NORMED MODULES AND THE CATEGORIZATION OF LEBESGUE INTEGRATION

YU-ZHE LIU, SHENGDA LIU, ZHAOYONG HUANG*, AND PANYUE ZHOU

Abstract. We explore the assignment of norms to Λ-modules over a finite-dimensional algebra Λ, resulting in the establishment of normed Λ-modules. Our primary contribution lies in constructing a new category $\mathscr{N}or^p$ related to normed modules along with its full subcategory $\mathscr{A}^p$. By examining the objects and morphisms in these categories, we establish a framework for understanding the categorization of Lebesgue integration.

Contents

MSC2020: 16G10; 46B99; 46M40.

Keywords: Categorization; Lebesgue integration; Normed modules, Finite-dimensional algebras; Banach spaces.

*Corresponding author.

1. Introduction

Lebesgue integration, introduced by Henri Lebesgue [15], is fundamentally pivotal in the field of mathematical analysis. The process of understanding the Lebesgue's integrability and its application to the real number line typically involves a series of methodical and incremental steps. This journey begins with defining measurable sets and null sets, followed by an exploration of convergence in measure. It then advances through the concepts of step functions and simple functions, along with their convergence sequences, culminating in the meticulous construction of spaces for integrable functions and the validation of consistent integration methods. While this path is comprehensive, it serves as an elaborate gateway to fully grasp the essence of Lebesgue integration, see [7,12] and so on. Indeed, building upon the foundational methods for defining integrals previously mentioned, our exploration extends well beyond traditional boundaries. The versatility and adaptability of these principles lay the groundwork for deriving more specialized forms of integration, designed to address the complex requirements of various fields. This notably includes the development of the Bochner integral [6], which is particularly effective in handling vector-valued functions and proves invaluable in the realm of functional analysis. In a similar vein, this framework also leads to the emergence of the Ito integral [14], a fundamental element in stochastic calculus that provides deep insights into the complex behavior of stochastic processes. These advancements are not merely extensions; they are crucial in bridging the theoretical concepts of integration with their practical applications across diverse domains, reflecting the dynamic interplay between theoretical constructs and their real-world implications.

As the landscape of integration theory expands, so too does the exploration into its algebraic facets, marking a significant evolution in the approach to integration. Algebraic approaches to integration can be traced back at least to Segal's work [20]. Building upon the foundational works of Escardó-Simpson [10] and Freyd [11], Leinster [16] constructed a special category $\mathscr{A}^p$, where p is a real number at least 1. In this category, objects are triples consisting of a Banach space V, an element v in V with $|v| \le 1$, and a $\Bbbk$-linear map $\delta : V \oplus_p V \to V$ that satisfies $\delta(v, v) = v$. Here, the notation "$V_1 \oplus_p V_2$" represents the direct sum of two normed spaces V_1 and V_2, where the norm is defined as $|(v_1, v_2)| = \big(\frac{1}{2}(|v_1|^p + |v_2|^p)\big)^{1/p}$. Furthermore, Leinster established three significant results as follows:

(1) $(L_p([0,1]), 1, \gamma)$ is the initial object in $\mathscr{A}^p$, where γ is a special $\Bbbk$-linear map from $L_p([0,1]) \oplus_p L_p([0,1])$ to $L_p([0,1])$ (indeed, γ is the map $\gamma_{\frac{1}{2}}$ given in Corollary 8.1);

(2) $(\mathbb{R}, 1, m)$ is an object in $\mathscr{A}^1$, where $m : \mathbb{R} \oplus_1 \mathbb{R} \to \mathbb{R}$ sends (x, y) to $\frac{1}{2}(x + y)$;

(3) there exists a unique morphism

$$H : (L_1([0,1]), 1, \gamma) \to (\mathbb{R}, 1, m)$$

in $\mathscr{A}^1$,

see [16, Theorem 2.1 and Proposition 2.2]. The map H is a $\Bbbk$-linear map from $L_1([0,1])$ to $\Bbbk$ that adheres to specific criteria enabling its interpretation as a morphism in the category $\mathscr{A}^1$. Significantly, H establishes a fundamental link between Lebesgue integration on $\mathbb{R}$ and the aforementioned category $\mathscr{A}^p$. Explicitly, for any function f in $L_1([0,1])$, the map is defined as

$$H(f) = \int_0^1 f \,\mathrm{d}\mu,$$

where μ denotes the Lebesgue measure on $\mathbb{R}$. This profound relationship illustrates Lebesgue integrability and integration are not merely abstract constructs; rather, they naturally emerge from the foundational principles of Banach spaces. Consequently, it can be logically inferred that the categorization of Lebesgue integration is inherently connected to, and can be derived from, the categorization of Banach spaces.

Some authors have been trying to characterize calculus by using the category theory, including differential algebras/categories [1,4,5,13,17,18] and integral algebras/categories [8, 9]. In this paper, we depart from the conventional trajectory and propose a novel approach. We extend the domain of Lebesgue integration beyond the real numbers to the broader framework of normed modules over normed finite-dimensional $\Bbbk$-algebras. At the core of our approach lies the aim to provide a categorical interpretation of traditional analytical methods, thus paving a novel categorical route to the underlying principles of Lebesgue integration. To establish this extended framework, we revisit pivotal results in the category theory and representation theory. These foundational elements enable us to elegantly circumvent traditional methodologies, offering a more direct and algebraically inclined understanding of integrable function spaces and the integration operator. Our exploration requires a foundational grasp of key concepts and conclusions in the category theory, representation theory, and the groundbreaking work of Leinster [16].

Firstly, we introduce functions defined on a finite-dimensional algebra $\mathit{\Lambda}$, along with the norm defined on $\mathit{\Lambda}$ and any $\mathit{\Lambda}$-module M. It is pertinent to note that all $\mathit{\Lambda}$-modules considered in this paper are left $\mathit{\Lambda}$-modules. The specifics of these structures are elaborated in Subsections 3.1 and 4.1, respectively. A pivotal motivation for us to introduce normed modules is the pursuit of an integration definition that transcends the conventional reliance on L_p spaces. This approach is rooted in the understanding that an equivalent definition of L_p spaces can emerge through the integration itself. However, as highlighted by Leinster, the notion of Lebesgue integrals is intrinsically linked to Banach spaces. Consequently, our investigation also necessitates considering the completions of normed finite-dimensional algebras and normed modules, see Subsections 3.2 and 4.2.

Secondly, for a special subset, denoted $\mathbb{I}_{\mathit{\Lambda}}$, of $\mathit{\Lambda}$, we construct the category $\mathscr{N}\!or^p$ in Subsection 5.1. Its object has the form (N, v, δ), where N is a normed $\mathit{\Lambda}$-module, v is an element in V satisfying $|v| \le \mu(\mathbb{I}_{\mathit{\Lambda}})$, and $\delta : V^{\oplus_p 2^n} \to V$ is a $\mathit{\Lambda}$-homomorphism sending $(v, \dots, v)$ to v. The morphism $h : (N, v, \delta) \to (N', v', \delta')$ is induced by a special $\mathit{\Lambda}$-homomorphism $V \to V'$ satisfying $h\delta = \delta'(h^{\oplus_p 2^n})$. Furthermore, we consider the full subcategory $\mathscr{A}^p$ of $\mathscr{N}\!or^p$ where each object (N, v, δ) consists of a Banach $\mathit{\Lambda}$-module N, an element $v \in N$, and a $\mathit{\Lambda}$-homomorphism $\delta : N^{\oplus_p 2^n} \to N$.

Thirdly, we investigate the set $\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}})$ of elementary simple functions (a special step function defined on $\mathit{\Lambda}$), where τ is a homomorphism between two $\Bbbk$-algebras. We demonstrate its structure as a $\mathit{\Lambda}$-module (Lemma 4.8). Consequently, we obtain an object $(\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}}), \mathbf{1}, \gamma_\xi)$ (Lemma 5.5) in $\mathscr{N}\!or^p$ and an object $(\widehat{\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}})}, \mathbf{1}, \widehat{\gamma}_\xi)$ in $\mathscr{A}^p$, where $\widehat{\mathbf{S}\tau(\mathbb{I}_{\mathit{\Lambda}})}$ is the completion of $\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}})$ and $\widehat{\gamma}_\xi$ is induced by γ_ξ.

Fourthly, we prove our main result in Section 6.

Theorem 1.1. (Theorem 6.3 and Remark 6.4) *The triple $(\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}}), \mathbf{1}, \gamma_\xi)$ is an object in $\mathscr{N}\!or^p$. For any object (N, v, δ) in $\mathscr{A}^p$, there exists a unique morphism*

$$h \in \mathrm{Hom}_{\mathscr{N}\!or^p}((\mathbf{S}_\tau(\mathbb{I}_{\mathit{\Lambda}}), \mathbf{1}, \gamma_\xi), (N, v, \delta))$$

such that the diagram

$$\begin{array}{ccc} (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) & \xrightarrow{h} & (N, v, \delta) \\ \subseteq \big\downarrow & \nearrow_{\widehat{h}} & \\ (\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) & & \end{array}$$

commutes, where $\widehat{h}$ *is given by the completion of* $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$*.*

Furthermore, we construct an object $(\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ in $\mathscr{A}^p$, where $m : \Bbbk^{\oplus_p 2^n} \to \Bbbk$ is a Λ-homomorphism whose definition is given in Section 7. Take $(N, v, \delta) = (\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ in Theorem 1.1, we obtain the following result.

Theorem 1.2. (Theorem 7.6) *If* $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ *is an extension of* $\mathbb{R}$*, then there exists a unique* Λ*-homomorphism* $T : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to \Bbbk$ *such that*

$$T : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (\Bbbk, \mu(\mathbb{I}_\Lambda), m)$$

is a morphism in $\mathrm{Hom}_{\mathscr{N}or^p}((\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi), (\Bbbk, \mu(\mathbb{I}_\Lambda), m))$ *and the diagram*

$$\begin{array}{ccc} (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) & \xrightarrow{T} & (\Bbbk, \mu(\mathbb{I}_\Lambda), m) \\ \subseteq \big\downarrow & \nearrow_{\widehat{T}} & \\ (\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) & & \end{array}$$

commutes, where $\widehat{T}$ *is the unique morphism lying in* $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (\Bbbk, \mu(\mathbb{I}_\Lambda), m))$*. Furthermore, we have the following three properties of* $\widehat{T}$ *by the direct limits* $\varinjlim T_i : \widehat{T} = \varinjlim E_i \to \Bbbk$ *(The definitions of* E_i *and* T_i *are given in Notation 5.3 and Section 7, respectively):*

(1) (The formula (7.1)) $\widehat{T}(1) = \mu(\mathbb{I}_\Lambda)$*;*
(2) (Lemma 7.1) $\widehat{T} : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to \Bbbk$ *is a homomorphism of* Λ*-modules;*
(3) (Proposition 7.5) $\widehat{T}(|f|) \le |\widehat{T}(f)|$.

The morphism $\widehat{T}$ provides the categorization for integration, that is,

$$\int_{\mathbb{I}_\Lambda} f \mathrm{d}\mu := \widehat{T}(f). \tag{1.1}$$

The above (1), (2) and (3) show that

$$\int_{\mathbb{I}_\Lambda} 1 \mathrm{d}\mu = \mu(\mathbb{I}_\Lambda),$$

$$\int_{\mathbb{I}_\Lambda} (\lambda_1 \cdot f_1 + \lambda_2 \cdot f_2)\mu = \lambda_1 \cdot \int_{\mathbb{I}_\Lambda} f_1\mu + \lambda_2 \cdot \int_{\mathbb{I}_\Lambda} f_2\mu \ \ (\lambda_1, \lambda_2 \in \Lambda), \tag{1.2}$$

and

$$\left| \int_{\mathbb{I}_\Lambda} f \mathrm{d}\mu \right| \le \int_{\mathbb{I}_\Lambda} |f| \mathrm{d}\mu,$$

respectively.

Finally, we provide two applications for our main results in Section 8. In Subsection 8.1, we assume $\Bbbk = \mathbb{R}$, $(\Lambda, \prec, \|\cdot\|_\Lambda) = (\mathbb{R}, \le, |\cdot|)$, $B_\mathbb{R} = \{1\}$, $\mathfrak{n} : B_\mathbb{R} \to \{1\} \subseteq \mathbb{R}^{\ge 0}$,

$\mathbb{I}_{\mathbb{R}} = [0,1]$, $\xi = \frac{1}{2}$, $\kappa_0(x) = \frac{x}{2}$, $\kappa_1(x) = \frac{x+1}{2}$ and $\tau = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, and let μ be the Lebesgue measure. Then (1.1) is a Lebesgue integration

$$\int_{\mathbb{I}_{\mathbb{R}}=[0,1]} f \mathrm{d}\mu = \int_0^1 f \mathrm{d}\mu,$$

and (1.2) shows that Lebesgue integration is $\mathbb{R}$-linear. This result provides a categorization of Lebesgue integration. In Subsection 8.2, we show that the functor $\widehat{T}$ satisfies the Cauchy-Schwarz inequality.

2. Preliminaries

In this section we recall some concepts in the category theory and representation theory of algebras. These concepts are familiar to algebraists, but may not be as familiar to those in the field of analysts.

2.1. Categories and limits.

Recall that a *category* $\mathcal{C}$ consists of three ingredients: a class of *objects*, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *morphisms* for any objects X and Y in $\mathcal{C}$, and the *composition* $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, denoted by

$$(f : X \to Y, g : Y \to Z) \mapsto gf : X \to Z,$$

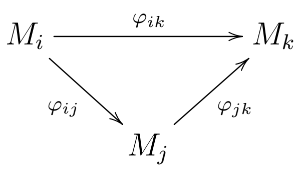
for any objects X, Y and Z in $\mathcal{C}$. These ingredients are subject to the following axioms:

(1) the Hom sets are pairwise disjoint;
(2) for any object X, the *identity morphism* $1_X : X \to X$ in $\mathrm{Hom}_{\mathcal{C}}(X, X)$ exists;
(3) the composition is associative: given morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} X$, we have

$$h(gf) = (hg)f.$$

Next, we review the limits in the category theory.

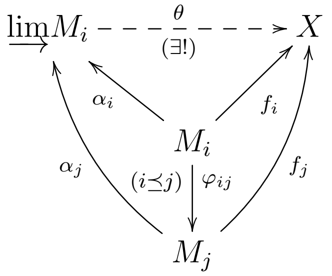
Definition 2.1 (c.f. [19, Chapter 5, Section 5.2])**.** Let $\mathfrak{I} = (\mathfrak{I}, \preceq)$ be a partially ordered set, and let $\mathcal{C}$ be a category. A *direct system* in $\mathcal{C}$ over $\mathfrak{I}$ is an ordered pair $((M_i)_{i\in\mathfrak{I}}, (\varphi_{ij})_{i\prec j})$, where $(M_i)_{i\in\mathfrak{I}}$ is an indexed family of objects in $\mathcal{C}$ and $(\varphi_{ij} : M_i \to M_j)_{i\prec j}$ is an indexed family of morphisms for which $\varphi_{ii} = 1_{M_i}$ for all i, such that the following diagram



commutes whenever $i \prec j \prec k$. Furthermore, for the above direct system $((M_i)_{i\in\mathfrak{I}}, (\varphi_{ij})_{i\prec j})$, the *direct limit* (also called *inductive limit* or *colimit*) is an object, say $\varinjlim M_i$, and *insertion morphisms* $(\alpha_i : M_i \to \varinjlim M_i)_{i\in\mathfrak{I}}$ such that

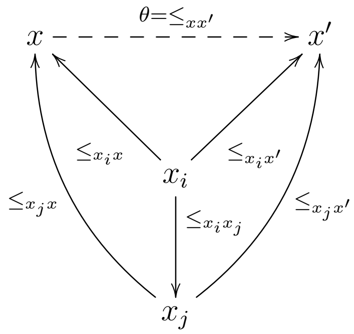
(1) $\alpha_j\varphi_{ij} = \alpha_i$ whenever $i \preceq j$;
(2) for any object X in $\mathcal{C}$ such that there are given morphisms $f_i : M_i \to X$ satisfying $f_j\varphi_{ij} = f_i$ for all $i \preceq j$, there exists a unique morphism $\theta : \varinjlim M_i \to X$ making

the following diagram



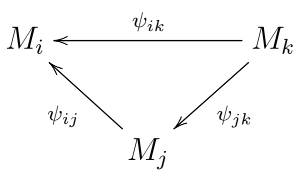
commutes.

Example 2.2. Let $\{x_n\}_{n\in\mathbb{N}^+}$ be a monotonically increasing sequence of real numbers, and let $\mathbb{R}$ be the partially ordered category $(\mathbb{R},\leq)$, in which the elements are real numbers and the morphisms are of the form $\leq_{r_2r_1}\colon r_1\to r_2$ $(r_2\leq r_1)$. If $\{x_n\}_{n\in\mathbb{N}^+}$ has the limit x in analysis, i.e., for any $\epsilon>0$, there exists $N\in\mathbb{N}^+$ such that $|x_n-x|<\epsilon$ holds for all $n>N$, then $x=\varinjlim x_n$. Indeed, for any $x'\in\mathbb{R}$ such that the morphisms $(\alpha_i=\leq_{x_ix'}\colon x_i\to x')_{i\in\mathbb{N}^+}$ exist, there is a morphism $\theta=\leq_{xx'}\colon x\to x'$ such that the following diagram



commutes. It is clear that the morphism θ is unique in this example. Furthermore, $x\leq x'$ holds because if $x'<x$ then we can find some x_t such that $x_t>x'$, i.e., $\alpha_t\in\mathrm{Hom}_{(\mathbb{R},\leq)}(x',x_t)=\varnothing$, this is a contradiction.

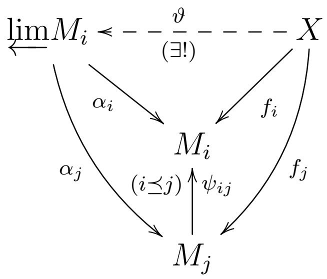
Definition 2.3 (c.f. [19, Chapter 5, Section 5.2])**.** Let $\mathfrak{I}=(\mathfrak{I},\preceq)$ be a partially ordered set, and let $\mathcal{C}$ be a category in this subsection. An *inverse system* in $\mathcal{C}$ over $\mathfrak{I}$ is an ordered pair $((M_i)_{i\in\mathfrak{I}},(\psi_{ij})_{j\prec i})$, where $(M_i)_{i\in\mathfrak{I}}$ is an indexed family of objects in $\mathcal{C}$ and $(\psi_{ij}:M_j\to M_i)_{j\prec i}$ is an indexed family of morphisms for which $\psi_{ii}=1_{M_i}$ for all i, such that the following diagram



commutes whenever $i\prec j\prec k$. Furthermore, for the above direct system $((M_i)_{i\in\mathfrak{I}},(\psi_{ij})_{j\prec i})$, the *inverse limit* (also called *projective limit* or *limit*) is an object, say $\varprojlim M_i$, and *projects morphisms* $(\alpha_i:\varprojlim M_i\to M_i)_{i\in\mathfrak{I}}$ such that

(1) $\psi_{ji}\alpha_j=\alpha_i$ whenever $i\preceq j$;
(2) for any object X in $\mathcal{C}$ such that there are given morphisms $f_i:X\to M_i$ satisfying $\psi_{ji}f_j=f_i$ for all $i\preceq j$, there exists a unique morphism $\vartheta:X\to\varinjlim M_i$ making

the following diagram



commutes.

Example 2.4. Let $\{x_n\}_{n\in\mathbb{N}^+}$ be a monotonically decreasing sequence of real numbers, and let $\mathbb{R}$ be the partially ordered category $(\mathbb{R}, \le)$. If $\{x_n\}_{n\in\mathbb{N}^+}$ has the limit x in analysis, then we have $x = \varprojlim x_n$ by a way similar to that in Example 2.3.

2.2. **$\Bbbk$-algebras and their completions.** Let $\Bbbk$ be a field. In this subsection we recall the definitions of $\Bbbk$-algebras and the completions of $\Bbbk$-algebras. All concepts in this subsection are parallel to those in [3, Chapter 10, Section 10.1] which extracts some important results about the completions of Abelian groups.

2.2.1. $\Bbbk$-*algebras.*

Definition 2.5. A $\Bbbk$-*algebra* A defined over $\Bbbk$ is both a ring and a $\Bbbk$-linear space such that

$$k(aa') = (ka)a' = a(ka').$$

Let $e_1, \ldots, e_t$ be the complete set of primitive orthogonal idempotents, i.e., any e_i is a primitive idempotent and $e_ie_j = 0$ holds for all $i \ne j$. Then A has a decomposition $A = \bigoplus_{i=1}^{t} Ae_i$, where each direct summand Ae_i is an indecomposable left A-module. We say A is *basic* if $Ae_i \not\cong Ae_j$ for all $1 \le i \ne j \le t$.

Example 2.6. The set $\mathbf{M}_n(\Bbbk)$ of all $n \times n$ matrices over $\Bbbk$, the polynomial ring $\Bbbk[x_1, \cdots, x_n]$, and the field $\Bbbk$ itself are $\Bbbk$-algebras. Aa $\Bbbk$-algebra Λ is called *finite-dimensional* if its $\Bbbk$-dimension $\dim_{\Bbbk} \Lambda$, i.e., the dimension of Λ as a $\Bbbk$-linear space, is finite.

Recall that a quiver is a quadruple $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$ where $\mathcal{Q}_0$ is the set of vertices, $\mathcal{Q}_1$ is the set of arrows, and $\mathfrak{s}, \mathfrak{t} : \mathcal{Q}_1 \to \mathcal{Q}_0$ are functions respectively sending each arrow to its starting point and ending point. Then any vertex $v \in \mathcal{Q}_0$ can be seen as a path on $\mathcal{Q}$ whose length is zero, and any arrow $\alpha \in \mathcal{Q}_1$ can be seen as a path on $\mathcal{Q}$ whose length is one. A path $\wp$ of length l, denoted $\ell(\wp)$, is the composition $\alpha_l \cdots \alpha_2\alpha_1$ of arrows $\alpha_1, \ldots, \alpha_l$, where $\mathfrak{t}(\alpha_i) = \mathfrak{s}(\alpha_{i+1})$ for all $1 \le i < l$. Then, naturally, we define the composition of two paths $\wp_1 = \alpha_l \cdots \alpha_1$ and $\wp_2 = \beta_\ell \cdots \beta_1$ as:

$$\wp_2\wp_1 = \beta_\ell \cdots \beta_1\alpha_l \cdots \alpha_1$$

provided that the ending point $\mathfrak{t}(\wp_1)$ of $\wp_1$ coincides with the starting point $\mathfrak{s}(\wp_2)$ of $\wp_2$, otherwise (i.e., $\mathfrak{t}(\wp_1) \ne \mathfrak{s}(\wp_2)$), then the composition is defined to be zero. Consequently, let $\mathcal{Q}_l$ be the set of all paths of length l. Then $\Bbbk\mathcal{Q} := \mathrm{span}_{\Bbbk}(\bigcup_{l\ge 0} \mathcal{Q}_l)$, known as the *path algebra* of $\mathcal{Q}$, is a $\Bbbk$-algebra whose multiplication defined as follows:

$$\Bbbk\mathcal{Q} \times \Bbbk\mathcal{Q} \to \Bbbk\mathcal{Q} \text{ via } (k_1\wp_1, k_2\wp_2) \mapsto \begin{cases} k_1k_2 \cdot \wp_2\wp_1, & \text{if } \mathfrak{t}(\wp_1) = \mathfrak{s}(\wp_2); \\ 0, & \text{otherwise.} \end{cases}$$

The following result shows that we can describe all finite-dimensional $\Bbbk$-algebras using quivers.

Theorem 2.7 (Gabriel)**.** *For any finite-dimensional* $\Bbbk$*-algebra* A*, there is a finite quiver* $\mathcal{Q}$*, i.e., the vertex set and arrow set are finite sets, and an admissible ideal*[1] $\mathcal{I}$ *of* $\Bbbk\mathcal{Q}$ *such that the module category of* A *is equivalent to that of* $\Bbbk\mathcal{Q}/\mathcal{I}$*. Furthermore, if* A *is basic, we have* $A \cong \Bbbk\mathcal{Q}/\mathcal{I}$*.*

Remark 2.8. We provide a remark for the isomorphism $A \cong \Bbbk\mathcal{Q}/\mathcal{I}$ given in Theorem 2.7 here: the existence of the quiver $\mathcal{Q}$ is unique if A is basic and $\mathcal{I}$ is admissible; the definition of admissible can be found in [2, Chapter I, Section I.6].

2.2.2. *Topologies on* $\Bbbk$*-algebras.* Now we recall the topologies of $\Bbbk$-algebras A (not necessarily basic or finite-dimensional). Let $\mathfrak{i}(A)$ be the set of all ideals of A, which forms a partially ordered set $\mathfrak{i}(A) = (\mathfrak{i}(A), \preceq)$ with the partial order defined by the inclusion. That is, for any $A_1, A_2 \in \mathfrak{i}(A)$, we have

$$A_1 \preceq A_2 \text{ if and only if } A_1 \subseteq A_2.$$

Naturally, we have at least one descending chain, denoted by $\mathcal{J}$, of ideals

$$A_0 = A \succeq A_1 \succeq A_2 \succeq \cdots$$

We say a subset U of A satisfies the *N-condition*, if it meets the following criteria:

(N1) U contains the zero of A;
(N2) there exists some $j \in \mathbb{N}$ such that $U \supseteq A_j$.

Furthermore, we denote by $\mathfrak{U}_A(0)$ the set of all subsets satisfying the N-condition, which forms a partially ordered set with the partial order "$\preceq$" given by "$\subseteq$".

Lemma 2.9. *The set* $\mathfrak{U}_A(0)$ *is a topology defined on* A*, in other words, it satisfies the following four conditions.*

(1) *For any* $U \in \mathfrak{U}_A(0)$*, we have* $0 \in U$*.*
(2) $\mathfrak{U}_A(0)$ *is closed under finite intersection, that is, for any* $U_1, \ldots, U_t \in \mathfrak{U}_A(0)$*, we have* $\bigcap_{1\le j\le t} U_j \in \mathfrak{U}_A(0)$*.*
(3) *If* $U \in \mathfrak{U}_A(0)$ *and* $U \subseteq V \subseteq A$*, then* $V \in \mathfrak{U}_A(0)$*.*
(4) *If* $U \in \mathfrak{U}_A(0)$*, then there is a set* $V \in \mathfrak{U}_A(0)$ *such that* $V \subseteq U$ *and* $U - y := \{u - y \mid u \in U\} \in \mathfrak{U}_A(0)$ *for all* $y \in V$*.*

Proof. First, (1) is trivial by the condition (N1).

Second, for arbitrary two subset U_1 and U_2, there are A_{j_1} and A_{j_2} such that $U_1 \supseteq A_{j_1}$ and $U_2 \supset A_{j_2}$. Then $U_1 \cap U_2 \supseteq A_{j_1} \cap A_{j_2}$. By the definition of A_j, we have $A_{j_1} \cap A_{j_2} = A_{\min\{j_1,j_2\}}$, that is,

$$U_1 \cap U_2 \supseteq A_{\min\{j_1,j_2\}}.$$

Since $0 \in U_1 \cap U_2$ trivially, we have $U_1 \cap U_2 \in \mathfrak{U}_A(0)$. By induction, we obtain (2).

Third, assume $U \in \mathfrak{U}_A(0)$ and $U \subseteq V \subseteq A$. By the definition of $\mathfrak{U}_A(0)$, we have $0 \in U$ and $U \supseteq A_j$ for some j. Thus, $0 \in V$ and $V \supseteq A_j$, so we obtain (3).

Finally, for each $U \in \mathfrak{U}_A(0)$, we can find V in the following way. There exists an index $\jmath$ such that $U \not\supseteq A_{\jmath-1}$ and $U \supseteq A_\jmath \supseteq A_{\jmath+1} \supseteq \cdots$. Take $V = \bigcap_{j\le\jmath} A_j$ $(= A_\jmath \subseteq U)$. For any $y \in V$, we have (N1), that is, $0 = y - y \in U - y = \{u - y \mid u \in U\}$ by $y \in V \subseteq U$; and have (N2) since $a = (a + y) - y$ holds for any $a \in V$ and $a + y \in V$. Then we obtain $U - y \in \mathfrak{U}_A(0)$, that is, (4) holds. □

Definition 2.10. The set $\mathfrak{U}_A(0)$ is called the $\mathcal{J}$*-topology* of A. Furthermore, we can define open sets on A.

[1]An admissible ideal $\mathcal{I}$ of $\Bbbk\mathcal{Q}$ is an ideal such that $R_{\mathcal{Q}}^m \subseteq \mathcal{I} \subseteq R_{\mathcal{Q}}^2$ holds for some $m \ge 2$, see [2, Chapter II, Section II.1, page 53], where $R_{\mathcal{Q}}^t$ is the ideal of $\Bbbk\mathcal{Q}$ generated by all paths of length $\ge t$.

(1) The subset in $\mathfrak{U}_A(0)$ is called a *neighborhood* of 0. For any $U \in \mathfrak{U}_A(0)$, the union $\bigcup_V V$ of all subsets V given in Lemma 2.9 (4) is called the *interior* of U and denote $\bigcup_V V$ by U°.
(2) A neighborhood U is called *open* if $U = U^\circ$. An *open set* O defined on A is one of the following cases:
  (a) O equals either A or $\varnothing$;
  (b) O is the intersection of a finite number of open neighborhoods;
  (c) O is the union of any number of open neighborhoods.

It induces the definitions of continuous homomorphisms of $\Bbbk$-algebras.

Definition 2.11. Let A_1 and A_2 be two $\Bbbk$-algebras, and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two descending chains of ideals in A_1 and A_2, respectively. Let $\mathfrak{U}_{A_1}(0)$ and $\mathfrak{U}_{A_2}(0)$ be the $\mathcal{J}_1$-topology $\mathcal{J}_2$-topology given by $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. A homomorphism $h : A_1 \to A_2$ of $\Bbbk$-algebras is called *continuous* if the preimage of arbitrary open set on A_2 is an open set on A_1.

Lemma 2.12. *Let A be a $\Bbbk$-algebra with a $\mathcal{J}$-topology. Then the addition $+ : A \times A \to A$ and each $\Bbbk$-linear transformation $h_\lambda : A \to A$ defined by $a \mapsto \lambda a$ ($\lambda \in A$) are continuous.*

Proof. It is obvious that $\mathrm{id}_A = h_1 : A \to A$ via $a \mapsto a$ is continuous. The continuity of h_λ can be given by id_A.

Let $\mathcal{J} =$

$$A = A_0 \succeq A_1 \succeq A_2 \succeq \cdots .$$

For any open neighborhood U of 0, its preimage is

$$+^{-1}(U) = \{(x_1, x_2) \mid x_1 + x_2 \in U\} =: \widetilde{U}.$$

We need show that $\widetilde{U} \in \mathfrak{U}_{A\times A}((0,0))$ and $\widetilde{U}^\circ = \widetilde{U}$ in the case for $A \times A$ being a $\Bbbk$-algebra, where the descending chain, say $\mathcal{J}_{A\times A}$, of $A \times A$ is induced by $\mathcal{J}$ as follows.

$$A \times A = A_0 \times A_0 \succeq A_1 \times A_1 \succeq A_2 \times A_2 \succeq \cdots .$$

First of all, the zero element of $A \times A$ is $(0,0)$ which satisfies that $0 \in U$ and $0 + 0 = 0 \in U$, then $(0,0) \in \widetilde{U}$.

Secondly, since U is a neighborhood of 0, there exists an ideal A_j of $\mathcal{J}$ such that $U \supseteq A_j$. Then for any $x_1, x_2 \in A_j$, we have $x_1 + x_2 \in A_j \subseteq U$, that is, $(x_1, x_2) \in \widetilde{U}$. It follows that $A_j \times A_j \subseteq \widetilde{U}$. We obtain $\widetilde{U} \in \mathfrak{U}_{A\times A}((0,0))$.

Thirdly, for any $(y_1, y_2) \in \widetilde{U}$, we have $y_1 + y_2 \in U$ by the definition of $\widetilde{U}$, then,

$$(0,0) = (y_1 - y_1, y_2 - y_2) \in \widetilde{U} - (y_1, y_2) = \{(x_1 - y_1, x_2 - y_2) \mid x_1 + x_2 \in U\},$$

that is, (N1) holds. On the other hand, for any $(z_1, z_2) \in A_j \times A_j$, we have

$$(z_1, z_2) = ((z_1 + y_1) - y_1, (z_2 + y_2) - y_2).$$

Note that $z_1 + y_1 + z_2 + y_2 = (y_1 + y_2) + (z_1 + z_2)$ is an element lying in $U + (z_1 + z_2)$. Since U is open, we have

$$U + (z_1 + z_2) = U^\circ - (-(z_1 + z_2)) = \{u + (z_1 + z_2) \mid u \in U\} \in \mathfrak{U}_A(0)$$

by Lemma 2.9 (4) and Definition 2.10, that is, $U + (z_1 + z_2)$ is a set satisfying Lemma 2.9 (4). Then

$$U^\circ = \bigcup_{\substack{V \subseteq U,\ V \text{ satisfies} \\ \text{Lemma 2.9 (4)}}} V \supseteq U + (z_1 + z_2),$$

and so, we obtain $(y_1+y_2)+(z_1+z_2) \in U+(z_1+z_2) \subseteq U^\circ$, that is, $(y_1+y_2)+(z_1+z_2) \in U$. Thus, $(z_1, z_2) \in \widetilde{U}$. It follows that $A_j \times A_j \subseteq \widetilde{U}-(y_1, y_2)$, and thus (N2) holds. Therefore, $\widetilde{U}-(y_1, y_2) \in \mathfrak{U}_{A\times A}((0,0))$. In summary, we have that $\widetilde{U}$ satisfies Lemma 2.9 (4), and so, by Definition 2.10, it is clear that $\widetilde{U}^\circ = \widetilde{U}$. □

Definition 2.13 (c.f. [3, Chapter 10, page 101])**.** A *topology* $\Bbbk$*-algebra* is a $\Bbbk$-algebra equipped with a topology such that the addition $+ : A \times A \to A$ and each $\Bbbk$-linear transformation $-h_1 : A \to A$ via $a \mapsto -a$ are continuous.

The following result is a consequence of Lemma 2.12.

Proposition 2.14. *Given an arbitrary* $\Bbbk$*-algebra* A *and its descending chain* $\mathcal{J}$ *of ideals. Then* A *becomes a topology* $\Bbbk$*-algebra with the* $\mathcal{J}$*-topology* $\mathfrak{U}_A(0)$*.*

In this paper, we refer to A as a *$\mathcal{J}$-topological* $\Bbbk$*-algebra*.

2.2.3. *Completions induced by $\mathcal{J}$-topologies.* Assume that $|\cdot| : \Bbbk \to \mathbb{R}^{\geq 0}$ be a norm defined on the field $\Bbbk$ in this subsection, that is, $|\cdot|$ is the map satisfying

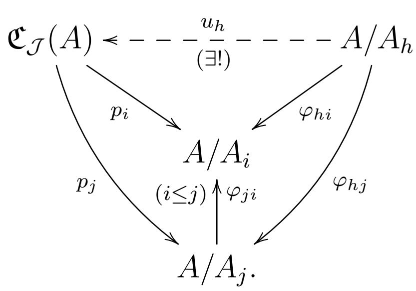
(1) $|k| = 0$ if and only if $k = 0$;
(2) $|k_1k_2| = |k_1||k_2|$ holds for all $k_1, k_2 \in \Bbbk$;
(3) and the triangle inequality $|k_1 + k_2| \leq |k_1| + |k_2|$ holds for all $k_1, k_2 \in \Bbbk$.

Then $\{\mathfrak{B}_r = \{a \in A \mid |a| < r\} \mid r \in \mathbb{R}^+\}$ induces a standard topology $\mathfrak{U}_\Bbbk(0)$ on $\Bbbk$ whose element is called the *neighborhood* of $0 \in \Bbbk$.

Let A be a $\mathcal{J}$-topological $\Bbbk$-algebra whose dimension is finite and let $B_A = \{b_1, \ldots, b_n\}$ be a basis of A. Then, naturally, we can define the Cauchy sequence by the $\mathcal{J}$-topology. More precisely, a sequence $\{x_i\}_{i\in\mathbb{N}}$ in A is called a *$\mathcal{J}$-Cauchy sequence* if for any U, lying in $\mathfrak{U}_A(0)$, containing some subset $\sum_{i=1}^n \mathfrak{u}_i b_i$ of A with $\mathfrak{u}_i \in \mathfrak{U}_\Bbbk(0)$ $(1 \leq i \leq n)$, there is $n \in \mathbb{N}$ such that $x_s - x_t \in U$ holds for all $s, t \geq n$. Two $\mathcal{J}$-Cauchy sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are called *equivalent*, denoted by $\{x_i\}_{i\in\mathbb{N}} \sim \{y_i\}_{i\in\mathbb{N}}$, if for any $U \in \mathfrak{U}_A(0)$, there is an integer $n \in \mathbb{N}$ such that $x_i - y_i \in U$ holds for all $i \geq N$. It is easy to see that "$\sim$" is an equivalence relation. We use $[\{x_i\}_{i\in\mathbb{N}}]$ to denote the equivalence class containing $\{x_i\}_{i\in\mathbb{N}}$, and use $\mathfrak{C}_\mathcal{J}(A)$ to denote the set of all equivalence classes of $\mathcal{J}$-Cauchy sequences. Then we have three families of A-homomorphisms:

(1) $(\varphi_{ji} : A/A_j \to A/A_i)_{j\geq i}$, where all φ_{ji} are naturally induced by $A_i \supseteq A_j$;
(2) $(p_i : \mathfrak{C}_\mathcal{J}(A) \to A/A_i)_{i\in\mathbb{N}}$, where $p_i(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots) = x_i$ (p_i is called the i-th *projection*);
(3) $(u_i : A/A_i \to \mathfrak{C}_\mathcal{J}(A))_{i\in\mathbb{N}}$, where $u_i(a + A_i) = (0, \ldots, \overset{i-1}{0}, a, \overset{i+1}{0}, 0 \ldots)$.

Let $\mathcal{X}$ be the category whose object set is $\{A/A_i \mid i \in \mathbb{N}\} \cup \{\mathfrak{C}_\mathcal{J}(A)\}$ and morphism set is the collection of all A-homomorphisms as above. Then we obtain the following commutative diagram



It follows from the above construction that the following proposition holds.

Proposition 2.15 (c.f. [3, Chapter 10, page 103]). *Using the notations as above, we have*

$$\varprojlim A/A_i \cong \mathfrak{C}_{\mathcal{J}}(A).$$

We write $\widehat{A} := \mathfrak{C}_{\mathcal{J}}(A)$ and call it the *completion* of A. In particular, if $A = \Bbbk$, then the descending chain $\mathcal{J}$:

$$A_0 = \Bbbk \succeq A_1 = 0$$

induces a $\mathcal{J}$-topology

$$\mathfrak{U}_A(0) = \{\text{the neighborhood of } 0\}$$

of A. In this case, the $\mathcal{J}$-Cauchy sequence coincides with the usual Cauchy sequence.

Proposition 2.16. *Let A be a basic finite-dimensional $\Bbbk$-algebra and let $\mathcal{J}$ be the descending chain*

$$A_0 = A = \mathrm{rad}^0 A \succeq A_1 = \mathrm{rad} A \succeq A_2 = \mathrm{rad}^2 A \succeq \cdots$$

Then A is complete (*in the sense of $\mathcal{J}$-topology*) *if and only if $\Bbbk$ is complete.*

Proof. Let A be a basic finite-dimensional $\Bbbk$-algebra. Then, by Theorem 2.7, there is a finite quiver $\mathcal{Q}$ and an ideal $\mathcal{I}$ of $\Bbbk\mathcal{Q}$ such that

$$A \cong \Bbbk\mathcal{Q}/\mathcal{I} = \bigoplus_{l\in\mathbb{N}} \Bbbk\mathcal{Q}_l.$$

Thus, up to isomorphism, each element $a \in A$ can be written as $\sum_{j=1}^{n} k_j \wp_j$, where n is the dimension of A, $k_u \in \Bbbk$ and $\wp_u$ is a path on $\mathcal{Q}$.

Assume that $\Bbbk$ is complete. Since A is finite-dimensional, we have $\mathrm{rad}^l A = \mathrm{span}_{\Bbbk}\{\mathcal{Q}_i \mid i \geq l\}$. Thus, $\mathrm{rad}^{L+1} A = 0$, where $L = \max_{\wp\in\mathcal{Q}_{\geq 0}} \ell(\wp)$, that is,

$$\mathcal{J} = \quad A \succeq \mathrm{rad} A \succeq \mathrm{rad}^2 A \succeq \cdots \mathrm{rad}^L A \succeq 0 \succeq 0 \succeq \cdots.$$

Let $\{x_i = \sum_{j=1}^{n} k_{ij}\wp_j\}_{i\in\mathbb{N}}$ be a $\mathcal{J}$-Cauchy sequence in A. Take

$$U = \left\{ \sum\nolimits_{\ell(\wp)=L} k_\wp \wp \mid k_\wp \text{ lie in some neighborhood in } \mathfrak{U}_{\Bbbk}(0) \right\} \;\; (\supsetneq \mathrm{rad}^{L+1} A = 0).$$

Then, there is $N(U) \in \mathbb{N}$ such that

$$x_s - x_t = \sum_{j=1}^{n} (k_{sj} - k_{tj})\wp_j \in \mathrm{rad}^L A \text{ holds for all } s, t \geq N(U).$$

Thus, $k_{sj} - k_{tj}$ lies in some neighborhood in $\mathfrak{U}_{\Bbbk}(0)$, and so, for all i, $\{k_{ij}\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\Bbbk$. Then it is clear that A is complete.

Conversely, if A is complete, we assume that $\Bbbk$ is not complete, and $\widehat{\Bbbk}$ be the completion of $\Bbbk$. Then we have a natural $\Bbbk$-linear embedding $\mathfrak{e} : \Bbbk \to \widehat{\Bbbk}$ sending $k \in \Bbbk$ to $\{k_i\}_{i\in\mathbb{N}}$, where $k_1 = k_2 = \cdots = k$. Then there is a Cauchy sequence $\{x_i\}_{i\in\mathbb{N}} \in \widehat{\Bbbk}\backslash\mathfrak{e}(\Bbbk)$. Consider the sequence $\{x_i \cdot \wp\}_{i\in\mathbb{N}}$ in A, where $\wp \in \mathrm{rad}^L A$ is a path of length L. Then $\{x_i \cdot \wp\}_{i\in\mathbb{N}}$ is a $\mathcal{J}$-Cauchy sequence in A. However, we have $\{x_i \cdot \wp\}_{i\in\mathbb{N}} \in \widehat{A}\backslash A$ in this case, which contradicts that A is complete. $\square$

2.3. **The total order of $\Bbbk$-algebras.** Recall that a field $\Bbbk$ equipped with a total order "$\preceq$" is a *ordered field* if it satisfies the following four conditions:

(1) for any $a, b \in \Bbbk$, either $a \preceq b$, $b \preceq a$ or $a = b$ holds;
(2) if $a \preceq b$, $b \preceq c$, then $a \preceq c$;
(3) if $a \preceq b$, then $a + c \preceq b + c$ for all $c \in \Bbbk$;
(4) if $a \preceq b$ and $0 \preceq c$, then $ac \preceq bc$.

In order to give the definition of integration defined on finite-dimensional $\Bbbk$-algebra Λ, we need to assume that $\Bbbk$ is a field with the total order "$\preceq$". However, it is well-known that $\Bbbk$ might not always be an ordered field, as the case for $\Bbbk$ being the complex field $\mathbb{C}$. Interestingly, for our purposes, the existence of such a total order is not a prerequisite. We only require that the finite-dimensional $\Bbbk$-algebra involved in our study, encompasses certain partially ordered subsets. Specifically, the subset $\mathbb{I}_\Lambda$ outlined in Subsection 3.3 is sufficient. For the sake of simplicity, we assume that $\Bbbk$ is fully ordered, although this assumption does not sacrifice generality. This simplification aids in our definition of integration within the context of category theory.

Remark 2.17. We provide a remark to show that if $\Bbbk$ is total ordered, then any finite-dimensional $\Bbbk$-algebra Λ can be endowed with a total order. Let $B_\Lambda = \{b_i \mid 1 \le i \le n\}$ be a $\Bbbk$-basis of Λ. If B_Λ is totally ordered (assuming $b_i \preceq b_j$ if and only if $i \le j$), then we can define a total order for Λ as follows.

Step 1. For any two arbitrary elements $a, a' \in \Lambda$, we define $a \prec_p a'$ if and only if $\varphi(a) \prec_p \varphi(a')$, where φ is a map from Λ to $\mathbb{R}^{\ge 0}$ (for example, φ is the norm $\|\cdot\|_p$ defined in Section 3).

Step 2. Assume $a = \sum_{i=1}^m k_ib_i$ and $a' = \sum_{i=1}^m k'_ib_i$ $(0 \le m \le n)$ such that $k_i = k'_i$ holds for all $i < m$. If $\varphi(a) = \varphi(a')$, then we define $a \preceq_p a'$ if and only if $k_m \preceq k'_m$.

3. Normed $\Bbbk$-algebras

In the sequel, let Λ be a finite-dimensional $\Bbbk$-algebra with a $\Bbbk$-basis $B_\Lambda = \{b_i \mid 1 \le i \le n\}$. Then any element $a \in \Lambda$ is of the form $a = \sum_{i=1}^n k_ib_i$. In this section, we define some algebraic structure for Λ.

3.1. **Norms of $\Bbbk$-algebras.** Take $\mathfrak{n} : B_\Lambda \to \mathbb{R}^+$ a map from Λ to $\mathbb{R}^+$ and, for any $p > 1$, $||\cdot||_p : \Lambda \to \mathbb{R}^{\ge 0}$ is the function defined by

$$\|a\|_p = \Big\| \sum_{i=1}^n k_ib_i \Big\|_p := \big((|k_1|\mathfrak{n}(b_1))^p + \cdots + (|k_n|\mathfrak{n}(b_n))^p\big)^{\frac{1}{p}}. \tag{3.1}$$

Proposition 3.1. *Any triple $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ (=Λ for short) is a normed $\Bbbk$-linear space.*

Proof. First of all, for any $a = \sum_{i=1}^n k_ib_i \in \Lambda$, we have $\|a\|_p \ge 0$ because $\mathfrak{n}(b_i) > 0$ and $|k_i| \ge 0$ $(1 \le i \le n)$. In particular, if $\|a\|_p = 0$, then

$$(|k_1|\mathfrak{n}(b_1))^p + \cdots + (|k_n|\mathfrak{n}(b_n))^p = 0.$$

Since $|k_i|\mathfrak{n}(b_i) \ge 0$ and $\mathfrak{n}(b_i) > 0$ hold for all $1 \le i \le n$, we obtain $|k_i|\mathfrak{n}(b_i) = 0$, and so $k_i = 0$. Thus, $a = \sum_{i=1}^n 0b_i = 0$. Then it is easy to see that $\|a\|_p = 0$ if and only if $a = 0$.

Next, for any $k \in \Bbbk$ and $a = \sum_{i=1}^n k_ib_i \in \Lambda$, we have

$$\begin{aligned} \|ka\|_p &= \|k(k_1b_1 + \cdots + k_nb_n)\|_p \\ &= \Big(\sum_{i=1}^n (|kk_i|\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^n |k|^p(|k_i|\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}} \end{aligned}$$

$$= |k|\Big(\sum_{i=1}^{n}(|k_i|\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}} = |k|\cdot\|a\|_p.$$

Finally, we prove the triangle inequality $\|a+a'\|_p \le \|a\|_p + \|a'\|_p$ for arbitrary two elements $a = \sum_{i=1}^{n} k_i b_i$ and $a' = \sum_{i=1}^{n} k'_i b_i$. It can be induced by the discrete Minkowski inequality $(\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} y_i^p)^{\frac{1}{p}} \ge (\sum_{i=1}^{n}(x_i+y_i)^p)^{\frac{1}{p}}$ as follows:

$$\begin{aligned}\|a\|_p + \|a'\|_p &= \Big(\sum_{i=1}^{n}(|k_i|\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{n}(|k'_i|\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}}\\ &\ge \Big(\sum_{i=1}^{n}(|k_i|\mathfrak{n}(b_i)+k'_i\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}}\\ &= \Big(\sum_{i=1}^{n}(|k_i+k'_i|\mathfrak{n}(b_i))^p\Big)^{\frac{1}{p}} = \|a+a'\|_p.\end{aligned}$$

Therefore, $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ is a normed space. □

Definition 3.2. A *normed* $\Bbbk$*-algebra* is a triple $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$, where $\mathfrak{n} : B_\Lambda \to \mathbb{R}^+$ and $\|\cdot\|_p : \Lambda \to \mathbb{R}^{\ge 0}$ are called the *normed basis function* and *norm* of Λ, respectively.

3.2. **Completions of normed $\Bbbk$-algebras.** We can define open neighborhoods $B(0,r)$ of 0 for any normed $\Bbbk$-algebra $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ by

$$B(0,r) := \{a \in \Lambda \mid \|a\|_p < r\}.$$

Let $\mathfrak{U}_\Lambda^B(0)$ be the class of all subsets U of Λ satisfying the following conditions.

(1) U is the intersection of a finite number of $B(0,r)$;
(2) U is the union of any number of $B(0,r)$.

Then $\mathfrak{U}_\Lambda^B(0)$ is a topology, say $\|\cdot\|_p$-topology, defined on Λ, and we can define the Cauchy sequence, say $\|\cdot\|_p$-Cauchy sequence, by the above topology.

Recall that Λ has a $\mathscr{J}$-topology $\mathfrak{U}_\Lambda(0)$ given by the descending chain

$$\Lambda = \mathrm{rad}^0\Lambda \succeq \mathrm{rad}^1\Lambda \succeq \mathrm{rad}^2\Lambda \succeq \cdots.$$

Thus, we obtain two completions $\widehat{\Lambda}^B$ and $\widehat{\Lambda}$ by $\|\cdot\|_p$-topology and $\mathscr{J}$-topology, respectively. The following lemma establishes the relation between $\widehat{\Lambda}^B$ and $\widehat{\Lambda}$.

Proposition 3.3. *Let $\Lambda = (\Lambda, \mathfrak{n}, \|\cdot\|_p)$ be an n-dimensional normed $\Bbbk$-algebra with the $\mathscr{J}$-topology $\mathfrak{U}_\Lambda(0)$ given by $\Lambda = \mathrm{rad}^0\Lambda \succeq \mathrm{rad}^1\Lambda \succeq \mathrm{rad}^2\Lambda \succeq \cdots$ ($\|\cdot\|_p$ is a norm defined on Λ given in Proposition 3.1). Then $\widehat{\Lambda}^B = \widehat{\Lambda}$.*

Proof. Similar to Proposition 2.16 we can show that $\widehat{\Lambda}^B = \Lambda$ (i.e., Λ is complete) if and only if $\widehat{\Bbbk} = \Bbbk$. By using Proposition 2.16 again, we have that $\widehat{\Lambda} = \Lambda$ if and only if $\widehat{\Bbbk} = \Bbbk$. Then $\widehat{\Bbbk} = \Bbbk$ if and only if $\widehat{\Lambda}^B = \Lambda = \widehat{\Lambda}$, Equivalently,

$$\widehat{\Lambda}^B = \Big(\widehat{\sum_{i=1}^{n}\Bbbk b_i}\Big)^B = \sum_{i=1}^{n}\widehat{\Bbbk} b_i = \widehat{\sum_{i=1}^{n}\Bbbk b_i} = \widehat{\Lambda}.$$

□

Remark 3.4. (1) Note that the norms defined on A is not unique. In Section 4, we will introduce normed Λ-modules N over any finite-dimensional normed $\Bbbk$-algebra Λ. In this case, we need a homomorphism $\tau : \Lambda \to \Lambda'$ between two finite-dimensional normed $\Bbbk$-algebras Λ and Λ', and the norms $\|\cdot\|$ and $\|\cdot\|'$ respectively defined on Λ and Λ' may not necessarily be the form of $\|\cdot\|_p$.

(2) If $A = \Bbbk$ and $\mathfrak{n}(1) = 1$, then the norm $\|\cdot\|_p$ given in Proposition 3.1 is the norm $|\cdot|$, i.e, $\|a\|_p = (|a|^p)^{\frac{1}{p}} = |a|$.

3.3. **Elementary simple functions.** Denote $\mathbb{I}_\Lambda$ by the subset

$$\left\{ \sum_{i=1}^{n} k_i b_i \mid k_i \in \mathbb{I} \right\} \overset{1-1}{\longleftrightarrow} \prod_{i=1}^{n} (\mathbb{I} \times \{b_i\})$$

of Λ. A *function defined on* $\mathbb{I}_\Lambda$ is a map $f : \mathbb{I}_\Lambda \to \Bbbk$ from $\mathbb{I}_\Lambda$ to $\Bbbk$. Since $(\Lambda, \mathfrak{n}, \|\cdot\|_p)$ is a normed space, Λ is also a topological space induced by the norm $\|\cdot\|_p$, and so is $\mathbb{I}_\Lambda$. Thus, we can define the open set for every subset of Λ, including $\mathbb{I}_\Lambda$. The function f is called *continuous* if the preimage of any open subset of $\Bbbk$ is an open set of $\mathbb{I}_\Lambda$.

An *elementary simple function* on $\mathbb{I}_\Lambda$ is a finite sum

$$\sum_{i=1}^{t} k_i \mathbf{1}_{I_i},$$

where

(1) for any $1 \le i \le t$, $k_i \in \Bbbk$;
(2) $I_i = I_{i1} \times \cdots \times I_{in}$, and, for any $1 \le j \le n$, I_{ij} is a subset of $\mathbb{I}$ which is one of the following forms
  (a) $(c_{ij}, d_{ij})_\Bbbk := \{k \in \Bbbk \mid c_{ij} \prec k \prec d_{ij}\}$,
  (b) $[c_{ij}, d_{ij})_\Bbbk := \{k \in \Bbbk \mid c_{ij} \preceq k \prec d_{ij}\}$,
  (c) $(c_{ij}, d_{ij}]_\Bbbk := \{k \in \Bbbk \mid c_{ij} \prec k \preceq d_{ij}\}$,
  (d) $[c_{ij}, d_{ij}]_\Bbbk := \{k \in \Bbbk \mid c_{ij} \preceq k \preceq d_{ij}\}$,
  where $a \preceq c_{ij} \prec d_{ij} \preceq b$;
(3) and $\mathbf{1}_{I_i}$ is the function $I_i \to \{1\}$ such that $I_i \cap I_j = \varnothing$ holds for all $1 \le i \ne j \le t$.

We denote $\mathbf{S}(\mathbb{I}_\Lambda)$ by the set of all elementary simple functions. Then $\mathbf{S}(\mathbb{I}_\Lambda)$ is a $\Bbbk$-linear space, and $\mathbf{S}(\mathbb{I}_\Lambda)$ induces the direct sum $\mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n}$ whose element can be seen as the sequence

$$\left\{ f_{(\delta_1,\ldots,\delta_n)}\left(\sum_{i=1}^{n} k_i b_i \right) \right\}_{(\delta_1,\ldots,\delta_n)\in\{a,b\}\times\cdots\times\{a,b\}} =: \boldsymbol{f}(k_1, \ldots, k_n),$$

$\sum_{i=1}^{n} k_i b_i$ is written as $(k_1, \ldots, k_n)$ since $\{b_1 \mid 1 \le i \le n\} = B_\Lambda$ is the $\Bbbk$-basis of Λ. Then we can characterize $\mathbf{S}(\mathbb{I}_\Lambda)$ together with two further pieces of data: the function $\mathbf{1}_{\mathbb{I}_\Lambda} : \mathbb{I}_\Lambda \to \{1\}$ (1 is the identity element of $\Bbbk$), and the map

$$\gamma_\xi : \mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n} \to \mathbf{S}(\mathbb{I}_\Lambda), \tag{3.2}$$

say *juxtaposition map*, sending $\boldsymbol{f}$ to the function

$$\gamma_\xi(\boldsymbol{f})(k_1, \ldots, k_n) = \sum_{(\delta_1,\ldots,\delta_n)} \mathbf{1}_{\kappa_{\delta_1}(\mathbb{I})\times\cdots\times\kappa_{\delta_n}(\mathbb{I})} \cdot f_{(\delta_1,\ldots,\delta_n)}(\kappa_{\delta_1}^{-1}(k_1), \ldots, \kappa_{\delta_n}^{-1}(k_n)),$$

$$(k_1 \ne \xi, \ \ldots, \ k_n \ne \xi),$$

where ξ is an element with $a \prec \xi \prec b$ such that the order preserving bijections

$$\kappa_a : \mathbb{I} \to [a, \xi]_\Bbbk \text{ and } \kappa_b : \mathbb{I} \to [\xi, b]_\Bbbk$$

exist.

Example 3.5. (1) Take Λ is the $\Bbbk$-algebra whose dimension is 2, and assume that basis of Λ is $\{b_1, b_2\}$. Then $\mathbb{I}_\Lambda \cong_\Bbbk [a, b]_\Bbbk b_1 \times [a, b]_\Bbbk b_2$.

For any element

$$\boldsymbol{f} = (f_{(a,a)}, f_{(b,a)}, f_{(a,b)}, f_{(b,b)}) \in \mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 4},$$

where $f_{(\delta_1,\delta_2)} : \mathbb{I}_\Lambda \to \Bbbk$ is a function in $\mathbf{S}(\mathbb{I}_\Lambda)$ sending each $k_1b_1 + k_2b_2$ to the element $f_{(\delta_1,\delta_2)}(k_1, k_2)$ in $\Bbbk$, $(\delta_1, \delta_2) \in \{a, b\} \times \{a, b\} = \{(a, a), (b, a), (a, b), (b, b)\}$, γ_ξ juxtapose $f_{(a,a)}$, $f_{(b,a)}$, $f_{(a,b)}$ and $f_{(b,b)}$ into a new function

$$\gamma_\xi(f_{(a,a)}, f_{(b,a)}, f_{(a,b)}, f_{(b,b)})(k_1, k_2) = \tilde{f}_{(a,a)}(k_1, k_2)+\tilde{f}_{(b,a)}(k_1, k_2)+\tilde{f}_{(a,b)}(k_1, k_2)+\tilde{f}_{(b,b)}(k_1, k_2)$$

as shown in Figure 3.1, where

$$\begin{aligned}
\tilde{f}_{(a,a)}(k_1, k_2) &= \mathbf{1}_{[a,\xi)\times[a,\xi)} \cdot f_{(a,a)}(\kappa_a^{-1}(k_1), \kappa_a^{-1}(k_2)),\\
\tilde{f}_{(b,a)}(k_1, k_2) &= \mathbf{1}_{(\xi,b]\times[a,\xi)} \cdot f_{(b,a)}(\kappa_b^{-1}(k_1), \kappa_a^{-1}(k_2)),\\
\tilde{f}_{(a,b)}(k_1, k_2) &= \mathbf{1}_{[a,\xi)\times(\xi,b]} \cdot f_{(a,b)}(\kappa_a^{-1}(k_1), \kappa_b^{-1}(k_2)),\\
\tilde{f}_{(b,b)}(k_1, k_2) &= \mathbf{1}_{(\xi,b]\times(\xi,b]} \cdot f_{(b,b)}(\kappa_b^{-1}(k_1), \kappa_b^{-1}(k_2)).
\end{aligned}$$

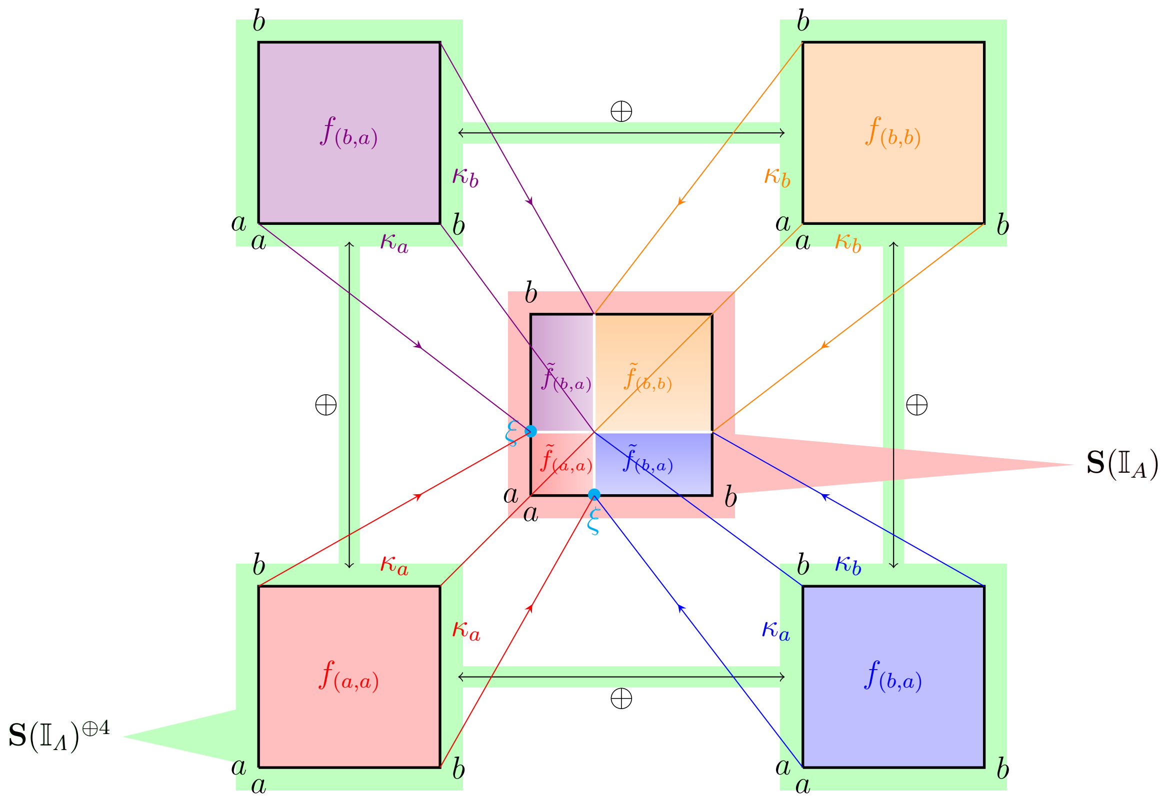


FIGURE 3.1. Juxtaposition map

(2) This example is used to establish the relation between Banach space and Lebesgue intersections in [16]. Take $\Bbbk = \mathbb{R}$, $\mathbb{I} = [0, 1]$, $\xi = \frac{1}{2}$, $\Lambda = \mathbb{R}$ and the order preserving bijections $\kappa_0 : \mathbb{I} = [0, 1] \to \Bbbk = \mathbb{R}$ and $\kappa_1 : \mathbb{I} = [0, 1] \to \Bbbk = \mathbb{R}$ are given by $x \mapsto \frac{x}{2}$ and $\frac{1+x}{2}$, respectively. Then $\mathbf{S}(\mathbb{I}_\mathbb{R}) = \mathbf{S}([0, 1])$ is a normed space together with two further pieces of data: the function $\mathbf{1}_{[0,1]} : [0, 1] \to \{1\}$ and the juxtaposition map

$$\gamma_{\frac{1}{2}} : \mathbf{S}([0, 1]) \oplus \mathbf{S}([0, 1]) \to \mathbf{S}([0, 1])$$

sending (f_1, f_2) to the following function

$$\gamma_{\frac{1}{2}}(f_1, f_2)(x) = \mathbf{1}_{\kappa_0([0,1))} \cdot f_1(\kappa_0^{-1}(x)) + \mathbf{1}_{\kappa_1((0,1])} \cdot f_1(\kappa_1^{-1}(x))$$

$$= \begin{cases} f_1(2x) & x \in \kappa_0([0,1)) = [0, \frac{1}{2}); \\ f_2(2x-1) & x \in \kappa_1((0,1]) = (\frac{1}{2}, 1]. \end{cases}$$

Lemma 3.6. *The map γ_ξ is a $\Bbbk$-linear map.*

Proof. Take $a, b \in \Bbbk$, $f, g \in \mathbf{S}(\mathbb{I}_\Lambda)$ and let $(k_i)_i$, $\mathbf{1}$ and $(\delta_i)_i$ be the element $(k_1, \ldots, k_n)$ in $\mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n}$, the identity function $\mathbf{1}_{\kappa_{\delta_1}(\mathbb{I}) \times \cdots \times \kappa_{\delta_n}(\mathbb{I})}$ and the n-multiple $(\delta_1 \times \cdots \times \delta_n)$, respectively. Then

$$\begin{aligned} \gamma_\xi(af + bg)((k_i)_i) &= \sum_{(\delta_i)_i} \mathbf{1} \cdot (af + bg)_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) \\ &= \sum_{(\delta_i)_i} \big(\mathbf{1} \cdot af_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) + \mathbf{1} \cdot bg_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i)\big) \\ &= a \sum_{(\delta_i)_i} \mathbf{1} \cdot f_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) + b \sum_{(\delta_i)_i} \mathbf{1} \cdot g_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) \\ &= a\gamma_\xi(f)((k_i)_i) + b\gamma_\xi(g)((k_i)_i). \end{aligned}$$

Thus, γ_ξ is a $\Bbbk$-linear map. □

Example 3.7. Take $\Bbbk = \mathbb{R}$, $\mathbb{I} = [0,1]$, $\xi = \frac{1}{2}$, $\Lambda = \mathbb{R}$ and the order preserving bijections $\kappa_0 : \mathbb{I} = [0,1] \to \Bbbk = \mathbb{R}$ and $\kappa_1 : \mathbb{I} = [0,1] \to \Bbbk = \mathbb{R}$ are given by $x \mapsto \frac{x}{2}$ and $\frac{x+1}{2}$, respectively. Then $\mathbf{S}(\mathbb{I}_\mathbb{R}) = \mathbf{S}([0,1])$ is a normed space together with two further pieces of data: the function $\mathbf{1}_{[0,1]} : [0,1] \to \{1\}$ and the juxtaposition map

$$\gamma_{\frac{1}{2}} : \mathbf{S}([0,1]) \oplus \mathbf{S}([0,1]) \to \mathbf{S}([0,1])$$

sending (f_1, f_2) to the following function

$$\begin{aligned} \gamma_{\frac{1}{2}}(f_1, f_2)(x) &= \mathbf{1}_{\kappa_0([0,1])} \cdot f_1(\kappa_0^{-1}(x)) + \mathbf{1}_{\kappa_1([0,1])} \cdot f_1(\kappa_1^{-1}(x)) \\ &= \begin{cases} f_1(2x) & x \in \kappa_0([0,1]) = [0, \frac{1}{2}); \\ f_2(2x-1) & x \in \kappa_1([0,1]) = (\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

4. Normed modules over $\Bbbk$-algebras

Let $\mathbb{I}$ be a subset of the field $\Bbbk = (\Bbbk, \preceq)$ with the totally ordered "$\preceq$". Then $\mathbb{I}$ is also a total ordered set. For simplicity, we denote by $[x, y]_\Bbbk$ the set of all elements $k \in \Bbbk$ with $x \preceq k \preceq y$ in our paper, that is,

$$[x, y]_\Bbbk := \{k \in \Bbbk \mid x \preceq k \preceq y\}.$$

In particular, if $x = y$ then $[x, y]_\Bbbk = \{x\} = \{y\}$ is a set containing only one element.

In our paper, assume that $\Bbbk$ and $[a, b]_\Bbbk$ are infinite sets and consider only the case for $\mathbb{I} = [a, b]_\Bbbk$ with $a \prec b$ such that there exists an element ξ with $a \prec \xi \prec b$ such that the order preserving bijections $\kappa_a : \mathbb{I} \to [a, \xi]_\Bbbk$ and $\kappa_b : \mathbb{I} \to [\xi, b]_\Bbbk$ exist (for example, the case of the cardinal number of $\mathbb{I}$ to be either $\aleph_0$ or $\aleph_1$). In this section, we introduce the category $\mathscr{N}or^p$, which is used to explore the categorization of integration.

4.1. **Norms of Λ-modules.** Recall that a *left A-module* (=A-module for short) over a $\Bbbk$-algebra A is a $\Bbbk$-linear space V with a $\Bbbk$-linear map $h : A \to \mathrm{End}_\Bbbk V$ sending a to h_a. Thus, h provides a right action $A \times V \to V$, $(a, v) \mapsto va := h_a(v)$ which satisfies the following properties:

1. $a(v + v') = av + av'$ for any $v, v' \in V$ and $a \in A$;
2. $(a + a')v = av + a'v$ for any $v \in V$ and $a, a' \in A$;
3. $a'(av) = (a'a)v$ for any $v \in V$ and $a, a' \in A$;

(4) $1v = v$ for any $v \in V$;
(5) $(ka)v = k(av) = a(kv)$ for any $v \in V$, $a \in A$ and $k \in \Bbbk$.

Take $A = \Lambda$ is the normed space with whose norm $\|\cdot\|_p : \Lambda \to \mathbb{R}^+$ given by (3.1), where the $\Bbbk$-basis of Λ is $B_\Lambda = \{b_i \mid 1 \le i \le n = \dim_{\Bbbk} \Lambda\}$.

Definition 4.1. Let $\tau : \Lambda \to \Bbbk$ be a homomorphism between two normed $\Bbbk$-algebras $(\Lambda, \|\cdot\|_p)$ and $(\Bbbk, |\cdot|)$. A *τ-normed Λ-module* is a Λ-module M with a norm $\|\cdot\| : M \to \mathbb{R}^{\ge 0}$ such that

$$\|am\| = |\tau(a)| \cdot \|m\| \text{ holds for all } a \in \Bbbk \text{ and } m \in M. \tag{4.1}$$

Thus, each normed Λ-module can be seen as a triple $(M, h, \|\cdot\|)$ of the $\Bbbk$-linear space M, the $\Bbbk$-linear map $h : M \to \mathrm{End}_{\Bbbk} M$ and $\|\cdot\| : M \to \mathbb{R}^{\ge 0}$ a norm.

The norms of Λ-modules yield that the following fact.

Fact 4.2.

(1) Note that $\|\cdot\|_p$ defined by (3.1) is the norm of Λ as a $\Bbbk$-linear space. It is easy to see that Λ is also a left Λ-module, say *regular module*, where the scalar multiplication is given by the multiplication $\Lambda \times \Lambda \to \Lambda, (a, x) \mapsto ax$ of Λ as a finite-dimensional $\Bbbk$-algebra. Thus, it is natural to ask whether $\|\cdot\|_p$ is a norm of Λ as a Λ-module. Indeed, the norm of Λ as a finite-dimensional $\Bbbk$-algebra may not be equal to the norm $\|\cdot\|$ of Λ as a regular module. However, if Λ as the left Λ-module defined by

$$\Lambda \times \Lambda \to \Lambda, (a, x) \mapsto a \star x := \tau(a)x, \tag{4.2}$$

where $\tau(a)x$ is defined by the scalar multiplication of Λ as the $\Bbbk$-linear space ${}_{\Bbbk}\Lambda$, then, for any $x = \sum_{i=1}^{n} k_i b_i \in \Lambda$, we obtain

$$\begin{aligned}\|a \star x\|_p &= \left\| \tau(a) \sum_{i=1}^{n} k_i b_i \right\|_p = \left(\sum_{i=1}^{n} |\tau(a) k_i|^p \mathfrak{n}(b_i)^p \right)^{\frac{1}{p}} \\ &= |\tau(a)| \left(\sum_{i=1}^{n} |k_i|^p \mathfrak{n}(b_i)^p \right)^{\frac{1}{p}} = |\tau(a)| \|x\|_p.\end{aligned}$$

To be more precise, Λ is a (Λ, Λ)-bimodule with two norms, and Λ is a normed module satisfying Definition 4.1 when it is considered as a module defined in (4.2).

(2) For any Λ-homomorphism $f : M \to N$ of two Λ-modules M and N, if M and N are normed Λ-modules, that is, $M = (M, h_M, \|\cdot\|_M)$ and $N = (N, h_N, \|\cdot\|_N)$, then we have

$$\|f(am)\|_N = \|af(m)\|_N = |\tau(a)| \cdot \|f(m)\|_N$$

Example 4.3. Let

$$\Lambda = \begin{pmatrix} \Bbbk & 0 \\ \Bbbk & \Bbbk \end{pmatrix}.$$

Then a $\Bbbk$-basis of Λ is $B_\Lambda = \{\boldsymbol{E}_{11}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}\}$, where $\boldsymbol{E}_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\boldsymbol{E}_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\boldsymbol{E}_{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Take $\mathfrak{n}$ be the map $B_\Lambda \to \mathbb{R}^+$ defined by $\mathfrak{n}(\boldsymbol{E}_{11}) = \mathfrak{n}(\boldsymbol{E}_{21}) = \mathfrak{n}(\boldsymbol{E}_{22}) = 1$, then for any element $x = \left(\begin{smallmatrix} k_{11} & 0 \\ k_{21} & k_{22} \end{smallmatrix}\right)$ in Λ, we have $\|x\|_p = (|k_{11}|^p + |k_{21}|^p + |k_{22}|^p)^{\frac{1}{p}}$.

There are three indecomposable Λ-modules up to Λ-isomorphism:

$$P(1) = \left(\begin{smallmatrix} \Bbbk & 0 \\ \Bbbk & 0 \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} 0 & \Bbbk \\ 0 & \Bbbk \end{smallmatrix}\right), P(2) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \Bbbk \end{smallmatrix}\right), \text{ and the cokernel } \mathrm{coker}\left(P(2) \to P(1) \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right).$$

Then each Λ-module M is isomorphic to the direct sum $P(1)^{\oplus t_1} \oplus P(2)^{\oplus t_2} \oplus (P(1)/P(2))^{\oplus t_3}$ for some $t_1, t_2, t_3 \in \mathbb{N}$. Assume that $M = (M, h_M, \|\cdot\|_M)$ and $N = (N, h_N, \|\cdot\|_N)$ are two normed Λ-modules. Then, naturally, $M \oplus N$ is also a Λ-module, where the left Λ-action is the map

$$h_M \oplus h_N := \begin{pmatrix} h_M & 0 \\ 0 & h_N \end{pmatrix} : \Lambda \times M \oplus N \to M \oplus N$$

which sends $(a, \binom{m}{n})$ to

$$\begin{pmatrix} h_M & 0 \\ 0 & h_N \end{pmatrix} \binom{m}{n} = \begin{pmatrix} (h_M)_a(m) \\ (h_N)_a(n) \end{pmatrix} = \binom{am}{an}.$$

Furthermore, we can use the τ-norms of M and N, that is, $\|\cdot\|_M$ and $\|\cdot\|_N$, to define a τ-norm $\|\cdot\|_{M\oplus N}$ of $M \oplus N$ by

$$\|(m,n)\|_{M\oplus N} := (|k|(\|m\|_M^p + \|n\|_N^p))^{\frac{1}{p}} \text{ for given } k \in \Bbbk\backslash\{0\}.$$

Then we have

$$\begin{aligned}\|a(m,n)\|_{M\oplus N} &= (|k|(\|am\|_M^p + \|an\|_N^p))^{\frac{1}{p}} = (|k|(|\tau(a)|^p\|m\|_M^p + |\tau(a)|^p\|n\|_N^p))^{\frac{1}{p}} \\ &= |\tau(a)|\,(|k|(\|m\|_M^p + \|n\|_N^p))^{\frac{1}{p}} = |\tau(a)|\|(m,n)\|_{M\oplus N}\end{aligned}$$

for any $a \in \Lambda$.

Example 4.4. The quiver of the $\Bbbk$-algebra Λ given in Example 4.3 is $\mathcal{Q} = 1 \xrightarrow{\alpha} 2$. By the representation theory all Λ-modules M can be represented by $M_1 \xrightarrow{\varphi_a} M_2$, where M_1 and M_2 are two $\Bbbk$-linear spaces and φ_a is a $\Bbbk$-linear map. Indeed, the identity element of Λ is $\boldsymbol{E} = \boldsymbol{E}_{11} + \boldsymbol{E}_{22}$, where $\boldsymbol{E}_{11}, \boldsymbol{E}_{22}$ are the complete set of primitive orthogonal idempotents. Thus, M, as a $\Bbbk$-linear space, has a decomposition $M = \boldsymbol{E}_{11}M \oplus \boldsymbol{E}_{22}M$ (because $\boldsymbol{E}_{11}\boldsymbol{E}_{22} = 0$ yields $\boldsymbol{E}_{11}M \cap \boldsymbol{E}_{22}M = 0$). For any $a = k_{11}\boldsymbol{E}_{11} + k_{22}\boldsymbol{E}_{22} + k_{21}\boldsymbol{E}_{21}$ and $m \in M$, we have

$$\begin{aligned} am &= (k_{11}\boldsymbol{E}_{11} + k_{22}\boldsymbol{E}_{22} + k_{21}\boldsymbol{E}_{21})(\boldsymbol{E}_{11}m + \boldsymbol{E}_{22}m) \\ &= k_{11}\boldsymbol{E}_{11}(\boldsymbol{E}_{11}m) + k_{22}\boldsymbol{E}_{22}(\boldsymbol{E}_{22}m) + k_{21}\boldsymbol{E}_{21}(\boldsymbol{E}_{11}m) \\ &= k_{11}(h_M)_{\boldsymbol{E}_{11}}(\boldsymbol{E}_{11}m) + k_{22}(h_M)_{\boldsymbol{E}_{22}}(\boldsymbol{E}_{22}m) + k_{21}(h_M)_{\boldsymbol{E}_{21}}(\boldsymbol{E}_{11}m) \\ &= (h_M)_{\boldsymbol{E}_{11}}(k_{11}\boldsymbol{E}_{11}m) + (h_M)_{k_{22}\boldsymbol{E}_{22}}(\boldsymbol{E}_{22}m) + (h_M)_{\boldsymbol{E}_{21}}(k_{21}\boldsymbol{E}_{11}m), \end{aligned} \tag{4.3}$$

where

(a) $h_M : \Lambda \to \mathrm{End}_{\Bbbk}M$ is a homomorphism of $\Bbbk$-algebras sending a to $(h_M)_a$, which satisfies $\mathbf{1}_M = (h_M)_{\boldsymbol{E}} = (h_M)_{\boldsymbol{E}_{11}} + (h_M)_{\boldsymbol{E}_{22}}$;
(b) $(h_M)_{\boldsymbol{E}_{ii}} = \mathbf{1}_{\boldsymbol{E}_{ii}M}$ $(i = 1, 2)$;
(c) $(h_M)_{\boldsymbol{E}_{12}}$ is a $\Bbbk$-linear map from $\boldsymbol{E}_{11}M$ to $\boldsymbol{E}_{22}M$ (this is equivalent to (4.3)).

Therefore, we obtain that the representation corresponding to $M = \boldsymbol{E}_{11}M \oplus \boldsymbol{E}_{22}M$ is

$$\boldsymbol{E}_{11}M \xrightarrow{\boldsymbol{E}_{21}} \boldsymbol{E}_{22}M.$$

Generally, $M_1 \xrightarrow{\varphi_a} M_2$ corresponds to the module $M_1 \oplus M_2$, where the Λ-action $\Lambda \times M_1 \oplus M_2 \to M_1 \oplus M_2$ is defined by

$$\boldsymbol{E}_{11}(m_1, m_2) = (m_1, 0),\ \boldsymbol{E}_{22}(m_1, m_2) = (0, m_2) \text{ and } \boldsymbol{E}_{12}(m_1, m_2) = (0, \varphi_\alpha(m_1)).$$

Without loss of generality, for any representation $M_1 \xrightarrow{\varphi_a} M_2$ of $\mathcal{Q}$, assume that $M_1 = \Bbbk^{\oplus t_1}$, $M_2 = \Bbbk^{\oplus t_2}$ and $\varphi_a \in \mathbf{Mat}_{t_2\times t_1}(\Bbbk)$ (up to Λ-isomorphism), and for any $i = 1, 2$, M_i is a normed space equipping with the norm $\|\cdot\|_{M_i} : M_i = \Bbbk^{\oplus t_i} \to \mathbb{R}^+$ sending $m_i = (m_{ij})_{1\le j\le t_i}$ to $\left(\sum_{j=1}^{t_i} |m_{ij}|^p\right)^{\frac{1}{p}}$. Then we can define a norm $\|\cdot\|_{M_1\oplus M_2}$ by

$$\|(m_1, m_2)\|_{M_1\oplus M_2} = (|k|(\|m_1\|_{M_1}^p + \|m_2\|_{M_2}^p))^{\frac{1}{p}},$$

where k is a given element in $\Bbbk\backslash\{0\}$. The direct sum "$\oplus$" of $\Bbbk$-linear spaces is the p powers of the norm preserving in the case for $k = 1$, that is, $\|(m_1, m_2)\|_{M_1\oplus M_2}^p =$

$\|m_1\|_{M_1}^p + \|m_2\|_{M_2}^p$. Furthermore, if $\|\cdot\|_{M_1}$ and $\|\cdot\|_{M_2}$ are τ-norms of M_1 and M_2, respectively, then, for any $a \in \Lambda$, we have

$$\begin{aligned}\|a(m_1, m_2)\|_{M_1\oplus M_2} &= \big(|k|(\|am_1\|_{M_1}^p + \|am_2\|_{M_2}^p)\big)^{\frac{1}{p}} \\ &= \big(|k|(|\tau(a)|^p\|m_1\|_{M_1}^p + |\tau(a)|^p\|m_2\|_{M_2}^p)\big)^{\frac{1}{p}} \\ &= |\tau(a)|\big(|k|(\|m_1\|_{M_1}^p + \|m_2\|_{M_2}^p)\big)^{\frac{1}{p}} \\ &= |\tau(a)|\|a(m_1, m_2)\|_{M_1\oplus M_2}.\end{aligned}$$

4.2. **Completions of normed Λ-modules.** Let $N = (N, h, \|\cdot\|)$ be a normed Λ-module. In this part we construct its completion. For us, we need the completion of the finite-dimensional $\Bbbk$-algebra Λ. Otherwise, there is at least one Λ-module which is not complete, for instance, Λ is a non-complete Λ-module. Therefore, we assume that $\Bbbk$ is complete in this subsection by Propositions 2.16 and 3.3.

Similar to finite-dimensional $\Bbbk$-algebras, we can define open neighborhoods $B(0, r)$ of 0 for any normed Λ-module $N = (N, h, \|\cdot\|)$ by

$$B(0, r) := \{x \in N \mid \|x\| < r\}.$$

Let $\mathfrak{U}_N^B(0)$ be the class of all subsets U of N satisfying the following conditions.

(1) U is the intersection of a finite number of $B(0, r)$;

(2) U is the union of any number of $B(0, r)$.

Then $\mathfrak{U}_N^B(0)$ is a topology defined on Λ, and we can define the Cauchy sequence by the above topology.

Lemma 4.5. *Let $\mathfrak{C}^*(N)$ be the set of all Cauchy sequences in the normed Λ-module $N = (N, h, \|\cdot\|)$. Then $\mathfrak{C}^*(N)$ is a Λ-module.*

Proof. First of all, $\mathfrak{C}^*(N)$ is a $\Bbbk$-linear space whose addition and $\Bbbk$-action are given by

$$\{x_i\}_{i\in\mathbb{N}} + \{y_i\}_{i\in\mathbb{N}} = \{x_i + y_i\}_{i\in\mathbb{N}} \ \ (\forall \{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}} \in \mathfrak{C}^*(N))$$
$$\text{and } k\{x_i\}_{i\in\mathbb{N}} = \{kx_i\}_{i\in\mathbb{N}} \ \ (\forall k \in \Bbbk),$$

respectively. Furthermore, define

$$\Lambda \times \mathfrak{C}^*(N) \to \mathfrak{C}^*(N),\ (a, \{x_i\}_{i\in\mathbb{N}}) \mapsto a \cdot \{x_i\}_{i\in\mathbb{N}} := \{a \cdot x_i\}_{i\in\mathbb{N}},$$

where $a \cdot x_i = h_a(x_i)$. Then $\mathfrak{C}^*(N)$ is a Λ-module. □

Two Cauchy sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ in N are called *equivalent*, denoted by $\{x_i\}_{i\in\mathbb{N}} \sim \{y_i\}_{i\in\mathbb{N}}$, if for any $U \in \mathfrak{U}_N^B(0)$, there is $r \in \mathbb{N}$ such that $x_s - x_t \in U$ holds for all $s, t \geq r$. It is easy to see that "$\sim$" is an equivalence relation. Let $[\{x_i\}_{i\in\mathbb{N}}]$ be the equivalent class of Cauchy sequences containing $\{x_i\}_{i\in\mathbb{N}}$ and let $\mathfrak{C}(N)$ be the set of all equivalent classes. We naturally obtain a map

$$h : \mathfrak{C}^*(N) \to \mathfrak{C}(N),\quad \{x_i\}_{i\in\mathbb{N}} \mapsto [\{x_i\}_{i\in\mathbb{N}}].$$

We can show that $\mathfrak{C}(N)$ is a Λ-module by using an argument similar to that in the proof of Lemma 4.5, and further obtain $\mathrm{Ker}(h : \mathfrak{C}^*(N) \to \mathfrak{C}(N)) = [\{0\}_{i\in\mathbb{N}}]$. Thus we have

$$\mathfrak{C}(N) \cong \mathfrak{C}^*(N)/[\{0\}_{i\in\mathbb{N}}].$$

Then $\mathfrak{C}(N)$ is complete, and we call it is the *completion* of N. We use $\widehat{N}$ to denote the completion $\mathfrak{C}(N)$ of N. The Λ-module $\widehat{N}$ is a normed Λ-module, where the norm defined on $\widehat{N}$ is induced by the norm $\|\cdot\| : N \to \mathbb{R}^{\geq 0}$ defined on N.

Definition 4.6. Assume that Λ is complete. A normed Λ-module N is called a *Banach Λ-module* if $\widehat{N} = N$ (i.e. N is complete).

4.3. **σ-algebras and the elementary simple function set $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$.** Take τ to be a homomorphism of $\Bbbk$-algebras $\tau : \Lambda \to \Bbbk$. Then the elementary simple function set $\mathbf{S}(\mathbb{I}_\Lambda)$ with the above homomorphism τ, denoted by $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$, is a Λ-module, where the Λ-action $\Lambda \times \mathbf{S}(\mathbb{I}_\Lambda) \to \mathbf{S}(\mathbb{I}_\Lambda)$ is given by

$$(a, f = \sum_{i=1}^t k_i \mathbf{1}_{I_i}) \mapsto af := \sum_{i=1}^t \tau(a)k_i\mathbf{1}_{I_i}$$

because, for all $a \in \Lambda$, $a' \in \Lambda$, $k \in \Bbbk$, $f = \sum_i k_i\mathbf{1}_{I_i} \in \mathbf{S}(\mathbb{I}_\Lambda)$ and $f' = \sum_j k'_j\mathbf{1}_{I'_j} \in \mathbf{S}(\mathbb{I}_\Lambda)$, the following conditions are satisfied:

(1) $a(f + f') = af + af'$ (trivial);
(2) $(a + a')f = af + a'f$ (trivial);
(3) $(aa')f = a(a'f)$ because

$$\begin{aligned}(aa')f = (aa')\sum_i k_i\mathbf{1}_{I_i} &= \sum_i \tau(aa')k_i\mathbf{1}_{I_i} = \sum_i \tau(a)\tau(a')k_i\mathbf{1}_{I_i}\\ &= a\sum_i \tau(a')k_i\mathbf{1}_{I_i} = a(a'\sum_i k_i\mathbf{1}_{I_i}) = a(a'f)\end{aligned}$$

(4) $1f = f$ (trivial);
(5) We have
   - $(ka)f = (ka)\sum_i k_i\mathbf{1}_{I_i} = \sum_i \tau(ka)(k_i\mathbf{1}_{I_i})$,
   - $k(af) = k(a\sum_i k_i\mathbf{1}_{I_i}) = k\sum_i \tau(a)k_i\mathbf{1}_{I_i} = \sum_i k(\tau(a)(k_i\mathbf{1}_{I_i}))$,
   - and $a(kf) = a\sum_i k(k_i\mathbf{1}_{I_i}) = \sum_i \tau(a)(k(k_i\mathbf{1}_{I_i}))$.

   Since τ is a homomorphism of $\Bbbk$-algebras, we have

$$\tau(ka)(k_i\mathbf{1}_{I_i}) = k(\tau(a)(k_i\mathbf{1}_{I_i})) = \sum_i \tau(a)(k(k_i\mathbf{1}_{I_i})) = \sum_i kk_i\tau(a)\mathbf{1}_{I_i},$$

   for all i. Then $(ka)f = k(af) = a(kf)$.

Now, we introduce a norm for $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ such that it is a normed Λ-module. To do this, we firstly recall the definition of σ-algebras.

Definition 4.7. Let S be a set and $P(S)$ the power set of S, that is, $P(S)$ is the set of all subsets of S. A *σ-algebra* is a subset $\mathcal{A}$ of $P(S)$ satisfying the following conditions:

(1) $\varnothing$ and S lie in $\mathcal{A}$;
(2) for any $X \in \mathcal{A}$, the complement set $X^c := S\backslash X$ of X lies in $\mathcal{A}$;
(3) for any $X_1, \ldots, X_n \ldots \in \mathcal{A}$, the union $\bigcup_{i=1}^\infty X_i$ is an element in $\mathcal{A}$.

For a class $\mathcal{C}$ of some sets lying in $P(S)$, we call $\mathcal{A}$ is a *σ-algebra generated by $\mathcal{C}$* if $\mathcal{A}$ is the minimal σ-algebra containing $\mathcal{C}$.

Let $\Sigma_\Bbbk$ be the σ-algebra generated by $\{(a,b)_\Bbbk, [a,b)_\Bbbk, (a,b]_\Bbbk, [a,b]_\Bbbk \mid a \preceq b\}$, and let $\mu : \Sigma_\Bbbk \to \mathbb{R}^{\geq 0}$ be a *measure* such that $\mu(\{k\}) = 0$ holds for any $k \in \Bbbk$, that is, μ is a function satisfying the following conditions:

(1) $\mu(\varnothing) = 0$;
(2) $\mu(\bigcup_{i\in\mathbb{N}} X_i) = \sum_{i\in\mathbb{N}} \mu(X_i)$ holds for all sets $X_1, X_2, \ldots$ satisfying $X_i \cap X_j = \varnothing$ $(i \neq j)$.

Any two functions f and g in $\mathbf{S}(\mathbb{I}_\Lambda)$ are called *equivalent* if

$$\mu(\{\boldsymbol{k} = (k_1, \ldots, k_n) \in \Bbbk^{\oplus n} \mid f(\boldsymbol{k}) \neq g(\boldsymbol{k})\}) = 0.$$

The equivalent class containing f is written as $[f]$. Then we obtain an epimorphism

$$\mathbf{S}(\mathbb{I}_\Lambda) \to \overline{\mathbf{S}(\mathbb{I}_\Lambda)} := \{[f] \mid f \in \mathbf{S}(\mathbb{I}_\Lambda)\}$$

sending each function to its equivalent classes. It is easy to see that the kernel of the above epimorphism is $[0]$, then we have

$$\overline{\mathbf{S}(\mathbb{I}_\Lambda)} \cong \mathbf{S}(\mathbb{I}_\Lambda)/[0].$$

For simplification, we do not differentiate between two equivalent functions under the above isomorphism. Therefore, we treat $\mathbf{S}(\mathbb{I}_\Lambda)$ and the quotient $\overline{\mathbf{S}(\mathbb{I}_\Lambda)}$ equivalently.

Lemma 4.8. *Let $\tau : \Lambda \to \Bbbk$ be a homomorphism between two $\Bbbk$-algebras. Then the Λ-module $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ with the map*

$$\|\cdot\|_p : \mathbf{S}_\tau(\mathbb{I}_\Lambda) \to \mathbb{R}^{\geq 0},\ f = \sum_{i=1}^{t} k_i \mathbf{1}_{I_i} \mapsto \left(\sum_{i=1}^{t} (|k_i|\mu(I_i))^p\right)^{\frac{1}{p}}$$

is normed.

Proof. Let f be an arbitrary function lying in $\mathbf{S}(\mathbb{I}_\Lambda)$. It is trivial that $\|f\|_p$ is non-negative. Let a be an arbitrary element in Λ and assume $f = \sum_{i=1}^{t} k_i \mathbf{1}_{I_i}$. We have

$$\begin{aligned}\|af\|_p &= \Big\|\sum_{i=1}^{t} \tau(a)k_i\mathbf{1}_{I_i}\Big\|_p = \left(\sum_{i=1}^{t} |\tau(a)k_i|^p \mu(\mathbf{1}_{I_i})^p\right)^{\frac{1}{p}} \\ &= |\tau(a)| \cdot \left(\sum_{i=1}^{t} |k_i|^p \mu(\mathbf{1}_{I_i})^p\right)^{\frac{1}{p}} = |\tau(a)| \cdot \|f\|_p\end{aligned}$$

which satisfies the formula (4.1) for the case $(\Lambda', \|\cdot\|_2) = (\Bbbk, |\cdot|)$. In particular, if $\|f\|_p = 0$, then so is $(|k_i|\mu(I_i))^p = 0$ for all i, and we have $|k_i| = 0$ in the case for $\mu(I_i) \neq 0$. If $\mu(I_j) = 0$ holds for some $j \in J$ $(\subseteq \{1, 2, \ldots, t\})$, then we have $f = \sum_{j\in J} k_j \mathbf{1}_{I_j}$. Clearly,

$$\mu(\{x \in \mathbb{I}_\Lambda \mid f(x) \neq 0\}) = \sum_{j\in J} \mu(I_j) = 0,$$

that is, $f = 0$ in treating $\mathbf{S}(\mathbb{I}_\Lambda)$ and the quotient $\overline{\mathbf{S}(\mathbb{I}_\Lambda)}$ equivalently. Thus, $\|f\|_p = 0$ if and only if $f = 0$.

Next, we prove the the triangle inequality. For two arbitrary functions $f = \sum_i k_i \mathbf{1}_{I_i}$ and $g = \sum_j l_j \mathbf{1}_{I'_j}$, we have

$$f + g = \sum\nolimits_i k_i \mathbf{1}_{I_i \backslash \bigcup_j I'_j} + \sum\nolimits_j l_j \mathbf{1}_{I'_j \backslash \bigcup_i I_i} + \sum_{I_i \cap I'_j \neq \varnothing} (k_i \mathbf{1}_{I_i \cap I'_j} + l_j \mathbf{1}_{I_i \cap I'_j}) \tag{4.4}$$

by $I_i \cap I_\imath = \varnothing$ $(\forall i \neq \imath)$ and $I'_j \cap I'_\jmath = \varnothing$ $(\forall j \neq \jmath)$. Then we can compute the norm of $f + g$ by (4.4) as the following formula:

$$\|f + g\|_p = (R + G + B)^{\frac{1}{p}},$$

where

$$\begin{aligned}R &= \sum\nolimits_i |k_i|^p \mu(I_i \backslash \bigcup\nolimits_j I'_j)^p; \\ G &= \sum\nolimits_j |l_j|^p \mu(I'_j \backslash \bigcup\nolimits_i I_i)^p; \\ B &= \sum_{I_i \cap I'_j \neq \varnothing} (|k_i|^p + |l_j|^p)\mu(I_i \cap I'_j)^p.\end{aligned}$$

On the other hand, we have the following inequality by the discrete Minkowski inequality:

$$\begin{aligned}\|f\|_p + \|g\|_p =& \left(\sum\nolimits_i |k_i|^p \mu(I_i)^p\right)^{\frac{1}{p}} + \left(\sum\nolimits_j |l_i|^p \mu(I'_i)^p\right)^{\frac{1}{p}} \\ \geq& \left(\sum\nolimits_i |k_i|^p \mu(I_i)^p + \sum\nolimits_j |l_i|^p \mu(I'_i)^p\right)^{\frac{1}{p}} =: \mathfrak{S}.\end{aligned} \tag{4.5}$$

Since, by the definition of measure, $\mu(X \cup Y) = \mu(X) + \mu(Y)$ holds for any X, Y with $X \cap Y = \varnothing$, we obtain

$$\mu(X \cup Y)^p \geq \mu(X)^p + \mu(Y)^p, \tag{4.6}$$

then

$$\mu(I_i)^p \geq \mu\big(I_i \backslash \bigcup\nolimits_j I'_j\big)^p + \mu\big(I_i \cap \bigcup\nolimits_j I'_j\big)^p.$$

Thus,

$$\begin{aligned} \sum\nolimits_i |k_i|^p \mu(I_i)^p &\geq \sum\nolimits_i |k_i|^p \mu\big(I_i \backslash \bigcup\nolimits_j I'_j\big)^p + \sum\nolimits_i |k_i|^p \mu\big(I_i \cap \bigcup\nolimits_j I'_j\big)^p \\ &= R + \sum\nolimits_i |k_i|^p \Bigg(\sum_{\substack{j \\ I_i \cap I'_j \neq \varnothing}} \mu(I_i \cap I'_j) \Bigg)^p \\ &\overset{(4.6)}{\geq} R + \sum_{I_i \cap I'_j \neq \varnothing} |k_i|^p \mu(I_i \cap I'_j)^p. \end{aligned} \tag{4.7}$$

Similarly,

$$\sum\nolimits_j |l_j|^p \mu(I'_j)^p \geq G + \sum_{I'_j \cap I_i \neq \varnothing} |l_j|^p \mu(I'_j \cap I_i)^p. \tag{4.8}$$

Notice that

$$\sum_{I_i \cap I'_j \neq \varnothing} |k_i|^p \mu(I_i \cap I'_j)^p + \sum_{I'_j \cap I_i \neq \varnothing} |l_j|^p \mu(I'_j \cap I_i)^p = \sum_{I_i \cap I'_j \neq \varnothing} (|k_i|^p + |l_j|^p) \mu(I_i \cap I'_j)^p = B,$$

then (4.7)+(4.8) induces $\mathfrak{S}^p \geq R + G + B$. Thus, the triangle inequality $\|f\|_p + \|g\|_p \geq \|f + g\|_p$ holds. □

5. Two categories

Let $\dim_{\Bbbk} \Lambda = n$, and let N be a normed Λ-module equipped with two additional pieces of data: an element $v \in N$ such that $\|v\| \leq \mu(\mathbb{I}_\Lambda)$, and a continuous Λ-homomorphism $\delta : N^{\oplus_p 2^n} \to N$. Here, $\oplus_p$ denotes the direct sum of 2^n Λ-modules $X_1, \ldots, X_{2^n}$ with the norm defined as follows:

$$\| \cdot \|_p : \bigoplus_{i=1}^{2^n}{}_p X_i \to \mathbb{R}^{\geq 0}, (x_1, x_2, \ldots, x_{2^n}) \mapsto \left(\left(\frac{\mu(\mathbb{I})}{\mu(\mathbb{I}_\Lambda)} \right)^n \sum_{i=1}^{2^n} \|x_i\|^p \right)^{\frac{1}{p}}.$$

5.1. **The categories $\mathscr{N}or^p$ and $\mathscr{A}^p$.** Let $\mathscr{N}or^p$ be a class of triples which are of the form (N, v, δ), where N is a normed Λ-module, $v \in N$ is an element with $\|v\|_p \leq \mu(\mathbb{I}_\Lambda)$ and $\delta : N^{\oplus_p 2^n} \to N$ is a Λ-homomorphism satisfying $\delta(v, v, \ldots, v) = v$ such that for any Cauchy sequence $\{x_i\}_{i \in \mathbb{N}} \in \widehat{N^{\oplus_p 2^n}} \cong \widehat{N}^{\oplus_p 2^n}$, the commutativity

$$\varprojlim \delta(x_i) = \delta(\varprojlim x_i) \tag{5.1}$$

of the inverse limit and the Λ-homomorphism holds. For any two triples (N, v, δ) and (N', v', δ') in $\mathscr{N}or^p$, we define the morphism $(N, v, \delta) \to (N', v', \delta')$ to be the Λ-homomorphism $\theta : N \to N'$ with $\theta(v) = v'$ such that the following diagram

$$\begin{array}{ccc} N^{\oplus_p 2^n} & \xrightarrow{\ \delta\ } & N \\ {\scriptstyle \theta^{\oplus 2^n} = \begin{pmatrix} \theta & & \\ & \ddots & \\ & & \theta \end{pmatrix}_{2^n \times 2^n}} \Big\downarrow & & \Big\downarrow {\scriptstyle \theta} \\ N'^{\oplus_p 2^n} & \xrightarrow[\ \delta'\ ]{} & N' \end{array}$$

commutes, that is, for any $(v_1, \dots, v_{2^n}) \in N^{\oplus_p 2^n}$, $\theta(\delta(v_1, \dots, v_{2^n})) = \delta'(\theta(v_1), \dots, \theta(v_{2^n}))$. Then it is easy to check that $\mathscr{N}or^p$ is a category.

Lemma 5.1. *Let*

(1) ξ *be an element in* $\mathbb{I} = [a,b]_{\Bbbk}$ *with* $a \prec \xi \prec b$ *such that there exists an element* ξ *with* $a \prec \xi \prec b$ *such that the order preserving linear bijections* $\kappa_a : \mathbb{I} \to [a,\xi]_{\Bbbk}$ *and* $\kappa_b : \mathbb{I} \to [\xi, b]_{\Bbbk}$ *exist,*
(2) $\mathbf{1}$ *be the identity function* $\mathbf{1}_{\mathbb{I}_\Lambda} : \mathbb{I}_\Lambda \to \{1\}$,
(3) γ_ξ *be the map given in (3.2),*
(4) $\tau : \Lambda \to \Bbbk$ *be the homomorphism of* $\Bbbk$*-algebras given in Lemma 4.8.*

Then the following statements hold.

(a) $\gamma_\xi(\mathbf{1}, \mathbf{1}, \cdots, \mathbf{1}) = \mathbf{1}$;
(b) γ_ξ *is a* Λ*-homomorphism.*

First, we provide a remark for the above lemma.

Remark 5.2. Indeed, $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ is an object in the category $\mathscr{N}or^p$. However, Lemma 5.1 points out that $\gamma_\xi(\mathbf{1}, \mathbf{1}, \cdots, \mathbf{1}) = \mathbf{1}$ and γ_ξ is a Λ-homomorphism. Thus, we need to show that the commutativity of the inverse limit and γ_ξ holds. We will prove this result in the following content, as shown in Lemma 5.5.

Next, we prove Lemma 5.1.

Proof. (a) We have that $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ is a normed Λ-module by Lemma 4.8, and γ_ξ is a $\Bbbk$-linear map by Lemma 3.6. The formula $\gamma_\xi(\mathbf{1}, \dots, \mathbf{1}) = \mathbf{1}$ can be directly induced by the definition of γ_ξ.

(b) Take $\lambda \in \Lambda$, $f \in \mathbf{S}(\mathbb{I}_\Lambda)$ and let $(k_i)_i$, $\mathbf{1}$ and $(\delta_i)_i$ be an arbitrary element $(k_1, \dots, k_n)$ in $\mathbf{S}(\mathbb{I}_\Lambda)^{\oplus 2^n}$, the identity function $\mathbf{1}_{\kappa_{\delta_1}(\mathbb{I}) \times \cdots \times \kappa_{\delta_n}(\mathbb{I})}$ and the n-multiple $(\delta_1 \times \cdots \times \delta_n)$, respectively. Then we have

$$
\begin{aligned}
&\gamma_\xi(\lambda \cdot f)((k_i)_i) \\
&= \sum_{(\delta_i)_i} \mathbf{1} \cdot (\tau(\lambda) f)_{(\delta_i)_i}((\kappa_{\delta_i}^{-1}(k_i))_i) \\
&= \tau(\lambda)\gamma_\xi(f)((k_i)_i) \quad \text{(similar to Lemma 3.6)} \\
&= \lambda \cdot \gamma_\xi(f)((k_i)_i).
\end{aligned}
$$

Thus γ_ξ is a Λ-homomorphism. □

Let $\mathscr{A}^p$ denote a class of triples which are of the form $(\widehat{N}, v, \widehat{\delta})$, where $\widehat{N}$ is a Banach Λ-module (see Definition 4.6), $v \in \widehat{N}$ is an element with $\|v\|_p \le \mu(\mathbb{I}_\Lambda)$ and $\delta : \widehat{N}^{\oplus_p 2^n} \to \widehat{N}$ is a Λ-homomorphism satisfying $\widehat{\delta}(v, v, \dots, v) = v$. Obviously, $\mathscr{A}^p$ is a full subcategory of $\mathscr{N}or^p$.

5.2. **The triple $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$.** Let (N, v, δ) be an object in $\mathscr{N}or^p$ and $\widehat{N}$ the completion of the Λ-module N. Then $\widehat{N}$, as a $\Bbbk$-linear space, is a Banach space which is a Banach Λ-module. And, naturally, we obtain the Λ-homomorphism

$$\widehat{\delta} : \widehat{N}^{\oplus_p 2^n} \to \widehat{N}$$

induced by the Λ-homomorphism δ. Furthermore, we have that $(\widehat{N}, v, \widehat{\delta})$ is also an object in $\mathscr{N}or^p$, and there is a naturally embedding morphism

$$\mathrm{emb} : (N, v, \delta) \hookrightarrow (\widehat{N}, v, \widehat{\delta})$$

which is induced by $N \subseteq \widehat{N}$.

Notation 5.3. Keep the notations $\xi =: \xi_{11}$, κ_a, κ_b, $\mathbf{1}$, γ_ξ and τ as in Lemma 5.1. Then ξ_{11} divides $\mathbb{I} =: \mathbb{I}^{(01)}$ to two subsets $[a, \xi_{11}]_{\Bbbk} =: \mathbb{I}^{(11)}$ and $[\xi_{11}, b]_{\Bbbk} =: \mathbb{I}^{(12)}$. Next, let $\xi_{22} = \xi_{11}$ $(= \xi)$, and denote by ξ_{21} and ξ_{23} the two elements in $\mathbb{I}_\Lambda$ such that

- $a \prec \xi_{21} = \kappa_a\kappa_a(b) = \kappa_a\kappa_b(a) = \kappa_b\kappa_a(a) = \kappa_a(\xi_{11}) \prec \xi_{22}$;
- $\xi_{22} \prec \xi_{23} = \kappa_b\kappa_b(a) = \kappa_b\kappa_a(b) = \kappa_b\kappa_a(b) = \kappa_b\xi_{11} \prec b$.

Then $\mathbb{I}_\Lambda$ is divided to four subsets which are of the form $\mathbb{I}^{(2t)} = [\xi_{2t}, \xi_{2\ t+1}]_{\Bbbk}$ $(0 \le t \le 3)$ by $a = \xi_{20} \prec \xi_{21} \prec \xi_{22} \prec \xi_{23} \prec \xi_{24} = b$. Repeating the above step t times, we obtain a sequence of $2^t - 1$ elements lying in $\mathbb{I}_\Lambda$

$$a = \xi_{t0} \prec \xi_{t1} \prec \xi_{t2} \prec \cdots \prec \xi_{t2^t} = b,$$

all 2^t subsets which are of the form $\mathbb{I}^{(t\ s+1)} = [\xi_{ts}, \xi_{t\ s+1}]_{\Bbbk}$, and 2^t order preserving bijections $\kappa_{\xi_{ts}}$ from $\mathbb{I}^{(t\ s+1)}$ to $\mathbb{I}^{(01)}$.

For any family of subsets $(\mathbb{I}^{(u_iv_i)})_{1\le i\le n}$ $(1 \le v_i \le 2^{u_i})$, we denote by $\mathbf{1}_{(u_iv_i)_i}$ the function

$$\mathbf{1}_{(u_iv_i)_i} := \mathbf{1}_{\mathbb{I}_\Lambda}\Big|_{\prod_{i=1}^n \mathbb{I}^{(u_iv_i)}} : \ \mathbb{I}_\Lambda \to \{0,1\},\ x \mapsto \begin{cases} 1, & x \in \prod_{i=1}^n \mathbb{I}^{(u_iv_i)}; \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbb{I}^{(u_iv_i)} \cong \mathbb{I}^{(u_iv_i)} \times \{b_i\} \subseteq \mathbb{I}_\Lambda$ holds for all i and $B_\Lambda = \{b_i \mid 1 \le i \le n\}$ is the $\Bbbk$-basis of Λ.

Let E_u be the set of all step functions constant on each of $\prod_{i=1}^n \mathbb{I}^{(u_iv_i)}$ $(1 \le v_i \le 2^{u_i}$ for all $i)$, that is, every step function in E_u is of the form

$$\sum_{(u_iv_i)_i} k_{(u_iv_i)_i}\mathbf{1}_{(u_iv_i)_i},$$

where each $k_{(u_iv_i)_i}$ lies in $\Bbbk$, the number of summands is $(2^u)^n = 2^{un}$, and each $(u_iv_i)_i$ corresponds to the Cartesian product $\prod_{i=1}^n \mathbb{I}^{(u_iv_i)}$. Then it is easy to check that each E_n is a normed submodule of $\mathbf{S}(\mathbb{I}_\Lambda)$, and $E_u \subseteq E_{u+1}$ because each step function constant on each of $\mathbb{I}^{(uv)}$ is equivalent to a step function constant on each of $\mathbb{I}^{(u+1\ v)}$. Thus,

$$\Bbbk \cong E_0 \subseteq E_1 \subseteq \ldots \subseteq E_t \subseteq \ldots \subseteq \mathbf{S}(\mathbb{I}_\Lambda) \subseteq \widehat{\mathbf{S}(\mathbb{I}_\Lambda)}.$$

Moreover, for any $\mathbb{I}^{(uv)} = [\xi_{u\ v-1}, \xi_{uv}]_{\Bbbk}$, we have two cases (i) $\xi_{uv} \preceq \xi$ and (ii) $\xi \preceq \xi_{u\ v-1}$ by the definition of E_u. Therefore, we obtain a map

$$\mathfrak{p} : \{\mathbb{I}^{(uv)} \mid u \in \mathbb{N}\} \to \{a, b\},\ \mathbb{I}^{(uv)} \mapsto \begin{cases} a, & \mathbb{I}^{(uv)} \text{ lies in case (i)}; \\ b, & \mathbb{I}^{(uv)} \text{ lies in case (ii)}. \end{cases}$$

Now we use the above map to prove the following lemma.

Lemma 5.4. *The map $\gamma_\xi : \mathbf{S}(\mathbb{I}_\Lambda)^{\oplus_p 2^n} \to \mathbf{S}(\mathbb{I}_\Lambda)$ induces the following $\Bbbk$-linear map*

$$\gamma_\xi : E_u^{\oplus_p 2^n} \xrightarrow{\cong} E_{u+1}$$

which is an isomorphism of Λ-modules.

Proof. The $\Bbbk$-linear space E_u is a Λ-module, where $\Lambda \times E_u \to E_u$ is defined by

$$(a, f = \textstyle\sum_i 1 \cdot \mathbf{1}_{I_i}) \mapsto a \cdot f = \sum_i \tau(a) \cdot \mathbf{1}_{I_i}.$$

Then it is easy to see that γ_ξ is a Λ-homomorphism. Since $\mathrm{Ker}(\gamma_\xi) = 0$, we have γ_ξ is injective. Next, we prove that it is surjective.

Any step function $f : \Bbbk^{\oplus n} \to \Bbbk$ lying in E_{u+1} can be written as

$$f(k_1, \ldots, k_n) = \sum_{(u_iv_i)_i} f_i = \sum_{(\omega_1,\ldots,\omega_n)\in\{a,b\}\times\cdots\times\{a,b\}} f_{(\omega_1,\ldots,\omega_n)}$$

where

- $f_i = k_{(u_iv_i)_i}\mathbf{1}_{(u_iv_i)_i}$;
- $$f_{(\omega_1,\ldots,\omega_n)}(k_1,\ldots,k_n) = \sum_{\prod_{i=1}^{n}\mathfrak{p}(\mathbb{I}^{(u_iv_i)})=(\omega_1,\ldots,\omega_n)} f_i,$$
  thus, the number of all summands of it is $(2^u)^n = 2^{un}$;
- the number of all summands of $\sum_{(\omega_1,\ldots,\omega_n)\in\{a,b\}\times\cdots\times\{a,b\}}$ is 2^n (thus the number of all summands of $\sum_{(u_iv_i)_i} f_i$ is $2^{un}\cdot 2^n = 2^{(u+1)n}$).

Then

$$\tilde{f}_{(\omega_1,\ldots,\omega_n)}(k_1,\ldots,k_n) = f_{(\omega_1,\ldots,\omega_n)}(\kappa_{\omega_1}^{-1}(k_1),\ldots,\kappa_{\omega_n}^{-1}(k_n)) \in E_u,$$

and γ_ξ sends $\{f_{(\omega_1,\ldots,\omega_n)}\}_{(\omega_1,\ldots,\omega_n)\in\{a,b\}\times\cdots\times\{a,b\}}$ to f by the definition of γ_ξ, see (3.2). We obtain that γ_ξ is surjective. Therefore, γ_ξ is a Λ-isomorphism. □

By Lemma 5.4, the following result holds.

Lemma 5.5. *The commutativity of the inverse limit and the map* $\widehat{\gamma}_\xi : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ *induced by the completion of* $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$ *holds, that is, for any sequence* $\{\boldsymbol{f}_i\}_{i\in\mathbb{N}^+}$ *in* $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n}$, *if its inverse limit exists, then we have*

$$\widehat{\gamma}_\xi(\varprojlim \boldsymbol{f}_i) = \varprojlim \widehat{\gamma}_\xi(\boldsymbol{f}_i).$$

Furthermore, $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ *is an object in* $\mathscr{N}\!or^p$.

Proof. Since γ_ξ is a Λ-isomorphism, it is clear that $\widehat{\gamma}_\xi$ is also a Λ-isomorphism. Then, the commutativity of the inverse limit and the map $\widehat{\gamma}_\xi$ holds. Thus, for any sequence $\{\boldsymbol{f}_i\}_{i\in\mathbb{N}^+}$ in $\mathbf{S}_\tau(\mathbb{I}_\Lambda)^{\oplus_p 2^n}$, if its inverse limit exists, then this inverse limit is also an element in $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n}$, and so

$$\gamma_\xi(\varprojlim \boldsymbol{f}_i) = \widehat{\gamma}_\xi(\varprojlim \boldsymbol{f}_i) \overset{\spadesuit}{=} \varprojlim \widehat{\gamma}_\xi(\boldsymbol{f}_i) = \varprojlim \gamma_\xi(\boldsymbol{f}_i),$$

where ♠ holds since $\widehat{\gamma}_\xi$ is a Λ-isomorphism (see Lemma 5.4). Therefore, by Lemma 5.1, $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ is an object in $\mathscr{N}\!or^p$. □

5.3. $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ is a direct limit.

Let $\mathsf{nor}\Lambda$ be the category of normed Λ-modules and Λ-homomorphisms between them. Then it is easy to check that all E_u are objects in $\mathsf{nor}\Lambda$. Furthermore, for any $u \le v$, we have a Λ-homomorphism $\varphi_{uv} : E_u \to E_v$ which is induced by $E_u \subseteq E_v$. Thus we obtain a direct system $((E_i)_{i\in\mathbb{N}}, (\varphi_{uv})_{u\le v})$ in $\mathsf{nor}\Lambda$ over $\mathbb{N}$. Let $\mathsf{Ban}\Lambda$ be the category of Banach Λ-modules and continuous Λ-homomorphisms between them. Then $\mathsf{Ban}\Lambda$ is a full subcategory of $\mathsf{nor}\Lambda$, and so, naturally, we obtain a direct system $((E_i)_{i\in\mathbb{N}}, (\varphi_{uv})_{u\le v})$ in $\mathsf{Ban}\Lambda$ if Λ is a complete $\Bbbk$-algebra.

The following lemma establishes the relation between E_n and $\mathbf{S}(\mathbb{I}_\Lambda)$.

Lemma 5.6. *Let* Λ *be a complete* $\Bbbk$*-algebra. Consider the category* $\mathsf{Ban}\Lambda$ *and take* $(\alpha_i : E_i \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)})_{i\in\mathbb{N}}$, *where every* α_i *is the embedding given by* $E_i \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. *Then* $\varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$.

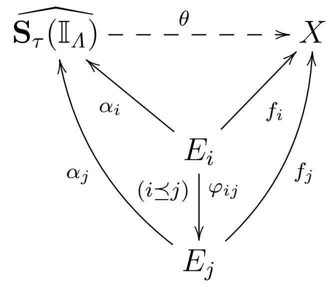
(Note that we assume that all morphisms in $\mathsf{Ban}(\Lambda)$ are continuous, which ensures the commutativity $\varprojlim \vartheta(x_i) = \vartheta(\varprojlim x_i)$ between the inverse limit and any morphism starting from $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$.)

Proof. Let X be an arbitrary object in $\mathsf{nor}\Lambda$ such that there is $(f_i : E_i \to X)_{i\in\mathbb{N}}$ satisfying $f_i\varphi_{ij} = f_j$ for all $i \le j$. Then we can find the Λ-homomorphism $\theta : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} \to X$ in the following way.

For any $x \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, there exists a sequence $\{x_t\}_{t\in\mathbb{N}}$ in $\bigcup_i E_i$ such that $\{\|x_t - x\|_p\}_t$ is a monotonically decreasing sequence of positive real numbers. Then we have $\varinjlim\{\|x_t - x\|_p\}_t = 0$ by Example 2.4 which induces $\varinjlim x_t = x$. Since α_i, α_j and φ_{ij} ($\forall i \le j$) are Λ-homomorphisms induced by "$\subseteq$" (thus they are $\Bbbk$-linear maps induced by "$\subseteq$") and every x_t has a preimage in some $E_{u(t)}$, then Λ-homomorphisms $(f_i)_{i\in\mathbb{N}}$ send $\{x_t\}_{t\in\mathbb{N}}$ to a sequence $\{f_{u(t)}(x_t)\}_{t\in\mathbb{N}}$ in X. By the completeness of X, $\varinjlim f_{u(t)}(x_t) \in X$ holds. Define

$$\theta(x) = \varinjlim f_{u(t)}(x_t) = \varinjlim f|_{E_{u(t)}}(x_t) = \varinjlim f(x_t),$$

where f is the map $\varinjlim E_u \to X$ induced by the direct limit of $((E_i)_{i\in\mathbb{N}}, (\varphi_{uv})_{u\le v})$. Then one can check that θ is well-defined and is a Λ-homomorphism making the following diagram commute.



Next, we show that the existence of θ is unique. Assume that θ' is also a Λ-homomorphism with $\theta'\alpha_i = f_i$ for all i. Then for any $x \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$, taking the sequence $\{x_t\}_{t\in\mathbb{N}}$ in $\bigcup_i E_i$ satisfying $\varinjlim x_t = x$, we have

$$\theta'(x) = \theta'\big(\varinjlim \alpha_i(x_t)\big) = \varinjlim \theta'(\alpha_i(x_t)) = \varinjlim f_i(x_t) = \varinjlim \theta(\alpha_i(x_t)) = \theta\big(\varinjlim \alpha_i(x_t)\big) = \theta(x),$$

that is, $\theta = \theta'$. Therefore, by the definition of direct limits, we have $\varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. □

6. The $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$

Let $\mathcal{C}$ be a category. Recall that an object O in $\mathcal{C}$ is an *initial object* if for any object Y we have $\mathrm{Hom}_{\mathcal{C}}(O, Y)$ contains only one morphism. Obviously, if $\mathcal{C}$ has initial objects, then the initial object is unique up to isomorphism. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. An object $C \in \mathcal{C}$ is called a *$\mathcal{D}$-initial object* if it is a subobject of the initial object in $\mathcal{D}$, that is, there is an object C' in $\mathcal{D}$ such that the following conditions hold:

- there is a monomorphism from C to C';
- for any $D \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(C', D)$ contains a unique morphism.

It is trivial that an initial object in $\mathcal{C}$ is a $\mathcal{C}$-initial object.

Let Λ is a complete $\Bbbk$-algebra. In this section, we show that $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$. The proof is divided to two parts: (1) there is at least one morphism from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ to any object in $\mathscr{A}^p$; (2) the above morphism is unique.

6.1. The existence of morphism from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$.

In this subsection we show that $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta))$ is not empty for every object (V, v, δ) in $\mathscr{A}^p$.

Lemma 6.1. *For any object $(V, v, \delta) \in \mathscr{A}^p$, we have*

$$\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta)) \ne \varnothing.$$

Proof. For each $u \in \mathbb{N}$, consider the map $\theta_n : E_n \to V$ as follows:

(i) $\theta_0 : E_0 \to V$ is a map induced by the $\Bbbk$-linear map $\Bbbk \to V$ sending 1 to v (note that $E_0 \cong \Bbbk$). Then one can check that θ is a $\mathit{\Lambda}$-homomorphism.

(ii) θ_{u+1} is induced by θ_u through the composition

$$\theta_{u+1} := \left(E_{u+1} \xrightarrow{\gamma_\xi^{-1}} E_u^{\oplus_p 2^n} \xrightarrow{\theta_u^{\oplus 2^n}} V^{\oplus_p 2^n} \xrightarrow{\delta} V \right),$$

where the inverse γ_ξ^{-1} of the map γ_ξ is given in Lemma 5.4.

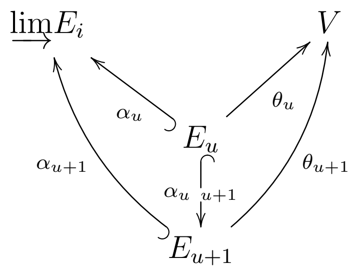
Notice that $\gamma_\xi^{-1}(f) \in E_{u-1}$ for any $f \in E_u \subseteq E_{u+1}$, then for the case $u = 0$, we have that $f = k\mathbf{1}_{E_0}$ is a constant defined on E_0, and

$$\theta_1(f) = \delta(\theta_0^{\oplus 2^n}(\gamma_\xi^{-1}(f))) = \delta(\theta_0(k\mathbf{1}_{E_0}), \theta_0(k\mathbf{1}_{E_0}), \ldots, \theta_0(k\mathbf{1}_{E_0})) = kv,$$

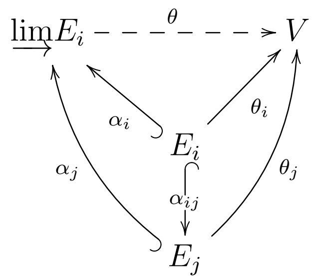
that is, θ_1 is an extension of θ_0. It yields $\theta_1(\mathbf{1}_{E_1}) = v$ by $\theta_0(\mathbf{1}_{E_0}) = v$ (see (i)). Furthermore, we can check that θ_{u+1} is an extension of θ_u and

$$\theta_u(\mathbf{1}_{E_u}) = v \ (\forall u \in \mathbb{N}) \tag{6.1}$$

by induction, that is, the following diagram



commutes, where $\alpha_i : E_i \to \varinjlim E_i$ and $\alpha_{ij} : E_i \to E_j$ $(i \le j)$ are the embeddings induced by $E_i \subseteq \varinjlim E_i$ and $E_i \subseteq E_j$, respectively. Then, for any $i \le j$, there is a unique $\mathit{\Lambda}$-homomorphism θ such that the following diagram



commutes. By Lemma 5.6, we have that $\theta : \varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\mathit{\Lambda})} \to V$ is a $\mathit{\Lambda}$-homomorphism in $\mathrm{Hom}_\mathit{\Lambda}(\widehat{\mathbf{S}_\tau(\mathbb{I}_\mathit{\Lambda})}, V)$.

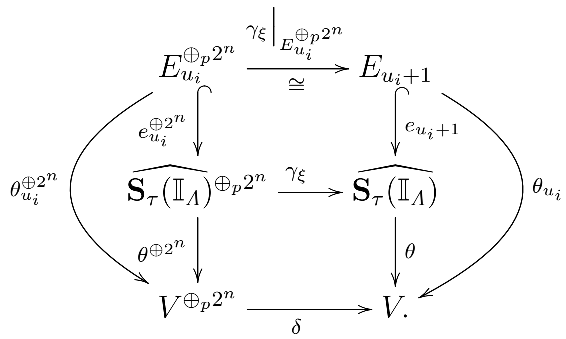
Next, we prove that θ is a morphism in $\mathscr{N}or^p$. First of all, we have

$$\theta(\mathbf{1}) = \varprojlim \theta|_{E_i}(\mathbf{1}_{E_i}) = \varprojlim \theta(\alpha_i(\mathbf{1}_{E_i})) = \varprojlim \theta_i(\mathbf{1}_{E_i}) \overset{(6.1)}{=\!=} \varprojlim v = v.$$

In the following, we show that the following diagram

$$\begin{array}{ccc} \widehat{\mathbf{S}_\tau(\mathbb{I}_\mathit{\Lambda})}^{\oplus_p 2^n} & \xrightarrow{\gamma_\xi} & \widehat{\mathbf{S}_\tau(\mathbb{I}_\mathit{\Lambda})} \\ {\scriptstyle \theta^{\oplus 2^n}} \downarrow & & \downarrow {\scriptstyle \theta} \\ V^{\oplus_p 2^n} & \xrightarrow[\delta]{} & V \end{array} \tag{6.2}$$

is commutative. Notice that each $\boldsymbol{f} = (f_1, \dots, f_{2^n}) \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n}$ can be seen as the inverse limit $\varprojlim \boldsymbol{f}_i$ of some sequence $\{\boldsymbol{f}_i = (f_{1i}, \dots, f_{2^n i})_{i\in\mathbb{N}}\}$ in $\bigcup_{u\in\mathbb{N}} E_u^{\oplus_p 2^n}$, where $f_{ji} \in E_{u_i}$ $(1 \le j \le 2^n)$, $u_i \in \mathbb{N}$, such that for any $i \le j$, we have $u_i \le u_j$. Thus, naturally, we need consider the following diagram ($e_{u_i} : E_{u_i} \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ is the embedding induced by $E_{u_i} \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$):



Since

$$\begin{aligned}
\theta(\gamma_\xi(\boldsymbol{f})) &= \varprojlim \theta(\gamma_\xi(e_{u_i}^{\oplus 2^n}(\boldsymbol{f}_i))) & \\
&= \varprojlim \theta(e_{u_i+1}(\gamma_\xi\big|_{E^{\oplus_p 2^n}}(\boldsymbol{f}_i))) & (\gamma_\xi e_{u_i}^{\oplus 2^n} = e_{u_i+1}\gamma_\xi\big|_{E^{\oplus_p 2^n}}) \\
&= \varprojlim \theta_{u_i}(\gamma_\xi\big|_{E^{\oplus_p 2^n}}(\boldsymbol{f}_i)) & (\theta e = \theta_{u_i}) \\
&= \varprojlim \delta(\theta_{u_i}^{\oplus 2^n}(\boldsymbol{f}_i)) & (\theta_{u_i}\gamma_\xi\big|_{E^{\oplus_p 2^n}} = \delta\theta_{u_i}^{\oplus 2^n}) \\
&= \varprojlim \delta(\theta^{\oplus 2^n}(e_{u_i}^{\oplus 2^n}(\boldsymbol{f}_i))) & (\theta_u^{\oplus 2^n} = \theta^{\oplus 2^n} e_{u_i}^{\oplus 2^n}) \\
&= \delta(\theta^{\oplus 2^n}(\varprojlim e_{u_i+1}^{\oplus 2^n}(\boldsymbol{f}_i))) = \delta(\theta^{\oplus 2^n}(\boldsymbol{f})), & (\text{by } (5.1))
\end{aligned}$$

the assertion follows. □

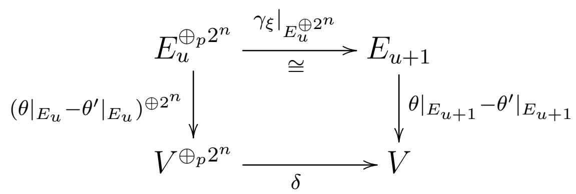
6.2. **The uniqueness of morphism from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$.** Now, we show that, for any object (V, v, δ) in $\mathscr{A}^p$, if the morphism in the category $\mathscr{A}^p$ from $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$ exists, then it is unique.

Lemma 6.2. *Let $(V, v, \delta) \in \mathscr{A}^p$ be an object in $\mathscr{A}^p$. If*

$$\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta)) \neq \varnothing,$$

then $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta))$ *contains a unique morphism.*

Proof. Let θ and θ' be two Λ-homomorphisms from $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ to (V, v, δ) in $\mathscr{A}^p$. Then $\theta(\mathbf{1}) = v = \theta'(\mathbf{1})$. Since θ and θ' are maps in $\mathscr{A}^p$, the square



commutes. Then for any $f \in E_{u+1}$, we have

$$(\theta|_{E_{u+1}} - \theta'|_{E_{u+1}})(f) = (\delta \circ (\theta|_{E_u} - \theta'|_{E_u})^{\oplus 2^n} \circ (\gamma_\xi|_{E_u^{\oplus 2^n}})^{-1})(f),$$

that is, $\theta|_{E_{u+1}} - \theta'|_{E_{u+1}}$ is determined by $\theta|_{E_u} - \theta'|_{E_u}$. Consider the case for $u = 0$, since $\theta|_{E_0}$ and $\theta'|_{E_0} : E_0 \to V$ are defined by $\theta_0(\mathbf{1}_{E_0}) = v$, we have

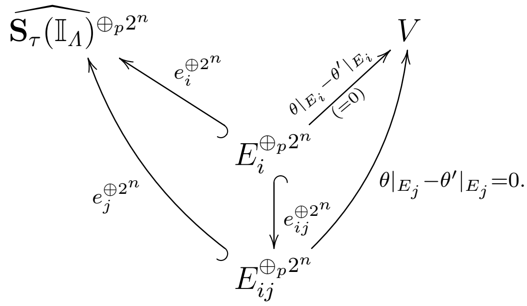
$$(\theta|_{E_0} - \theta'|_{E_0})(k\mathbf{1}_{E_0}) = k(\theta|_{E_0}(\mathbf{1}_{E_0}) - \theta'|_{E_0}(\mathbf{1}_{E_0})) = k(v - v) = 0.$$

Therefore $\theta|_{E_u} - \theta'|_{E_u} = 0$ for all $u \in \mathbb{N}$ by induction.

On the other hand, consider the embeddings $e_u : E_u \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ and $e_{uv} : E_u \to E_v$ $(u \leq v)$ induced by "$\subseteq$" and the direct system

$$\big(E_u^{\oplus_p 2^n}, (e_u^{\oplus 2^n} : E_u^{\oplus_p 2^n} \to \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n})_{u\in\mathbb{N}}\big),$$

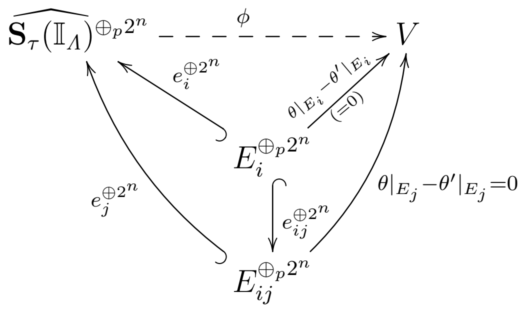
we have the following commutative diagram



Since

$$\varinjlim E_i^{\oplus_p 2^n} \cong (\varinjlim E_i)^{\oplus_p 2^n} \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n},$$

there is a unique Λ-homomorphism $\phi : \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}^{\oplus_p 2^n} \to V$ such that the following diagram



commutes. Since $(\theta - \theta')e_u^{\oplus 2^n} = \theta|_{E_i} - \theta'|_{E_j}$, we know that the case for $\phi = \theta - \theta'$ making the above diagram commute. On the other hand, the case for $\phi = 0$ making the above diagram commute. Thus $\theta - \theta' = 0$ and $\theta = \theta'$. □

6.3. **The $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$.** Now, we can prove the following result of this paper.

Theorem 6.3. *The triple $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi)$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$.*

Proof. For any object (V, v, δ) in $\mathscr{A}^p$, the existence of morphisms in $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi), (V, v, \delta))$ is proved in Lemma 6.1, and its uniqueness is proved in Lemma 6.2. Thus, the triple $(\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi)$, as an object in $\mathscr{A}^p$, is an initial object in $\mathscr{A}^p$. It follows that $(\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \widehat{\gamma}_\xi)$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}or^p$. □

We give a remark for Theorem 6.3.

Remark 6.4. For any object (V, v, δ) in $\mathscr{A}^p$, there is a unique morphism

$$h : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (V, v, \delta)$$

in $\mathscr{N}or^p$, which can be extended to $\widehat{h} : (\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) \to (V, v, \delta)$, In other words, if there exists a morphism h making the diagram

$$\begin{array}{ccc} (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) & \xrightarrow{\ h\ } & (V, v, \delta) \\ {\scriptstyle\subseteq}\downarrow & \nearrow_{\widehat{h}} & \\ (\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) & & \end{array}$$

commute, then the existence of h is guaranteed to be unique.

7. The categorization of Lebesgue integration

Take $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ to be an extension of $\mathbb{R}$. Recall the symbols given in Notation 5.3, any step function f in E_u can be written as

$$f = \sum_{(u_i v_i)_i} k_{(u_i v_i)_i} \mathbf{1}_{(u_i v_i)_i}.$$

We define the map $T_u : E_u \to \Bbbk$ by

$$T_u(f) = \sum_{(u_i v_i)_i} k_{(u_i v_i)} \mu\bigg(\prod\nolimits_i \mathbb{I}^{(u_i v_i)} \bigg) \tag{7.1}$$

(note that if all coefficients $k_{(u_i v_i)}$ equal to 1, then $T_u(f) = \mu(E_u)$).

Then the Λ-isomorphism γ_ξ shown in Lemma 5.4 points out the following fact: there is a map $m_u : \Bbbk^{\oplus_p 2^n} \to \Bbbk$ such that the following diagram

$$\begin{array}{ccc} E_u^{\oplus_p 2^n} & \xrightarrow{\ \gamma_\xi\ } & E_{u+1} \\ {\scriptstyle T_u^{\oplus 2^n}}\downarrow & & \downarrow{\scriptstyle T_{u+1}} \\ \Bbbk^{\oplus_p 2^n} & \xrightarrow[\ m_u\ ]{} & \Bbbk \end{array} \tag{7.2}$$

commutes. Indeed, for the function $f_k = \frac{k}{\mu(\mathbb{I}_\Lambda)} \mathbf{1}_{\mathbb{I}_\Lambda}$ with $k \in \Bbbk$, we have

$$T_u(f_k) = T_u(\tfrac{k}{\mu(\mathbb{I}_\Lambda)} \mathbf{1}_{\mathbb{I}_\Lambda}) = \tfrac{k}{\mu(\mathbb{I}_\Lambda)} T_u(\mathbf{1}_{\mathbb{I}_\Lambda}) = k$$

by (7.1). Then for any $\boldsymbol{k} = (k_1, \ldots, k_{2^n}) \in \Bbbk^{\oplus_p 2^n}$, $\boldsymbol{f_k} = (f_{k_1}, \ldots, f_{k_{2^n}}) \in E_u^{\oplus_p 2^n}$ is a preimage of $\boldsymbol{k}$ under the $\Bbbk$-linear map $T_u^{\oplus 2^n}$. We define

$$m_u(\boldsymbol{k}) = T_{u+1}(\gamma_\xi(\boldsymbol{f_k})).$$

It is easy to see that m_u is a $\Bbbk$-linear map. In particular, for the constant function $\mathbf{1}_{\mathbb{I}_\Lambda}$ given by the measure $\mu(\mathbb{I}_\Lambda)$ of $\mathbb{I}_\Lambda$, we obtain that $f_{\mu(E_u)}$ is a preimage of $\mu(\mathbb{I}_\Lambda) \in \Bbbk$, and then

$$m_u(\mu(\mathbb{I}_\Lambda), \ldots, \mu(\mathbb{I}_\Lambda)) = T_{u+1}\gamma_\xi(\mathbf{1}_{\mathbb{I}_\Lambda}, \ldots, \mathbf{1}_{\mathbb{I}_\Lambda}) = \sum_{(u_i v_i)_i} 1 \cdot \mu\bigg(\prod\nolimits_i \mathbb{I}^{(u_i v_i)} \bigg) = \mu(\mathbb{I}_\Lambda).$$

Lemma 7.1. *Let $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ be an extension of $\mathbb{R}$. Then $T_u : E_u \to \Bbbk$ is a Λ-homomorphism.*

Proof. Note that $\Bbbk$ is a Λ-module given by

$$\Lambda \times \Bbbk \to \Bbbk, (\lambda, k) \mapsto \lambda \cdot k := \tau(\lambda) k$$

For arbitrary two elements $\lambda_1, \lambda_2 \in \Lambda$ and arbitrary two functions $f = \sum_i k_i \mathbf{1}_{I_i}$, $g = \sum_j k'_j \mathbf{1}_{I'_j} \in E_u$, we have

$$\begin{aligned} T_u(\lambda_1 \cdot f + \lambda_2 \cdot g) &= T_u\Big(\sum\nolimits_i \tau(\lambda_1) k_i \mathbf{1}_{I_i} + \sum\nolimits_j \tau(\lambda_2) k'_j \mathbf{1}_{I'_j}\Big) \\ &= \tau(\lambda_1) T_u\Big(\sum\nolimits_i k_i \mathbf{1}_{I_i}\Big) + \tau(\lambda_2) T_u\Big(\sum\nolimits_j k'_j \mathbf{1}_{I'_j}\Big) \\ &= \tau(\lambda_1) T_u(f) + \tau(\lambda_2) T_u(g) \\ &= \lambda_1 \cdot T_u(f) + \lambda_2 \cdot T_u(g). \end{aligned}$$

□

Lemma 7.2. *Let $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ be an extension of $\mathbb{R}$ and let m_u be the $\Bbbk$-linear map given in the diagram (7.2). Then m_u is a Λ-homomorphism.*

Proof. We can prove that m_u is a Λ-homomorphism by using an argument similar to proving that m_u is a $\Bbbk$-linear mapping (note that this proof needs to use Lemma 7.1). □

Remark 7.3. Since $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_u \subseteq \cdots \subseteq \mathbf{S}_\tau(\mathbb{I}_\Lambda) \subseteq \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} = \varinjlim E_i$, we have that μ is independent on u. Thus, we can use m to present all maps m_i $(i \in \mathbb{N})$ because $m_0 = m_1 = m_2 = \ldots$.

Proposition 7.4. *Let $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ be an extension of $\mathbb{R}$. Then the triple $(\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ is an object in $\mathscr{N}or^p$. Furthermore, since Λ is complete, so is $\Bbbk$. Then $\Bbbk^{\oplus_p 2n}$ is a Banach Λ-module, and so $(\Bbbk, \mu(\mathbb{I}_\Lambda), m)$ is an object in $\mathscr{A}^p$.*

Proof. It follows from Lemmas 7.1 and 7.2 and Remark 7.3. □

The following proposition shows that T_u satisfies the triangle inequality.

Proposition 7.5. *If $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ is an extension of $\mathbb{R}$, then for any $f \in E_u$, the following inequality holds for all $u \in \mathbb{N}$.*

$$|T_u(f)| \le T_u(|f|). \tag{7.3}$$

Proof. Assume that $f = \sum_i k_i \mathbf{1}_{I_i} \in E_u$, where $I_i \cap I_j = \varnothing$ for all $i \ne j$. Then $|f| = |\sum_i k_i \mathbf{1}_{I_i}|$ is also a step function in E_u, and we have

$$\begin{aligned} T_u(|f|) = T_u\Big(\Big|\sum\nolimits_i k_i \mathbf{1}_{I_i}\Big|\Big) &\overset{(\star)}{=\!=} T_u\Big(\sum\nolimits_i |k_i| \mathbf{1}_{I_i}\Big) \\ &= \sum\nolimits_i |k_i| \mu\Big(\prod\nolimits_i \mathbb{I}^{(u_i v_i)_i}\Big) \\ &\ge \Big|\sum\nolimits_i k_i \mu\Big(\prod\nolimits_i \mathbb{I}^{(u_i v_i)_i}\Big)\Big| = |T_u(f)|, \end{aligned}$$

where $(\star)$ is given by $I_i \cap I_j = \varnothing$. □

Theorem 7.6. *If $\Bbbk = (\Bbbk, |\cdot|, \preceq)$ is an extension of $\mathbb{R}$, then there exists a unique morphism*

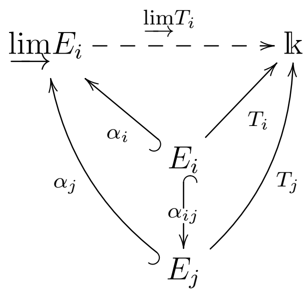
$$T : (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) \to (\Bbbk, \mu(\mathbb{I}_\Lambda), m)$$

in $\mathrm{Hom}_{\mathscr{N}or^p}((\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi), (\Bbbk, \mu(\mathbb{I}_\Lambda), m))$ such that the diagram

$$\begin{array}{ccc} (\mathbf{S}_\tau(\mathbb{I}_\Lambda), \mathbf{1}, \gamma_\xi) & \xrightarrow{\quad T \quad} & (\Bbbk, \mu(\mathbb{I}_\Lambda), m) \\ \subseteq \big\downarrow & \nearrow_{\widehat{T}} & \\ (\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}, \mathbf{1}, \widehat{\gamma}_\xi) & & \end{array}$$

commutes, where $\widehat{T}$ *is the morphism in* $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}_\tau(\mathbb{I}_A)}, \mathbf{1}, \widehat{\gamma}_\xi), (\Bbbk, \mu(\mathbb{I}_A), m))$ *whose existence is unique. Furthermore,* $\widehat{T}$ *is given by the direct limit* $\varinjlim T_i : \varinjlim E_i \to \Bbbk$.

Proof. Denote by $\alpha_{ij} : E_i \to E_j$ $(i \le j)$ and $\alpha_i : E_i \to \varinjlim E_i$ the monomorphism induced by $E_i \subseteq E_j \subseteq \varinjlim E_i$. Then there is a unique morphism $\varinjlim T_i : \varinjlim E_i \to \Bbbk$ such that the following diagram



commutes. By Lemma 5.6, we have $\varinjlim E_i \cong \widehat{\mathbf{S}_\tau(\mathbb{I}_A)}$, then $\varinjlim T_i$ induces a morphism in $\mathscr{A}^p$ from $(\mathbf{S}_\tau(\mathbb{I}_A), \mathbf{1}, \gamma_\xi)$ to $(\Bbbk, \mu(\mathbb{I}_A), m)$. Theorem 6.3 and its remark show that $\varinjlim T_i = \widehat{T}$ and $T = \widehat{T}|_{\mathbf{S}_\tau(\mathbb{I}_A)}$. □

Definition 7.7. Let $\Bbbk$ be a field with the norm $|\cdot| : \Bbbk \to \mathbb{R}^{\ge 0}$ and the total ordered "$\preceq$", and let $f : A \to \Bbbk$ be a function in $\mathbf{S}_\tau(\mathbb{I}_A)$. If $\Bbbk$ is an extension of $\mathbb{R} = (\mathbb{R}, |\cdot|, \le)$, then we call that f is a *integrable function* on $\mathbb{I}_A$ and whose integration, denoted by $\int_{\mathbb{I}_A} f \mathrm{d}\mu$, is defined by

$$\int_{\mathbb{I}_A} f \mathrm{d}\mu := \widehat{T}(f).$$

By using the limit $\varinjlim T_i : \varinjlim E_i \to \Bbbk$ given in Theorem 7.6, the formula (7.1), Lemma 7.1 and Proposition 7.5 show that

$$\int_{\mathbb{I}_A} 1 \mathrm{d}\mu = \mu(\mathbb{I}_A),$$

$$\int_{\mathbb{I}_A} (\lambda_1 \cdot f_1 + \lambda_2 \cdot f_2)\mu = \lambda_1 \cdot \int_{\mathbb{I}_A} f_1 \mu + \lambda_2 \cdot \int_{\mathbb{I}_A} f_2 \mu \quad (\lambda_1, \lambda_2 \in A)$$

and

$$\left| \int_{\mathbb{I}_A} f \mathrm{d}\mu \right| \le \int_{\mathbb{I}_A} |f| \mathrm{d}\mu,$$

respectively.

In the subsection 8.1 of Section 8, we point out that Theorem 7.6 and Definition 7.7 provide a categorization of Lebesgue integration.

8. Applications

8.1. **Lebesgue integration.** Take $\Bbbk = \mathbb{R}$, $(A, \prec, \|\cdot\|_A) = (\mathbb{R}, \le, \|\cdot\|_{\mathbb{R}})$, $B_{\mathbb{R}} = \{1\}$ and $\mathfrak{n} : B_{\mathbb{R}} \to \{1\} \subseteq \mathbb{R}^{\ge 0}$. Then $\dim \mathbb{R} = 1$, $\mathbb{R}$ is a normed $\mathbb{R}$-algebra with the norm $\|\cdot\|_{\mathbb{R}} = |\cdot| : \mathbb{R} \to \mathbb{R}^{\ge 0}$ sending each real number r to its absolute value $|r|$, and any normed $\mathbb{R}$-module is a normed $\Bbbk$-linear space. Take $\mathbb{I}_{\mathbb{R}} = [0, 1]$, $\xi = \frac{1}{2}$, $\kappa_0(x) = \frac{x}{2}$, $\kappa_1(x) = \frac{x+1}{2}$ and $\tau = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ in this subsection. Then any object (N, v, δ) in $\mathscr{N}\!or^p$ is a triple of a normed $\Bbbk$-module $N = (N, h_N, \|\cdot\|)$, an element $v \in N$ with $\|v\|_1$ and the $\Bbbk$-linear map $\delta : N \oplus_1 N \to N$, where the norm $\|\cdot\|$ satisfies

$$\|rx\| = |\tau(r)| \cdot \|x\| = |r| \cdot \|x\|$$

for any $r \in A = \mathbb{R}$ and $x \in N$. In this case, we have the following properties for $\mathscr{N}\!or^p$.

(L1) The normed $\Bbbk$-module $\mathbf{S}_\tau(\mathbb{I}_\Lambda) = \mathbf{S}_{\mathbf{1}_{\mathbb{R}}}([0,1])$ ($= \mathbf{S}$ for short) is a $\Bbbk$-linear space of all elementary simple functions which are of the form

$$f = \sum_{x=i}^{t} k_i \mathbf{1}_{[x_i, y_i]},$$

where $[x_i, y_i] \cap [x_j, y_j] = \varnothing$ for any $i \neq j$, and for any $f(r), g(r) \in \mathbf{S}$, it holds that

$$\gamma_{\frac{1}{2}}(f, g) = \begin{cases} f(2r), & 1 \leq r < \frac{1}{2}, \\ g(2r-1), & \frac{1}{2} < r \leq 1, \end{cases}$$

by the definition of γ_ξ, see (3.2).

(L2) $\mathscr{A}^p$ is a full subcategory, $(\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}})$ is an object in $\mathscr{N}\!or^p$, but is not an object in $\mathscr{A}^p$ because $\mathbf{S}$ is not complete.

(L3) Let $\widehat{\mathbf{S}}$ be the completion of $\mathbf{S}$, and let $\widehat{\gamma}_{\frac{1}{2}}$ be the map $\widehat{\mathbf{S}} \oplus_1 \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$ induced by $\gamma_{\frac{1}{2}}$. Then $(\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}})$ is an object in $\mathscr{A}^p$.

By Theorem 6.3, we obtain the following result directly.

Corollary 8.1. *The triple $(\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}})$ is an $\mathscr{A}^p$-initial object in $\mathscr{N}\!or^p$.*

Remark 8.2. It follows from Theorem 6.3 that $(\widehat{\mathbf{S}}, \mathbf{1}[0,1], \widehat{\gamma}_{\frac{1}{2}})$ is an initial object in $\mathscr{A}^p$, and then Corollary 8.1 holds. In [16], Leinster showed that the initial object in $\mathscr{A}^p$ is $(L^p([0,1]), \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}})$. Then we obtain $L^p([0,1]) \cong \widehat{\mathbf{S}}$ since the uniqueness (up to isomorphism) of initial objects in arbitrary categories.

Consider the triple $(\mathbb{R}, 1, m)$ of the normed $\mathbb{R}$-module $\mathbb{R}$, the constant function and the map

$$m : \mathbb{R} \oplus_p \mathbb{R} \to \mathbb{R}$$

sending (x, y) to $\frac{1}{2}(x+y)$. Then $(\mathbb{R}, 1, m)$ is an object in $\mathscr{A}^p$, and there are a family of $\mathbb{R}$-linear maps $(L_i : E_i \to \Bbbk)_{i \in \mathbb{N}}$ such that the diagram

$$\begin{array}{ccc} E_i \oplus_p E_i & \xrightarrow{\gamma_{\frac{1}{2}}} & E_{i+1} \\ {\scriptstyle \left(\begin{smallmatrix} L_i & 0 \\ 0 & L_i \end{smallmatrix}\right)} \downarrow & & \downarrow {\scriptstyle L_{i+1}} \\ \Bbbk \oplus_p \Bbbk & \xrightarrow[m_i]{} & \Bbbk \end{array}$$

commutes, where E_i is the set of all step function constants on each $(\frac{t-1}{2^i}, \frac{t}{2^i})$, L_i sends $f = \sum_i k_i \mathbf{1}_{[a_i, b_i]}$ to $\sum_i k_i |b_i - a_i|$, and $m = \varinjlim m_i$. Furthermore, we have the following result.

Corollary 8.3. *There exists a unique morphism*

$$L : (\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}}) \to (\mathbb{R}, 1, m)$$

in $\mathrm{Hom}_{\mathscr{N}\!or^1}((\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}}), (\mathbb{R}, 1, m))$ *such that the diagram*

$$\begin{array}{ccc} (\mathbf{S}, \mathbf{1}_{[0,1]}, \gamma_{\frac{1}{2}}) & \xrightarrow{L} & (\mathbb{R}, 1, m) \\ {\scriptstyle \subseteq} \downarrow & \nearrow_{\widehat{L}} & \\ (\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}}) & & \end{array}$$

commutes, where $\widehat{L}$ is the morphism in $\mathrm{Hom}_{\mathscr{A}^p}((\widehat{\mathbf{S}}, \mathbf{1}_{[0,1]}, \widehat{\gamma}_{\frac{1}{2}}), (\mathbb{R}, 1, m))$ *whose existence is unique. Furthermore, $\widehat{L}$ is given by the direct limit $\varinjlim L_i : \varinjlim E_i \to \Bbbk$.*

Proof. It is an immediate consequence of Theorem 7.6. □

Remark 8.4. The morphism $\widehat{L}$ induces a $\Bbbk$-linear map sending f to $\widehat{L}(f)$. Indeed, $\widehat{L}(f)$ is Lebesgue integration of f, that is,

$$\widehat{L}(f) = \int_0^1 f\mathrm{d}\mu,$$

where μ is the Lebesgue measure in this case, see [16, Proposition 2.2].

8.2. **Cauchy-Schwarz inequality.** Take $\Bbbk = \mathbb{R}$. In this subsection we will establish the Cauchy-Schwarz inequality for the morphism $\widehat{T}$ in $\mathscr{N}\!or^p$. We need the following lemma.

Lemma 8.5. *If $f \in \widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$ is non-negative, then so is $\widehat{T}(f)$. That is, $f \geq 0$ yields*

$$\int_{\mathbb{I}_\Lambda} f\mathrm{d}\mu \geq 0.$$

Proof. By $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)} = \varinjlim E_u$, there is a monotonically increasing sequence $\{f_t\}_{t\in\mathbb{N}^+}$ of non-negative functions with $f_t = \sum_{i=1}^{2^{u_t}} k_{ti}\mathbf{1}_{I_{ti}} \in E_{u_t}$, such that $I_{ti} \cap I_{tj} = \varnothing$ for any $i \neq j$; $t_1 < t_2$ yields $u_{t_1} < u_{t_2}$ and $f_{t_1} \leq f_{t_2}$; and $f = \varinjlim f_t$. Thus, for any $1 \leq i \leq 2^{u_t}$ and $t \in \mathbb{N}^+$, we have $k_{ti\geq 0}$, and then the following inequality

$$\widehat{T}(f_t) = T_{u_t}(f_t) = \sum_{i=1}^{2^{u_t}} k_{ti}\mu(I_{ti}) \geq 0$$

holds. Furthermore, we obtain

$$\widehat{T}(f) = \varinjlim T_{u_t}(f_t) = \varinjlim T|_{E_{u_t}}(f_t) = \varinjlim T(f_t) \geq 0$$

as required, where $\varinjlim T(f_t) = \lim\limits_{t\to+\infty} T(f_t)$ is the usual limit in $\mathbb{R}$ in analysis. □

Theorem 8.6 (Cauchy-Schwarz inequality)**.** *Let f and g be two functions lying in $\widehat{\mathbf{S}_\tau(\mathbb{I}_\Lambda)}$. Then*

$$\left(\int_{\mathbb{I}_\Lambda} fg\right)^2 \leq \left(\int_{\mathbb{I}_\Lambda} f^2\mathrm{d}\mu\right)\left(\int_{\mathbb{I}_\Lambda} g^2\mathrm{d}\mu\right).$$

Proof. Take the quadratic function

$$\varphi(t) = \widehat{T}(f^2)\cdot t^2 + 2\widehat{T}(fg)\cdot t + \widehat{T}(g^2)\ (t \in \mathbb{R}).$$

Notice that $\widehat{T}$ is a Λ-homomorphism, thus it is also an $\mathbb{R}$-linear map. Then

$$\begin{aligned}\varphi(t) &= \widehat{T}(f^2\cdot(t\mathbf{1}_\mathbb{R})^2 + 2fg\cdot(t\mathbf{1}_\mathbb{R}) + g^2)\\ &= \widehat{T}((f\cdot(t\mathbf{1}_\mathbb{R}) + g)^2).\end{aligned}$$

Notice that $(f\cdot(t\mathbf{1}_\mathbb{R})+g)^2$, written as h, is also a function defined on $\mathbb{I}_\Lambda$ lying in $\mathbf{S}_\tau(\mathbb{I}_\Lambda)$, thus for any $x \in \mathbb{I}_\Lambda$, we have $h(x) = (tf(x) + g(x))^2 \geq 0$. Then $\varphi(t) \geq 0$ by Lemma 8.5. It follows that the discriminant $(2\widehat{T}(fg))^2 - 4\widehat{T}(f^2)\widehat{T}(g^2)$ of $\varphi(x)$ is at most zero, that is,

$$\left(\int_{\mathbb{I}_\Lambda} fg\mathrm{d}\mu\right)^2 \leq \left(\int_{\mathbb{I}_\Lambda} f^2\mathrm{d}\mu\right)\left(\int_{\mathbb{I}_\Lambda} g^2\mathrm{d}\mu\right).$$

□

The above inequality yields the Cauchy-Schwarz inequality

$$\left(\int_0^1 fg\mathrm{d}\mu\right)^2 \leq \left(\int_0^1 f^2\mathrm{d}\mu\right)\left(\int_0^1 g^2\mathrm{d}\mu\right)$$

for Lebesgue integration if $\mathscr{N}\!or^p$ satisfies the conditions (L1)–(L3) given in Subsection 8.1.

Acknowledgements

Yu-Zhe Liu is supported by the National Natural Science Foundation of China (Grant No. 12171207), Guizhou Provincial Basic Research Program (Natural Science) (Grant No. ZK[2024]YiBan066) and Scientific Research Foundation of Guizhou University (Grant Nos. [2022]53, [2022]65). Shenda Liu is supported by the National Natural Science Foundation of China (Grant No. 62203442). Zhaoyong Huang is supported by the National Natural Science Foundation of China (Grant Nos. 12371038, 12171207). Panyue Zhou is supported by the National Natural Science Foundation of China (Grant No. 12371034) and the Hunan Provincial Natural Science Foundation of China (Grant No. 2023JJ30008). The authors are greatly indebted to Zhenxing Di, Jian He, Yajun Ma, Yucheng Wang and Yuan Yuan for helpful suggestions.

References

[1] M. Alvarez-Picallo and J.-S. P. Lemay. Cartesian difference categories. *Log. Methods Comput. Sci.*, 17(3): Paper No.23, 48, 2021. DOI:10.46298/lmcs-17(3:23)2021.

[2] I. Assem, D. Simson and A. Skowroński. *Elements of the Representation Theory of Associative Algebras, Volume 1, Techniques of Representation Theory.* Cambridge University Press, 2006. DOI:10.1017/CBO9780511614309.

[3] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. CRC Press, 2018.

[4] R. F. Blute, J. R. B. Cockett and R. A. G. Seely. Differential categories. *Math. Struct. Comput. Sci.*, 16(6): 1049–1083, 2006. DOI:10.1017/S0960129506005676.

[5] R. F. Blute, J. R. B. Cockett and R. A. G. Seely. Cartesian differential storage categories. *Theor. Appl. Categ.*, 30: 620–687, 2015. AMS:MR3346979.

[6] S. Bochner. Integration von funktionen, deren werte die elemente eines vektorraumes sind *Fund. Math.*, 20(1): 262–176, 1933. https://bibliotekanauki.pl/articles/1385181.

[7] F. Burk. *Lebesgue Measure and Integration: An Introduction.* John Wiley & Sons, Inc., 2011.

[8] J. R. B. Cockett and J.-S. P. Lemay. Cartesian integral categories and contextual integral categories. In *Proceedings of the Thirty-Fourth Conference on the Mathematical Foundations of Programming Semantics (MFPS XXXIV)*, pp.45–72, Elsevier Science Publishers B.V., 2018. DOI:10.1016/j.entcs.2018.11.004.

[9] J. R. B. Cockett and J.-S. P. Lemay. Integral categories and calculus categories. *Math. Struct. Comp. Sci.*, 29(2): 243–308, 2018. DOI:10.1017/S0960129518000014.

[10] M. H. Escardó and A. K. Simpson. A universal characterization of the closed euclidean interval. In *Proceedings 16th Annual IEEE Symposium on Logic in Computer Science*, pp.108–126. IEEE Computer Society, 2001. DOI: 10.1109/LICS.2001.932488.

[11] P. Freyd. Algebraic real analysis. *Theor. Appl. Catego.*, 20(10): 215–306, 2008. https://emis.dsd.sztaki.hu/journals/TAC/volumes/20/10/20-10abs.html.

[12] S. Hartman and J. Mikusinski. *The theory of Lebesgue measure and integration.* Elsevier, 2014.

[13] S. Ikonicoff and J.-S. P. Lemay. Cartesian differential comonads and new models of Cartesian differential categories. *Cah. Topol. Géom. Differ. Catég.*, 64(2): 198–239, 2023. AMS:MR4605864.

[14] K. Ito. Stochastic integral. *Proc. Imp. Acad.*, 20(8): 519–524, 2008. DOI:10.3792/pia/1195572786.

[15] H. L. Lebesgue. *Leçons sur l'intégration et la recherche des fonctions primitives: professees au college de France (deuxiemeedition).* Collection de monographies sur la theorie des fonctions, 1928.

[16] T. Leinster. A categorical derivation of lebesgue integration. *J. Lond. Math. Soc.*, 107(6): 1959–1982, 2023. DOI:10.1112/jlms.12730.

[17] J.-S. P. Lemay. Differential algebras in codifferential categories. *J. Pure. Appl. Algebra*, 223(10): 4191–4225, 2019. DOI:10.1016/j.jpaa.2019.01.005.

[18] J.-S. P. Lemay. Cartesian Differential Kleisli Categories. *Electronic Notes in Theoretical Informatics and Computer Science*, 3, 2023.
[19] J. J. Rotman. *An Introduction to Homological Algebra (Second Edition)*. Springer, 2009.
[20] I. Segal. Algebraic integration theory. *Bull. Amer. Math. Soc.*, 71(3): 419–489, 1965. DOI:S0002-9904-1965-11284-8.

School of Mathematics and Statistics, Guizhou University, Guiyang 550025, Guizhou, P. R. China
E-mail address: liuyz@gzu.edu.cn / yzliu3@163.com

The State Key Laboratory of Multimodal Artificial Intelligence Systems, Institute of Automation, Chinese Academy of Sciences, Beijing 100190, P. R. China.
E-mail address: thinksheng@foxmail.com

Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu, P. R. China
E-mail address: huangzy@nju.edu.cn

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, Hunan, P. R. China
E-mail address: panyuezhou@163.com