RSS equivalences between bimodule monomorphism and epimorphism subcategories [*†]

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Abstract

Let A and B be arbitrary rings and ${}_AM_B$ an A-B-bimodule, and let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of A-Mod and B-Mod, respectively. We construct the generalized kernel and cokernel functors between the bimodule monomorphism subcategory $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and the bimodule epimorphism subcategory $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$, which generalize the classical ones by Ringel–Schmidmeier. If M is $\mathcal{A}$-$\mathcal{B}$-compatible, we prove that the generalized cokernel functor is an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$ if and only if the functors $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ induce quasi-inverse equivalences between $\mathcal{A}$ and $\mathcal{B}$. This equivalence not only recovers the classical Ringel–Schmidmeier's equivalence, but also serves to construct RSS equivalences between various monomorphism and epimorphism subcategories associated with Morita equivalence, tilting torsion pairs and Foxby equivalence.

1 Introduction

The notion of the monomorphism category originated with Birkhoff [6], who posed the problem of classifying the subgroups of a finite abelian group. This category was later revisited by Ringel and Schmidmeier [24, 25], where it was known as the submodule category. The monomorphism category has attracted considerable attention in various areas, including Auslander–Reiten theory [13, 25, 34], classification problems and the representation type [14, 24, 26, 28, 29, 30], Gorenstein projective modules and tilting theory [18, 36], weighted projective lines and singularity categories [9, 19, 20]. The monomorphism category has a dual structure, known as the epimorphism category. A close connection between these two categories is given by the so-called Ringel–Schmidmeier–Simson equivalence (RSS equivalence for short), which was established by Xiong–Zhang–Zhang [35] in honor of the work of Ringel–Schmidmeier [25] and Simson [28]. Recently, the study of RSS equivalences has been further developed in contexts such as acyclic quivers with monomial relations [37] and Morita rings [15].

Let A and B be arbitrary rings and ${}_AM_B$ an A-B-bimodule, and let $\Lambda := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the triangular matrix ring. Given a subcategory $\mathcal{A}$ of A-Mod and a subcategory $\mathcal{B}$ of B-Mod, the *bimodule monomorphism subcategory* $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ associated to $\mathcal{A}$ and $\mathcal{B}$ is defined as the subcategory of Λ-Mod consisting of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$ such that $X_2 \in \mathcal{B}$ and $\varphi^X : M \otimes_B X_2 \to X_1$ is a monomorphism with

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$\operatorname{Coker}\varphi^X \in \mathcal{A}$. While, the *bimodule epimorphism subcategory* $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$ associated to $\mathcal{A}$ and $\mathcal{B}$ is defined dually, see Definition 3.1 for details. In this paper, we are mainly concerned with the bimodule monomorphism subcategory $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and the bimodule epimorphism subcategory $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$ associated to $\mathcal{A}$ and $\mathcal{B}$. Note that $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ (resp. $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$) could be viewed as a certain subcategory of the monomorphism (resp. epimorphism) category induced by the bimodule ${}_AM_B$ in the sense of [35]. Moreover, these two categories are close related to cotorsion pairs, approximation theory and Frobenius categories [21, 23].

This paper is motivated by the work of Ringel–Schmidmeier [25] and Xiong–Zhang–Zhang [35], both of which established equivalences between the monomorphism category and the epimorphism category. Our purpose is to develop quasi-equivalences between $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$.

The classical equivalences between the monomorphism category and the epimorphism category are given by the kernel and cokernel functors due to Ringel–Schmidmeier [25]. This leads us to consider how to adapt these functors to the contexts of $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$. For this reason, we first introduce the notion of a bimodule ${}_AM_B$ to be left (resp. right) compatible with respect to $\mathcal{A}$ (resp. $\mathcal{B}$) (see Definition 3.3). After that, the generalized kernel functor $\mathbf{K}_{\mathcal{A},\mathcal{B}} : \mathcal{E}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{M}(\mathcal{A}, M, \mathcal{B})$ as well as the generalized cokernel functor $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ are defined. We prove that the pair $(\mathbf{K}_{\mathcal{A},\mathcal{B}}, \mathbf{C}_{\mathcal{A},\mathcal{B}})$ forms an adjoint pair when the bimodule ${}_AM_B$ is strongly $\mathcal{A}$-$\mathcal{B}$-compatible (Theorem 3.8).

Furthermore, inspired by the RSS equivalence between the monomorphism category and the epimorphism category due to Xiong–Zhang–Zhang [35], we define an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$. Precisely, a functor $F : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is called an *RSS equivalence with respect to the subcategories* $\mathcal{A}$ *and* $\mathcal{B}$ if F is an equivalence such that

$$F\begin{pmatrix} X \\ 0 \end{pmatrix} \cong \begin{pmatrix} X \\ \operatorname{Hom}_A(M, X) \end{pmatrix}_{\widetilde{id}} \text{ and } F\begin{pmatrix} M \otimes_B Y \\ Y \end{pmatrix}_{id} \cong \begin{pmatrix} 0 \\ Y \end{pmatrix}$$

for any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ (Definition 3.9).

The main theorem of this paper is as follows.

Theorem 1.1. (Theorem 3.10) *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be closed under extensions. Assume that* M *is* $\mathcal{A}$-$\mathcal{B}$-*compatible. Then the generalized cokernel functor*

$$\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$$

is an RSS equivalence with respect to the subcategories $\mathcal{A}$ *and* $\mathcal{B}$ *with* $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ *as its quasi-inverse, if and only if there is an equivalence of categories*

$$\mathcal{A} \underset{M\otimes_B -}{\overset{\operatorname{Hom}_A(M,-)}{\rightleftarrows}} \mathcal{B}.$$

We point out that the generalized cokernel functor in Theorem 1.1 is usually different from the functor given by Xiong–Zhang–Zhang [35], even though the latter is an RSS equivalence (Remark 3.11). What's more, Theorem 1.1 recovers the classical Ringel–Schmidmeier's equivalence (Proposition 4.1). Finally, we apply Theorem 1.1 to establish certain RSS equivalences between various monomorphism and epimorphism subcategories associated with the Morita equivalence, tilting torsion pairs and Foxby equivalence.

The paper is organized as follows. In Section 2, we fix some notations and recall some basic definitions and facts. In Section 3, we introduce the generalized kernel and cokernel functors, and prove Theorem 1.1. In Section 4, we apply Theorem 1.1 to construct certain RSS equivalences between various monomorphism and epimorphism subcategories. Note that, in contrast to the classical RSS setting, the RSS equivalence in Theorem 1.1 is of quite a different type. It establishes a direct connection with tilting theory and involves tilted algebras (Theorem 4.3).

2 Preliminaries

Throughout this paper, all rings are associative with identity and all subcategories involved are full, additive and closed under isomorphisms. For any ring A, we use $A\text{-}\mathrm{Mod}$ (resp. $A\text{-}\mathrm{mod}$) to denote the category of (resp. finitely generated) left A-modules, and use $A\text{-}\mathrm{Proj}$ (resp. $A\text{-}\mathrm{Flat}$, $A\text{-}\mathrm{Inj}$) to denote the subcategory of $A\text{-}\mathrm{Mod}$ consisting of projective (resp. flat, injective) modules. For a module $M \in A\text{-}\mathrm{Mod}$, we use $\mathrm{pd}_A M$ to denote the projective dimension of M, and use $\mathrm{Add}\, M$ to denote the subcategory of $A\text{-}\mathrm{Mod}$ consisting of direct summands of direct sums of copies of M.

Given an additive category $\mathscr{X}$ and an object $X \in \mathscr{X}$, we use id_X, or simply id, to denote the identity morphism of X. Two additive categories $\mathscr{X}$ and $\mathscr{X}'$ are said to be *equivalent* if there exist additive functors $F : \mathscr{X} \to \mathscr{X}'$ and $F' : \mathscr{X}' \to \mathscr{X}$ such that the composition functors $F'F$ and FF' are naturally isomorphic to the identity functors $\mathrm{Id}_{\mathscr{X}}$ and $\mathrm{Id}_{\mathscr{X}'}$, respectively. In this case, we call F' a *quasi-inverse* of F.

Definition 2.1. ([10, 12]) Let A be an arbitrary ring. A module $T \in A\text{-}\mathrm{Mod}$ is a *tilting module* provided the following conditions are satisfied:

(T1) $\mathrm{pd}_A T \le 1$.

(T2) $\mathrm{Ext}^1_A(T, T^{(\kappa)}) = 0$ for any cardinal κ.

(T3) There is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ in $A\text{-}\mathrm{Mod}$ such that $T_0, T_1 \in \mathrm{Add}\, T$.

Recall that a left A-module M is a *progenerator* if M is a finitely generated projective generator; equivalently, there exist integers m and n, along with left A-modules M' and A', such that $A^{(m)} \cong M \oplus M'$ and $M^{(n)} \cong A \oplus A'$. It is known that the notion of a tilting module is a generalization of a progenerator.

Let ${}_AT$ be a tilting module with $B = \mathrm{End}({}_AT)$. Consider the following subcategories of $A\text{-}\mathrm{Mod}$ and $B\text{-}\mathrm{Mod}$, respectively:

$$\mathcal{T} = \{X \in A\text{-}\mathrm{Mod} \mid \mathrm{Ext}^1_A(T, X) = 0\} \text{ and } \mathcal{F} = \{X \in A\text{-}\mathrm{Mod} \mid \mathrm{Hom}_A(T, X) = 0\},$$

$$\mathcal{X} = \{Y \in B\text{-}\mathrm{Mod} \mid T \otimes_B Y = 0\} \text{ and } \mathcal{Y} = \{Y \in B\text{-}\mathrm{Mod} \mid \mathrm{Tor}^B_1(T, Y) = 0\}.$$

It is well known that $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion pairs in $A\text{-}\mathrm{Mod}$ and $B\text{-}\mathrm{Mod}$, respectively. We call them the torsion pairs induced by T, which are widely studied, see [7, 8, 10, 11, 12] and references therin. The following theorem is known as the Brenner–Butler theorem or the tilting theorem.

Theorem 2.2. ([3, Section VI.3] and [10, Theorem 1.4]) *Suppose that ${}_AT$ is a tilting module and $B = \mathrm{End}({}_AT)$.*

(1) *For any $X \in A\text{-}\mathrm{Mod}$ and $Y \in B\text{-}\mathrm{Mod}$, it holds*

$$T \otimes_B \mathrm{Ext}^1_A(T, X) = 0 = \mathrm{Tor}^B_1(T, \mathrm{Hom}_A(T, X)) \text{ and}$$

$$\mathrm{Hom}_A(T, \mathrm{Tor}^B_1(T, Y)) = 0 = \mathrm{Ext}^1_A(T, T \otimes_B Y).$$

(2) *The functors* $\mathrm{Hom}_A(T,-)$ *and* $T\otimes_B -$ *induce quasi-inverse equivalences between* $\mathcal{T}$ *and* $\mathcal{Y}$.

(3) *The functors* $\mathrm{Ext}^1_A(T,-)$ *and* $\mathrm{Tor}^B_1(T,-)$ *induce quasi-inverse equivalences between* $\mathcal{F}$ *and* $\mathcal{X}$.

Let A and B be arbitrary rings and ${}_AM_B$ an A-B-bimodule. For any modules $X\in A\text{-}\mathrm{Mod}$ and $Y\in B\text{-}\mathrm{Mod}$, we use

$$\theta=\theta_{Y,X}:\mathrm{Hom}_A(M\otimes_B Y,X)\to\mathrm{Hom}_B(Y,\mathrm{Hom}_A(M,X))$$

to denote the adjunction isomorphism, where $\theta(f)(y)(m)=f(m\otimes y)$ for any $f\in\mathrm{Hom}_A(M\otimes_B Y,X), m\in M$ and $y\in Y$. We write μ and ν for the unit and counit of the adjoint pair $(M\otimes_B -,\mathrm{Hom}_A(M,-))$. For any $X\in A\text{-}\mathrm{Mod}$ and $Y\in B\text{-}\mathrm{Mod}$, we have the following natural evaluation homomorphisms:

$$\mu_Y:Y\to\mathrm{Hom}_A(M,M\otimes_B Y)\text{ via }\mu_Y(y)(m)=m\otimes y\text{ for any }y\in Y\text{ and }m\in M,\text{ and}$$

$$\nu_X:M\otimes_B\mathrm{Hom}_A(M,X)\to X\text{ via }\nu_X(m\otimes f)=f(m)\text{ for any }m\in M\text{ and }f\in\mathrm{Hom}_A(M,X).$$

Definition 2.3. (See [11, Section 2.1]) Let A and B be arbitrary rings and ${}_AM_B$ an A-B-bimodule. A module ${}_AX$ is called *ν-reflexive* if the natural evaluation homomorphism ν_X is an isomorphism. A module ${}_BY$ is called *μ-reflexive* if the natural evaluation homomorphism μ_Y is an isomorphism.

Definition 2.4. ([2, 17]) Let A and B be arbitrary rings. An A-B-bimodule ${}_AM_B$ is called *semidualizing* if the following conditions are satisfied.

(a1) ${}_AM$ admits a degreewise finite A-projective resolution.

(a2) M_B admits a degreewise finite B^{op}-projective resolution.

(b1) The homothety morphism ${}_AA_A\xrightarrow{{}_A\gamma}\mathrm{Hom}_{B^{op}}(M,M)$ is an isomorphism.

(b2) The homothety morphism ${}_BB_B\xrightarrow{\gamma_B}\mathrm{Hom}_A(M,M)$ is an isomorphism.

(c1) $\mathrm{Ext}^{\geq 1}_A(M,M)=0$.

(c2) $\mathrm{Ext}^{\geq 1}_{B^{op}}(M,M)=0$.

Wakamatsu in [31] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*, see [5, 22]. Note that a bimodule ${}_AM_B$ is semidualizing if and only if ${}_AM$ is Wakamatsu tilting with $B=\mathrm{End}({}_AM)$, and if and only if M_B is Wakamatsu tilting with $A=\mathrm{End}(M_B)$ ([33, Corollary 3.2]). Typical examples of semidualizing bimodules include the free module of rank one and the dualizing module over a Cohen-Macaulay local ring. More examples of semidualizing bimodules are referred to [17, 27, 32].

Definition 2.5. ([17, Definition 4.1]) Let A and B be arbitrary rings and ${}_AM_B$ a semidualizing bimodule.

(1) The *Auslander class* $\mathcal{A}_M(B)$ with respect to M consists of all left B-modules Y satisfying

  (A1) $\mathrm{Tor}^B_{\geq 1}(M,Y)=0$;

  (A2) $\mathrm{Ext}^{\geq 1}_A(M,M\otimes_B Y)=0$;

(A3) Y is μ-reflexive.

(2) The *Bass class* $\mathcal{B}_M(A)$ with respect to M consists of all left A-modules X satisfying

(B1) $\mathrm{Ext}_A^{\geq 1}(M,X)=0$;

(B2) $\mathrm{Tor}_{\geq 1}^B(M,\mathrm{Hom}_A(M,X))=0$;

(B3) X is ν-reflexive.

Lemma 2.6. ([17, Proposition 4.1]) *Let ${}_AM_B$ be a semidualizing bimodule. Then the functors* $\mathrm{Hom}_A(M,-)$ *and* $M\otimes_B -$ *induce the following Foxby equivalence:*

$$\mathcal{A}_M(B) \underset{\mathrm{Hom}_A(M,-)}{\overset{M\otimes_B-}{\rightleftarrows}} \mathcal{B}_M(A).$$

3 RSS equivalences between monomorphism and epimorphism subcategories

In this section, A and B are arbitrary rings. Given an A-B-bimodule ${}_AM_B$, we use $\Lambda=\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}$ to denote the triangular matrix ring.

Recall that a left Λ-module is identified with a triple $\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\varphi^X}$, where $X_1\in A\text{-}\mathrm{Mod}$, $X_2\in B\text{-}\mathrm{Mod}$ and $\varphi^X: M\otimes_B X_2\to X_1$ is a homomorphism in A-Mod. Analogously, the left Λ-module $\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\varphi^X}$ is also identified with the triple $\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\widetilde{\varphi^X}}$ such that $\widetilde{\varphi^X}=\theta(\varphi^X): X_2\to \mathrm{Hom}_A(M,X_1)$ is a homomorphism in B-Mod, where $\theta:\mathrm{Hom}_A(M\otimes_B X_2, X_1)\to \mathrm{Hom}_B(X_2,\mathrm{Hom}_A(M,X_1))$ is the adjunction isomorphism.

A homomorphism $\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\varphi^X}\to\begin{pmatrix} Y_1 \\ Y_2\end{pmatrix}_{\varphi^Y}$ in Λ-Mod is identified with a pair $\begin{pmatrix} f \\ g\end{pmatrix}$, where $f\in\mathrm{Hom}_A(X_1,Y_1)$ and $g\in\mathrm{Hom}_B(X_2,Y_2)$, such that the following diagram

$$\begin{tikzcd}
M\otimes_B X_2 \arrow[r, "\varphi^X"] \arrow[d, "id\otimes g"'] & X_1 \arrow[d, "f"] \\
M\otimes_B Y_2 \arrow[r, "\varphi^Y"] & Y_1
\end{tikzcd}$$

commutes. A sequence

$$0\to\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\varphi^X}\xrightarrow{\begin{pmatrix} f_1 \\ g_1\end{pmatrix}}\begin{pmatrix} Y_1 \\ Y_2\end{pmatrix}_{\varphi^Y}\xrightarrow{\begin{pmatrix} f_2 \\ g_2\end{pmatrix}}\begin{pmatrix} Z_1 \\ Z_2\end{pmatrix}_{\varphi^Z}\to 0$$

in Λ-Mod is exact if and only if $0\to X_1\xrightarrow{f_1}Y_1\xrightarrow{f_2}Z_1\to 0$ and $0\to X_2\xrightarrow{g_1}Y_2\xrightarrow{g_2}Z_2\to 0$ are exact in A-Mod and B-Mod, respectively. We refer the reader to [4, 16] for more details.

The *monomorphism category* $\mathcal{M}(A,M,B)$ induced by the bimodule ${}_AM_B$ is the subcategory of Λ-Mod consisting of $\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\varphi^X}$ such that $\varphi^X: M\otimes_B X_2\to X_1$ is monic in A-Mod. Dually, the *epimorphism category* $\mathcal{E}(A,M,B)$ induced by the bimodule ${}_AM_B$ is the subcategory of Λ-Mod consisting of $\begin{pmatrix} X_1 \\ X_2\end{pmatrix}_{\widetilde{\varphi^X}}$ such that $\widetilde{\varphi^X}: X_2\to\mathrm{Hom}_A(M,X_1)$ is epic in B-Mod.

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of A- Mod and B- Mod, respectively.

(1) The *bimodule monomorphism subcategory* $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ associated to subcategories $\mathcal{A}$ and $\mathcal{B}$ is defined as the subcategory of Λ- Mod consisting of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$ such that $X_2 \in \mathcal{B}$ and $\varphi^X : M \otimes_B X_2 \to X_1$ is a monomorphism with $\operatorname{Coker} \varphi^X \in \mathcal{A}$.

(2) The *bimodule epimorphism subcategory* $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$ associated to subcategories $\mathcal{A}$ and $\mathcal{B}$ is defined as the subcategory of Λ- Mod consisting of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\widetilde{\varphi^X}}$ such that $X_1 \in \mathcal{A}$ and $\widetilde{\varphi^X} : X_2 \to \operatorname{Hom}_A(M, X_1)$ is an epimorphism with $\operatorname{Ker} \widetilde{\varphi^X} \in \mathcal{B}$.

Note that $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ (resp. $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$) is the subcategory of $\mathcal{M}(A, M, B)$ (resp. $\mathcal{E}(A, M, B)$). We will write $\mathcal{M}(A, M, \mathcal{B})$ (resp. $\mathcal{E}(A, M, \mathcal{B})$) for the bimodule monomorphism (resp. epimorphism) subcategory associated to A- Mod and $\mathcal{B}$. Similarly, we have the notations for $\mathcal{M}(\mathcal{A}, M, B)$ and $\mathcal{E}(\mathcal{A}, M, B)$.

Remark 3.2. Let A be an arbitrary ring. We write $\mathcal{M}(A, A, A)$ (resp. $\mathcal{E}(A, A, A)$) for the monomorphism (resp. epimorphism) category, whose object is identified with a triple $\begin{pmatrix} X \\ Y \end{pmatrix}_f$, where $f : Y \to X$ is monic (resp. epic) in A-Mod. Such two categories were originated from Ringel–Schmidmeier [25] and they mainly studied the Auslander–Reiten translation in these two categories. Furthermore, due to Ringel–Schmidmeier, we have the following kernel and cokernel functors:

$$\operatorname{Ker} : \mathcal{E}(A, A, A) \to \mathcal{M}(A, A, A) \text{ via } \operatorname{Ker} \begin{pmatrix} X \\ Y \end{pmatrix}_f = \begin{pmatrix} Y \\ \operatorname{Ker} f \end{pmatrix}_{\iota_f},$$

$$\operatorname{Cok} : \mathcal{M}(A, A, A) \to \mathcal{E}(A, A, A) \text{ via } \operatorname{Cok} \begin{pmatrix} X \\ Y \end{pmatrix}_f = \begin{pmatrix} \operatorname{Coker} f \\ X \end{pmatrix}_{\pi_f},$$

where $\iota_f : \operatorname{Ker} f \to Y$ and $\pi_f : X \to \operatorname{Coker} f$ are the canonical homomorphisms induced by f. As was shown in [25, Lemma 1.2], these two functors form a pair of quasi-inverse equivalences.

We will investigate whether the kernel and cokernel functors can be adapted to the contexts of $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$. To this end, we introduce the following notions.

Definition 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of A- Mod and B- Mod, respectively.

(1) The bimodule ${}_AM_B$ is said to be *left compatible* with respect to $\mathcal{A}$, or simply *left $\mathcal{A}$-compatible*, if the following conditions are satisfied:

(L1) $\operatorname{Tor}_1^B(M, \operatorname{Hom}_A(M, \mathcal{A})) = 0$,

(L2) $M \otimes_B \operatorname{Hom}_A(M, \mathcal{A}) \subseteq \mathcal{A}$.

(2) The bimodule ${}_AM_B$ is said to be *right compatible* with respect to $\mathcal{B}$, or simply *right $\mathcal{B}$-compatible*, if the following conditions are satisfied:

(R1) $\operatorname{Ext}_A^1(M, M \otimes_B \mathcal{B}) = 0$,

(R2) $\operatorname{Hom}_A(M, M \otimes_B \mathcal{B}) \subseteq \mathcal{B}$.

(3) The bimodule ${}_AM_B$ is said to be *compatible* with respect to $\mathcal{A}$ and $\mathcal{B}$, or simply *$\mathcal{A}$-$\mathcal{B}$-compatible*, if M is both left $\mathcal{A}$- and right $\mathcal{B}$-compatible.

Example 3.4. (1) Let A be an arbitrary ring. For any subcategories $\mathcal{A}$ and $\mathcal{B}$ of A-Mod, it is trivial that ${}_AA_A$ is $\mathcal{A}$-$\mathcal{B}$-compatible.

(2) Any bimodule ${}_AM_B$ is $\mathcal{A}$-$\mathcal{B}$-compatible, where $\mathcal{A} := \{X \in A\text{-Mod} \mid \mathrm{Hom}_A(M, X) = 0\}$ and $\mathcal{B} := \{Y \in B\text{-Mod} \mid M \otimes_B Y = 0\}$.

(3) If ${}_AM$ is projective and M_B is flat, then ${}_AM_B$ is $\mathcal{A}$-$\mathcal{B}$-compatible for $\mathcal{A} = A\text{-Mod}$ and $\mathcal{B} = B\text{-Mod}$.

(4) Let ${}_AM$ be finitely presented with $B = \mathrm{End}({}_AM)$. It follows from [1, Lemma 29.4] that $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ induce quasi-inverse equivalences between $\mathrm{Add}\, M$ and $B\text{-Proj}$. If $\mathrm{Ext}^1_A(M, M) = 0$, then M is $\mathcal{A}$-$\mathcal{B}$-compatible in case that $\mathcal{A} = \mathrm{Add}\, M$ and $\mathcal{B} = B\text{-Proj}$.

(5) Assume that ${}_AM$ is a tilting module with $B = \mathrm{End}({}_AM)$. Let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ be the torsion pairs induced by M in A-Mod and B-Mod, respectively. By Theorem 2.2, we have

$$\mathrm{Ext}^1_A(M, M \otimes_B Y) = 0 = \mathrm{Tor}^B_1(M, \mathrm{Hom}_A(M, X)) = 0$$

for any $Y \in B\text{-Mod}$ and $X \in A\text{-Mod}$. Thus ${}_AM_B$ is $\mathcal{A}$-$\mathcal{B}$-compatible for $\mathcal{A} = A\text{-Mod}$ and $\mathcal{B} = B\text{-Mod}$. Furthermore, the functors $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ induce quasi-inverse equivalences between $\mathcal{T}$ and $\mathcal{Y}$. It follows that ${}_AM_B$ is $\mathcal{A}$-$\mathcal{B}$-compatible for $\mathcal{A} = \mathcal{T}$ and $\mathcal{B} = \mathcal{Y}$.

(6) Let ${}_AM$ be a progenerator with $B = \mathrm{End}({}_AM)$. By the Morita theorem (see [1, Theorem 22.1]), the functors $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ induce quasi-inverse equivalences between A-Mod and B-Mod. Setting $\mathcal{A} := A\text{-Mod}$ and $\mathcal{B} := B\text{-Mod}$, it is easy to see that the conditions (L1), (L2), (R1) and (R2) are satisfied. Thus, M is $\mathcal{A}$-$\mathcal{B}$-compatible.

(7) Assume that ${}_AM_B$ is a semidualizing bimodule. If we take $\mathcal{A} := \mathcal{B}_M(A)$ and $\mathcal{B} := \mathcal{A}_M(B)$ as the corresponding Bass and Auslander classes, respectively, then by the conditions (A2), (A3), (B2) and (B3) in Definition 2.5, we have that M is $\mathcal{A}$-$\mathcal{B}$-compatible.

In the following, we introduce the so-called generalized kernel and cokernel functors between $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$.

Assume that M is left $\mathcal{A}$-compatible and $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\widetilde{\varphi^X}} \in \mathcal{E}(\mathcal{A}, M, \mathcal{B})$. Then we have the following exact sequence

$$0 \to \mathrm{Ker}\, \widetilde{\varphi^X} \xrightarrow{\iota^X} X_2 \xrightarrow{\widetilde{\varphi^X}} \mathrm{Hom}_A(M, X_1) \to 0 \tag{3.1}$$

in B-Mod such that $X_1 \in \mathcal{A}$, $\mathrm{Ker}\, \widetilde{\varphi^X} \in \mathcal{B}$ and $\widetilde{\varphi^X} : X_2 \to \mathrm{Hom}_A(M, X_1)$ is epic. Since M is left $\mathcal{A}$-compatible, we have $\mathrm{Tor}^B_1(M, \mathrm{Hom}_A(M, \mathcal{A})) = 0$. Applying the functor $M \otimes_B -$ to (3.1) yields the following exact sequence:

$$0 \to M \otimes_B \mathrm{Ker}\, \widetilde{\varphi^X} \xrightarrow{id \otimes \iota^X} M \otimes_B X_2 \xrightarrow{id \otimes \widetilde{\varphi^X}} M \otimes_B \mathrm{Hom}_A(M, X_1) \to 0. \tag{3.2}$$

In view of (3.2), it follows that $\begin{pmatrix} M \otimes_B X_2 \\ \mathrm{Ker}\, \widetilde{\varphi^X} \end{pmatrix}_{id \otimes \iota^X} \in \mathcal{M}(\mathcal{A}, M, \mathcal{B})$. We define the *generalized kernel functor* as follows. We first define

$$\mathbf{K}_{\mathcal{A},\mathcal{B}} : \mathcal{E}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \text{ via } \mathbf{K}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\widetilde{\varphi^X}} := \begin{pmatrix} M \otimes_B X_2 \\ \mathrm{Ker}\, \widetilde{\varphi^X} \end{pmatrix}_{id \otimes \iota^X}.$$

Next, for any morphism $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\widetilde{\varphi^X}} \to \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\widetilde{\varphi^Y}}$ in $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \widetilde{\varphi^X} \arrow[r, "\iota^X"] \arrow[d, dashed, "\iota_f"'] & X_2 \arrow[r, "\widetilde{\varphi^X}"] \arrow[d, "f_2"'] & \operatorname{Hom}_A(M, X_1) \arrow[r] \arrow[d, "(f_1)_*"] & 0 \\
0 \arrow[r] & \operatorname{Ker} \widetilde{\varphi^Y} \arrow[r, "\iota^Y"] & Y_2 \arrow[r, "\widetilde{\varphi^Y}"] & \operatorname{Hom}_A(M, Y_1) \arrow[r] & 0,
\end{tikzcd}$$

where $(f_1)_* = \operatorname{Hom}_A(M, f_1)$ is the induced morphism of f_1. Applying the functor $M \otimes_B -$ to it yields the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_B \operatorname{Ker} \widetilde{\varphi^X} \arrow[r, "id\otimes\iota^X"] \arrow[d, dashed, "id\otimes\iota_f"'] & M \otimes_B X_2 \arrow[r, "id\otimes\widetilde{\varphi^X}"] \arrow[d, "id\otimes f_2"'] & M \otimes_B \operatorname{Hom}_A(M, X_1) \arrow[r] \arrow[d, "id\otimes(f_1)_*"] & 0 \\
0 \arrow[r] & M \otimes_B \operatorname{Ker} \widetilde{\varphi^Y} \arrow[r, "id\otimes\iota^Y"] & M \otimes_B Y_2 \arrow[r, "id\otimes\widetilde{\varphi^Y}"] & M \otimes_B \operatorname{Hom}_A(M, Y_1) \arrow[r] & 0.
\end{tikzcd}$$

In view of the left square in the above commutative diagram, we now define

$$\mathbf{K}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} id \otimes f_2 \\ \iota_f \end{pmatrix}.$$

Now, assume that M is right $\mathcal{B}$-compatible and $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_{\varphi^U} \in \mathcal{M}(\mathcal{A}, M, \mathcal{B})$. Then we have an exact sequence

$$0 \to M \otimes_B U_2 \xrightarrow{\varphi^U} U_1 \xrightarrow{\pi^U} \operatorname{Coker} \varphi^U \to 0 \tag{3.3}$$

in A-Mod such that $U_2 \in \mathcal{B}$, $\operatorname{Coker} \varphi^U \in \mathcal{A}$ and φ^U is monic. Since M is right $\mathcal{B}$-compatible, we have $\operatorname{Ext}^1_A(M, M \otimes_B \mathcal{B}) = 0$. Applying the functor $\operatorname{Hom}_A(M, -)$ to (3.3) yields the following exact sequence:

$$0 \to \operatorname{Hom}_A(M, M \otimes_B U_2) \xrightarrow{(\varphi^U)_*} \operatorname{Hom}_A(M, U_1) \xrightarrow{(\pi^U)_*} \operatorname{Hom}_A(M, \operatorname{Coker} \varphi^U) \to 0. \tag{3.4}$$

In view of (3.4), we define the *generalized cokernel functor* as follows. We first define

$$\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B}) \text{ via } \mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_{\varphi^U} := \begin{pmatrix} \operatorname{Coker} \varphi^U \\ \operatorname{Hom}_A(M, U_1) \end{pmatrix}_{(\pi^U)_*}.$$

Next, for any morphism $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} : \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_{\varphi^U} \to \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_{\varphi^V}$ in $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_B U_2 \arrow[r, "\varphi^U"] \arrow[d, "id\otimes g_2"'] & U_1 \arrow[r, "\pi^U"] \arrow[d, "g_1"'] & \operatorname{Coker} \varphi^U \arrow[r] \arrow[d, dashed, "\pi_g"] & 0 \\
0 \arrow[r] & M \otimes_B V_2 \arrow[r, "\varphi^V"] & V_1 \arrow[r, "\pi^V"] & \operatorname{Coker} \varphi^V \arrow[r] & 0.
\end{tikzcd}$$

By applying the functor $\operatorname{Hom}_A(M, -)$ to it yields the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(M, M \otimes_B U_2) \arrow[r, "(\varphi^U)_*"] \arrow[d, "(id\otimes g_2)_*"'] & \operatorname{Hom}_A(M, U_1) \arrow[r, "(\pi^U)_*"] \arrow[d, "(g_1)_*"'] & \operatorname{Hom}_A(M, \operatorname{Coker} \varphi^U) \arrow[r] \arrow[d, dashed, "(\pi_g)_*"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(M, M \otimes_B V_2) \arrow[r, "(\varphi^V)_*"] & \operatorname{Hom}_A(M, V_1) \arrow[r, "(\pi^V)_*"] & \operatorname{Hom}_A(M, \operatorname{Coker} \varphi^V) \arrow[r] & 0.
\end{tikzcd}$$

In view of the right square of the above commutative diagram, we now define

$$\mathbf{C}_{\mathcal{A},\mathcal{B}}\left(\begin{array}{c} g_1 \\ g_2 \end{array}\right) := \left(\begin{array}{c} \pi_g \\ (g_1)_* \end{array}\right).$$

Remark 3.5. (1) Let A be an arbitrary ring. Setting $\mathcal{A} = \mathcal{B} := A\text{-}\mathrm{Mod}$ and $M := {}_AA_A$, the functor $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ (resp. $\mathbf{C}_{\mathcal{A},\mathcal{B}}$) coincides with the kernel (resp. cokernel) functor by Ringel–Schmidmeier [25], which is mentioned in Remark 3.2.

(2) The functor $\mathbf{K}_{\mathcal{A},\mathcal{B}} : \mathcal{E}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{M}(\mathcal{A}, M, \mathcal{B})$ is well-defined provided that M is left $\mathcal{A}$-compatible. Meanwhile, the functor $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is well-defined provided that M is right $\mathcal{B}$-compatible. If $\mathcal{A} = A\text{-}\mathrm{Mod}$, then we write $\mathbf{K}_{A,\mathcal{B}}$ (resp. $\mathbf{C}_{A,\mathcal{B}}$) for $\mathbf{K}_{A\text{-}\mathrm{Mod},\mathcal{B}}$ (resp. $\mathbf{C}_{A\text{-}\mathrm{Mod},\mathcal{B}}$). Similarly, we define $\mathbf{K}_{\mathcal{A},B}$, $\mathbf{C}_{\mathcal{A},B}$, $\mathbf{K}_{A,B}$ and $\mathbf{C}_{A,B}$.

Proposition 3.6. (1) *Let M be left $\mathcal{A}$-compatible. Then $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ carries any exact sequence*

$$0 \to \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\widetilde{\varphi^X}} \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\widetilde{\varphi^Y}} \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{\widetilde{\varphi^Z}} \to 0$$

in $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$ to an exact sequence

$$\mathbf{K}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\widetilde{\varphi^X}} \xrightarrow{\mathbf{K}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \mathbf{K}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\widetilde{\varphi^Y}} \xrightarrow{\mathbf{K}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \mathbf{K}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{\widetilde{\varphi^Z}} \to 0$$

in $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$.

(2) *Let M be right $\mathcal{B}$-compatible. Then $\mathbf{C}_{\mathcal{A},\mathcal{B}}$ carries any exact sequence*

$$0 \to \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\varphi^Y} \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{\varphi^Z} \to 0$$

in $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$ to an exact sequence

$$0 \to \mathbf{C}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \xrightarrow{\mathbf{C}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \mathbf{C}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\varphi^Y} \xrightarrow{\mathbf{C}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \mathbf{C}_{\mathcal{A},\mathcal{B}}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{\varphi^Z}$$

in $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$.

Proof. We only prove (2), the proof of (1) is dual.

Let

$$0 \to \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\varphi^Y} \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{\varphi^Z} \to 0$$

be an exact sequence in $\mathcal{M}(\mathcal{A}, M, \mathcal{B})$. Then we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d, dashed] & \\
0 \arrow[r] & M \otimes_B X_2 \arrow[r, "\varphi^X"] \arrow[d, "id\otimes f_2"'] & X_1 \arrow[r, "\pi^X"] \arrow[d, "f_1"'] & \operatorname{Coker} \varphi^X \arrow[r] \arrow[d, "\pi_f"'] & 0 \\
0 \arrow[r] & M \otimes_B Y_2 \arrow[r, "\varphi^Y"] \arrow[d, "id\otimes g_2"'] & Y_1 \arrow[r, "\pi^Y"] \arrow[d, "g_1"'] & \operatorname{Coker} \varphi^Y \arrow[r] \arrow[d, "\pi_g"'] & 0 \\
0 \arrow[r] & M \otimes_B Z_2 \arrow[r, "\varphi^Z"] \arrow[d] & Z_1 \arrow[r, "\pi^Z"] \arrow[d] & \operatorname{Coker} \varphi^Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

By the snake lemma, we have that π_f is monic. Since $\operatorname{Ext}^1_A(M, M \otimes_B \mathcal{B}) = 0$, applying the functor $\operatorname{Hom}_A(M, -)$ to the above diagram yields the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_A(M, M \otimes_B X_2) \arrow[r, "(\varphi^X)_*"] \arrow[d, "(id\otimes f_2)_*"'] & \operatorname{Hom}_A(M, X_1) \arrow[r, "(\pi^X)_*"] \arrow[d, "(f_1)_*"'] & \operatorname{Hom}_A(M, \operatorname{Coker} \varphi^X) \arrow[r] \arrow[d, "(\pi_f)_*"'] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(M, M \otimes_B Y_2) \arrow[r, "(\varphi^Y)_*"] \arrow[d, "(id\otimes g_2)_*"'] & \operatorname{Hom}_A(M, Y_1) \arrow[r, "(\pi^Y)_*"] \arrow[d, "(g_1)_*"'] & \operatorname{Hom}_A(M, \operatorname{Coker} \varphi^Y) \arrow[r] \arrow[d, "(\pi_g)_*"'] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(M, M \otimes_B Z_2) \arrow[r, "(\varphi^Z)_*"] & \operatorname{Hom}_A(M, Z_1) \arrow[r, "(\pi^Z)_*"] & \operatorname{Hom}_A(M, \operatorname{Coker} \varphi^Z) \arrow[r] & 0.
\end{tikzcd}$$

From the above two commutative diagrams, we deduce that

$$0 \to \begin{pmatrix} \operatorname{Coker} \varphi^X \\ \operatorname{Hom}_A(M, X_1) \end{pmatrix}_{(\pi^X)_*} \xrightarrow{\begin{pmatrix} \pi_f \\ (f_1)_* \end{pmatrix}} \begin{pmatrix} \operatorname{Coker} \varphi^Y \\ \operatorname{Hom}_A(M, Y_1) \end{pmatrix}_{(\pi^Y)_*} \xrightarrow{\begin{pmatrix} \pi_g \\ (g_1)_* \end{pmatrix}} \begin{pmatrix} \operatorname{Coker} \varphi^Z \\ \operatorname{Hom}_A(M, Z_1) \end{pmatrix}_{(\pi^Z)_*}$$

is exact in $\mathcal{E}(\mathcal{A}, M, \mathcal{B})$. It follows that

$$0 \to \mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \xrightarrow{\mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} \mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\varphi^Y} \xrightarrow{\mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} \mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{\varphi^Z}$$

is exact. □

Definition 3.7. A bimodule ${}_AM_B$ is called *strongly left $\mathcal{A}$-compatible* if the condition (L1) in Definition 3.3(1) is satisfied, and any object $X \in \mathcal{A}$ is ν-reflexive. Similarly, ${}_AM_B$ is called *strongly right $\mathcal{B}$-compatible* if the condition (R1) in Definition 3.3(2) is satisfied, and any object $Y \in \mathcal{B}$ is μ-reflexive. If M is both strongly left $\mathcal{A}$- and strongly right $\mathcal{B}$-compatible, then we call M *strongly $\mathcal{A}$-$\mathcal{B}$-compatible.*

It is is easy to see that a strongly left $\mathcal{A}$-compatible bimodule is left $\mathcal{A}$-compatible and that a strongly right $\mathcal{B}$-compatible bimodule is right $\mathcal{B}$-compatible. However, neither of the converses does hold true in general. For example, let $\mathcal{A} = \mathcal{B} = A\text{-}\mathrm{Mod}$ for a ring A, and let $M = A^2$ be a free A-module of rank 2. It is easy to see that ${}_AM_A$ is both left $\mathcal{A}$- and right $\mathcal{B}$-compatible. Whereas, for any module $X \in A\text{-}\mathrm{Mod}$, we have $M \otimes_A \mathrm{Hom}_A(M, X) \cong X^4 \not\cong X$. Consequently, X is not ν-reflexive, and hence M is not strongly left $\mathcal{A}$-compatible. Similarly, we have that M is not strongly right $\mathcal{B}$-compatible either.

Theorem 3.8. *If ${}_AM_B$ is strongly $\mathcal{A}$-$\mathcal{B}$-compatible, then $(\mathbf{K}_{\mathcal{A},\mathcal{B}}, \mathbf{C}_{\mathcal{A},\mathcal{B}})$ is an adjoint pair.*

Proof. Let $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \in \mathcal{M}(\mathcal{A}, M, \mathcal{B})$ and $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\widetilde{\varphi^Y}} \in \mathcal{E}(\mathcal{A}, M, \mathcal{B})$. Note that

$$\mathrm{Hom}_\Lambda(\mathbf{K}_{\mathcal{A},\mathcal{B}}\ \mathbf{Y}, \mathbf{X}) = \mathrm{Hom}_\Lambda(\begin{pmatrix} M \otimes_B Y_2 \\ \mathrm{Ker}\,\widetilde{\varphi^Y} \end{pmatrix}_{id\otimes\iota^Y}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X})$$

and

$$\mathrm{Hom}_\Lambda(\mathbf{Y}, \mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X}) = \mathrm{Hom}_\Lambda(\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\widetilde{\varphi^Y}}, \begin{pmatrix} \mathrm{Coker}\,\varphi^X \\ \mathrm{Hom}_A(M, X_1) \end{pmatrix}_{(\pi^X)_*}).$$

For any morphism $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} : \begin{pmatrix} M \otimes_B Y_2 \\ \mathrm{Ker}\,\widetilde{\varphi^Y} \end{pmatrix}_{id\otimes\iota^Y} \to \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$, it corresponds to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_B \mathrm{Ker}\,\widetilde{\varphi^Y} \arrow[r, "id\otimes\iota^Y"] \arrow[d, "id\otimes\alpha_2"'] & M \otimes_B Y_2 \arrow[r, "id\otimes\widetilde{\varphi^Y}"] \arrow[d, "\alpha_1"'] & M \otimes_B \mathrm{Hom}_A(M, Y_1) \arrow[r] \arrow[d, "\pi_\alpha"] & 0 \\
0 \arrow[r] & M \otimes_B X_2 \arrow[r, "\varphi^X"] & X_1 \arrow[r, "\pi^X"] & \mathrm{Coker}\,\varphi^X \arrow[r] & 0.
\end{tikzcd}$$

For any $X \in A\text{-}\mathrm{Mod}$ and $Y \in B\text{-}\mathrm{Mod}$, since

$$\theta = \theta_{Y,X} : \mathrm{Hom}_A(M \otimes_B Y, X) \to \mathrm{Hom}_B(Y, \mathrm{Hom}_A(M, X))$$

is the adjunction isomorphism, we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,\widetilde{\varphi^Y} \arrow[r, "\iota^Y"] \arrow[d, "\theta(id\otimes\alpha_2)"'] & Y_2 \arrow[r, "\widetilde{\varphi^Y}"] \arrow[d, "\theta(\alpha_1)"'] & \mathrm{Hom}_A(M, Y_1) \arrow[r] \arrow[d, "\theta(\pi_\alpha)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M, M \otimes_B X_2) \arrow[r, "(\varphi^X)_*"] & \mathrm{Hom}_A(M, X_1) \arrow[r, "(\pi^X)_*"] & \mathrm{Hom}_A(M, \mathrm{Coker}\,\varphi^X) \arrow[r] & 0.
\end{tikzcd}$$

Since M is strongly left $\mathcal{A}$-compatible, we have that Y_1 is ν-reflexive. Hence, the natural evaluation homomorphism $\nu_{Y_1} : M \otimes_B \mathrm{Hom}_A(M, Y_1) \to Y_1$ is an isomorphism. Let $\widehat{\alpha}$ be the composition of the morphisms

$$Y_1 \xrightarrow{\nu_{Y_1}^{-1}} M \otimes_B \mathrm{Hom}_A(M, Y_1) \xrightarrow{\pi_\alpha} \mathrm{Coker}\,\varphi^X.$$

We infer that $\theta(\pi_\alpha) = (\widehat{\alpha})_*$. In fact, for any $g \in \mathrm{Hom}_A(M, Y_1)$ and $m \in M$, we have

$$(\widehat{\alpha})_*(g)(m) = \widehat{\alpha}g(m) = \pi_\alpha \circ \nu_{Y_1}^{-1} \circ g(m) = \pi_\alpha \circ \nu_{Y_1}^{-1} \circ \nu_{Y_1}(m \otimes g) = \pi_\alpha(m \otimes g) = \theta(\pi_\alpha)(g)(m).$$

Define

$$\Phi_{\mathbf{Y},\mathbf{X}} : \mathrm{Hom}_\Lambda(\mathbf{K}_{\mathcal{A},\mathcal{B}}\ \mathbf{Y}, \mathbf{X}) \to \mathrm{Hom}_\Lambda(\mathbf{Y}, \mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X}) \text{ via } \Phi_{\mathbf{Y},\mathbf{X}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \theta(\alpha_1) \end{pmatrix}.$$

By the foregoing argument, it is well-defined. This defines the homomorphism $\Phi_{\mathbf{Y},\mathbf{X}}$. It is easy to check that $\Phi_{\mathbf{Y},\mathbf{X}}$ is functorial on $\mathbf{Y}$ and $\mathbf{X}$.

Conversely, for any morphism $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} : \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{\widetilde{\varphi^Y}} \to \begin{pmatrix} \mathrm{Coker}\, \varphi^X \\ \mathrm{Hom}_A(M, X_1) \end{pmatrix}_{(\pi^X)_*}$, it corresponds to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, \widetilde{\varphi^Y} \arrow[r, "\iota^Y"] \arrow[d, "\iota(\beta)"'] & Y_2 \arrow[r, "\widetilde{\varphi^Y}"] \arrow[d, "\beta_2"'] & \mathrm{Hom}_A(M, Y_1) \arrow[r] \arrow[d, "(\beta_1)_*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(M, M \otimes_B X_2) \arrow[r, "(\varphi^X)_*"] & \mathrm{Hom}_A(M, X_1) \arrow[r, "(\pi^X)_*"] & \mathrm{Hom}_A(M, \mathrm{Coker}\, \varphi^X) \arrow[r] & 0.
\end{tikzcd}$$

Then we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_B \mathrm{Ker}\, \widetilde{\varphi^Y} \arrow[r, "id\otimes\iota^Y"] \arrow[d, "\theta^{-1}(\iota(\beta))"'] & M \otimes_B Y_2 \arrow[r, "id\otimes\widetilde{\varphi^Y}"] \arrow[d, "\theta^{-1}(\beta_2)"'] & M \otimes_B \mathrm{Hom}_A(M, Y_1) \arrow[r] \arrow[d, "\theta^{-1}((\beta_1)_*)"] & 0 \\
0 \arrow[r] & M \otimes_B X_2 \arrow[r, "\varphi^X"] & X_1 \arrow[r, "\pi^X"] & \mathrm{Coker}\, \varphi^X \arrow[r] & 0.
\end{tikzcd}$$

In view that M is strongly right $\mathcal{B}$-compatible, we have that X_2 is μ-reflexive. Then the natural evaluation homomorphism $\mu_{X_2} : X_2 \to \mathrm{Hom}_A(M, M \otimes_B X_2)$ is an isomorphism. Let $\widehat{\beta}$ be the composition of the morphisms

$$\mathrm{Ker}\, \widetilde{\varphi^Y} \xrightarrow{\iota(\beta)} \mathrm{Hom}_A(M, M \otimes_B X_2) \xrightarrow{\mu_{X_2}^{-1}} X_2.$$

Similarly, we have $id \otimes \widehat{\beta} = \theta^{-1}(\iota(\beta))$. Define

$$\Psi_{\mathbf{Y},\mathbf{X}} : \mathrm{Hom}_\Lambda(\mathbf{Y}, \mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X}) \to \mathrm{Hom}_\Lambda(\mathbf{K}_{\mathcal{A},\mathcal{B}}\ \mathbf{Y}, \mathbf{X}) \text{ via } \Psi_{\mathbf{Y},\mathbf{X}} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \theta^{-1}(\beta_2) \\ \widehat{\beta} \end{pmatrix}.$$

It is not hard to see that it is well-defined. This defines the homomorphism $\Psi_{\mathbf{Y},\mathbf{X}}$ which is functorial on $\mathbf{Y}$ and $\mathbf{X}$.

Furthermore, it is verified directly that

$$\Psi_{\mathbf{Y},\mathbf{X}} \circ \Phi_{\mathbf{Y},\mathbf{X}} = \mathrm{Id}_{\mathrm{Hom}_\Lambda(\mathbf{K}_{\mathcal{A},\mathcal{B}}\ \mathbf{Y},\mathbf{X})} \text{ and } \Phi_{\mathbf{Y},\mathbf{X}} \circ \Psi_{\mathbf{Y},\mathbf{X}} = \mathrm{Id}_{\mathrm{Hom}_\Lambda(\mathbf{Y},\mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X})}.$$

To sum up, $\Phi_{\mathbf{Y},\mathbf{X}}$ is a functorial isomorphism $\mathrm{Hom}_\Lambda(\mathbf{K}_{\mathcal{A},\mathcal{B}}\ \mathbf{Y}, \mathbf{X}) \cong \mathrm{Hom}_\Lambda(\mathbf{Y}, \mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X})$ with inverse $\Psi_{\mathbf{Y},\mathbf{X}}$. Therefore, $(\mathbf{K}_{\mathcal{A},\mathcal{B}}, \mathbf{C}_{\mathcal{A},\mathcal{B}})$ is an adjoint pair. □

Inspired by [35, Definition 5.1], we introduce the following notion.

Definition 3.9. A functor $F : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is called an *RSS equivalence with respect to the subcategories* $\mathcal{A}$ *and* $\mathcal{B}$ if F is an equivalence such that

$$F \begin{pmatrix} X \\ 0 \end{pmatrix} \cong \begin{pmatrix} X \\ \mathrm{Hom}_A(M, X) \end{pmatrix}_{\widetilde{id}} \text{ and } F \begin{pmatrix} M \otimes_B Y \\ Y \end{pmatrix}_{id} \cong \begin{pmatrix} 0 \\ Y \end{pmatrix}$$

for any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

Note that an RSS equivalence with respect to $\mathcal{A} = A\text{-}\mathrm{Mod}$ and $\mathcal{B} = B\text{-}\mathrm{Mod}$ is exactly the usual RSS equivalence [35].

Theorem 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be closed under extensions. Assume that M is $\mathcal{A}$-$\mathcal{B}$-compatible. Then*

$$\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$$

is an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$ with $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ as its quasi-inverse, if and only if there is an equivalence of categories:

$$\mathcal{A} \underset{M\otimes_B -}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \mathcal{B}.$$

In this case, the bimodule M is strongly $\mathcal{A}$-$\mathcal{B}$-compatible, hence by Theorem 3.8, the kernel and cokernel functors $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ and $\mathbf{C}_{\mathcal{A},\mathcal{B}}$ form an adjoint pair.

Proof. We first prove the sufficiency. Assume that there is an equivalence of categories:

$$\mathcal{A} \underset{M\otimes_B -}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \mathcal{B}.$$

Then the natural evaluation homomorphisms $\nu_X : M \otimes_B \mathrm{Hom}_A(M, X) \to X$ and $\mu_Y : Y \to \mathrm{Hom}_A(M, M \otimes_B Y)$ are isomorphisms for any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

Let $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} \in \mathcal{M}(\mathcal{A}, M, \mathcal{B})$. We have $\mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X} = \begin{pmatrix} \mathrm{Coker}\,\varphi^X \\ \mathrm{Hom}_A(M, X_1) \end{pmatrix}_{(\pi^X)_*}$, where $(\pi^X)_*$ is given by the exact sequence:

$$0 \to \mathrm{Hom}_A(M, M \otimes_B X_2) \xrightarrow{(\varphi^X)_*} \mathrm{Hom}_A(M, X_1) \xrightarrow{(\pi^X)_*} \mathrm{Hom}_A(M, \mathrm{Coker}\,\varphi^X) \to 0.$$

Since $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ induce quasi-inverse equivalences between $\mathcal{A}$ and $\mathcal{B}$, we have $\mathrm{Hom}_A(M, \mathrm{Coker}\,\varphi^X) \in \mathcal{B}$. Applying the functor $M \otimes_B -$ to the above exact sequence yields the following exact sequence:

$$0 \to M\otimes_B\mathrm{Hom}_A(M, M\otimes_B X_2) \xrightarrow{id\otimes(\varphi^X)_*} M\otimes_B\mathrm{Hom}_A(M, X_1) \xrightarrow{id\otimes(\pi^X)_*} M\otimes_B\mathrm{Hom}_A(M, \mathrm{Coker}\,\varphi^X) \to 0.$$

Then $\mathbf{K}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} \mathrm{Coker}\,\varphi^X \\ \mathrm{Hom}_A(M, X_1) \end{pmatrix}_{(\pi^X)_*} = \begin{pmatrix} M \otimes_B \mathrm{Hom}_A(M, X_1) \\ \mathrm{Hom}_A(M, M \otimes_B X_2) \end{pmatrix}_{id\otimes(\varphi^X)_*}$. Consider the following commutative diagram:

$$\begin{CD}
M \otimes_B \mathrm{Hom}_A(M, M \otimes_B X_2) @>{id\otimes(\varphi^X)_*}>> M \otimes_B \mathrm{Hom}_A(M, X_1) \\
@V{\nu_{M\otimes_B X_2}}VV @VV{\nu_{X_1}}V \\
M \otimes_B X_2 @>{\varphi^X}>> X_1.
\end{CD}$$

Again since $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ are quasi-inverse, we have $M \otimes_B X_2 \in \mathcal{A}$. Notice that $\mathcal{A}$ is closed under extensions, it is trivial to see $X_1 \in \mathcal{A}$. Hence, $\nu_{M\otimes_B X_2}$ and ν_{X_1} are isomorphisms. Thus

$$\begin{pmatrix} M \otimes_B \mathrm{Hom}_A(M, X_1) \\ \mathrm{Hom}_A(M, M \otimes_B X_2) \end{pmatrix}_{id\otimes(\varphi^X)_*} \cong \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X} = \mathbf{X}.$$

Consequently, we have $\mathbf{K}_{\mathcal{A},\mathcal{B}} \cdot \mathbf{C}_{\mathcal{A},\mathcal{B}}\ \mathbf{X} \cong \mathbf{X}$. This implies that there is a natural transformation $\mathbf{K}_{\mathcal{A},\mathcal{B}} \cdot \mathbf{C}_{\mathcal{A},\mathcal{B}} \simeq \mathrm{Id}_{\mathcal{M}(\mathcal{A},M,\mathcal{B})}$. Similarly, we also have a natural transformation $\mathbf{C}_{\mathcal{A},\mathcal{B}} \cdot \mathbf{K}_{\mathcal{A},\mathcal{B}} \simeq \mathrm{Id}_{\mathcal{E}(\mathcal{A},M,\mathcal{B})}$. Thus $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is an equivalence of categories with $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ as its quasi-inverse.

Now for any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, we have

$$\mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X \\ \mathrm{Hom}_A(M, X) \end{pmatrix}_{\widetilde{id}}$$

and

$$\mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} M \otimes_B Y \\ Y \end{pmatrix}_{id} = \begin{pmatrix} 0 \\ \mathrm{Hom}_A(M, M \otimes_B Y) \end{pmatrix} \cong \begin{pmatrix} 0 \\ Y \end{pmatrix}.$$

Therefore, $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$.

In the following, we prove the necessity. Assume that $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$, and $\mathbf{K}_{\mathcal{A},\mathcal{B}}$ is its quasi-inverse. For any $X \in \mathcal{A}$, we have $\begin{pmatrix} X \\ 0 \end{pmatrix} \in \mathcal{M}(\mathcal{A}, M, \mathcal{B})$. Hence we get a functorial isomorphism:

$$\mathbf{K}_{\mathcal{A},\mathcal{B}} \cdot \mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} X \\ 0 \end{pmatrix} \cong \begin{pmatrix} X \\ 0 \end{pmatrix}.$$

Note that $\mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X \\ \mathrm{Hom}_A(M, X) \end{pmatrix}_{\widetilde{id}}$ and $\mathbf{K}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} X \\ \mathrm{Hom}_A(M, X) \end{pmatrix}_{\widetilde{id}} = \begin{pmatrix} M \otimes_B \mathrm{Hom}_A(M, X) \\ 0 \end{pmatrix}$. Then we have a functorial isomorphism $\begin{pmatrix} M \otimes_B \mathrm{Hom}_A(M, X) \\ 0 \end{pmatrix} \cong \begin{pmatrix} X \\ 0 \end{pmatrix}$. Thus we get a functorial isomorphism $M \otimes_B \mathrm{Hom}_A(M, X) \cong X$, this induces a natural transformation $(M \otimes_B -) \circ \mathrm{Hom}_A(M, -) \simeq \mathrm{Id}_{\mathcal{A}}$.

Analogously, for any $Y \in \mathcal{B}$, we have $\begin{pmatrix} 0 \\ Y \end{pmatrix} \in \mathcal{E}(\mathcal{A}, M, \mathcal{B})$. Hence we get a functorial isomorphism:

$$\mathbf{C}_{\mathcal{A},\mathcal{B}} \cdot \mathbf{K}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} 0 \\ Y \end{pmatrix} \cong \begin{pmatrix} 0 \\ Y \end{pmatrix}.$$

Note that $\mathbf{K}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} 0 \\ Y \end{pmatrix} = \begin{pmatrix} M \otimes_B Y \\ Y \end{pmatrix}_{id}$ and $\mathbf{C}_{\mathcal{A},\mathcal{B}} \begin{pmatrix} M \otimes_B Y \\ Y \end{pmatrix}_{id} = \begin{pmatrix} 0 \\ \mathrm{Hom}_A(M, M \otimes_B Y) \end{pmatrix}$. Thus there is a functorial isomorphism $\begin{pmatrix} 0 \\ Y \end{pmatrix} \cong \begin{pmatrix} 0 \\ \mathrm{Hom}_A(M, M \otimes_B Y) \end{pmatrix}$, and then we get a functorial isomorphism $Y \cong \mathrm{Hom}_A(M, M \otimes_B Y)$. This induces a natural transformation $\mathrm{Hom}_A(M, -) \circ (M \otimes_B -) \simeq \mathrm{Id}_{\mathcal{B}}$.

To sum up, there is an equivalence of categories

$$\mathcal{A} \underset{M \otimes_B -}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \mathcal{B}.$$

This completes the proof. □

Remark 3.11. Let A and B be Artin algebras and D the standard Matlis duality. Recall from [35, Section 1.6] that a bimodule ${}_AM_B$ is *exchangeable* if both ${}_AM$ and M_B are finitely generated and projective, and there is an A-B-bimodule isomorphism $D({}_AA_A) \otimes_A M \cong M \otimes_B D({}_BB_B)$. If ${}_AM_B$

is exchangeable, it is easy to see that M is $\mathcal{A}$-$\mathcal{B}$-compatible for $\mathcal{A} = A\text{-}\mathrm{mod}$ and $\mathcal{B} = B\text{-}\mathrm{mod}$. By Remark 3.5(2), the functor

$$\mathbf{C}_{A\text{-}\mathrm{mod},B\text{-}\mathrm{mod}} : \mathcal{M}(A\text{-}\mathrm{mod}, M, B\text{-}\mathrm{mod}) \to \mathcal{E}(A\text{-}\mathrm{mod}, M, B\text{-}\mathrm{mod})$$

is well-defined. For simplicity, we denote it by $\mathbf{c}_{A,B}$.

On the other hand, by [35, Theorem 1.4], the functor

$$D\,\mathrm{Hom}_\Lambda(-,T) : \mathcal{M}(A\text{-}\mathrm{mod}, M, B\text{-}\mathrm{mod}) \to \mathcal{E}(A\text{-}\mathrm{mod}, M, B\text{-}\mathrm{mod})$$

is an RSS equivalence, where $T = \begin{pmatrix} D(A) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M\otimes_B D(B) \\ D(B) \end{pmatrix}_{id}$ has a Λ-Λ-bimodule structure such that both ${}_\Lambda T$ and T_Λ are cotilting modules in the sense of [3, p.242].

Let $Y \in B\text{-}\mathrm{mod}$. Since $D\,\mathrm{Hom}_\Lambda(-,T)$ is an RSS equivalence, we have

$$D\,\mathrm{Hom}_\Lambda(-,T) \begin{pmatrix} M\otimes_B Y \\ Y \end{pmatrix}_{id} \cong \begin{pmatrix} 0 \\ Y \end{pmatrix}.$$

However, we have $\mathbf{c}_{A,B} \begin{pmatrix} M\otimes_B Y \\ Y \end{pmatrix}_{id} = \begin{pmatrix} 0 \\ \mathrm{Hom}_A(M, M\otimes_B Y) \end{pmatrix}$. Notice that

$$\begin{pmatrix} 0 \\ \mathrm{Hom}_A(M, M\otimes_B Y) \end{pmatrix} \ncong \begin{pmatrix} 0 \\ Y \end{pmatrix}$$

in general, thus the functor $\mathbf{c}_{A,B}$ differs from $D\,\mathrm{Hom}_\Lambda(-,T)$, even though ${}_A M_B$ is exchangeable.

4 Applications

In this section, we apply Theorem 3.10 to establish certain RSS equivalences between various monomorphism and epimorphism subcategories. Firstly, we revisit the classical Ringel–Schmidmeier's equivalence (see [25, Lemma 1.2]).

Proposition 4.1. (Compare [25, Lemma 1.2]) *Let A be an arbitrary ring. For any subcategory $\mathcal{A}$ of $A\text{-}\mathrm{Mod}$ that is closed under extensions,*

$$\mathbf{C}_{\mathcal{A},\mathcal{A}} : \mathcal{M}(\mathcal{A}, A, \mathcal{A}) \to \mathcal{E}(\mathcal{A}, A, \mathcal{A})$$

is an RSS equivalence with respect to the subcategory $\mathcal{A}$.

Proof. For any subcategory $\mathcal{A}$ of $A\text{-}\mathrm{Mod}$, it is easy to see that A is $\mathcal{A}$-$\mathcal{A}$-compatible. Since

$$\mathcal{A} \underset{A\otimes_A -}{\overset{\mathrm{Hom}_A(A,-)}{\rightleftarrows}} \mathcal{A}$$

is an equivalence of categories, it follows from Theorem 3.10 that $\mathbf{C}_{\mathcal{A},\mathcal{A}} : \mathcal{M}(\mathcal{A}, A, \mathcal{A}) \to \mathcal{E}(\mathcal{A}, A, \mathcal{A})$ is an RSS equivalence with respect to the subcategory $\mathcal{A}$. $\square$

By Proposition 4.1, it is easy to see that $\mathbf{C}_{A,A} : \mathcal{M}(A, A, A) \to \mathcal{E}(A, A, A)$ is an RSS equivalence. This recovers the classical Ringel–Schmidmeier's equivalence.

In the following, let A be an arbitrary ring and $M \in A\text{-}\mathrm{Mod}$ with $B = \mathrm{End}({}_A M)$. We will study RSS equivalences with respect to certain subcategories of $A\text{-}\mathrm{Mod}$ and $B\text{-}\mathrm{Mod}$.

Proposition 4.2. *If ${}_AM$ is a progenerator, then*

$$\mathbf{C}_{A,B} : \mathcal{M}(A\text{-}\mathrm{Mod}, M, B\text{-}\mathrm{Mod}) \to \mathcal{E}(A\text{-}\mathrm{Mod}, M, B\text{-}\mathrm{Mod})$$

is an RSS equivalence.

Proof. As illustrated in Example 3.4(6), since ${}_AM$ is a progenerator, we have that M is $\mathcal{A}$-$\mathcal{B}$-compatible for $\mathcal{A} = A\text{-}\mathrm{Mod}$ and $\mathcal{B} = B\text{-}\mathrm{Mod}$. What's more, by the Morita theorem, there is an equivalence of categories

$$A\text{-}\mathrm{Mod} \underset{M\otimes_B -}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} B\text{-}\mathrm{Mod}\,.$$

It follows from Theorem 3.10 that $\mathbf{C}_{A,B} : \mathcal{M}(A\text{-}\mathrm{Mod}, M, B\text{-}\mathrm{Mod}) \to \mathcal{E}(A\text{-}\mathrm{Mod}, M, B\text{-}\mathrm{Mod})$ is an RSS equivalence. □

The following theorem lifts the Hom-tensor equivalence associated with a tilting module in Theorem 2.2 to an RSS equivalence between the corresponding bimodule monomorphism and epimorphism subcategories.

Theorem 4.3. *Let ${}_AM$ be a tilting module. Assume that $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion pairs induced by M in $A\text{-}\mathrm{Mod}$ and $B\text{-}\mathrm{Mod}$, respectively. Then $\mathbf{C}_{\mathcal{T},\mathcal{Y}} : \mathcal{M}(\mathcal{T}, M, \mathcal{Y}) \to \mathcal{E}(\mathcal{T}, M, \mathcal{Y})$ is an RSS equivalence with respect to the subcategories $\mathcal{T}$ and $\mathcal{Y}$.*

Proof. Firstly, it is easy to check that $\mathcal{T}$ and $\mathcal{Y}$ are closed under extensions. Because ${}_AM$ is a tilting module, by Theorem 2.2, M is $\mathcal{T}$-$\mathcal{Y}$-compatible. Moreover, by Theorem 2.2 again, the functors $\mathrm{Hom}_A(M, -)$ and $M \otimes_B -$ induce quasi-inverse equivalences between $\mathcal{T}$ and $\mathcal{Y}$. It follows from Theorem 3.10 that $\mathbf{C}_{\mathcal{T},\mathcal{Y}} : \mathcal{M}(\mathcal{T}, M, \mathcal{Y}) \to \mathcal{E}(\mathcal{T}, M, \mathcal{Y})$ is an RSS equivalence with respect to the subcategories $\mathcal{T}$ and $\mathcal{Y}$. □

In the rest of this section, assume that A and B are arbitrary rings and ${}_AM_B$ is a semidualizing bimodule. Recall from [17] that a left A-module is *M-projective* (resp. *M-flat*) if it has the form $M \otimes_B P$ for some projective (resp. flat) B-module ${}_BP$. A left B-module is *M-injective* if it has the form $\mathrm{Hom}_A(M, I)$ for some injective A-module ${}_AI$. We use $\mathcal{P}_M(A)$ (resp. $\mathcal{F}_M(A)$) to denote the subcategory of $A\text{-}\mathrm{Mod}$ consisting of M-projective (resp. M-flat) modules, and use $\mathcal{I}_M(B)$ to denote the subcategory of $B\text{-}\mathrm{Mod}$ consisting of M-injective modules.

Finally, we establish RSS equivalences associated to the Foxby equivalence in the sense of [17].

Proposition 4.4. *Let A and B be arbitrary rings such that ${}_AM_B$ is a semidualizing bimodule. Consider the following conditions:*

(1) $\mathcal{A} = \mathcal{B}_M(A)$ *and* $\mathcal{B} = \mathcal{A}_M(B)$*;*

(2) $\mathcal{A} = \mathcal{F}_M(A)$ *and* $\mathcal{B} = B\text{-}\mathrm{Flat}$*;*

(3) $\mathcal{A} = \mathcal{P}_M(A)$ *and* $\mathcal{B} = B\text{-}\mathrm{Proj}$*;*

(4) $\mathcal{A} = A\text{-}\mathrm{Inj}$ *and* $\mathcal{B} = \mathcal{I}_M(B)$.

If any of these conditions holds, then $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$.

Proof. Set $\mathcal{A} := \mathcal{B}_M(A)$ and $\mathcal{B} := \mathcal{A}_M(B)$. From [17, Theorem 6.2], $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions. Since ${}_AM_B$ is a semidualizing bimodule, it is trivial that M is $\mathcal{A}$-$\mathcal{B}$-compatible. It follows from Lemma 2.6 that $\mathrm{Hom}_A(M,-)$ and $M \otimes_B -$ induce quasi-inverse equivalences between $\mathcal{B}_M(A)$ and $\mathcal{A}_M(B)$. Hence, from Theorem 3.10, we get that $\mathbf{C}_{\mathcal{A},\mathcal{B}} : \mathcal{M}(\mathcal{A}, M, \mathcal{B}) \to \mathcal{E}(\mathcal{A}, M, \mathcal{B})$ is an RSS equivalence with respect to the subcategories $\mathcal{A} = \mathcal{B}_M(A)$ and $\mathcal{B} = \mathcal{A}_M(B)$.

Now suppose that one of the conditions (2)–(4) holds. It follows from [17, Proposition 5.2] that $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions. By [17, Theorem 1], we have that M is $\mathcal{A}$-$\mathcal{B}$-compatible. Furthermore, there is an equivalence of categories

$$\mathcal{A} \underset{M\otimes_B -}{\overset{\mathrm{Hom}_A(M,-)}{\rightleftarrows}} \mathcal{B}.$$

It follows from Theorem 3.10 that $\mathbf{C}_{\mathcal{A},\mathcal{B}}$ is an RSS equivalence with respect to the subcategories $\mathcal{A}$ and $\mathcal{B}$. $\square$

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Data availability

No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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