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Homological Dimensions Relative to Special Subcategories

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Abstract. Let $\mathscr{A}$ be an abelian category, $\mathscr{C}$ an additive, full and self-orthogonal subcategory of $\mathscr{A}$ closed under direct summands, $r\mathcal{G}(\mathscr{C})$ the right Gorenstein subcategory of $\mathscr{A}$ relative to $\mathscr{C}$, and ${}^{\perp}\mathscr{C}$ the left orthogonal class of $\mathscr{C}$. For an object A in $\mathscr{A}$, we prove that if A is in the right 1-orthogonal class of $r\mathcal{G}(\mathscr{C})$, then the $\mathscr{C}$-projective and $r\mathcal{G}(\mathscr{C})$-projective dimensions of A are identical; if the $r\mathcal{G}(\mathscr{C})$-projective dimension of A is finite, then the $r\mathcal{G}(\mathscr{C})$-projective and ${}^{\perp}\mathscr{C}$-projective dimensions of A are identical. We also prove that the supremum of the $\mathscr{C}$-projective dimensions of objects with finite $\mathscr{C}$-projective dimension and that of the $r\mathcal{G}(\mathscr{C})$-projective dimensions of objects with finite $r\mathcal{G}(\mathscr{C})$-projective dimension coincide. Then we apply these results to the category of modules.

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1 Introduction

In homological theory, homological dimensions are fundamental invariants and every homological dimension of objects is defined relative to a certain subcategory. For example, projective and injective dimensions of modules are defined relative to the categories of projective and injective modules, respectively, and Gorenstein projective and injective dimensions of modules are defined relative to the categories

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of Gorenstein projective and injective modules, respectively; see, e.g., [6–8, 11–13, 24, 26, 27]. Because projective modules are Gorenstein projective, the Gorenstein projective dimension of a module is at most its projective dimension. A natural question is when they are identical. Holm studied this question in [6, 7].

Let $\mathscr{A}$ be an abelian category and let $\mathscr{C}$ be an additive and full subcategory of $\mathscr{A}$. As a common generalization of Gorenstein projective and injective modules, Sather-Wagstaff, Sharif and White [16] introduced the Gorenstein subcategory $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$ relative to $\mathscr{C}$. Huang studied in [12] when the $\mathscr{C}$-projective dimension and the $\mathcal{G}(\mathscr{C})$-projective dimension of an object in $\mathscr{A}$ are identical. From the definition of the Gorenstein subcategory $\mathcal{G}(\mathscr{C})$, it is known that $\mathscr{C}$ should be simultaneously a generator and a cogenerator for $\mathcal{G}(\mathscr{C})$, and both functors $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)$ and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{C})$ should possess certain exactness. These assumptions seem to be strong to some extent. In [18], by modifying the definition of Gorenstein subcategories, the so-called right Gorenstein subcategory $r\mathcal{G}(\mathscr{C})$ and left Gorenstein subcategory $l\mathcal{G}(\mathscr{C})$ were introduced such that for a self-orthogonal subcategory $\mathscr{C}$ of $\mathscr{A}$, an object $A \in \mathscr{A}$ is in $\mathcal{G}(\mathscr{C})$ if and only if it is in $r\mathcal{G}(\mathscr{C}) \cap l\mathcal{G}(\mathscr{C})$. According to the ideas above, we will study when the $\mathscr{C}$-projective dimension and the $r\mathcal{G}(\mathscr{C})$-projective dimension of an object in $\mathscr{A}$ are identical. Our main result and its dual extend [12, Corollary 3.12 and Theorem 3.14] and their duals [12, Corollary 4.12 and Theorem 4.14], respectively, and the strong assumptions on $\mathscr{C}$ are not needed for the one-sided Gorenstein categories $r\mathcal{G}(\mathscr{C})$ and $l\mathcal{G}(\mathscr{C})$.

The paper is organized as follows. In Section 2, we give some terminology and notations. Let $\mathscr{A}$ be an abelian category, $\mathscr{C}$ an additive, full and self-orthogonal subcategory of $\mathscr{A}$ closed under direct summands, and ${}^{\perp}\mathscr{C}$ the left orthogonal class of $\mathscr{C}$. In Section 3, for an object A in $\mathscr{A}$ we prove that if A is in the right 1-orthogonal class of $r\mathcal{G}(\mathscr{C})$, then the $\mathscr{C}$-projective and $r\mathcal{G}(\mathscr{C})$-projective dimensions of A are identical; if the $r\mathcal{G}(\mathscr{C})$-projective dimension of A is finite, then the $r\mathcal{G}(\mathscr{C})$-projective and ${}^{\perp}\mathscr{C}$-projective dimensions of A are identical (Theorem 3.3). Moreover, we prove that the supremum of the $\mathscr{C}$-projective dimensions of objects with finite $\mathscr{C}$-projective dimension and that of the $r\mathcal{G}(\mathscr{C})$-projective dimensions of objects with finite $r\mathcal{G}(\mathscr{C})$-projective dimension coincide (Theorem 3.10). The dual versions of these results are also given. In Section 4, we apply the results obtained to the category of modules. Let R, S be rings and ${}_RC_S$ a semidualizing bimodule. For a left R-module A, we prove that if either the C-projective dimension of A is finite or $A \in {}_RC^{\perp_1}$ and the injective dimension of A is finite, then the C-projective and C-Gorenstein projective dimensions of A are identical (Corollary 4.4). It generalizes [6, Proposition 2.27] and [7, Theorem 2.2]. As a consequence, if $R \in {}_RC^{\perp_1}$ and the left self-injective dimension of R is finite (in particular, if R is left self-injective), then the category of C-projective modules is projectively resolving; further, if the projective dimension of a left R-module A is finite, then the projective, C-projective and C-Gorenstein projective dimensions of A are identical (Proposition 4.6).

2 Preliminaries

In this paper, $\mathscr{A}$ is an abelian category and all subcategories of $\mathscr{A}$ are additive, full and closed under isomorphisms. Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$. We write

$$\begin{aligned}
{}^{\perp}\mathscr{X} &= \{A \in \mathscr{A} \mid \operatorname{Ext}^{\geq 1}_{\mathscr{A}}(A, X) = 0 \text{ for any } X \in \mathscr{X}\},\\
\mathscr{X}^{\perp} &= \{A \in \mathscr{A} \mid \operatorname{Ext}^{\geq 1}_{\mathscr{A}}(X, A) = 0 \text{ for any } X \in \mathscr{X}\},\\
{}^{\perp_1}\mathscr{X} &= \{A \in \mathscr{A} \mid \operatorname{Ext}^{1}_{\mathscr{A}}(A, X) = 0 \text{ for any } X \in \mathscr{X}\},\\
\mathscr{X}^{\perp_1} &= \{A \in \mathscr{A} \mid \operatorname{Ext}^{1}_{\mathscr{A}}(X, A) = 0 \text{ for any } X \in \mathscr{X}\}.
\end{aligned}$$

For subcategories $\mathscr{X}, \mathscr{Y}$ of $\mathscr{A}$, we write $\mathscr{X} \perp \mathscr{Y}$ if $\operatorname{Ext}^{\geq 1}_{\mathscr{A}}(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$; we say that $\mathscr{X}$ is *self-orthogonal* if $\mathscr{X} \perp \mathscr{X}$.

For an object $A \in \mathscr{A}$, the *$\mathscr{X}$-projective dimension* of A, denoted by $\mathscr{X}\text{-pd}\, A$, is defined as

$$\begin{aligned}
&\inf\{n \mid \text{there exists an exact sequence}\\
&\qquad 0 \to X_n \to \cdots \to X_1 \to X_0 \to A \to 0 \text{ in } \mathscr{A} \text{ with all } X_i \text{ in } \mathscr{X}\},
\end{aligned}$$

and we set $\mathscr{X}\text{-pd}\, A$ infinite if no such integer exists. Dually, the *$\mathscr{X}$-injective dimension* of A is defined, which is denoted by $\mathscr{X}\text{-id}\, A$. For a ring R and a left R-module A, we use $\operatorname{pd}_R A$ and $\operatorname{id}_R A$ to denote the projective and injective dimensions of A, respectively.

A sequence $\mathbb{E}$ in $\mathscr{A}$ is said to be $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X}, -)$*-exact* (resp., $\operatorname{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact*) if it is exact after we apply the functor $\operatorname{Hom}_{\mathscr{A}}(X, -)$ (resp., $\operatorname{Hom}_{\mathscr{A}}(-, X)$) for any $X \in \mathscr{X}$. Following [16], we write

$$\begin{aligned}
\operatorname{res}\widetilde{\mathscr{X}} = \{A \in \mathscr{A} \mid\ &\text{there exists a } \operatorname{Hom}_{\mathscr{A}}(\mathscr{X}, -)\text{-exact exact sequence}\\
&\cdots \to X_i \to \cdots \to X_1 \to X_0 \to A \to 0 \text{ in } \mathscr{A} \text{ with all } X_i \text{ in } \mathscr{X}\}.
\end{aligned}$$

Dually, $\operatorname{cores}\widetilde{\mathscr{X}}$ is defined.

Definition 2.1. [16, Definition 4.1] Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. The *Gorenstein subcategory* $\mathcal{G}(\mathscr{C})$ of $\mathscr{A}$ (relative to $\mathscr{C}$) is defined as

$$\begin{aligned}
\{G \in \mathscr{A} \mid\ &\text{there exists a } \operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)\text{-exact and } \operatorname{Hom}_{\mathscr{A}}(-, \mathscr{C})\text{-exact}\\
&\text{exact sequence } \cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots\\
&\text{in } \mathscr{A} \text{ with all } C_i, C^i \text{ in } \mathscr{C} \text{ such that } G \cong \operatorname{Im}(C_0 \to C^0)\}.
\end{aligned}$$

The Gorenstein subcategory unifies the following notions: modules of Gorenstein dimension zero [1], Gorenstein projective modules, Gorenstein injective modules [3], V-Gorenstein projective modules, V-Gorenstein injective modules [4], $\mathscr{W}$-Gorenstein modules [5], and so on; see [11] for details.

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. Following [11, Lemma 5.7], if $\mathscr{C} \perp \mathscr{C}$, then the Gorenstein subcategory

$$\mathcal{G}(\mathscr{C}) = ({}^{\perp}\mathscr{C} \cap \operatorname{cores}\widetilde{\mathscr{C}}) \cap (\mathscr{C}^{\perp} \cap \operatorname{res}\widetilde{\mathscr{C}}).$$

Motivated by this fact, we introduced the following definition in [18].

Definition 2.2. $r\mathcal{G}(\mathscr{C}) := {}^{\perp}\mathscr{C} \cap \operatorname{cores}\widetilde{\mathscr{C}}$ (resp., $l\mathcal{G}(\mathscr{C}) := \mathscr{C}^{\perp} \cap \operatorname{res}\widetilde{\mathscr{C}}$) is called the *right* (resp., *left*) *Gorenstein subcategory* of $\mathscr{A}$ (relative to $\mathscr{C}$).

By the explanation above and this definition, we observe that if $\mathscr{C}\perp\mathscr{C}$, then we have $\mathcal{G}(\mathscr{C}) = r\mathcal{G}(\mathscr{C}) \cap l\mathcal{G}(\mathscr{C})$.

3 Main results

In this section, we fix $\mathscr{C}$ a self-orthogonal subcategory of $\mathscr{A}$ closed under direct summands. We begin with the following easy observation.

Lemma 3.1. *We have $r\mathcal{G}(\mathscr{C}) \perp \mathscr{C}\text{-pd}^{<\infty}$, where $\mathscr{C}\text{-pd}^{<\infty}$ is the subcategory of $\mathscr{A}$ consisting of objects having finite $\mathscr{C}$-projective dimension.*

Proposition 3.2. *Let $A \in \mathscr{A}$ with $r\mathcal{G}(\mathscr{C})\text{-pd}\, A < \infty$. Then the following statements are equivalent for any $n \geq 0$:*

(1) *$r\mathcal{G}(\mathscr{C})\text{-pd}\, A \leq n$.*
(2) *$\mathrm{Ext}_{\mathscr{A}}^{\geq n+1}(A, C) = 0$ for any $C \in \mathscr{C}$.*
(3) *$\mathrm{Ext}_{\mathscr{A}}^{n+1}(A, H) = 0$ for any $H \in \mathscr{A}$ with $\mathscr{C}\text{-pd}\, H < \infty$.*
(4) *$\mathrm{Ext}_{\mathscr{A}}^{\geq n+1}(A, H) = 0$ for any $H \in \mathscr{A}$ with $\mathscr{C}\text{-pd}\, H < \infty$.*

Proof. The implications (4)$\Rightarrow$(2) and (4)$\Rightarrow$(3) are trivial, and the implications (1)$\Rightarrow$(2)$\Rightarrow$(4) follow from dimension shifting.

(3)$\Rightarrow$(1) Let $r\mathcal{G}(\mathscr{C})\text{-pd}\, A = m$ $(< \infty)$. By [18, Theorem 3.11], there exists an exact sequence

$$0 \longrightarrow C_m \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow G_0 \longrightarrow A \longrightarrow 0$$

in $\mathscr{A}$ with all $C_i \in \mathscr{C}$ and $G_0 \in r\mathcal{G}(\mathscr{C})$. We need to prove $m \leq n$. Otherwise, suppose $m > n$. Set $H_{n+1} = \mathrm{Im}(C_{n+1} \to C_n)$, $H_n = \mathrm{Coker}(C_{n+1} \to C_n)$ (note that $C_0 = G_0$). Then $\mathscr{C}\text{-pd}\, H_{n+1} \leq m - n - 1 < \infty$. Since $\mathscr{C}$ is self-orthogonal, we have $\mathscr{C} \subseteq r\mathcal{G}(\mathscr{C}) \subseteq {}^{\perp}H_{n+1}$ by Lemma 3.1. So $\mathrm{Ext}_{\mathscr{A}}^{1}(H_n, H_{n+1}) \cong \mathrm{Ext}_{\mathscr{A}}^{n+1}(A, H_{n+1}) = 0$ by (3). Hence, the exact sequence

$$0 \longrightarrow H_{n+1} \longrightarrow C_n \longrightarrow H_n \longrightarrow 0$$

splits. Thus H_n is isomorphic to a direct summand of C_n, and therefore $H_n \in r\mathcal{G}(\mathscr{C})$ by [18, Proposition 3.3(1)]. It implies $r\mathcal{G}(\mathscr{C})\text{-pd}\, A \leq n$, which is a contradiction. $\square$

Because $\mathscr{C} \subseteq r\mathcal{G}(\mathscr{C}) \subseteq {}^{\perp}\mathscr{C}$, we have ${}^{\perp}\mathscr{C}\text{-pd}\, A \leq r\mathcal{G}(\mathscr{C})\text{-pd}\, A \leq \mathscr{C}\text{-pd}\, A$ for any $A \in \mathscr{A}$. It is natural to ask when these two inequalities are equalities. The following result gives some partial answer to this question, which extends [12, Corollary 3.12 and Theorem 3.14]. It provides some relatively simple methods for computing the $r\mathcal{G}(\mathscr{C})$-projective dimension of objects under certain conditions.

Theorem 3.3. *For an object $A \in \mathscr{A}$, the following statements hold:*

(1) *If $A \in r\mathcal{G}(\mathscr{C})^{\perp_1}$, then $r\mathcal{G}(\mathscr{C})\text{-pd}\, A = \mathscr{C}\text{-pd}\, A$.*
(2) *If $r\mathcal{G}(\mathscr{C})\text{-pd}\, A < \infty$, then $r\mathcal{G}(\mathscr{C})\text{-pd}\, A = {}^{\perp}\mathscr{C}\text{-pd}\, A$.*

Proof. (1) Notice that $r\mathcal{G}(\mathscr{C})\text{-pd}\, A \leq \mathscr{C}\text{-pd}\, A$, so the case $r\mathcal{G}(\mathscr{C})\text{-pd}\, A = \infty$ clearly implies the equality $\mathscr{C}\text{-pd}\, A = r\mathcal{G}(\mathscr{C})\text{-pd}\, A$. Now let $A \in r\mathcal{G}(\mathscr{C})^{\perp_1}$ and

$r\mathcal{G}(\mathscr{C})$-pd $A = n$ $(< \infty)$. By [18, Theorem 3.10], there exists an exact sequence $0 \to H \to G \to A \to 0$ in $\mathscr{A}$ with $\mathscr{C}$-pd $H \leq n-1$ and $G \in r\mathcal{G}(\mathscr{C})$. By Lemma 3.1, we have $H \in r\mathcal{G}(\mathscr{C})^{\perp}$. Then $G \in r\mathcal{G}(\mathscr{C})^{\perp_1}$. Because $G \in r\mathcal{G}(\mathscr{C})$, there exists an exact sequence $0 \to G \to C \to G' \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $G' \in r\mathcal{G}(\mathscr{C})$. This exact sequence splits since $G \in r\mathcal{G}(\mathscr{C})^{\perp_1}$, and so G is isomorphic to a direct summand of C. Because $\mathscr{C}$ is closed under direct summands, we have $G \in \mathscr{C}$ and $\mathscr{C}$-pd $A \leq n$.

(2) Notice that ${}^{\perp}\mathscr{C}$-pd $A \leq r\mathcal{G}(\mathscr{C})$-pd A, so the case ${}^{\perp}\mathscr{C}$-pd $A = \infty$ clearly implies the equality $r\mathcal{G}(\mathscr{C})$-pd $A = {}^{\perp}\mathscr{C}$-pd A. Let ${}^{\perp}\mathscr{C}$-pd $A = n$ $(< \infty)$. Then there exists an exact sequence

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

in $\mathscr{A}$ with all X_i in ${}^{\perp}\mathscr{C}$. So $\mathrm{Ext}_{\mathscr{A}}^{\geq n+1}(A, C) = 0$ for any $C \in \mathscr{C}$. It follows from Proposition 3.2 that $r\mathcal{G}(\mathscr{C})$-pd $A \leq n$. □

Dual to Theorem 3.3, we have the following result, which extends [12, Corollary 4.12 and Theorem 4.14].

Theorem 3.4. *For an object $B \in \mathscr{A}$, the following statements hold:*

(1) *If $B \in {}^{\perp_1}l\mathcal{G}(\mathscr{C})$, then $l\mathcal{G}(\mathscr{C})$-id $B = \mathscr{C}$-id B.*

(2) *If $l\mathcal{G}(\mathscr{C})$-id $B < \infty$, then $l\mathcal{G}(\mathscr{C})$-id $B = \mathscr{C}^{\perp}$-id B.*

In the following, we give an application of Theorem 3.3. Before proceeding, we note the lemma below.

Lemma 3.5.

(1) *$r\mathcal{G}(\mathscr{C})^{\perp}$ is closed under extensions and cokernels of monomorphisms.*

(2) *Let $0 \to A \to M \to N \to 0$ be an exact sequence in $\mathscr{A}$ with M, N in $r\mathcal{G}(\mathscr{C})^{\perp}$. If $A \in \mathscr{C}^{\perp_1}$, then $A \in r\mathcal{G}(\mathscr{C})^{\perp}$.*

Proof. (1) Obvious.

(2) Let $G \in r\mathcal{G}(\mathscr{C})$. Then $\mathrm{Ext}_{\mathscr{A}}^{\geq 2}(G, A) = 0$. Because $G \in r\mathcal{G}(\mathscr{C})$, there exists an exact sequence $0 \to G \to C \to G' \to 0$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $G' \in r\mathcal{G}(\mathscr{C})$. Then $\mathrm{Ext}_{\mathscr{A}}^{\geq 2}(G', A) = 0$ by the above argument. If $A \in \mathscr{C}^{\perp_1}$, then we have a monomorphism $\mathrm{Ext}_{\mathscr{A}}^{1}(G, A) \rightarrowtail \mathrm{Ext}_{\mathscr{A}}^{2}(G', A)\,(= 0)$. Therefore, $\mathrm{Ext}_{\mathscr{A}}^{1}(G, A) = 0$ and $A \in r\mathcal{G}(\mathscr{C})^{\perp}$. □

Now we give the following consequence of Theorem 3.3.

Corollary 3.6. *For an object $A \in \mathscr{A}$, if one of the following conditions is satisfied, then $r\mathcal{G}(\mathscr{C})$-pd $A = \mathscr{C}$-pd A:*

(1) *$\mathscr{C}$-pd $A < \infty$.*

(2) *$A \in \mathscr{C}^{\perp}$ and id $A < \infty$.*

Proof. If $\mathscr{C}$-pd $A < \infty$, then $A \in r\mathcal{G}(\mathscr{C})^{\perp}$ by Proposition 3.2. On the other hand, note that $\mathscr{C} \cup \{\text{all injectives in } \mathscr{A}\} \subseteq r\mathcal{G}(\mathscr{C})^{\perp}$. So, if $A \in \mathscr{C}^{\perp}$ and id $A < \infty$, then $A \in r\mathcal{G}(\mathscr{C})^{\perp}$ by Lemma 3.5 and dimension shifting. Thus, $\mathscr{C}$-pd $A = r\mathcal{G}(\mathscr{C})$-pd A in both cases by Theorem 3.3(1). □

Dually, we have the following consequence of Theorem 3.4.

Corollary 3.7. *For an object $B \in \mathscr{A}$, if one of the following conditions is satisfied, then $l\mathcal{G}(\mathscr{C})$-id $B = \mathscr{C}$-id B:*
(1) *$\mathscr{C}$-id $B < \infty$.*
(2) *$B \in {}^{\perp}\mathscr{C}$ and $\operatorname{pd} B < \infty$.*

Let R be a ring and M a left R-module. We use $\operatorname{Prod} M$ to denote the class consisting of all left R-modules isomorphic to direct summands of direct products of copies of M.

Example 3.8. (1) For an object $A \in {}^{\perp}r\mathcal{G}(\mathscr{C})$, we have $\mathscr{C}$-pd $A > r\mathcal{G}(\mathscr{C})$-pd A in general. For example, let R be a ring which is not left self-injective and let

$$0 \to R \xrightarrow{f^0} E^0(R) \xrightarrow{f^1} E^1(R) \xrightarrow{f^2} \cdots \xrightarrow{f^i} E^i(R) \xrightarrow{f^{i+1}} \cdots$$

be a minimal injective resolution of ${}_RR$, that is, it is an exact sequence and $E^i(R)$ is the injective envelope of $\operatorname{Im} f^i$ for any $i \geq 0$. Put $\mathscr{C} = \operatorname{Prod}\big(\prod_{i\geq 0} E^i(R)\big)$. Then $\mathscr{C}\perp\mathscr{C}$ and ${}_RR \in {}^{\perp}r\mathcal{G}(\mathscr{C})$. Since R is not left self-injective, we see that ${}_RR \notin \mathscr{C}$ and $\mathscr{C}$-pd $R > 0$. On the other hand, it is clear that ${}_RR \in r\mathcal{G}(\mathscr{C})$ and $r\mathcal{G}(\mathscr{C})$-pd $R = 0$.

(2) For an object $A \in \mathscr{A}$, whether $\operatorname{pd} A$, the projective dimension of A, is finite or infinite, we may have $\operatorname{pd} A \neq \mathscr{C}$-pd A and $\operatorname{pd} A \neq r\mathcal{G}(\mathscr{C})$-pd A in general. For example, let R be a left and right Artinian ring with $\operatorname{id}_{R^{op}} R = n$, where n is a positive integer or infinity, and let A be an injective cogenerator for the category of left R-modules. Put $\mathscr{C} = \operatorname{Prod} A$. Then $\mathscr{C}\perp\mathscr{C}$ and $\mathscr{C}$-pd $A = r\mathcal{G}(\mathscr{C})$-pd $A = 0$. But $\operatorname{pd}_R A = n$ by [10, Lemma 17.2.4(1)].

We need the following lemma.

Lemma 3.9. *Let $0 \to A \to B \to G \to 0$ be an exact sequence in $\mathscr{A}$. If $G \in r\mathcal{G}(\mathscr{C})$, then $r\mathcal{G}(\mathscr{C})$-pd $A \leq r\mathcal{G}(\mathscr{C})$-pd B.*

Proof. Let $r\mathcal{G}(\mathscr{C})$-pd $B = n$ $(< \infty)$. Then we see that there exists an exact sequence $0 \to K \to G_0 \to B \to 0$ in $\mathscr{A}$ with $G_0 \in r\mathcal{G}(\mathscr{C})$ and $r\mathcal{G}(\mathscr{C})$-pd $K \leq n - 1$. Consider the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d] & & \\
 & K \arrow[r, equal, dashed] \arrow[d, dashed] & K \arrow[d] & & \\
0 \arrow[r, dashed] & G_0' \arrow[r, dashed] \arrow[d, dashed] & G_0 \arrow[r, dashed] \arrow[d] & G \arrow[r, dashed] \arrow[d, equal, dashed] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, dashed] & B \arrow[r] \arrow[d] & G \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By [18, Proposition 3.3(2)] and by the middle row in this diagram, we can obtain $G'_0 \in r\mathcal{G}(\mathscr{C})$. Therefore, the exactness of the leftmost column in the above diagram yields $r\mathcal{G}(\mathscr{C})\text{-pd}\, A \leq n$. $\square$

We write

$$\mathscr{C}\text{-FPD} = \sup\{\mathscr{C}\text{-pd}\, A \mid A \in \mathscr{A} \text{ with } \mathscr{C}\text{-pd}\, A < \infty\},$$
$$\mathscr{C}\text{-FrGPD} = \sup\{r\mathcal{G}(\mathscr{C})\text{-pd}\, A \mid A \in \mathscr{A} \text{ with } r\mathcal{G}(\mathscr{C})\text{-pd}\, A < \infty\}.$$

The following result unifies some known results about absolute and Gorenstein big (resp., small) finitistic dimensions (see Proposition 4.7).

Theorem 3.10. $\mathscr{C}$-FPD $=$ $\mathscr{C}$-FrGPD.

Proof. By Corollary 3.6(1), we have $\mathscr{C}\text{-FPD} \leq \mathscr{C}\text{-FrGPD}$. Let $A \in \mathscr{A}$ with $r\mathcal{G}(\mathscr{C})\text{-pd}\, A = n < \infty$. By [18, Theorem 3.10], there exists an exact sequence

$$0 \longrightarrow A \longrightarrow H' \longrightarrow G' \longrightarrow 0$$

in $\mathscr{A}$ with $G' \in r\mathcal{G}(\mathscr{C})$ and $\mathscr{C}\text{-pd}\, H' \leq n$. If $\mathscr{C}\text{-pd}\, H' \leq n-1$, then

$$r\mathcal{G}(\mathscr{C})\text{-pd}\, A \leq r\mathcal{G}(\mathscr{C})\text{-pd}\, H' \leq \mathscr{C}\text{-pd}\, H' \leq n-1$$

by Lemma 3.9, which is a contradiction. So $\mathscr{C}\text{-pd}\, H' = n$ and $\mathscr{C}\text{-FPD} \geq n$, which implies $\mathscr{C}\text{-FrGPD} \leq \mathscr{C}\text{-FPD}$. $\square$

Now we write

$$\mathscr{C}\text{-FID} = \sup\{\mathscr{C}\text{-id}\, B \mid B \in \mathscr{A} \text{ with } \mathscr{C}\text{-id}\, B < \infty\},$$
$$\mathscr{C}\text{-FlGID} = \sup\{l\mathcal{G}(\mathscr{C})\text{-id}\, B \mid B \in \mathscr{A} \text{ with } l\mathcal{G}(\mathscr{C})\text{-id}\, B < \infty\}.$$

The following result is the dual version of Theorem 3.10.

Theorem 3.11. $\mathscr{C}$-FID $=$ $\mathscr{C}$-FlGID.

4 Applications to Module Categories

In all that follows all rings are associative rings with identity. For a ring R, $\operatorname{Mod} R$ is the category of left R-modules and $\operatorname{mod} R$ is the category of finitely generated left R-modules.

Definition 4.1. [9, Definition 2.1] Let R and S be rings. An (R, S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied:

(a1) ${}_RC$ admits a degreewise finite R-projective resolution; that is, there exists an exact sequence $\cdots \to P_1 \to P_0 \to {}_RC \to 0$ in $\operatorname{mod} R$ with all P_i projective.

(a2) C_S admits a degreewise finite S^{op}-projective resolution; that is, there exists an exact sequence $\cdots \to Q_1 \to Q_0 \to C_S \to 0$ in $\operatorname{mod} S^{op}$ with all Q_i projective.

(b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \operatorname{Hom}_{S^{op}}(C, C)$ is an isomorphism.

(b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_R(C, C)$ is an isomorphism.

(c1) $\operatorname{Ext}_R^{\geq 1}(C,C)=0$.

(c2) $\operatorname{Ext}_{S^{op}}^{\geq 1}(C,C)=0$.

Wakamatsu [20] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*; see [2, Chapter IV, Section 3] and [14, Section 2]. Note that an (R,S)-bimodule ${}_RC_S$ is semidualizing if and only if ${}_RC$ (resp., C_S) is Wakamatsu tilting with $S=\operatorname{End}({}_RC)$ (resp., $R=\operatorname{End}(C_S)$), and if and only if both ${}_RC$ and C_S are Wakamatsu tilting with $S=\operatorname{End}({}_RC)$ and $R=\operatorname{End}(C_S)$ (see [22, Corollary 3.2]). For examples of semidualizing bimodules, we refer the reader to [9, Example 2.1] and [21, Section 3]. In particular, ${}_RR_R$ is a semidualizing (R,R)-bimodule.

From now on, R, S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. By $\operatorname{Add}_R C$ we denote the subcategory of $\operatorname{Mod} R$ consisting of direct summands of direct sums of copies of C, and write

$$\mathcal{P}_C(R)=\{C\otimes_S P\mid P \text{ is projective in } \operatorname{Mod} S\},$$
$$\mathcal{I}_C(S)=\{\operatorname{Hom}_R(C,I)\mid I \text{ is injective in } \operatorname{Mod} R\}.$$

The modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called *C-projective* and *C-injective*, respectively. When ${}_RC_S={}_RR_R$, C-projective and C-injective modules are exactly projective and injective modules, respectively.

Definition 4.2. [8, Definition 2.7], [13, Definition 2.5] A module $M\in\operatorname{Mod} R$ is called *C-Gorenstein projective* if $M\in{}^{\perp}\mathcal{P}_C(R)$ and there exists a $\operatorname{Hom}_R(-,\mathcal{P}_C(R))$-exact exact sequence

$$0\longrightarrow M\longrightarrow G^0\longrightarrow G^1\longrightarrow\cdots\longrightarrow G^i\longrightarrow\cdots$$

in $\operatorname{Mod} R$ with all G^i in $\mathcal{P}_C(R)$. Dually, the notion of *C-Gorenstein injective modules* in $\operatorname{Mod} S$ is defined.

We use $\mathcal{GP}_C(R)$ (resp., $\mathcal{GI}_C(S)$) to denote the subcategory of $\operatorname{Mod} R$ (resp., $\operatorname{Mod} S$) consisting of C-Gorenstein projective (resp., injective) modules. When ${}_RC_S={}_RR_R$, C-Gorenstein projective and injective modules are exactly Gorenstein projective and injective modules, respectively.

• **C-Gorenstein projective dimension.** We have the following facts:

(i) $\mathcal{P}_C(R)=\operatorname{Add}_R C$ [13, Proposition 2.4(1)];

(ii) $\mathcal{GP}_C(R)=r\mathcal{G}(\mathcal{P}_C(R))$ [17, Lemma 4.7(1)];

(iii) $\mathcal{P}_C(R)\perp\mathcal{P}_C(R)$ [19, Lemma 2.5(1)].

So, putting $\mathscr{C}=\mathcal{P}_C(R)\,(=\operatorname{Add}_R C)$ in Theorem 3.3, we have the following.

Corollary 4.3. *For a module $A\in\operatorname{Mod} R$, we have the following statements:*

(1) *If $A\in\mathcal{GP}_C(R)^{\perp_1}$, then $\mathcal{GP}_C(R)$-$\operatorname{pd}_R A=\mathcal{P}_C(R)$-$\operatorname{pd}_R A$.*

(2) *If $\mathcal{GP}_C(R)$-$\operatorname{pd}_R A<\infty$, then $\mathcal{GP}_C(R)$-$\operatorname{pd}_R A={}^{\perp}\mathcal{P}_C(R)$-$\operatorname{pd}_R A$.*

Since $\mathcal{P}_C(R) = \mathrm{Add}_R\, C$, we have $\mathcal{P}_C(R)^{\perp} = {}_RC^{\perp}$ by [15, Proposition 7.21]. So, putting $\mathscr{C} = \mathcal{P}_C(R)\,(= \mathrm{Add}_R\, C)$ in Corollary 3.6, we have the next result.

Corollary 4.4. *For a module $A \in \mathrm{Mod}\, R$, if one of the following conditions is satisfied, then $\mathcal{GP}_C(R)\text{-pd}_R A = \mathcal{P}_C(R)\text{-pd}_R A$:*
(1) $\mathcal{P}_C(R)\text{-pd}_R A < \infty$.
(2) $A \in {}_RC^{\perp}$ *and* $\mathrm{id}_R\, A < \infty$.

Putting ${}_RC_S = {}_RR_R$ in Corollary 4.4, we obtain the following corollary.

Corollary 4.5. [6, Proposition 2.27], [7, Theorem 2.2] *For a module $A \in \mathrm{Mod}\, R$, if one of the following conditions is satisfied, then $\mathcal{GP}_R(R)\text{-pd}_R A = \mathrm{pd}_R\, A$:*
(1) $\mathrm{pd}_R\, A < \infty$.
(2) $\mathrm{id}_R\, A < \infty$.

Recall from [6] that a subcategory of $\mathrm{Mod}\, R$ is called *projectively resolving* if it contains $\mathcal{P}_R(R)$ and is closed under extensions and kernels of epimorphisms. Holm and White stated in [9, Corollary 6.4] that $\mathcal{P}_C(R)$ is projectively resolving if ${}_RC_S$ is faithful. But it is not true and the problem is that $\mathcal{P}_C(R)$ does not contain $\mathcal{P}_R(R)$ in general (see [9, Example 4.7(1)]). The first assertion in the following result gives a sufficient condition for $\mathcal{P}_C(R)$ to be projectively resolving. We compare the second assertion with Example 3.8(2).

Proposition 4.6. *If $R \in {}_RC^{\perp}$ and $\mathrm{id}_R\, R < \infty$ (in particular, if R is left self-injective), then we have the following:*
(1) *$\mathcal{P}_C(R)$ is projectively resolving.*
(2) *For any module $A \in \mathrm{Mod}\, R$ with $\mathrm{pd}_R\, A < \infty$, we have*

$$\mathcal{GP}_C(R)\text{-pd}_R A = \mathcal{P}_C(R)\text{-pd}_R A = \mathrm{pd}_R\, A.$$

Proof. (1) By [13, Corollary 2.10] we have $\mathcal{P}_R(R) \subseteq \mathcal{GP}_C(R)$. Since $\mathrm{id}_R\, R < \infty$, we have $\mathcal{P}_C(R)\text{-pd}_R R = \mathcal{GP}_C(R)\text{-pd}_R R = 0$ by Corollary 4.4(2). So $R \in \mathcal{P}_C(R)$, and hence $\mathcal{P}_R(R) \subseteq \mathcal{P}_C(R)$. By [9, Proposition 5.2(b)] and the proof of [9, Corollary 6.4], $\mathcal{P}_C(R)$ is closed under extensions and kernels of epimorphisms. Thus, we conclude that $\mathcal{P}_C(R)$ is projectively resolving.

(2) By (1) and [19, Proposition 4.8]. □

We define

$$C\text{-FPD}(R) = \sup\{\mathcal{P}_C(R)\text{-pd}_R A \mid A \in \mathrm{Mod}\, R \text{ with } \mathcal{P}_C(R)\text{-pd}_R A < \infty\},$$
$$C\text{-FGPD}(R) = \sup\{\mathcal{GP}_C(R)\text{-pd}_R A \mid A \in \mathrm{Mod}\, R \text{ with } \mathcal{GP}_C(R)\text{-pd}_R A < \infty\}.$$

When ${}_RC_S = {}_RR_R$, we write $\mathrm{FPD}(R) = C\text{-FPD}(R)$ and $\mathrm{FGPD}(R) = C\text{-FGPD}(R)$. In addition, we write

$$\mathrm{fPD}(R) = \sup\{\mathrm{pd}_R\, A \mid A \in \mathrm{mod}\, R \text{ with } \mathrm{pd}_R\, A < \infty\}.$$

If R is a left noetherian ring, then by [25, Proposition 1.4] we have

$$\begin{aligned}&\mathcal{GP}_R(R)\cap \operatorname{mod} R\\ &=\{G\in \operatorname{mod} R \mid \text{there exists an exact sequence } 0\to P_1\to P_0\to P^0\to P^1\to\cdots\\ &\quad \text{in } \operatorname{mod} R \text{ with all } P_i, P^i \text{ projective such that it remains exact after we apply}\\ &\quad \text{the functor } \operatorname{Hom}_R(-,P) \text{ for any projective module } P \text{ in } \operatorname{mod} R \text{ and}\\ &\quad G\cong \operatorname{Im}(P_0\to P^0)\}.\end{aligned}$$

In this case, this category is still denoted by $\mathcal{GP}_R(R)$, and we write

$$\operatorname{fGPD}(R)=\sup\{\mathcal{GP}_R(R)\text{-}\operatorname{pd}_R A \mid A\in \operatorname{mod} R \text{ with } \mathcal{GP}_R(R)\text{-}\operatorname{pd}_R A<\infty\}.$$

In the following result, the assertion (2) is [6, Theorem 2.28], and the assertion (3) was proved in [23, Lemma 4.4] when R is an artin algebra.

Proposition 4.7.
(1) C-FPD$(R)=C$-FGPD(R).
(2) FPD$(R)=$ FGPD(R).
(3) *If R is a left noetherian ring, then* fPD$(R)=$ fGPD(R).

Proof. If we put $\mathscr{C}=\operatorname{Add}_R C$ and $\mathscr{A}=\operatorname{Mod} R$ in Theorem 3.10, then the assertion (1) follows. The assertion (2) is the special case of (1) for ${}_RC_S={}_RR_R$. If we put $\mathscr{C}=\{$finitely generated projective left R-modules$\}$ and $\mathscr{A}=\operatorname{mod} R$ in Theorem 3.10, then the assertion (3) follows. □

• **C-Gorenstein injective dimension.** The results in this part and their proofs are completely dual to those for C-Gorenstein projective dimension above, so we only list the results without proofs. We fix an injective cogenerator ${}_RE$ for $\operatorname{Mod} R$ and write $(-)^+=\operatorname{Hom}_R(-,E)$. Then we have the following facts:
(i) $\mathcal{I}_C(S)=\operatorname{Prod}_S C^+$ [13, Proposition 2.4(2)];
(ii) $\mathcal{GI}_C(S)=l\mathcal{G}(\mathcal{I}_C(S))$ [17, Lemma 4.7(1)];
(iii) $\mathcal{I}_C(S)\perp\mathcal{I}_C(S)$ [19, Lemma 2.5(2)].
So, putting $\mathscr{C}=\mathcal{I}_C(S)\,(=\operatorname{Prod}_S C^+)$ in Theorem 3.4, we have the following.

Corollary 4.8. *For a module $B\in\operatorname{Mod} S$, the following statements hold:*
(1) *If $B\in{}^{\perp_1}\mathcal{GI}_C(S)$, then $\mathcal{GI}_C(S)$-$\operatorname{id}_S B=\mathcal{I}_C(S)$-$\operatorname{id}_S B$.*
(2) *If $\mathcal{GI}_C(S)$-$\operatorname{id}_S B<\infty$, then $\mathcal{GI}_C(S)$-$\operatorname{id}_S B=\mathcal{I}_C(S)^{\perp}$-$\operatorname{id}_S B$.*

Now we define $C_S^{\top}=\{B\in\operatorname{Mod} S\mid \operatorname{Tor}^S_{\geq 1}(C,B)=0\}$.

Observation. ${}^{\perp}\mathcal{I}_C(S)={}^{\perp}(C^+)=C_S^{\top}$.

Indeed, since $\mathcal{I}_C(S)=\operatorname{Prod}_S C^+$, we have ${}^{\perp}\mathcal{I}_C(S)={}^{\perp}(C^+)$ by [15, Proposition 7.22]. By [15, Corollary 10.63], we have the natural isomorphism

$$\operatorname{Ext}^i_S(B,C^+)\cong[\operatorname{Tor}^S_i(C,B)]^+$$

for any $B\in\operatorname{Mod} S$ and $i\geq 1$. It follows that $\operatorname{Ext}^i_S(B,C^+)=0$ if and only if $[\operatorname{Tor}^S_i(C,B)]^+=0$ if and only if $\operatorname{Tor}^S_i(C,B)=0$ since ${}_RE$ is an injective cogenerator for $\operatorname{Mod} R$. Thus, we have ${}^{\perp}(C^+)=C_S^{\top}$.

Putting $\mathscr{C}=\mathcal{I}_C(S)\,(=\operatorname{Prod}_S C^+)$ in Corollary 3.7, we have the following.

Corollary 4.9. *For a module $B \in \operatorname{Mod} S$, if one of the following conditions is satisfied, then $\mathcal{GI}_C(S)$-$\mathrm{id}_S B = \mathcal{I}_C(S)$-$\mathrm{id}_S B$:*
(1) $\mathcal{I}_C(S)$-$\mathrm{id}_S B < \infty$.
(2) $B \in C_S^{\top}$ *and* $\mathrm{pd}_S B < \infty$.

Putting ${}_RC_S = {}_SS_S$ in Corollary 4.9, we get the next result, in which the assertion (2) is [7, Theorem 2.1].

Corollary 4.10. *For a module $B \in \operatorname{Mod} S$, if one of the following conditions is satisfied, then $\mathcal{GI}_S(S)$-$\mathrm{id}_S B = \mathrm{id}_S B$:*
(1) $\mathrm{id}_S B < \infty$.
(2) $\mathrm{pd}_S B < \infty$.

Recall from [6] that a subcategory of $\operatorname{Mod} S$ is called *injectively coresolving* if it contains $\mathcal{I}_S(S)$ and is closed under extensions and cokernels of monomorphisms.

Proposition 4.11. *If $Q \in C_S^{\top}$ and $\mathrm{pd}_S Q < \infty$ for an injective cogenerator Q for* $\operatorname{Mod} S$, *then the following statements hold:*
(1) $\mathcal{I}_C(S)$ *is injectively coresolving.*
(2) $\mathcal{GI}_C(S)$-$\mathrm{id}_S B = \mathcal{I}_C(S)$-$\mathrm{id}_S B = \mathrm{id}_S B$ *for every module $B \in \operatorname{Mod} S$ with* $\mathrm{id}_S B < \infty$.

We set

$$
\begin{aligned}
C\text{-FID}(S) &= \sup\{\mathcal{I}_C(S)\text{-}\mathrm{id}_S B \mid B \in \operatorname{Mod} S \text{ with } \mathcal{I}_C(S)\text{-}\mathrm{id}_S B < \infty\},\\
C\text{-FGID}(S) &= \sup\{\mathcal{GI}_C(S)\text{-}\mathrm{id}_S B \mid B \in \operatorname{Mod} S \text{ with } \mathcal{GI}_C(S)\text{-}\mathrm{id}_S B < \infty\}.
\end{aligned}
$$

When ${}_RC_S = {}_SS_S$, we write $\mathrm{FID}(S) = C\text{-FID}(S)$ and $\mathrm{FGID}(S) = C\text{-FGID}(S)$. The assertion (2) in the following result is [6, Theorem 2.29].

Proposition 4.12. (1) C-FID$(S) = C$-FGID(S) *and* (2) FID$(S) =$ FGID(S).

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