = \frac{d(x,y)^2}{2t}.$$

2) For the Tonelli Lagrangian $L_V : TM \to \mathbb{R}$ defined by

$$L_V(x,v) = \frac{1}{2}\|v\|_x^2 - V(x),$$

where $V : M \to \mathbb{R}$ is a bounded C^r function, with $r \geq 2$, we have

$$\frac{d(x,y)^2}{2t} - \sup V \leq h_t^V(x,y) \leq \frac{d(x,y)^2}{2t} - \inf V.$$

Properties of minimal action

Some of the properties of the h_t's that we will use are the following ones:

(a) For every $K \in [0, \infty[, t > 0$ and every $x, y \in M$, we have:

$$Kd(x,y) - C(K)t \leq h_t(x,y) \leq tA\left(\frac{d(x,y)}{t}\right).$$

(b) (semi-group property) For every $t, t' > 0$ and every $x, y \in M$, we have:

$$h_{t+t'}(x,y) = \inf_{z \in M} h_t(x,z) + h_{t'}(z,y).$$

Proof.

To prove part (a), consider a curve $\gamma : [0,t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$, then by the superlinearity property of the Tonelli Lagrangian L, we have

$$L(\gamma(s), \dot\gamma(s)) \geq K\|\dot\gamma(s)\|_{\gamma(s)} - C(K), \text{ for all } s \in [0,t].$$

Integrating the inequality

$$L(\gamma(s), \dot\gamma(s)) \geq K\|\dot\gamma(s)\|_{\gamma(s)} - C(K)$$

between 0 and t yields

$$\mathbb{L}(\gamma) \geq K\ell_g(\gamma) - C(K)t \geq Kd(x, y) - C(K)t,$$

where we used $\ell_g(\gamma) \geq d(x, y)$, for the last inequality.
Taking the infimum over all curves $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$, we obtain the inequality

$$Kd(x, y) - C(K)t \leq h_t(x, y),$$

which is the left hand side of (a). To finish the proof of (a), consider a (length-)minimizing geodesic $\gamma : [0, t] \to M$, with $\gamma(0) = x$ and $\gamma(t) = y$. Since $\ell_g(\gamma) = d(x, y)$ and $\|\dot\gamma(s)\|_{\gamma(s)}$ is constant, we obtain

$$\|\dot\gamma(s)\|_{\gamma(s)} = \frac{d(x, y)}{t}.$$

Therefore by the boundedness of the Tonelli Lagrangian L, we have

$$L(\gamma(s), \dot\gamma(s)) \leq A\left(\frac{d(x, y)}{t}\right), \text{ for all } s \in [0, t].$$

Integrating the inequality

$$L(\gamma(s), \dot{\gamma}(s)) \leq A\left(\frac{d(x,y)}{t}\right)$$

between 0 and t yields

$$\mathbb{L}(\gamma) \leq tA\left(\frac{d(x,y)}{t}\right).$$

Since $h_t(x,y) \leq \mathbb{L}(\gamma)$, we obtain

$$h_t(x,y) \leq tA\left(\frac{d(x,y)}{t}\right),$$

which is the right hand side of (a).
Part (b) $h_{t+t'}(x,y) = \inf_{z \in M} h_t(x,z) + h_{t'}(z,y)$ is left to the reader. $\square$

The Lax-Oleinik semi-group

We now come to the definition of the (negative) Lax-Oleinik semi-group $T_t^-, t \geq 0$.

If $u : M \to [-\infty, +\infty]$ is a function and $t > 0$, the function $T_t^- u : M \to [-\infty, +\infty]$ is defined by

$$T_t^- u(x) = \inf_\gamma u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, t] \to M$, with $\gamma(t) = x$. Using that

$$h_t(y, x) = \inf_\gamma \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, t] \to M$, with $\gamma(0) = y$ and $\gamma(t) = x$. We can equivalently define $T_t^- u$ by

$$T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y, x).$$

We also set $T_0^- u = u$.

First properties of the Lax-Oleinik semi-group

Let $u : M \to [-\infty, +\infty]$ be a function, we have:

(a) $T_t^- u(x) \leq u(x) + A(0)t$, for $x \in M$ and every $t \geq 0$.

(b) If $u < +\infty$ at one point in M, then $T_t^- u < +\infty$ everywhere $t > 0$.

(c) If $u = -\infty$ at one point in M, then $T_t^- u = -\infty$ everywhere, for $t > 0$.

(d) $T_t^-(u + c) = T_t^-(u) + c$, for $c \in \mathbb{R}$.

(e) If $u \leq v$ everywhere, then $T_t^- u \leq T_t^- v$.

(f) $-\|u - v\|_\infty + T_t^- v \leq T_t^- u \leq T_t^- v + \|u - v\|_\infty$, where $\|u - v\|_\infty = \sup_{x \in M} |u(x) - v(x)|$.

(g) (semi-group property) $T_{t+t'}^- = T_t^- \circ T_{t'}^-$ for $t, t' \geq 0$.

Sketch of Proof

From the properties of h_t shown above, we obtain $h_t(x,x) \leq tA(d(x,x)/t) = A(0)t$. Therefore $T_t^- u(x) \leq u(x) + h_t(x,x) \leq u(x) + A(0)t$. This proves (a).

Note that from the definition $T_t^- u(x) = \inf_{y \in M} u(y) + h_t(y,x)$, we get $T_t^- u(x) \leq u(y) + h_t(y,x)$, for every $y \in M$. Since $h_t(y,x)$ is finite everywhere, this proves (b), namely: $T_t^- u < +\infty$ everywhere if $u < +\infty$ at one point in M, and (c), namely: $T_t^- u = -\infty$ everywhere if $u = -\infty$ at one point in M.

Parts (d), namely: $T_t^-(u+c) = T_t^-(u) + c$, and (e), namely $T_t^- u \leq T_t^- v$ if $u \leq v$, are clear from the definition of $T_t^- u$.

Part (f), namely $-\|u-v\|_\infty + T_t^- v \leq T_t^- u \leq T_t^- v + \|u-v\|_\infty$, is a consequence of (d) and (e) since $-\|u-v\|_\infty + v \leq u \leq v + \|u-v\|_\infty$.

(g) is a consequence of the semi-group property of h_t. $\square$

Lax-Oleinik evolution, evolution domination

Definition 9

For a function $u : M \to [-\infty, +\infty]$, its *Lax-Oleinik evolution* $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$ is defined by $\hat{u}(t, x) = T_t^- u(x)$.

At this point it is useful to recall here the notion of evolution domination.

Definition 10 (Evolution dominated)

We will say that a function $U : [0, +\infty[\times M \to [-\infty, +\infty]$ is evolution dominated by the Lagrangian L if for every piecewise C^1 curve $\gamma : [a, b] \to M$, with $0 \leq a < b$, we have

$$U(b, \gamma(b)) \leq U(a, \gamma(a)) + \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds,$$

or equivalently

$$U(t+s, x) \leq U(t, y) + h_s(y, x), \text{ for all } x, y \in M, t \geq 0, s > 0.$$

Lemma 11

For any function $u : M \to [-\infty, +\infty]$*, its Lax-Oleinik evolution* $\hat{u} : [0, +\infty[\times M \to [-\infty, +\infty]$ *is evolution dominated by* L*.*

Proof.

The semi-group property $T^-_{t+s} = T^-_s \circ T^-_t$ for $s, t \geq 0$ and the definition of T^-_s, for $s > 0$ imply

$$T^-_{t+s}u(x) = T^-_s\big(T^-_t u(x)\big) \leq T^-_t u(y) + h_s(y, x).$$

By the definition of $\hat{u}$, this translates to

$$\hat{u}(t+s, x) \leq \hat{u}(t, y) + h_s(y, x), \text{ for all } x, y \in M, t \geq 0, s > 0,$$

which precisely mean that $\hat{u}$ is evolution dominated by L $\square$

Lax-Oleinik and Viscosity

We now explain some of the relationship between the Lax-Oleinik semi-group and viscosity. More on that later on.

Theorem 12

Suppose $U : [0, +\infty[\times M \to [-\infty, +\infty]$ is evolution dominated by L. If U is finite on $]0, \tau[\times M$, for some $\tau \in]0, +\infty]$, then U is a viscosity subsolution of

$$\frac{\partial U}{\partial t}(t, x) + H(x, \frac{\partial U}{\partial x}(t, x)) = 0,$$

on the open subset $]0, \tau[\times M$.

Proof.

Suppose $\phi \geq U$ on $]0, \tau[\times M$, with ϕ of class C^1 and $\phi(t_0, x_0) = U(t_0, x_0)$, where $t_0 \in]0, \tau[$.
Fix $v \in T_{x_0}M$, and pick a C^1 curve $\gamma : [0, t_0] \to M$ such that $(\gamma(t_0), \dot\gamma(t_0)) = (x_0, v)$.

If $0 \leq t \leq t_0 < \tau$, by the domination inequality, we have

$$U(t_0, \gamma(t_0)) - U(t, \gamma(t)) \leq \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds, \text{ for all } t \in [0, t_0].$$

Since $\phi \geq U$, with equality at $(t_0, x_0) = (t_0, \gamma(t_0))$, we obtain

$$\phi(t_0, \gamma(t_0)) - \phi(t, \gamma(t)) \leq \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds, \text{ for all } t \in [0, t_0].$$

Dividing both sides of this last inequality by $t_0 - t > 0$, and letting $t \to t_0$, we get

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + \frac{\partial \phi}{\partial x}(t_0, x_0)(v) \leq L(x_0, v).$$

Since this is true for all $v \in T_{x_0}M$, and

$$H(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)) = \sup_{v \in T_{x_0}M} \frac{\partial \phi}{\partial x}(t_0, x_0)(v) - L(x_0, v),$$

we obtain

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)) \leq 0.$$

This finishes to show that U is a viscosity subsolution. $\square$

Theorem 13

Suppose $u : M \to [-\infty, +\infty]$ *is a function for which there exists* $\tau \in]0, +\infty]$ *such that its Lax-Oleinik evolution* $\hat{u}$ *is finite on* $]0, \tau[\times M$ *and, for every* $(t, x) \in]0, \tau[\times M$*, the infimum in the definition of the Lax-Oleinik evolution*

$$\hat{u}(t, x) = \inf_{y \in M} u(y) + h_t(y, x)$$

is attained at some point $y \in M$*.*
Then $\hat{u}$ *is a viscosity solution of*

$$\frac{\partial \hat{u}}{\partial t}(t, x) + H(x, \frac{\partial \hat{u}}{\partial x}(t, x)) = 0,$$

on the open subset $]0, \tau[\times M$*.*

Proof.

Since $\hat{u}$ is evolution dominated by L, from the previous theorem, it is a viscosity subsolution on $]0, \tau[\times M$.

We now prove that $\hat{u}$ is a supersolution.
Suppose that $\psi :]0, \tau[\times M$ is of class C^1, with $\psi \leq \hat{u}$ and $\hat{u}(t_0, x_0) = \psi(t_0, x_0)$, with $(t_0, x_0) \in]0, \tau[\times M$.
By the hypothesis, we can find a $y \in M$ such that

$$\hat{u}(t_0, x_0) = u(y) + h_{t_0}(y, x_0).$$

By Tonelli's theorem, we can find a curve $\gamma : [0, t_0] \to M$, with $\gamma(t_0) = x_0, \gamma(0) = y$, and whose action is precisely $h_{t_0}(y, x_0)$.
Therefore

$$\hat{u}(t_0, x_0) = u(\gamma(0)) + \int_0^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Note that $u(\gamma(0))$ is finite, both since both $\hat{u}(t_0, x_0)$ and $\int_0^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds$ are finite.
Using $\hat{u}(0, \gamma(0)) = u(\gamma(0))$, the inequality above can be rewritten as

$$\hat{u}(t_0, x_0) = \hat{u}(0, \gamma(0)) + \int_0^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds.$$

We have thus obtained

$$\hat{u}(t_0, x_0) = \hat{u}(0, \gamma(0)) + \int_0^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds. \tag{4}$$

Since $\hat{u}$ is evolution dominated by L, for every $t \in]0, t_0[$, we have

$$\hat{u}(t_0, x_0) \leq \hat{u}(t, \gamma(t)) + \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds$$

$$\hat{u}(t, \gamma(t)) \leq \hat{u}(0, \gamma(0)) + \int_0^{t} L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Note that $\hat{u}(t, \gamma(t))$ is finite since $(t, \gamma(t)) \in]0, \tau[\times M$.
Adding the two inequalities above, we get in fact the equality (4).
Therefore both inequalities must be equalities. Hence

$$\hat{u}(t_0, \gamma(t_0)) = \hat{u}(t, \gamma(t)) + \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds, \text{ for every } t \in [0, t_0].$$

Since $\psi \leq \hat{u}$, with equality at $(t_0, \gamma(t_0)$, we obtain

$$\psi(t_0, \gamma(t_0)) \geq \psi(t, \gamma(t)) + \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds, \text{ for every } t \in [0, t_0].$$

The inequality

$$\psi(t_0, \gamma(t_0)) \geq \psi(t, \gamma(t)) + \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds$$

obtained above, for all $t \in]0, t_0[$, can be rewritten as

$$\psi(t_0, \gamma(t_0)) - \psi(t, \gamma(t)) \geq \int_t^{t_0} L(\gamma(s), \dot{\gamma}(s))\, ds, \text{ for every } t \in [0, t_0].$$

Dividing by $t_0 - t > 0$, and letting $t \to t_0$, we get

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) \geq L(x_0, \dot{\gamma}(t_0)).$$

By definition of L, we have

$$L(x_0, \dot{\gamma}(t_0)) \geq \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) - H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)).$$

It follows that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) \geq \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot{\gamma}(t_0)) - H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)).$$

Therefore

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)) \geq 0.$$

□

Remark 14

As we will later see, if the Lax-Oleinik $\hat{u}$ of $u : M \to [-\infty, +\infty]$ is finite on $]0, \tau[\times M$, then $\hat{u}$ is automatically continuous on $]0, \tau[\times M$. In fact $\hat{u}$ is even locally semi-concave on $]0, \tau[\times M$. Moreover, up to replacing u by its lower semi-continuous regularization if necessary, for every $(t, x) \in]0, \tau[\times M$, the infimum in the definition of the Lax-Oleinik evolution

$$\hat{u}(t, x) = \inf_{y \in M} u(y) + h_t(y, x)$$

is attained at some point $y \in M$.