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Left Frobenius Pairs, Cotorsion Pairs and Weak Auslander-Buchweitz Contexts in Triangulated Categories

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Abstract. Let $\mathcal{T}$ be a triangulated category with a proper class ξ of triangles. We introduce the notions of left Frobenius pairs, left (n-)cotorsion pairs and left (weak) Auslander-Buchweitz contexts with respect to ξ in $\mathcal{T}$. We show how to construct left cotorsion pais from left n-cotorsion pairs, and establish a one-to-one correspondence between left Frobenius pairs and left (weak) Auslander-Buchweitz contexts. Some applications are given in the Gorenstein homological theory of triangulated categories.

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1 Introduction

An important branch of relative homological algebra was developed by Auslander and Buchweitz in their paper [4]. Based on this, the so-called "Auslander-Buchweitz context" for abelian categories was defined by Hashimoto [12], and Auslander-Buchweitz approximation theory is the prerequisite for computing relative dimensions. On the other hand, cotorsion pairs, developed in [9–11], are important in the

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study of the algebraic and geometric structures of abelian categories. This notion provides a good setting for investigating relative homological dimensions (see [1]). Moreover, Huerta et al. [16] introduced the notion of n-cotorsion pairs in abelian categories. They described several properties of n-cotorsion pairs and established a relation between n-cotorsion pairs and (complete) cotorsion pairs. Becerril et al. [6] introduced Frobenius pairs in abelian categories, and presented one-to-one correspondences between left Frobenius pairs, Auslander-Buchweitz contexts and relative cotorsion pairs in abelian categories.

Recently, triangulated categories entered into the subject in a relevant way. Let $\mathcal{T}$ be a triangulated category with the class Δ of triangles. In analogy with relative homological algebra in abelian categories, Beligiannis developed in [7] a relative version of homological algebra in triangulated categories, in which the notion of a proper class of exact sequences is replaced by that of a proper class of triangles $\xi \subseteq \Delta$. Later on, by combining it with Gorenstein homological theory in abelian categories, many authors developed relative homological theory, especially Gorenstein homological theory, in triangulated categories (see [2, 3, 8, 17, 18, 21, 23]). Recently, Ma and Zhao [17] introduced and developed the Auslander-Buchweitz approximation theory with respect to a proper class ξ of triangles in triangulated categories, which is an analog of the approximation theory in abelian categories [4].

Throughout, unless otherwise stated, we always assume that $\mathcal{T}$ is a triangulated category with enough ξ-projective and ξ-injective objects. This paper is devoted to developing relative homological theory along with the Auslander-Buchweitz approximation theory in triangulated categories. Moreover, some applications are given in the context of Gorenstein homological algebra in triangulated categories.

This paper is organized as follows. In Section 2, we give some terminologies and some preliminary results. In Section 3, we recall the notion of left (n-)cotorsion pairs in $\mathcal{T}$ with respect to ξ, and then by virtue of an equivalent characterization of n-cotorsion pairs [13], we establish a relation between n-cotorsion pairs and cotorsion pairs (Proposition 3.10). In Section 4, we introduce the notions of left Frobenius pairs and left (weak) Auslander-Buchweitz contexts in $\mathcal{T}$. For a subcategory $\mathcal{X}$ of $\mathcal{T}$, $\mathcal{X}^\wedge$ denotes the subcategory of $\mathcal{T}$ consisting of objects with finite $\mathcal{X}$-resolution dimension. Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$. We show that $\mathcal{X}^\wedge$ is closed under ξ-extensions, hokernels of ξ-proper epimorphisms, hocokernels of ξ-proper monomorphisms and direct summands (Theorem 4.9 and Proposition 4.12). Then we show how to obtain (left) cotorsion pairs from left Frobenius pairs (Theorem 4.14). Finally, we introduce the notion of left (weak) Auslander-Buchweitz context, and establish a one-to-one correspondence between left weak Auslander-Buchweitz contexts and left Frobenius pairs as follows (Theorem 4.22): Let $n \geq 1$ be an integer, and consider the classes

$\mathfrak{A} := \{(\mathcal{X}, \omega) \mid (\mathcal{X}, \omega) \text{ is a left Frobenius pair in } \mathcal{T}\}$,

$\mathfrak{B} := \{(\mathcal{A}, \mathcal{B}) \mid (\mathcal{A}, \mathcal{B}) \text{ is a left weak Auslander-Buchweitz context}\}$,

$\mathfrak{C} := \{(\mathcal{U}, \mathcal{V}) \mid (\mathcal{U}, \mathcal{V}) \text{ is a cotorsion pair in } \mathcal{T} \text{ with } \mathcal{U} \text{ resolving, } \mathcal{V} \subseteq \mathcal{U}^\wedge\}$,

$\mathfrak{D} := \{(\mathcal{U}, \mathcal{V}) \mid (\mathcal{U}, \mathcal{V}) \text{ is an } n\text{-cotorsion pair in } \mathcal{T} \text{ with } \mathcal{U} \text{ resolving, } \mathcal{V} \subseteq \mathcal{U}^\wedge\}$.

Then (1) there is a one-to-one correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ given by

$$\begin{aligned}\Phi\colon \mathfrak{A} &\longrightarrow \mathfrak{B}, \quad (\mathcal{X},\omega) \longmapsto (\mathcal{X},\omega^\wedge),\\ \Psi\colon \mathfrak{B} &\longrightarrow \mathfrak{A}, \quad (\mathcal{A},\mathcal{B}) \longmapsto (\mathcal{A},\, \mathcal{A}\cap\mathcal{B});\end{aligned}$$

(2) $\mathfrak{C} \subseteq \mathfrak{B}$; (3) $\mathfrak{C} = \mathfrak{D}$.

2 Preliminaries

Let $\mathcal{T}$ be an additive category and $\Sigma\colon \mathcal{T} \to \mathcal{T}$ be an additive functor. Define the category $\mathrm{Diag}(\mathcal{T},\Sigma)$ as follows:

- An object of $\mathrm{Diag}(\mathcal{T},\Sigma)$ is a diagram in $\mathcal{T}$ of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$.
- A morphism in $\mathrm{Diag}(\mathcal{T},\Sigma)$ between $X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} \Sigma X_i$, $i = 1, 2$, is a triple (α,β,γ) of morphisms in $\mathcal{T}$ such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "u_1"] \arrow[d, "\alpha"] & Y_1 \arrow[r, "v_1"] \arrow[d, "\beta"] & Z_1 \arrow[r, "w_1"] \arrow[d, "\gamma"] & \Sigma X_1 \arrow[d, "\Sigma\alpha"] \\ X_2 \arrow[r, "u_2"] & Y_2 \arrow[r, "v_2"] & Z_2 \arrow[r, "w_2"] & \Sigma X_2 \end{tikzcd}$$

A triangulated category is a triple $(\mathcal{T},\Sigma,\Delta)$, where $\mathcal{T}$ is an additive category and $\Sigma\colon \mathcal{T} \to \mathcal{T}$ is an autoequivalence of $\mathcal{T}$ (called the *suspension* functor), and Δ is a full subcategory of $\mathrm{Diag}(\mathcal{T},\Sigma)$ which is closed under isomorphisms and satisfies the axioms (T_1)–(T_4) in [7, Section 2.1] (also see [20]), where (T_4) is called the octahedral axiom. The elements in Δ are called *triangles*.

The following well-known result is an efficient tool.

Remark 2.1. [7, Proposition 2.1] Let $\mathcal{T}$ be an additive category, $\Sigma\colon \mathcal{T} \to \mathcal{T}$ an autoequivalence of $\mathcal{T}$, and Δ a full subcategory of $\mathrm{Diag}(\mathcal{T},\Sigma)$ which is closed under isomorphisms. Suppose that the triple $(\mathcal{T},\Sigma,\Delta)$ satisfies all axioms of a triangulated category except possibly the octahedral axiom. Then the following statements are equivalent:

(1) *Octahedral axiom.* For any two morphisms $u\colon X \to Y$ and $v\colon Y \to Z$, there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "u'"] \arrow[d, "v"] & Z' \arrow[r, "u''"] \arrow[d, "\alpha"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "vu"] & Z \arrow[r, "w"] \arrow[d, "v'"] & Y' \arrow[r, "w'"] \arrow[d, "\beta"] & \Sigma X \arrow[d, "\Sigma u"] \\ & Z'' \arrow[r, equal] \arrow[d, "v''"] & Z'' \arrow[r, "v''"] \arrow[d, "\gamma"] & \Sigma Y \\ & \Sigma Y \arrow[r, "\Sigma u'"] & \Sigma Z' & \end{tikzcd}$$

in which the first two rows and the middle two columns are triangles in Δ.

(2) *Base change.* For any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in Δ and any morphism $\alpha\colon Z' \to Z$, there exists the following commutative diagram:

$$\begin{tikzcd}
 & X' \arrow[r, equal] \arrow[d, "\beta'"] & X' \arrow[d, "\beta"] & \\
X \arrow[r, "u'"] \arrow[d, equal] & Y' \arrow[r, "v'"] \arrow[d, "\alpha'"] & Z' \arrow[r, "w'"] \arrow[d, "\alpha"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "\gamma'"] & Z \arrow[r, "w"] \arrow[d, "\gamma"] & \Sigma X \arrow[d, "\Sigma u'"] \\
 & \Sigma X' \arrow[r, equal] & \Sigma X' \arrow[r, "-\Sigma\beta'"] & \Sigma Y'
\end{tikzcd}$$

in which the middle two rows and the middle two columns are triangles in Δ.

(3) *Cobase change*. For any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in Δ and any morphism $\beta\colon X \to X'$, there exists the following commutative diagram:

$$\begin{tikzcd}
 & \Sigma^{-1}Z' \arrow[r, equal] \arrow[d, "-\Sigma^{-1}\gamma"] & \Sigma^{-1}Z' \arrow[d, "-\Sigma^{-1}\gamma'"] & \\
\Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}w"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, "\beta"] & Y \arrow[r, "v"] \arrow[d, "\beta'"] & Z \arrow[d, equal] \\
\Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}w'"] & X' \arrow[r, "u'"] \arrow[d, "\alpha"] & Y' \arrow[r, "v'"] \arrow[d, "\alpha'"] & Z \arrow[d, "-w"] \\
 & Z' \arrow[r, equal] & Z' \arrow[r, "\gamma"] & \Sigma X
\end{tikzcd}$$

in which the middle two rows and the middle two columns are triangles in Δ.

In all that follows, let $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ be a triangulated category, and $\mathcal{A}b$ be the category of abelian groups. Recall that a triangle $X \to Y \to Z \to \Sigma X$ is called *split* if it is isomorphic to the triangle

$$X \xrightarrow{\binom{1}{0}} X \oplus Z \xrightarrow{(0,\ 1)} Z \xrightarrow{0} \Sigma X.$$

We use Δ_0 to denote the full subcategory of Δ consisting of all split triangles.

Definition 2.2. [7] Let ξ be a class of triangles in $\mathcal{T}$.

(1) ξ is said to be *closed under base change* (resp., *cobase change*) providing that for any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in ξ and any morphism $\alpha\colon Z' \to Z$ (resp., $\beta\colon X \to X'$) as in Remark 2.1(2) (resp., Remark 2.1(3)), the triangle $X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$ (resp., $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X'$) is in ξ.

(2) ξ is said to be *closed under suspension* providing that for any integer i and for any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in ξ, the triangle

$$\Sigma^i X \xrightarrow{(-1)^i\Sigma^i u} \Sigma^i Y \xrightarrow{(-1)^i\Sigma^i v} \Sigma^i Z \xrightarrow{(-1)^i\Sigma^i w} \Sigma^{i+1} X$$

is in ξ.

(3) ξ is called *saturated* if in the situation of base change as in Remark 2.1(2), whenever the third vertical and the second horizontal triangles are in ξ, the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is in ξ.

Definition 2.3. [7, Definition 2.2] A class ξ of triangles in $\mathcal{T}$ is called *proper* if the following conditions are satisfied:

(1) ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi$.
(2) ξ is closed under suspensions and is saturated.
(3) ξ is closed under base and cobase change.

Example 2.4. [7, Example 2.3] (1) Let $\mathcal{T}$ be a triangulated category. There are two trivial proper classes of triangles: Δ_0 and Δ.

(2) Let $F\colon \mathcal{T} \to \mathcal{T}'$ be an exact functor of triangulated categories and let ξ' be a proper class of triangles in $\mathcal{T}'$. Let ξ be the class of triangles δ in $\mathcal{T}$ such that $F(\delta) \in \xi'$. Then ξ is a proper class of triangles in $\mathcal{T}$.

(3) Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}$ an abelian category, and $F\colon \mathcal{T} \to \mathcal{A}$ a (co)homological functor. Define ξ_F as follows: a triangle $X \to Y \to Z \to \Sigma X$ is in ξ_F if and only if the induced sequence $0 \to F^i(X) \to F^i(Y) \to F^i(Z) \to 0$ for any integer i is exact in $\mathcal{A}$, where $F^i = F\Sigma^i$. Then ξ_F is a proper class of triangles in $\mathcal{T}$.

(4) Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{T}$ with $\Sigma\mathcal{X} = \mathcal{X}$. Define $\xi_{\mathcal{X}}$ (resp., $\xi_{\mathcal{X}}^{\mathrm{op}}$) as follows: a triangle $A \to B \to C \to \Sigma A$ is in $\xi_{\mathcal{X}}$ if and only if the induced sequence $0 \to \mathrm{Hom}_{\mathcal{T}}(X,A) \to \mathrm{Hom}_{\mathcal{T}}(X,B) \to \mathrm{Hom}_{\mathcal{T}}(X,C) \to 0$ (resp., $0 \to \mathrm{Hom}_{\mathcal{T}}(C,X) \to \mathrm{Hom}_{\mathcal{T}}(B,X) \to \mathrm{Hom}_{\mathcal{T}}(A,X) \to 0$) for any $X \in \mathcal{X}$ is exact in $\mathcal{A}b$. Then $\xi_{\mathcal{X}}$ (resp., $\xi_{\mathcal{X}}^{\mathrm{op}}$) is a proper class of triangles in $\mathcal{T}$.

In what follows we always assume that ξ is a proper class of triangles in $\mathcal{T}$.

Definition 2.5. [7, Definition 2.4] Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a triangle in ξ. Then the morphism u (resp., v) is called *ξ-proper monic* (resp., *ξ-proper epic*), and u (resp., v) is called the *hokernel* of v (resp., the *hocokernel* of u).

For any triangle $X \to Y \to Z \to \Sigma X$ in ξ, we say that $\mathcal{X}$ is *closed under ξ-extensions* if $X, Z \in \mathcal{X}$ implies $Y \in \mathcal{X}$; $\mathcal{X}$ is *closed under hokernels of ξ-proper epimorphisms* (resp., *hocokernels of ξ-proper monomorphisms*) if $Y, Z \in \mathcal{X}$ (resp., $X, Y \in \mathcal{X}$) implies $X \in \mathcal{X}$ (resp., $Z \in \mathcal{X}$).

Definition 2.6. [7, Definition 4.1] An object P (resp., I) in $\mathcal{T}$ is called *ξ-projective* (resp., *ξ-injective*) if for any triangle $X \to Y \to Z \to \Sigma X$ in ξ, the induced complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P,X) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P,Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P,Z) \longrightarrow 0$$
$$(\text{resp., } 0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z,I) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Y,I) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X,I) \longrightarrow 0)$$

is exact in $\mathcal{A}b$. We use $\mathcal{P}(\xi)$ (resp., $\mathcal{I}(\xi)$) to denote the full subcategory of $\mathcal{T}$ consisting of ξ-projective (resp., ξ-injective) objects.

We say that $\mathcal{T}$ has *enough ξ-projective objects* if for any object $M \in \mathcal{T}$ there exists a triangle $K \to P \to M \to \Sigma K$ in ξ with $P \in \mathcal{P}(\xi)$. Dually, we say that $\mathcal{T}$ has *enough ξ-injective objects* if for any object $M \in \mathcal{T}$ there exists a triangle $M \to I \to K \to \Sigma M$ in ξ with $I \in \mathcal{I}(\xi)$.

From now on, we always assume that $\mathcal{T}$ is a triangulated category with enough ξ-projective and ξ-injective objects.

Definition 2.7. [2, Section 3] Let $\mathcal{T}$ be a triangulated category.

(1) A *ξ-exact* complex is a complex

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \tag{2.1}$$

in $\mathcal{T}$ such that for any $n \in \mathbb{Z}$ there exists a triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1} \tag{2.2}$$

in ξ and the differential d_n is defined as $d_n = g_{n-1}f_n$.

(2) A triangle $X \to Y \to Z \to \Sigma X$ in ξ is called $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P}(\xi))$-*exact* if the induced complex $0 \to \mathrm{Hom}_{\mathcal{T}}(Z, P) \to \mathrm{Hom}_{\mathcal{T}}(Y, P) \to \mathrm{Hom}_{\mathcal{T}}(X, P) \to 0$ for any object $P \in \mathcal{P}(\xi)$ is exact in $\mathcal{A}b$.

(3) A ξ-exact complex as (2.1) is called $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P}(\xi))$-*exact* if the triangle (2.2) is $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P}(\xi))$-exact for any $n \in \mathbb{Z}$.

Asadollahi and Salarian [2] introduced the notion of ξ-Gorenstein projective objects.

Definition 2.8. [2, Definition 3.6] Let $\mathcal{T}$ be a triangulated category and X an object in $\mathcal{T}$. A *complete ξ-projective resolution* is a $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P}(\xi))$-exact ξ-exact complex $\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ in $\mathcal{T}$ with all P_i ξ-projective objects. The objects K_n as in (2.2) are called *ξ-Gorenstein projective* objects. We use $\mathcal{GP}(\xi)$ to denote the full subcategory of $\mathcal{T}$ consisting of all ξ-Gorenstein projective objects. Dually, ξ-Gorenstein injective objects and $\mathcal{GI}(\xi)$ are defined.

Suppose that M is an object in $\mathcal{T}$. Beligiannis [7] defined the ξ-extension groups $\xi xt^n_\xi(-, M)$ to be the nth right ξ-derived functor of the functor $\mathrm{Hom}_{\mathcal{T}}(-, M)$, that is, $\xi xt^n_\xi(-, M) := \mathcal{R}^n_\xi \mathrm{Hom}_{\mathcal{T}}(-, M)$.

Remark 2.9. Let $X \to Y \to Z \to \Sigma X$ be a triangle in ξ. For any objects $M, N \in \mathcal{T}$, by [7, Corollary 4.12] there exist long exact sequences of "ξxt" functors

$$\begin{aligned}
&0 \longrightarrow \xi xt^0_\xi(Z, M) \longrightarrow \xi xt^0_\xi(Y, M) \longrightarrow \xi xt^0_\xi(X, M)\\
&\quad\longrightarrow \xi xt^1_\xi(Z, M) \longrightarrow \xi xt^1_\xi(Y, M) \longrightarrow \xi xt^1_\xi(X, M) \longrightarrow \cdots,\\
&0 \longrightarrow \xi xt^0_\xi(N, X) \longrightarrow \xi xt^0_\xi(N, Y) \longrightarrow \xi xt^0_\xi(N, Z)\\
&\quad\longrightarrow \xi xt^1_\xi(N, X) \longrightarrow \xi xt^1_\xi(N, Y) \longrightarrow \xi xt^1_\xi(N, Z) \longrightarrow \cdots.
\end{aligned}$$

Following Remark 2.9, we usually use the strategy of "dimension shifting", which is an important tool in relative homological theory of triangulated categories. Set

$$\begin{aligned}
&\mathcal{X}^{\perp_n} := \{M \in \mathcal{T} \mid \xi xt^n_\xi(X, M) = 0 \text{ for all } X \in \mathcal{X}\},\\
&\mathcal{X}^{\perp} := \{M \in \mathcal{T} \mid \xi xt^n_\xi(X, M) = 0 \text{ for all } X \in \mathcal{X} \text{ and all } n \geq 1\} = \bigcap_{n\geq 1} \mathcal{X}^{\perp_n}.
\end{aligned}$$

Dually, ${}^{\perp_n}\mathcal{X}$ and ${}^{\perp}\mathcal{X}$ are defined.

The notion of a contravariantly (or covariantly) finite subcategory of the category of finitely generated modules, which is also called a precovering (or preenveloping) class, was first introduced over artin algebras by Auslander and Smalø [5]. It plays an important role in homological algebra and representation theory of algebra. Here we recall the corresponding notions in the setting relative to a proper class of triangles.

Definition 2.10. [17, Definition 3.8] Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and M be an object in $\mathcal{T}$. A *right $\mathcal{X}$-approximation* of M is a ξ-proper epimorphism $X \to M$ such that the induced complex $\mathrm{Hom}_{\mathcal{T}}(\widetilde{X}, X) \to \mathrm{Hom}_{\mathcal{T}}(\widetilde{X}, M) \to 0$ is exact in $\mathcal{A}b$ for any $\widetilde{X} \in \mathcal{X}$. In this case, there is a triangle $K \to X \to M \to \Sigma K$ in ξ. Dually, a *left $\mathcal{X}$-approximation* of M is defined.

The subcategory $\mathcal{X}$ is said to be *contravariantly finite* if any object $T \in \mathcal{T}$ admits a right $\mathcal{X}$-approximation, and dually $\mathcal{X}$ is said to be *covariantly finite* if any object $T \in \mathcal{T}$ admits a left $\mathcal{X}$-approximation (cf. [15, Definition 3.9]). The subcategory $\mathcal{X}$ is called *functorially finite* if it is both contravariantly finite and covariantly finite.

Definition 2.11. [17, Definition 2.11] Let $(\mathcal{X}, \omega)$ be a pair of subcategories in $\mathcal{T}$ with $\omega \subseteq \mathcal{X}$.

(1) ω is called a *ξ-cogenerator* of $\mathcal{X}$ if for any object X in $\mathcal{X}$, there exists a triangle $X \to W \to X' \to \Sigma X$ in ξ with $W \in \omega$ and $X' \in \mathcal{X}$.

(2) ω is called *$\mathcal{X}$-injective* if $\omega \subseteq \mathcal{X}^{\perp}$.

Definition 2.12. [17, Definition 2.12] Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{T}$. Then $\mathcal{X}$ is called a *resolving* subcategory of $\mathcal{T}$ if the following conditions are satisfied:

(1) $\mathcal{P}(\xi) \subseteq \mathcal{X}$.

(2) $\mathcal{X}$ is closed under ξ-extensions.

(3) $\mathcal{X}$ is closed under hokernels of ξ-proper epimorphisms.

Dually, a *coresolving* subcategory is defined.

Definition 2.13. [17, Definition 3.1] Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and T be an object in $\mathcal{T}$. The *$\mathcal{X}$-resolution dimension* of T is defined as

$$\begin{aligned}\mathcal{X}\text{-res.}\dim T := \inf\{n \geq 0 \mid & \text{ there exists a } \xi\text{-exact complex} \\ & 0 \to X_n \to \cdots \to X_1 \to X_0 \to T \to 0 \text{ in } \mathcal{T} \text{ with all } X_i \in \mathcal{X}\}.\end{aligned}$$

If no such integer n exists, then set $\mathcal{X}$-res. $\dim T = \infty$. The *$\mathcal{X}$-resolution dimension* of $\mathcal{T}$ is defined by $\mathcal{X}\text{-res.}\dim \mathcal{T} := \sup\{\mathcal{X}\text{-res.}\dim T \mid T \in \mathcal{T}\}$.

The *$\mathcal{X}$-coresolution dimensions* $\mathcal{X}$-cores. $\dim T$ and $\mathcal{X}$-cores. $\dim \mathcal{T}$ are defined dually.

When $\mathcal{X} = \mathcal{P}(\xi)$, we write $\xi\text{-pd}\, T := \mathcal{X}\text{-res.}\dim T$, and when $\mathcal{X} = \mathcal{I}(\xi)$, we write $\xi\text{-id}\, T := \mathcal{X}\text{-cores.}\dim T$. In the case $\mathcal{X} = \mathcal{GP}(\xi)$, $\mathcal{X}$-res. $\dim T$ coincides with $\xi\text{-}\mathcal{G}\mathrm{pd}\, T$ defined in [2] as ξ-Gorenstein projective dimensions.

We use $\mathcal{X}^{\wedge}$ (resp., $\mathcal{X}^{\vee}$) to denote the subcategory of $\mathcal{T}$ consisting of objects having finite $\mathcal{X}$-resolution (resp., $\mathcal{X}$-coresolution) dimension, and use $\mathcal{X}^{\wedge}_n$ (resp., $\mathcal{X}^{\vee}_n$) to denote the subcategory of $\mathcal{T}$ consisting of objects having $\mathcal{X}$-resolution (resp., $\mathcal{X}$-coresolution) dimension at most n.

3 Left n-Cotorsion Pairs

We first introduce the notion of left (resp., right) cotorsion pair in triangulated categories with respect to a proper class of triangles.

Definition 3.1. Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{T}$. We say that $(\mathcal{U},\mathcal{V})$ is a *left cotorsion pair* in $\mathcal{T}$ if the following conditions are satisfied:
(L1) $\mathcal{U}$ is closed under direct summands.
(L2) $\xi xt^1_\xi(\mathcal{U},\mathcal{V})=0$.
(L3) Every object $T\in\mathcal{T}$ admits a triangle $V\to U\to T\to\Sigma V$ in ξ with $U\in\mathcal{U}$ and $V\in\mathcal{V}$.
Dually, we say that $(\mathcal{U},\mathcal{V})$ is a *right cotorsion pair* in $\mathcal{T}$ if the following conditions are satisfied:
(R1) $\mathcal{V}$ is closed under direct summands.
(R2) $\xi xt^1_\xi(\mathcal{U},\mathcal{V})=0$.
(R3) Every object $T\in\mathcal{T}$ admits a triangle $T\to V'\to U'\to\Sigma T$ in ξ with $U'\in\mathcal{U}$ and $V'\in\mathcal{V}$.

Remark 3.2. Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{T}$.

(1) If $(\mathcal{U},\mathcal{V})$ is a left cotorsion pair in $\mathcal{T}$, then $\mathcal{U}={}^{\perp_1}\mathcal{V}$. Moreover, $\mathcal{P}(\xi)\subseteq\mathcal{U}$, $\mathcal{U}$ is closed under ξ-extensions, and $\mathcal{U}$ is a contravariantly finite subcategory of $\mathcal{T}$.

(2) If $(\mathcal{U},\mathcal{V})$ is a right cotorsion pair in $\mathcal{T}$, then $\mathcal{V}=\mathcal{U}^{\perp_1}$. Moreover, $\mathcal{I}(\xi)\subseteq\mathcal{V}$, $\mathcal{V}$ is closed under ξ-extensions, and $\mathcal{V}$ is a covariantly finite subcategory of $\mathcal{T}$.

We say that $(\mathcal{U},\mathcal{V})$ is a *cotorsion pair* in $\mathcal{T}$ if $(\mathcal{U},\mathcal{V})$ is both a left and right cotorsion pair in $\mathcal{T}$, which is essentially a ξ-complete cotorsion theory in the sense of Asadollahi and Salarian [3].

In what follows, we always assume that n is a positive integer. In [13], Zhou introduced the notion of n-cotorsion pairs in extriangulated categories (see [19]). Notice that a triangulated category with respect to a proper class of triangles is an extriangulated category (see [14, Remark 3.3]). Now we rewrite the notion of n-cotorsion pairs with respect to a proper class of triangles in triangulated categories.

Definition 3.3. (cf. [13, Definition 3.1]) Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{T}$. We say that $(\mathcal{U},\mathcal{V})$ is a *left n-cotorsion pair* in $\mathcal{T}$ if the following conditions are satisfied:
(LN1) $\mathcal{U}$ is closed under direct summands.
(LN2) $\xi xt^i_\xi(\mathcal{U},\mathcal{V})=0$ for every $1\leq i\leq n$.
(LN3) Every object $T\in\mathcal{T}$ admits a triangle $K\to U\to T\to\Sigma K$ in ξ with $U\in\mathcal{U}$ and $K\in\mathcal{V}^{\wedge}_{n-1}$.
Dually, we say that $(\mathcal{U},\mathcal{V})$ is a *right n-cotorsion pair* in $\mathcal{T}$ if the following conditions are satisfied:
(RN1) $\mathcal{V}$ is closed under direct summands.
(RN2) $\xi xt^i_\xi(\mathcal{U},\mathcal{V})=0$ for every $1\leq i\leq n$.
(RN3) Every object $T\in\mathcal{T}$ admits a triangle $T\to V'\to K'\to\Sigma T$ in ξ with $V'\in\mathcal{V}$ and $K'\in\mathcal{U}^{\vee}_{n-1}$.
We say that $(\mathcal{U},\mathcal{V})$ is an *n-cotorsion pair* in $\mathcal{T}$ if $(\mathcal{U},\mathcal{V})$ is both a left and right n-cotorsion pair in $\mathcal{T}$.

We remark that left (resp., right) 1-cotorsion pairs are exactly left (resp., right) cotorsion pairs in $\mathcal{T}$.

Proposition 3.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{T}$ satisfying $\xi xt^i_\xi(\mathcal{U}, \mathcal{V}) = 0$ for every $1 \leq i \leq n$. If $Y \in \mathcal{V}^\wedge_k$ with $0 \leq k \leq n-1$, then $\xi xt^i_\xi(\mathcal{U}, Y) = 0$ for every $1 \leq i \leq n-k$. In particular, $\xi xt^1_\xi(\mathcal{U}, \mathcal{V}^\wedge_{n-1}) = 0$.*

Proof. The case $n = 1$ is clear. Now suppose $n \geq 2$. We will proceed by induction on k. The case $k = 0$ is also clear, so we suppose $1 \leq k \leq n-1$. Let $U \in \mathcal{U}$ and $Y \in \mathcal{V}^\wedge_k$. For the case $k = 1$, there is a triangle $V_1 \to V_0 \to Y \to \Sigma V_1$ in ξ with $V_1, V_0 \in \mathcal{V}$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(U, -)$ to the above triangle yields the exact sequence $\cdots \to \xi xt^i_\xi(U, V_0) \to \xi xt^i_\xi(U, Y) \to \xi xt^{i+1}_\xi(U, V_1) \to \cdots$. For every $1 \leq i \leq n-1$, since $\xi xt^i_\xi(U, V_0) = 0 = \xi xt^{i+1}_\xi(U, V_1)$, we have $\xi xt^i_\xi(U, Y) = 0$.

Suppose $2 \leq k \leq n-1$. Consider a triangle $Y' \to V_0' \to Y \to \Sigma Y'$ in ξ with $Y' \in \mathcal{V}^\wedge_{k-1}$ and $V_0' \in \mathcal{V}$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(U, -)$ to the above triangle yields the exact sequence $\cdots \to \xi xt^i_\xi(U, V_0') \to \xi xt^i_\xi(U, Y) \to \xi xt^{i+1}_\xi(U, Y') \to \cdots$. Since $\xi xt^i_\xi(U, V_0') = 0$ for any $1 \leq i \leq n-k$ by assumption and since $\xi xt^i_\xi(U, Y') = 0$ for every $2 \leq i \leq n-k+1$ by the induction hypothesis, we have $\xi xt^i_\xi(U, Y) = 0$ for every $1 \leq i \leq n-k$. $\square$

Corollary 3.5. (cf. [13, Lemma 3.3]) *Let $\mathcal{V}$ be a subcategory of $\mathcal{T}$. Then we have $\bigcap_{i=1}^n {}^{\perp_i}\mathcal{V} \subseteq {}^{\perp_1}\mathcal{V}^\wedge_{n-1}$.*

The following result gives an equivalent characterization of left n-cotorsion pairs.

Lemma 3.6. (cf. [13, Lemma 3.4]) *Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{T}$. Then the following statements are equivalent:*

(1) *$(\mathcal{U}, \mathcal{V})$ is a left n-cotorsion pair in $\mathcal{T}$.*

(2) *$\mathcal{U} = \bigcap_{i=1}^n {}^{\perp_i}\mathcal{V}$, and for any object $T \in \mathcal{T}$ there is a triangle $K \to U \to T \to \Sigma K$ in ξ with $U \in \mathcal{U}$ and $K \in \mathcal{V}^\wedge_{n-1}$.*

If one of the above conditions holds true, $(\mathcal{U}, \mathcal{V}^\wedge_{n-1})$ is a left cotorsion pair in $\mathcal{T}$.

In the rest of this section, we give some properties related to (left) n-cotorsion pairs.

Proposition 3.7. *Let $(\mathcal{U}, \mathcal{V})$ be an n-cotorsion pair in $\mathcal{T}$. Then the following statements are equivalent:* (1) *$\mathcal{U} \subseteq \mathcal{V}$.* (2) *$\mathcal{T} = \mathcal{V}^\wedge_n$.* (3) *$\xi xt^1_\xi(\mathcal{U}^\vee_{n-1}, \mathcal{U}) = 0$.*

Proof. (1)$\Rightarrow$(2) This is clear.

(2)$\Rightarrow$(1) Let $U \in \mathcal{U} \subseteq \mathcal{T}$. By assumption, there is a triangle $K \to V_0 \to U \to \Sigma K$ in ξ with $K \in \mathcal{V}^\wedge_{n-1}$ and $V_0 \in \mathcal{V}$. By Lemma 3.6, the above triangle is split, so U is a direct summand of V_0, and hence $U \in \mathcal{V}$. Thus, $\mathcal{U} \subseteq \mathcal{V}$.

(1)$\Leftrightarrow$(3) This follows from the dual of Lemma 3.6. $\square$

Note that $\mathcal{V}^\wedge = \mathcal{V}$ if $\mathcal{V}$ is coresolving, and $\mathcal{U}^\vee = \mathcal{U}$ if $\mathcal{U}$ is resolving.

Corollary 3.8. *Let $(\mathcal{U}, \mathcal{V})$ be an n-cotorsion pair in $\mathcal{T}$ with $\mathcal{U}$ resolving. Then the following statements are equivalent:* (1) *$\mathcal{U} \subseteq \mathcal{V}$.* (2) *$\mathcal{T} = \mathcal{V}$.* (3) *$\xi xt^1_\xi(\mathcal{U}, \mathcal{U}) = 0$.*

Applying Lemma 3.6, we also have the following result.

Proposition 3.9. *Let $(\mathcal{U}, \mathcal{V})$ be a left n-cotorsion pair in $\mathcal{T}$ with $\xi xt_\xi^{n+1}(\mathcal{U}, \mathcal{V}) = 0$. Then $\mathcal{U}$ is resolving.*

Proof. By Lemma 3.6, $(\mathcal{U}, \mathcal{V}^{\wedge}_{n-1})$ is a left cotorsion pair in $\mathcal{T}$. By Remark 3.2, $\mathcal{U}$ is closed under ξ-extensions and $\mathcal{P}(\xi) \subseteq \mathcal{U}$. Let $U \to U' \to U'' \to \Sigma U$ be a triangle in ξ with $U', U'' \in \mathcal{U}$. For any $V \in \mathcal{V}$, applying the functor $\mathrm{Hom}_{\mathcal{T}}(-, V)$ to this triangle yields the exact sequence $\cdots \to \xi xt_\xi^i(U', V) \to \xi xt_\xi^i(U, V) \to \xi xt_\xi^{i+1}(U'', V) \to \cdots$. Notice that $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $1 \le i \le n+1$ by assumption, so $\xi xt_\xi^i(U, V) = 0$ for every $1 \le i \le n$. Thus $U \in \bigcap_{i=1}^n {}^{\perp_i}\mathcal{V} = \mathcal{U}$ by Lemma 3.6, and hence $\mathcal{U}$ is closed under hokernels of ξ-proper epimorphisms. Therefore, $\mathcal{U}$ is resolving. □

The following result establishes a relation between n-cotorsion pairs and cotorsion pairs.

Proposition 3.10. *Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories in $\mathcal{T}$. Then the following statements are equivalent:*
(1) *$(\mathcal{U}, \mathcal{V})$ is an n-cotorsion pair with $\xi xt_\xi^{n+1}(\mathcal{U}, \mathcal{V}) = 0$ in $\mathcal{T}$.*
(2) *$(\mathcal{U}, \mathcal{V})$ is an n-cotorsion pair in $\mathcal{T}$ and $\mathcal{U}$ is resolving.*
(3) *$(\mathcal{U}, \mathcal{V})$ is an n-cotorsion pair in $\mathcal{T}$ and $\mathcal{V}$ is coresolving.*
(4) *$(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in $\mathcal{T}$ and $\mathcal{U}$ is resolving.*
Moreover, if one of the above conditions holds true, $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $i \ge 1$.

Proof. (1)⇒(2) This follows from Proposition 3.9.

(2)⇒(1) It suffices to show $\xi xt_\xi^{n+1}(\mathcal{U}, \mathcal{V}) = 0$. Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $\mathcal{T}$ has enough ξ-projective objects, there is a triangle $U' \to P \to U \to \Sigma U'$ in ξ with $P \in \mathcal{P}(\xi)$. Since $\mathcal{U}$ is resolving, we have $U' \in \mathcal{U}$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(-, V)$ to the above triangle yields the following exact sequence:

$$\cdots \longrightarrow \xi xt_\xi^i(U', V) \longrightarrow \xi xt_\xi^{i+1}(U, V) \longrightarrow \xi xt_\xi^{i+1}(P, V) \longrightarrow \cdots .$$

Since $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for any $1 \le i \le n$, we have $\xi xt_\xi^{n+1}(\mathcal{U}, \mathcal{V}) = 0$.

(1)⇔(3) This is a dual of (1)⇔(2).

(2)⇒(4) or (3)⇒(4) By Lemma 3.6, $(\mathcal{U}, \mathcal{V}^{\wedge}_{n-1})$ is a left cotorsion pair in $\mathcal{T}$. Since $\mathcal{V}$ is coresolving, $\mathcal{V} = \mathcal{V}^{\wedge}_{n-1}$. So $(\mathcal{U}, \mathcal{V})$ is a left cotorsion pair in $\mathcal{T}$. Dually, $(\mathcal{U}, \mathcal{V})$ is a right cotorsion pair in $\mathcal{T}$. Thus, $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in $\mathcal{T}$, and $\mathcal{U}$ is resolving.

(4)⇒(2) By using an argument similar to that of the implication (2)⇒(1), we get $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $1 \le i \le n$.

Moreover, by an argument similar to that of the implication (2)⇒(1), we get $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $i \ge n+1$. Then $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $i \ge 1$. □

Corollary 3.11. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\mathcal{T}$. Then the following statements are equivalent:* (1) *$\xi xt_\xi^2(\mathcal{U}, \mathcal{V}) = 0$.* (2) *$\mathcal{U}$ is resolving.* (3) *$\mathcal{V}$ is coresolving. Moreover, if one of the above conditions holds true, $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $i \ge 1$.*

Lemma 3.12. *Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{T}$ such that $\xi xt_\xi^i(\mathcal{U}, \mathcal{V}) = 0$ for every $1 \le i \le n$. Then $\mathcal{U}^{\wedge}_k \subseteq {}^{\perp_{k+1}}\mathcal{V}$ for any $0 \le k \le n-1$.*

Proof. We will proceed by induction on k. The case $k = 0$ is clear. Let $X \in \mathcal{U}^{\wedge}_k$ and $V \in \mathcal{V}$. For the case $k = 1$, there is a triangle $U_1 \to U_0 \to X \to \Sigma U_1$ in ξ

with $U_1, U_0 \in \mathcal{U}$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(-,V)$ to the above triangle yields the exact sequence $\cdots \to \xi xt^1_\xi(U_1,V) \to \xi xt^2_\xi(X,V) \to \xi xt^2_\xi(U_0,V) \to \cdots$. Since $\xi xt^1_\xi(U_1,V) = 0 = \xi xt^2_\xi(U_0,V)$ by assumption, $\xi xt^2_\xi(X,V) = 0$ and $X \in {}^{\perp_2}\mathcal{V}$.

Suppose $k \geq 2$. Consider a triangle $K \to U_0' \to X \to \Sigma K$ in ξ with $U_0' \in \mathcal{U}$ and $K \in \mathcal{U}^\wedge_{k-1}$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(-,V)$ to the above triangle yields the exact sequence $\cdots \to \xi xt^k_\xi(K,V) \to \xi xt^{k+1}_\xi(X,V) \to \xi xt^{k+1}_\xi(U_0',V) \to \cdots$. Since $\xi xt^k_\xi(K,V) = 0$ by the induction hypothesis, and since $\xi xt^{k+1}_\xi(U_0',V) = 0$ by assumption, we have $\xi xt^{k+1}_\xi(X,V) = 0$ and $X \in {}^{\perp_{k+1}}\mathcal{V}$. Thus, $\mathcal{U}^\wedge_k \subseteq {}^{\perp_{k+1}}\mathcal{V}$ for any $0 \leq k \leq n-1$. □

As a consequence, we get the following proposition.

Proposition 3.13. *Let $(\mathcal{U},\mathcal{V})$ be a left n-cotorsion pair in $\mathcal{T}$. Then the following statements are equivalent:* (1) $\mathcal{U} = {}^{\perp_1}\mathcal{V}$. (2) $\mathcal{U}^\wedge_k = {}^{\perp_{k+1}}\mathcal{V}$ *for any* $0 \leq k \leq n-1$.

Proof. (2)⇒(1) This is trivial by setting $k = 0$ in (2).

(1)⇒(2) The case $k = 0$ is clear. Suppose $k \geq 1$. By Lemma 3.12, $\mathcal{U}^\wedge_k \subseteq {}^{\perp_{k+1}}\mathcal{V}$. Conversely, assume $Y \in {}^{\perp_{k+1}}\mathcal{V}$. Consider a triangle $K_1 \to U_0 \to Y \to \Sigma K_1$ in ξ with $U_0 \in \mathcal{U}$ and $K_1 \in \mathcal{V}^\wedge_{n-1}$. Repeating this process, we get the ξ-exact complex $0 \to K_k \to U_{k-1} \to \cdots \to U_1 \to U_0 \to Y \to 0$ with $U_i \in \mathcal{U}$ for $0 \leq i \leq k-1$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(-,V)$ to it, we have $\xi xt^1_\xi(K_k,V) \cong \xi xt^{k+1}_\xi(Y,V) = 0$ by dimension shifting. This implies $K_k \in {}^{\perp_1}\mathcal{V} = \mathcal{U}$ by assumption. Hence, $Y \in \mathcal{U}^\wedge_k$ and ${}^{\perp_{k+1}}\mathcal{V} \subseteq \mathcal{U}^\wedge_k$. Thus, $\mathcal{U}^\wedge_k = {}^{\perp_{k+1}}\mathcal{V}$. □

Corollary 3.14. *Let $(\mathcal{U},\mathcal{V})$ be a left n-cotorsion pair in $\mathcal{T}$. If $\mathcal{U} = {}^{\perp_1}\mathcal{V}$, then for any $0 \leq k \leq n-1$ the following conditions are equivalent:* (1) $\mathcal{U}$-res. dim $\mathcal{T} \leq k$. (2) $\mathcal{T} = {}^{\perp_{k+1}}\mathcal{V}$.

As an application of Proposition 3.10, along with Proposition 3.13 and its dual, the following result describes the subcategories $\mathcal{U}^\wedge$ and $\mathcal{V}^\vee$ if $(\mathcal{U},\mathcal{V})$ is a cotorsion pair with $\mathcal{U}$ resolving.

Corollary 3.15. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair with $\mathcal{U}$ resolving. Then for any $m, n \geq 0$, we have $\mathcal{U}^{\perp_{m+1}} = \mathcal{V}^\vee_m$ and ${}^{\perp_{n+1}}\mathcal{V} = \mathcal{U}^\wedge_n$.*

4 Left Frobenius Pairs and Weak Auslander-Buchweitz Contexts

We begin with the following easy observation.

Proposition 4.1. *Let $(\mathcal{X},\omega)$ be a pair of subcategories in $\mathcal{T}$ such that ω is $\mathcal{X}$-injective. Then we have the following:*

(1) $\mathcal{X} \subseteq {}^{\perp}(\omega^\wedge)$.

(2) *If ω is a ξ-cogenerator for $\mathcal{X}$ and ω is closed under direct summands in $\mathcal{T}$, then $\omega = \mathcal{X} \cap \omega^\wedge = \mathcal{X} \cap \mathcal{X}^\perp$.*

Proof. (1) This follows from [17, Lemma 3.9].

(2) By (1), we have $\omega \subseteq \mathcal{X} \cap \omega^\wedge \subseteq \mathcal{X} \cap \mathcal{X}^\perp$, so it suffices to show $\mathcal{X} \cap \mathcal{X}^\perp \subseteq \omega$. Now let $X \in \mathcal{X} \cap \mathcal{X}^\perp$. Since ω is a ξ-cogenerator in $\mathcal{X}$, there exists a triangle $X \to W \to X' \to \Sigma X$ in ξ with $W \in \omega$ and $X' \in \mathcal{X}$. Since $\xi xt^1_\xi(X',X) = 0$ by

assumption, the above triangle is split. So X is a direct summand of W and $X \in \omega$. Thus, we get the desired assertion. $\square$

The following result gives the so-called Auslander-Buchweitz approximation triangles, which play a crucial role in what follows.

Proposition 4.2. [17, Proposition 3.10] *Let $(\mathcal{X}, \omega)$ be a pair of subcategories in $\mathcal{T}$ such that $\mathcal{X}$ is closed under ξ-extensions and ω is a ξ-cogenerator in $\mathcal{X}$. Then for any $C \in \mathcal{X}_n^\wedge$, there exist triangles $Y_C \to X_C \to C \to \Sigma Y_C$, $C \to Y^C \to X^C \to \Sigma C$ in ξ with $Y_C \in \omega_{n-1}^\wedge$, $Y^C \in \omega_n^\wedge$ and $X_C, X^C \in \mathcal{X}$. In particular, if ω is $\mathcal{X}$-injective, then $X_C \to C$ is a right $\mathcal{X}$-approximation of C.*

Corollary 4.3. *Let $(\mathcal{X}, \omega)$ be a pair of subcategories in $\mathcal{T}$ such that $\mathcal{X}$ is closed under ξ-extensions and direct summands, and that ω is a ξ-cogenerator of $\mathcal{X}$. Then $\{C \in \mathcal{T} \mid \mathcal{X}\text{-res.}\dim C \leq 1\} \cap {}^{\perp_1}\omega \subseteq \mathcal{X}$.*

Proof. Suppose that $\mathcal{X}$-res. dim $C \leq 1$. By Proposition 4.2, we have a triangle $K \to X \to C \to \Sigma K$ in ξ with $X \in \mathcal{X}$ and $K \in \omega$. Notice that $\xi xt_\xi^1(C, K) = 0$ by assumption, so the above triangle is split, and thus C is a direct summand of X, which implies $C \in \mathcal{X}$. $\square$

Corollary 4.4. *Let $(\mathcal{X}, \omega)$ be a pair of subcategories in $\mathcal{T}$ such that $\mathcal{X}$ is closed under ξ-extensions and ω is closed under direct summands in $\mathcal{T}$. If ω is $\mathcal{X}$-injective and a ξ-cogenerator for $\mathcal{X}$, then $\omega^\wedge = \mathcal{X}^\perp \cap \mathcal{X}^\wedge$.*

Proof. By Proposition 4.1 we get $\omega^\wedge \subseteq \mathcal{X}^\perp$. Clearly, $\omega^\wedge \subseteq \mathcal{X}^\wedge$. So $\omega^\wedge \subseteq \mathcal{X}^\perp \cap \mathcal{X}^\wedge$.

Conversely, let $C \in \mathcal{X}^\perp \cap \mathcal{X}^\wedge$. Then by Proposition 4.2 there exists a triangle $Y \to X \to C \to \Sigma Y$ in ξ with $X \in \mathcal{X}$ and $Y \in \omega^\wedge \subseteq \mathcal{X}^\perp$. Since $C \in \mathcal{X}^\perp$, we have $X \in \mathcal{X}^\perp$. Then $X \in \mathcal{X} \cap \mathcal{X}^\perp$. It follows from Proposition 4.1 that $X \in \omega$. So $C \in \omega^\wedge$, and thus $\mathcal{X}^\perp \cap \mathcal{X}^\wedge \subseteq \omega^\wedge$. $\square$

For a pair $(\mathcal{X}, \omega)$ of subcategories in $\mathcal{T}$, if $\omega \subseteq \mathcal{X}$, then $\omega^\wedge \subseteq \mathcal{X}^\wedge$. We establish a more specific relation between them under some conditions.

Proposition 4.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{T}$ such that $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands and $\mathcal{Y} \subseteq \mathcal{X}^\wedge$. Assume that*

(a) *$\mathcal{X}$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms, and*

(b) *$\mathcal{Y}$ is closed under ξ-extensions and hocokernels of ξ-proper monomorphisms.*

Suppose that $\omega := \mathcal{X} \cap \mathcal{Y}$ is $\mathcal{X}$-injective and a ξ-cogenerator for $\mathcal{X}$. Then we have $\mathcal{Y} = \omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp = \mathcal{X}^\wedge \cap \mathcal{X}^{\perp_1}$.

Proof. By Corollary 4.4, we know that $\omega^\wedge = \mathcal{X}^\perp \cap \mathcal{X}^\wedge$.

Since $\mathcal{Y}$ is closed under hocokernels of ξ-proper monomorphisms, we get $\mathcal{Y}^\wedge = \mathcal{Y}$. It follows that $\omega^\wedge \subseteq \mathcal{Y}$ since $\omega \subseteq \mathcal{Y}$. Now let $Y \in \mathcal{Y}$. Since by assumption $\mathcal{Y} \subseteq \mathcal{X}^\wedge$, by Proposition 4.2 there is a triangle $K \to X \to Y \to \Sigma K$ in ξ with $X \in \mathcal{X}$ and $K \in \omega^\wedge \subseteq \mathcal{Y}$. Since $\mathcal{Y}$ is closed under ξ-extensions, we have $X \in \mathcal{Y}$. So $X \in \mathcal{X} \cap \mathcal{Y} = \omega$, and hence $Y \in \omega^\wedge$ and $\mathcal{Y} \subseteq \omega^\wedge$. Thus, $\mathcal{Y} = \omega^\wedge$.

Clearly, $\mathcal{X}^\wedge \cap \mathcal{X}^\perp \subseteq \mathcal{X}^\wedge \cap \mathcal{X}^{\perp_1}$. Now let $Z \in \mathcal{X}^\wedge \cap \mathcal{X}^{\perp_1}$. By Proposition 4.2 there is a triangle $Z \to W \to X \to \Sigma Z$ in ξ with $X \in \mathcal{X}$ and $W \in \omega^\wedge$. Since $Z \in \mathcal{X}^{\perp_1}$, the above triangle is split. So Z is a direct summand of W. Noticing that

$\omega^\wedge$ $(=\mathcal{Y})$ is closed under direct summands, we have $Z \in \omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp$. Thus, we get the third equality. $\square$

4.1. Left Frobenius pairs. Inspired by the definition of left Frobenius pairs in abelian categories [6], we introduce the notion of left Frobenius pairs with respect to ξ in triangulated categories as follows.

Definition 4.6. A pair of subcategories $(\mathcal{X}, \omega)$ in $\mathcal{T}$ is called a *left Frobenius pair* if the following conditions hold:
(LF1) $\mathcal{X}$ and ω are closed under direct summands.
(LF2) $\mathcal{X}$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms.
(LF3) ω is $\mathcal{X}$-injective and a ξ-cogenerator of $\mathcal{X}$.

Example 4.7. (1) We have the following facts. (i) $\mathcal{GP}(\xi)$ is closed under direct summands (see [2, Proposition 3.13]). (ii) $\mathcal{GP}(\xi)$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms. In particular, $\mathcal{GP}(\xi)$ is a resolving subcategory of $\mathcal{T}$ (see [18, Corollary 4.4] or [17, Theorem 5.3]). (iii) $\mathcal{P}(\xi)$ is $\mathcal{GP}(\xi)$-injective and is a ξ-cogenerator of $\mathcal{GP}(\xi)$ since $\mathcal{P}(\xi) \subseteq \mathcal{GP}(\xi) \cap \mathcal{GP}(\xi)^\perp$ (see [2, Lemma 3.7 and Proposition 3.19]). Therefore, $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left Frobenius pair in $\mathcal{T}$.

(2) Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$ such that $\mathcal{X}\text{-res.}\dim \mathcal{T} = n$. By Proposition 4.2 we see that $(\mathcal{X}, \omega)$ is a left n-cotorsion pair in $\mathcal{T}$. In particular, if $\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\, T \mid T \in \mathcal{T}\} = n$, then $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left n-cotorsion pair in $\mathcal{T}$.

Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$. In the following, we will study the homological behavior of $\mathcal{X}^\wedge$, involving $\omega^\wedge$.

Lemma 4.8. *Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$, and let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{4.1}$$

be a triangle in ξ.
(1) *If $Z \in \mathcal{X}$, then $X \in \mathcal{X}^\wedge$ if and only if $Y \in \mathcal{X}^\wedge$.*
(2) *If $Y \in \mathcal{X}$, then $X \in \mathcal{X}^\wedge$ if and only if $Z \in \mathcal{X}^\wedge$.*

Proof. (1) Let $Y \in \mathcal{X}^\wedge$ and $\mathcal{X}\text{-res.}\dim Y = m$. We proceed by induction on m. The case $m = 0$ is clear. Suppose $m \geq 1$. Consider a triangle $K \to X_0 \to Y \to \Sigma K$ in ξ with $X_0 \in \mathcal{X}$ and $K \in \mathcal{X}^\wedge_{m-1}$. Applying base change to the triangle (4.1) along the morphism $X_0 \to Y$ yields the following commutative diagram:

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d, "h"] & K \arrow[d, "f"] & \\
\Sigma^{-1}Z \arrow[r] \arrow[d, equal] & X' \arrow[r, "g"] \arrow[d] & X_0 \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\Sigma^{-1}Z \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \\
 & \Sigma K \arrow[r, equal] & \Sigma K &
\end{tikzcd} \tag{4.2}$$

By [22, Proposition 2.4], one can see that the triangle $X' \to X_0 \to Z \to \Sigma X'$ is in ξ. Since $gh = f$ is ξ-proper monic, h is ξ-proper monic by [22, Proposition 2.7]. So the second vertical triangle is in ξ. It follows that $X' \in \mathcal{X}$ since $\mathcal{X}$ is closed under hokernels of ξ-proper epimorphisms. Thus, $X \in \mathcal{X}^{\wedge}$.

On the other hand, assume that $X \in \mathcal{X}^{\wedge}$ and $\mathcal{X}$-res. dim $X = n$. The case $n = 0$ is clear. By Proposition 4.2 there exists a triangle $K' \to X_0' \to X \to \Sigma K'$ in ξ with $X_0' \in \mathcal{X}$ and $K' \in \omega^{\wedge}$. Applying $\mathrm{Hom}_{\mathcal{T}}(Z, -)$ to this triangle yields the exact sequence $\cdots \to \xi xt^1_{\xi}(Z, K') \to \xi xt^1_{\xi}(Z, X_0') \to \xi xt^1_{\xi}(Z, X) \to \xi xt^2_{\xi}(Z, K') \to \cdots$. Since $\xi xt^1_{\xi}(Z, K') = 0 = \xi xt^2_{\xi}(Z, K')$ by Proposition 4.1, $\xi xt^1_{\xi}(Z, X_0') \cong \xi xt^1_{\xi}(Z, X)$. We get the following commutative diagram:

$$\begin{tikzcd}
 & K' \arrow[r, equal] \arrow[d] & K' \arrow[d] & \\
\Sigma^{-1}Z \arrow[r] \arrow[d, equal] & X_0' \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\Sigma^{-1}Z \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \\
 & \Sigma K' \arrow[r, equal] & \Sigma K' &
\end{tikzcd}$$

Notice that the triangle $X_0' \to X'' \to Z \to \Sigma X_0'$ is in ξ, so the third vertical triangle is also in ξ by [22, Proposition 2.4]. Since $\mathcal{X}$ is closed under ξ-extensions, we have $X'' \in \mathcal{X}$ and $Y \in \mathcal{X}^{\wedge}$.

(2) When $X \in \mathcal{X}^{\wedge}$, the assertion $Z \in \mathcal{X}^{\wedge}$ is clear. Conversely, assume that $Z \in \mathcal{X}^{\wedge}$ and $\mathcal{X}$-res. dim $Z = m$. We proceed by induction on m. The case $m = 0$ is clear. Suppose $m \geq 1$. Consider a triangle $K \to X_0 \to Z \to \Sigma K$ in ξ with $X_0 \in \mathcal{X}$ and $K \in \mathcal{X}^{\wedge}_{m-1}$. Applying base change to the triangle (4.1) along the morphism $X_0 \to Z$, we get the following commutative diagram:

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d, "h"] & K \arrow[d, "f"] & \\
X \arrow[r] \arrow[d, equal] & U \arrow[r, "g"] \arrow[d] & X_0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \\
 & \Sigma K \arrow[r, equal] & \Sigma K &
\end{tikzcd} \tag{4.3}$$

Since ξ is closed under base change, the second horizontal triangle is in ξ. Since $gh = f$ is ξ-proper monic, h is ξ-proper monic by [22, Proposition 2.7]. So the second vertical triangle is in ξ. By (1), we have $U \in \mathcal{X}^{\wedge}$ and then $X \in \mathcal{X}^{\wedge}$. $\square$

The following result shows that $\mathcal{X}^{\wedge}$ satisfies the two-out-of-three property relating to ξ-proper triangles.

Theorem 4.9. *Let $(\mathcal{X},\omega)$ be a left Frobenius pair in $\mathcal{T}$. Then $\mathcal{X}^{\wedge}$ is closed under ξ-extensions, hokernels of ξ-proper epimorphisms and hocokernels of ξ-proper monomorphisms.*

Proof. Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{4.4}$$

be a triangle in ξ.

Claim 1. $\mathcal{X}^{\wedge}$ is closed under ξ-extensions.

Let $X, Z \in \mathcal{X}^{\wedge}$ with $\mathcal{X}$-res. dim $X = m$ and $\mathcal{X}$-res. dim $Z = n$. By Proposition 4.2, there exist the following triangles in ξ:

$$\begin{gathered} K \longrightarrow X_0 \longrightarrow X \longrightarrow \Sigma K, \\ K' \longrightarrow X_0' \longrightarrow Z \longrightarrow \Sigma K' \end{gathered} \tag{4.5}$$

with $X_0, X_0' \in \mathcal{X}$ and $K \in \omega^{\wedge}_{m-1}$, $K' \in \omega^{\wedge}_{n-1}$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(X_0', -)$ to the triangle (4.5) yields $\xi xt^1_{\xi}(X_0', X_0) \cong \xi xt^1_{\xi}(X_0', X)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] \arrow[dd, bend left] & X'' \arrow[r] \arrow[d] \arrow[dd, bend left] & X_0' \arrow[r] \arrow[d, equal] \arrow[dd, bend left] & \Sigma X_0 \arrow[d] \arrow[dd, bend left] \\
X \arrow[r] \arrow[d, equal] & U' \arrow[r] \arrow[d] & X_0' \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X
\end{tikzcd}$$

where all triangles are in ξ and the bottom commutative diagram follows by base change. Since $\mathcal{X}$ is closed under ξ-extensions, $X'' \in \mathcal{X}$. Since Σ is an automorphism, by using the 3×3 lemma, one can get the following commutative diagram except the right square at the bottom, which anticommutes:

$$\begin{tikzcd}
\Sigma^{-1} X_0' \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & X_0' \arrow[d] \\
\Sigma^{-1} Z \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
K' \arrow[r] \arrow[d] & \Sigma K \arrow[r] \arrow[d] & \Sigma K'' \arrow[r] \arrow[d] & \Sigma K' \arrow[d] \\
X_0' \arrow[r] & \Sigma X_0 \arrow[r] & \Sigma X'' \arrow[r] & \Sigma X_0'
\end{tikzcd}$$

Here all the rows and columns are in Δ. One can get the following triangles:

$$K \longrightarrow K'' \longrightarrow K' \longrightarrow \Sigma K, \qquad K'' \longrightarrow X'' \longrightarrow Y \longrightarrow \Sigma K'',$$

which are in ξ by [18, Lemma 3.10]. By the induction hypothesis, we have $K'' \in \mathcal{X}^{\wedge}$. Notice that $X'' \in \mathcal{X}$, so $Y \in \mathcal{X}^{\wedge}$.

Claim 2. $\mathcal{X}^\wedge$ is closed under hokernels of ξ-proper epimorphisms.

Let $Y, Z \in \mathcal{X}^\wedge$ with $\mathcal{X}$-res. dim $Z = m$. We proceed by induction on m. The case $m = 0$ follows from Lemma 4.8. Now suppose $m \geq 1$. Consider the following triangle in ξ: $K \to X_0 \to Z \to \Sigma K$ with $X_0 \in \mathcal{X}$ and $K \in \mathcal{X}^\wedge_{m-1}$. Applying base change to the triangle (4.4) along the morphism $X_0 \to Z$, we get the following commutative diagram:

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
X \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \\
 & \Sigma K \arrow[r, equal] & \Sigma K &
\end{tikzcd}$$

By a similar argument to that of the diagram (4.3), one can see that the second vertical and the second horizontal triangles are in ξ. By Claim 1, we have $U \in \mathcal{X}^\wedge$. So $X \in \mathcal{X}^\wedge$ by Lemma 4.8.

Claim 3. $\mathcal{X}^\wedge$ is closed under hocokernels of ξ-proper monomorphisms.

Let $X, Y \in \mathcal{X}^\wedge$ with $\mathcal{X}$-res. dim $Y = n$. We proceed by induction on n. The case $n = 0$ is clear. Suppose $n \geq 1$. Consider a triangle $K \to X_0 \to Y \to \Sigma K$ in ξ with $X_0 \in \mathcal{X}$ and $K \in \mathcal{X}^\wedge_{n-1}$. Applying base change to the triangle (4.4) along the morphism $X_0 \to Y$ yields the following commutative diagram:

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
\Sigma^{-1} Z \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\Sigma^{-1} Z \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \\
 & \Sigma K \arrow[r, equal] & \Sigma K &
\end{tikzcd}$$

By a similar argument to that of the diagram (4.2), one can see that the second vertical triangle and the triangle $U \to X_0 \to Z \to \Sigma U$ are in ξ. By Claim 1 we have $U \in \mathcal{X}^\wedge$, and thus $Z \in \mathcal{X}^\wedge$ by Lemma 4.8. $\square$

The following result is also a consequence of Proposition 4.2.

Proposition 4.10. *Suppose that $(\mathcal{X}, \omega)$ is a left Frobenius pair in $\mathcal{T}$. Then we have $\mathcal{X}^\wedge \cap {}^\perp\omega = \mathcal{X} = \mathcal{X}^\wedge \cap {}^\perp(\omega^\wedge)$.*

Proof. Clearly, $\mathcal{X} \subseteq \mathcal{X}^\wedge \cap {}^\perp\omega$ and $\mathcal{X}^\wedge \cap {}^\perp(\omega^\wedge) \subseteq \mathcal{X}^\wedge \cap {}^\perp\omega$. By [17, Lemma 3.9] we have ${}^\perp\omega \subseteq {}^\perp(\omega^\wedge)$, and hence $\mathcal{X}^\wedge \cap {}^\perp\omega \subseteq \mathcal{X}^\wedge \cap {}^\perp(\omega^\wedge)$.

Now, let $M \in \mathcal{X}^\wedge \cap {}^\perp\omega$. By Proposition 4.2, there is a triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{4.6}$$

in ξ with $X \in \mathcal{X}$ and $K \in \omega^\wedge$. Then $K \in {}^\perp\omega$, and so $K \in \omega^\wedge \cap {}^\perp\omega = \omega$ by [17, Lemma 3.12]. Noticing that $\xi xt^1_\xi(M, K) = 0$, we find that the triangle (4.6) is split, and hence $X \cong K \oplus M$. Since $\mathcal{X}$ is closed under direct summands, it follows that $M \in \mathcal{X}$. Thus, $\mathcal{X}^\wedge \cap {}^\perp\omega \subseteq \mathcal{X}$. $\square$

The following result provides a standard criterion for computing the $\mathcal{X}$-resolution dimension of an object in $\mathcal{T}$.

Proposition 4.11. *Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$. Then for any $T \in \mathcal{T}$, the following statements are equivalent:*

(1) $\mathcal{X}$-res. dim $T \leq n$.

(2) *If $0 \to K_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to T \to 0$ is a ξ-exact complex in $\mathcal{T}$ with $X_i \in \mathcal{X}$ for any $0 \leq i \leq n-1$, then $K_n \in \mathcal{X}$.*

Proof. (2)⇒(1) This is obvious.

(1)⇒(2) By Lemma 4.8 we see that $K_n \in \mathcal{X}^\wedge$. For any $W \in \omega$, applying the functor $\mathrm{Hom}_\mathcal{T}(-, W)$ to the ξ-exact complex in (2), by dimension shifting we can find that $\xi xt^i_\xi(K_n, W) \cong \xi xt^{n+i}_\xi(T, W) = 0$ for all $i \geq 1$. Then $K_n \in {}^\perp\omega$, and hence $K_n \in \mathcal{X}^\wedge \cap {}^\perp\omega = \mathcal{X}$ by Proposition 4.10. $\square$

Applying Proposition 4.11, we get the following result.

Proposition 4.12. *Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$. Then $\mathcal{X}^\wedge$ is closed under direct summands.*

Proof. Let $M \in \mathcal{X}^\wedge$ and $M = M_1 \oplus M_2$. Suppose that $\mathcal{X}$-res. dim $M \leq n$. We proceed by induction on n. The case $n = 0$ is trivial. Suppose $n \geq 1$. By Proposition 4.2 we have the ξ-exact complex $0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\mathcal{T}$ with all X_i objects in ω for $1 \leq i \leq n$ and $X_0 \in \mathcal{X}$.

Applying base change to the triangle $\Sigma^{-1}M_1 \xrightarrow{0} M_2 \to M \to M_1$ along the morphism $X_0 \to M$ yields the following commutative diagram:

$$\begin{tikzcd}
 & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d] & \\
\Sigma^{-1}M_1 \arrow[r] \arrow[d, equal] & W_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & M_1 \arrow[d, equal] \\
\Sigma^{-1}M_1 \arrow[r, "0"] & M_2 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_1 \\
 & \Sigma K_1 \arrow[r, equal] & \Sigma K_1 &
\end{tikzcd}$$

By a similar argument to that of the diagram (4.2), one can see that the second vertical triangle and the triangle $W_1 \to X_0 \to M_1 \to \Sigma W_1$ are in ξ. Similarly, we get a triangle $U_1 \to X_0 \to M_2 \to \Sigma U_1$ in ξ.

Since $\xi xt_{\xi}^{1}(X_0,K_1)=0$ by Proposition 4.1, $\mathrm{Hom}_{\mathcal{T}}(X_0,W_1)\to\mathrm{Hom}_{\mathcal{T}}(X_0,M_2)\to 0$ is epic. By [24, Lemma 2.2] and [18, Lemma 3.11], we have the following commutative diagram except the middle square at the top, which anticommutes:

$$\begin{tikzcd}
K_2 \arrow[r, dashed] \arrow[d] & W_2 \arrow[r, dashed] \arrow[d, dashed] & U_1 \arrow[r, dashed] \arrow[d] & \Sigma K_2 \arrow[d] \\
X_1 \arrow[r] \arrow[d] & X_1\oplus X_0 \arrow[r] \arrow[d, dashed] & X_0 \arrow[r, "0"] \arrow[d] & \Sigma X_1 \arrow[d] \\
K_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d, dashed] & M_2 \arrow[r] \arrow[d] & \Sigma K_1 \arrow[d] \\
\Sigma K_2 \arrow[r, dashed] & \Sigma W_2 \arrow[r, dashed] & \Sigma U_1 \arrow[r, dashed] & \Sigma^2 K_2
\end{tikzcd}$$

Here the first horizontal and the second vertical triangles are in ξ. Similarly, we get a triangle $U_2\to X_1\oplus X_0\to U_1\to\Sigma U_2$ in ξ. Repeating this process, we get the following two ξ-exact complexes:

$$0\longrightarrow W_n\longrightarrow\bigoplus_{i=1}^{n-1}X_i\longrightarrow\bigoplus_{i=1}^{n-2}X_i\longrightarrow\cdots\longrightarrow X_0\oplus X_1\longrightarrow X_0\longrightarrow M_1\longrightarrow 0,$$
$$0\longrightarrow U_n\longrightarrow\bigoplus_{i=1}^{n-1}X_i\longrightarrow\bigoplus_{i=1}^{n-2}X_i\longrightarrow\cdots\longrightarrow X_0\oplus X_1\longrightarrow X_0\longrightarrow M_2\longrightarrow 0.$$

Since ξ is closed under finite coproducts, we get a ξ-exact complex

$$\begin{aligned}0\longrightarrow W_n\oplus U_n\longrightarrow\bigoplus_{i=1}^{n-1}X_i\oplus\bigoplus_{i=1}^{n-1}X_i\longrightarrow\bigoplus_{i=1}^{n-2}X_i\oplus\bigoplus_{i=1}^{n-2}X_i\\ \longrightarrow\cdots\longrightarrow X_0\oplus X_1\oplus X_0\oplus X_1\longrightarrow X_0\oplus X_0\longrightarrow M\longrightarrow 0.\end{aligned}$$

By Proposition 4.11, $W_n\oplus U_n\in\mathcal{X}$. Because $\mathcal{X}$ is closed under direct summands by assumption, both W_n and U_n are objects in $\mathcal{X}$. Therefore, $\mathcal{X}\text{-res.}\dim M_1\leq n$ and $\mathcal{X}\text{-res.}\dim M_2\leq n$. □

Recall from [7, Section 4.3] that a subcategory $\mathcal{X}$ of $\mathcal{T}$ is called Σ-*stable* if $\Sigma\mathcal{X}=\mathcal{X}$, and $\mathcal{X}$ is called a *generating subcategory* of $\mathcal{T}$ if $\mathcal{X}$ is Σ-stable and for all $X\in\mathcal{X}$ the condition $\mathrm{Hom}_{\mathcal{T}}(X,C)=0$ implies $C=0$. Dually, a subcategory $\mathcal{Y}$ is called a *cogenerating subcategory* of $\mathcal{T}$ if $\mathcal{Y}$ is Σ-stable and for all $Y\in\mathcal{Y}$ the condition $\mathrm{Hom}_{\mathcal{T}}(C,Y)=0$ implies $C=0$. We need the following fact.

Lemma 4.13. [7, Corollary 4.15 and Proposition 4.17] *Let X be an object of $\mathcal{T}$.*

1. *If $\mathcal{P}(\xi)$ is a generating subcategory of $\mathcal{T}$, then $\xi\text{-pd}\,X\leq n$ if and only if $\xi xt_{\xi}^{n+1}(X,Y)=0$ for any $Y\in\mathcal{T}$.*
2. *If $\mathcal{I}(\xi)$ is a cogenerating subcategory of $\mathcal{T}$, then $\xi\text{-id}\,X\leq n$ if and only if $\xi xt_{\xi}^{n+1}(Y,X)=0$ for any $Y\in\mathcal{T}$.*

The following result shows how to obtain cotorsion pairs from left Frobenius pairs in $\mathcal{T}$.

Theorem 4.14. *Let $(\mathcal{X},\omega)$ be a left Frobenius pair in $\mathcal{T}$. Assume that $\mathcal{I}(\xi)$ is a cogenerating subcategory of $\mathcal{T}$. Then the following statements are equivalent:*

(1) $\mathcal{X}^\wedge = \mathcal{T}$.
(2) $(\mathcal{X}, \omega^\wedge)$ *is a cotorsion pair in* $\mathcal{T}$ *with* ξ-$\mathrm{id}\,\omega < \infty$.
(3) $(\mathcal{X}, \omega^\wedge)$ *is a left cotorsion pair in* $\mathcal{T}$ *with* ξ-$\mathrm{id}\,\omega < \infty$.
(4) $\mathcal{X} = {}^\perp\omega$ *and* ξ-$\mathrm{id}\,\omega < \infty$.

Moreover, if one of the equivalent conditions holds, then $\mathcal{X}$-$\mathrm{res.\,dim}\,\mathcal{T} = \xi$-$\mathrm{id}\,\omega$.

Proof. (1)⇒(2) By Corollary 4.4, we have $\omega^\wedge = \mathcal{X}^\perp \cap \mathcal{X}^\wedge$. Note that $\mathcal{X}^\wedge$ is closed under direct summands by Proposition 4.12, so $\omega^\wedge$ is closed under direct summands. By Proposition 4.1, we have $\xi xt^1_\xi(\mathcal{X}, \omega^\wedge) = 0$. On the other hand, by Proposition 4.2 we can get two desired triangles as in Definition 3.1. Thus, $(\mathcal{X}, \omega^\wedge)$ is a cotorsion pair in $\mathcal{T}$.

Let $W \in \omega$ and $T \in \mathcal{T}$ with $\mathcal{X}$-$\mathrm{res.\,dim}\,T \leq n$. Then we have a ξ-exact complex $0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to T \to 0$ in $\mathcal{T}$ with $X_i \in \mathcal{X}$ for any $0 \leq i \leq n$. Applying the functor $\mathrm{Hom}_\mathcal{T}(-, W)$, we have $\xi xt^{n+1}_\xi(T, W) \cong \xi xt^1_\xi(X_n, W) = 0$ by dimension shifting. So ξ-$\mathrm{id}\,W \leq n$ by Lemma 4.13, and thus ξ-$\mathrm{id}\,\omega < \infty$.

(2)⇒(3) This is obvious.

(3)⇒(4) Note $\mathcal{X} = {}^{\perp_1}(\omega^\wedge)$ by Remark 3.2. It is clear that ${}^\perp(\omega^\wedge) \subseteq {}^{\perp_1}(\omega^\wedge) = \mathcal{X}$. On the other hand, we have $\mathcal{X} \subseteq {}^\perp(\omega^\wedge)$ by Proposition 4.1. Thus, ${}^{\perp_1}(\omega^\wedge) = {}^\perp(\omega^\wedge)$. Clearly, ${}^\perp(\omega^\wedge) = {}^\perp\omega$. So $\mathcal{X} = {}^{\perp_1}(\omega^\wedge) = {}^\perp\omega$.

(4)⇒(1) Suppose ξ-$\mathrm{id}\,\omega = n$. For any $T \in \mathcal{T}$, since $\mathcal{T}$ has enough ξ-projective objects, there exists a ξ-exact complex $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to T \to 0$ in $\mathcal{T}$ with $P_i \in \mathcal{P}(\xi)$ for any $0 \leq i \leq n-1$. For any $W \in \omega$, applying the functor $\mathrm{Hom}_\mathcal{T}(-, W)$, by dimension shifting we have $\xi xt^i_\xi(K, W) \cong \xi xt^{n+i}_\xi(T, W) = 0$ for any $i \geq 1$ since ξ-$\mathrm{id}\,W \leq n$. So $K \in {}^\perp\omega$. Since $\mathcal{X} = {}^\perp\omega$ by assumption, $K \in \mathcal{X}$. Notice that all P_i are in $\mathcal{X}$, so $\mathcal{X}$-$\mathrm{res.\,dim}\,T \leq n$ and $T \in \mathcal{X}^\wedge$, and thus $\mathcal{T} = \mathcal{X}^\wedge$. □

Putting $\mathcal{X} = \mathcal{GP}(\xi)$ and $\omega = \mathcal{P}(\xi)$ in Theorem 4.14, we get the following result, in which part of the implication (1)⇒(2) was proved in [21, Proposition 3.7].

Corollary 4.15. *If* $\mathcal{I}(\xi)$ *is a cogenerating subcategory of* $\mathcal{T}$*, then the following statements are equivalent:*

(1) $\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\,T \mid T \in \mathcal{T}\} < \infty$.
(2) $(\mathcal{GP}(\xi), \mathcal{P}(\xi)^\wedge)$ *is a cotorsion pair in* $\mathcal{T}$ *and* ξ-$\mathrm{id}\,\mathcal{P}(\xi) < \infty$.
(3) $(\mathcal{GP}(\xi), \mathcal{P}(\xi)^\wedge)$ *is a left cotorsion pair in* $\mathcal{T}$ *and* ξ-$\mathrm{id}\,\mathcal{P}(\xi) < \infty$.
(4) $\mathcal{GP}(\xi) = {}^\perp\mathcal{P}(\xi)$ *and* ξ-$\mathrm{id}\,\mathcal{P}(\xi) < \infty$.

If one of these equivalent conditions holds, then $\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\,T \mid T \in \mathcal{T}\} = \xi$-$\mathrm{id}\,\mathcal{P}(\xi)$.

Furthermore, we have the following result.

Proposition 4.16. *Assume that* $\mathcal{P}(\xi)$ *is a generating subcategory of* $\mathcal{T}$ *and* $\mathcal{I}(\xi)$ *is a cogenerating subcategory of* $\mathcal{T}$.

(1) *If* $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ *is a left* n*-cotorsion pair in* $\mathcal{T}$*, then*

$$\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\,T \mid T \in \mathcal{T}\} = \xi\text{-}\mathrm{id}\,\mathcal{P}(\xi) \leq n.$$

Dually, if $(\mathcal{I}(\xi), \mathcal{GI}(\xi))$ *is a right* m*-cotorsion pair in* $\mathcal{T}$*, then*

$$\sup\{\xi\text{-}\mathcal{G}\mathrm{id}\,T \mid T \in \mathcal{T}\} = \xi\text{-}\mathrm{pd}\,\mathcal{I}(\xi) \leq m.$$

(2) *If there are integers $n, m \geq 1$ such that $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left n-cotorsion pair and $(\mathcal{I}(\xi), \mathcal{GI}(\xi))$ is a right m-cotorsion pair in $\mathcal{T}$, then we can choose $n = m = \xi\text{-id}\,\mathcal{P}(\xi) = \xi\text{-pd}\,\mathcal{I}(\xi)$.*

Proof. (1) Suppose that $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left n-cotorsion pair in $\mathcal{T}$. Accordingly, every object in $\mathcal{T}$ has ξ-Gorenstein projective dimension at most n, and hence $\sup\{\xi\text{-}\mathcal{G}\text{pd}\,T \mid T \in \mathcal{T}\} \leq n$. By Corollary 4.15, $\sup\{\xi\text{-}\mathcal{G}\text{pd}\,T \mid T \in \mathcal{T}\} = \xi\text{-id}\,\mathcal{P}(\xi) \leq n$. Dually, we get the other assertion.

(2) By (1), $\sup\{\xi\text{-}\mathcal{G}\text{pd}\,T \mid T \in \mathcal{T}\} = \xi\text{-id}\,\mathcal{P}(\xi) \leq n$ and $\sup\{\xi\text{-}\mathcal{G}\text{id}\,T \mid T \in \mathcal{T}\} = \xi\text{-pd}\,\mathcal{I}(\xi) \leq m$. Hence, it follows from [21, Theorem 4.6 and Corollary 4.7] that $\sup\{\xi\text{-}\mathcal{G}\text{pd}\,T \mid T \in \mathcal{T}\} = \sup\{\xi\text{-}\mathcal{G}\text{id}\,T \mid T \in \mathcal{T}\} = \xi\text{-pd}\,\mathcal{I}(\xi) = \xi\text{-id}\,\mathcal{P}(\xi)$. □

The following result is a consequence of Proposition 4.16.

Corollary 4.17. *Let $\mathcal{P}(\xi)$ be a generating subcategory of $\mathcal{T}$ and $\mathcal{I}(\xi)$ a cogenerating subcategory of $\mathcal{T}$. Assume that $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left n-cotorsion pair in $\mathcal{T}$.*

(1) $\mathcal{T} = \mathcal{GP}(\xi)^{\wedge}_{n} = {}^{\perp_n}(\mathcal{P}(\xi)^{\wedge}_{n-1})$.

(2) $\mathcal{T} = \mathcal{GI}(\xi)^{\vee}_{n} = (\mathcal{I}(\xi)^{\vee}_{n-1})^{\perp_n}$.

Proof. We only need to prove (1), since (2) is a dual of (1).

Since $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left n-cotorsion pair in $\mathcal{T}$, we see that $\mathcal{T} = \mathcal{GP}(\xi)^{\wedge}_{n}$ by Proposition 4.16(1), and that $\mathcal{GP}(\xi) = {}^{\perp_1}(\mathcal{P}(\xi)^{\wedge}_{n-1})$ by Theorem 3.6. Notice that $\mathcal{P}(\xi)^{\wedge}_{n-1} \subseteq (\mathcal{P}(\xi)^{\wedge}_{n-1})^{\wedge}_{n}$, so $(\mathcal{GP}(\xi), (\mathcal{P}(\xi)^{\wedge}_{n-1})^{\wedge}_{n})$ is a left $(n+1)$-cotorsion pair in $\mathcal{T}$. Then $\mathcal{T} = {}^{\perp_n}(\mathcal{P}(\xi)^{\wedge}_{n-1})$ by Corollary 3.14. □

4.2. Left weak Auslander-Buchweitz contexts, corresponding with cotorsion pairs and left Frobenius pairs. In [12], Hashimoto introduced and studied relative Auslander-Buchweitz contexts in abelian categories. Motivated by this, we now introduce left (weak) Auslander-Buchweitz contexts with respect to ξ in a triangulated category $\mathcal{T}$, and establish a one-to-one correspondence between left weak Auslander-Buchweitz contexts and left Frobenius pairs.

Definition 4.18. Let $(\mathcal{A}, \mathcal{B})$ be a pair of subcategories in $\mathcal{T}$ and $\omega = \mathcal{A} \cap \mathcal{B}$. We say that $(\mathcal{A}, \mathcal{B})$ is a *left weak Auslander-Buchweitz context* (*left weak AB context* for short) in $\mathcal{T}$ if the following conditions are satisfied:

(AB1) The pair $(\mathcal{A}, \omega)$ is a left Frobenius pair in $\mathcal{T}$.

(AB2) $\mathcal{B}$ is closed under direct summands, ξ-extensions and hocokernels of ξ-proper monomorphisms.

(AB3) $\mathcal{B} \subseteq \mathcal{A}^{\wedge}$.

A left weak AB context $(\mathcal{A}, \mathcal{B})$ is called a *left AB context* if the following condition is satisfied:

(AB4) $\mathcal{A}^{\wedge} = \mathcal{T}$.

The next result shows how to obtain left (weak) AB contexts from left Frobenius pairs in $\mathcal{T}$.

Proposition 4.19. *Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{T}$. Then $(\mathcal{X}, \omega^{\wedge})$ is a left weak AB context in $\mathcal{T}$. Moreover, if $\mathcal{X}^{\wedge} = \mathcal{T}$, $(\mathcal{X}, \omega^{\wedge})$ is a left AB context in $\mathcal{T}$.*

Proof. By Proposition 4.1, we have $\mathcal{X} \cap \omega^\wedge = \omega$. Since $(\mathcal{X}, \omega)$ is a left Frobenius pair in $\mathcal{T}$, we have $\omega^\wedge = \mathcal{X}^\perp \cap \mathcal{X}^\wedge$ by Corollary 4.4. Since $\mathcal{X}^\wedge$ is closed under direct summands by Proposition 4.12, and since $\mathcal{X}^\wedge$ is closed under ξ-extensions and hocokernels of ξ-proper monomorphisms by Theorem 4.9, we find that $\omega^\wedge$ is closed under direct summands, ξ-extensions and hocokernels of ξ-proper monomorphisms. Clearly, $\omega^\wedge \subseteq \mathcal{X}^\wedge$. Thus, $(\mathcal{X}, \omega^\wedge)$ is a left weak AB context in $\mathcal{T}$. The last assertion is clear. $\square$

The following result shows how to obtain cotorsion pairs from left AB contexts in $\mathcal{T}$.

Proposition 4.20. *Let $(\mathcal{A}, \mathcal{B})$ be a left weak AB context in $\mathcal{T}$ and $\omega := \mathcal{A} \cap \mathcal{B}$. Then $\omega = \mathcal{A} \cap \mathcal{A}^\perp$ and $\omega^\wedge = \mathcal{B}$. In this case, we have the following equivalent statements:*
(1) *$\mathcal{A}^\wedge = \mathcal{T}$.*
(2) *$(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{T}$.*
Moreover, if one of the above conditions holds, then $\mathcal{A}$ is resolving.

Proof. By assumption, we know that $(\mathcal{A}, \omega)$ is a left Frobenius pair in $\mathcal{T}$. Then by Proposition 4.1 and Corollary 4.4, we have $\omega = \mathcal{A} \cap \omega^\wedge$ and $\omega^\wedge = \mathcal{A}^\perp \cap \mathcal{A}^\wedge$. Thus, $\omega = \mathcal{A} \cap \mathcal{A}^\perp \cap \mathcal{A}^\wedge = \mathcal{A} \cap \mathcal{A}^\perp$.

Since $\omega \subseteq \mathcal{B}$ and $\mathcal{B}$ is closed under hocokernels of ξ-proper monomorphisms, we have $\omega^\wedge \subseteq \mathcal{B}$. Conversely, let $X \in \mathcal{B} \subseteq \mathcal{A}^\wedge$. By Proposition 4.2, there is a triangle $K \to A \to X \to \Sigma K$ in ξ with $A \in \mathcal{A}$ and $K \in \omega^\wedge \subseteq \mathcal{B}$. It follows that $A \in \mathcal{B}$ since $\mathcal{B}$ is closed under ξ-extensions. So $A \in \mathcal{A} \cap \mathcal{B} = \omega$, and hence $X \in \omega^\wedge$ and $\mathcal{B} \subseteq \omega^\wedge$. Thus, $\mathcal{B} = \omega^\wedge$.

(1)⇒(2) By Proposition 4.1, we have $\mathcal{A} \subseteq {}^\perp(\omega^\wedge)$ and $\xi xt^1_\xi(\mathcal{A}, \mathcal{B}) = 0$. Since $\mathcal{T} = \mathcal{A}^\wedge$ by assumption, for any $T \in \mathcal{T}$, there exist triangles

$$B \longrightarrow A \longrightarrow T \longrightarrow \Sigma B \quad \text{and} \quad T \longrightarrow B' \longrightarrow A' \longrightarrow \Sigma T$$

in ξ with $A, A' \in \mathcal{A}$ and $B, B' \in \omega^\wedge = \mathcal{B}$ by Proposition 4.2. Thus, $(\mathcal{A},\, \mathcal{B} = \omega^\wedge)$ is a cotorsion pair in $\mathcal{T}$.

(2)⇒(1) This is clear.

The last assertion follows by the fact that $\mathcal{A} = {}^{\perp_1}\mathcal{B} \supseteq \mathcal{P}(\xi)$ (see Remark 3.2). $\square$

The following result provides a way to obtain left Frobenius pairs and left (weak) AB contexts from cotorsion pairs in $\mathcal{T}$.

Proposition 4.21. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\mathcal{T}$ with $\mathcal{U}$ resolving. Then $(\mathcal{U}, \omega)$ is a left Frobenius pair in $\mathcal{T}$, where $\omega := \mathcal{U} \cap \mathcal{V}$. Moreover, the following assertions hold true:*
(1) *If $\mathcal{V} \subseteq \mathcal{U}^\wedge$, then $(\mathcal{U}, \mathcal{V})$ is a left weak AB context in $\mathcal{T}$.*
(2) *If $\mathcal{U}^\wedge = \mathcal{T}$, then $(\mathcal{U}, \mathcal{V})$ is a left AB context in $\mathcal{T}$.*

Proof. By assumption, we know that $\mathcal{U}$ and $\mathcal{V}$ are closed under direct summands, and $\mathcal{U}$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms. So $\omega := \mathcal{U} \cap \mathcal{V}$ is closed under direct summands. It follows from Corollary 3.11 that $\mathcal{V} \subseteq \mathcal{U}^\perp$ and $\omega \subseteq \mathcal{U} \cap \mathcal{U}^\perp$, which implies that ω is $\mathcal{U}$-injective. Now, let $U \in \mathcal{U}$.

Consider the following triangle in ξ: $U \to V' \to U' \to \Sigma U$ with $U' \in \mathcal{U}$ and $V' \in \mathcal{V}$. It follows that $V' \in \mathcal{U} \cap \mathcal{V} = \omega$ from the fact that $\mathcal{U}$ is closed under ξ-extensions, and so ω is a ξ-cogenerator in $\mathcal{U}$. Thus, $(\mathcal{U}, \omega)$ is a left Frobenius pair in $\mathcal{T}$.

(1) By Corollary 3.11, $\mathcal{V}$ is closed under ξ-extensions and hocokernels of ξ-proper monomorphisms. Since $\mathcal{V} \subseteq \mathcal{U}^\wedge$ by assumption, we infer that $(\mathcal{U}, \mathcal{V})$ is a left weak AB context in $\mathcal{T}$.

(2) The result is clear by (1). □

Our main result is the following correspondence theorem.

Theorem 4.22. *For an integer $n \geq 1$, consider the following classes:*

$$\mathfrak{A} := \{(\mathcal{X}, \omega) \,|\, (\mathcal{X}, \omega) \text{ is a left Frobenius pair in } \mathcal{T}\},$$
$$\mathfrak{B} := \{(\mathcal{A}, \mathcal{B}) \,|\, (\mathcal{A}, \mathcal{B}) \text{ is a left weak AB context}\},$$
$$\mathfrak{C} := \{(\mathcal{U}, \mathcal{V}) \,|\, (\mathcal{U}, \mathcal{V}) \text{ is a cotorsion pair in } \mathcal{T} \text{ with } \mathcal{U} \text{ resolving, } \mathcal{V} \subseteq \mathcal{U}^\wedge\},$$
$$\mathfrak{D} := \{(\mathcal{U}, \mathcal{V}) \,|\, (\mathcal{U}, \mathcal{V}) \text{ is an } n\text{-cotorsion pair in } \mathcal{T} \text{ with } \mathcal{U} \text{ resolving, } \mathcal{V} \subseteq \mathcal{U}^\wedge\}.$$

Then we have the following:

(1) *There is a one-to-one correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ given by*

$$\Phi: \ \mathfrak{A} \longrightarrow \mathfrak{B}, \quad (\mathcal{X}, \omega) \longmapsto (\mathcal{X}, \omega^\wedge),$$
$$\Psi: \ \mathfrak{B} \longrightarrow \mathfrak{A}, \quad (\mathcal{A}, \mathcal{B}) \longmapsto (\mathcal{A}, \mathcal{A} \cap \mathcal{B}).$$

(2) $\mathfrak{C} \subseteq \mathfrak{B}$.

(3) $\mathfrak{C} = \mathfrak{D}$.

Proof. (1) Following Proposition 4.19, we know that Φ is well-defined. It suffices to prove $\Phi\Psi = 1_{\mathfrak{B}}$ and $\Psi\Phi = 1_{\mathfrak{A}}$. Let $(\mathcal{A}, \mathcal{B})$ be a left weak AB context. Then we have $\Phi\Psi(\mathcal{A}, \mathcal{B}) = \Phi(\mathcal{A}, \mathcal{A} \cap \mathcal{B}) = (\mathcal{A}, (\mathcal{A} \cap \mathcal{B})^\wedge)$. By Proposition 4.20, we see that $\mathcal{B} = (\mathcal{A} \cap \mathcal{B})^\wedge$. It follows that $\Phi\Psi(\mathcal{A}, \mathcal{B}) = (\mathcal{A}, \mathcal{B})$ and $\Phi\Psi = 1_{\mathfrak{B}}$. Conversely, let $(\mathcal{X}, \omega)$ be a left Frobenius pair. Then $\Psi\Phi(\mathcal{X}, \omega) = \Psi(\mathcal{X}, \omega^\wedge) = (\mathcal{X}, \mathcal{X} \cap \omega^\wedge)$. Since $\mathcal{X} \cap \omega^\wedge = \omega$ by Proposition 4.1, we have $\Psi\Phi(\mathcal{X}, \omega) = (\mathcal{X}, \omega)$ and $\Psi\Phi = 1_{\mathfrak{A}}$.

(2) This follows from Proposition 4.21.

(3) This follows from Proposition 3.10. □

Furthermore, we get the following result.

Theorem 4.23. *For an integer $n \geq 1$, consider the following classes:*

$$\mathfrak{A} := \{(\mathcal{X}, \omega) \,|\, (\mathcal{X}, \omega) \text{ is a left Frobenius pair with } \mathcal{X}^\wedge = \mathcal{T}\},$$
$$\mathfrak{B} := \{(\mathcal{A}, \mathcal{B}) \,|\, (\mathcal{A}, \mathcal{B}) \text{ is a left AB context}\},$$
$$\mathfrak{C} := \{(\mathcal{U}, \mathcal{V}) \,|\, (\mathcal{U}, \mathcal{V}) \text{ is a cotorsion pair in } \mathcal{T} \text{ with } \mathcal{U} \text{ resolving, } \mathcal{U}^\wedge = \mathcal{T}\},$$
$$\mathfrak{D} := \{(\mathcal{U}, \mathcal{V}) \,|\, (\mathcal{U}, \mathcal{V}) \text{ is an } n\text{-cotorsion pair in } \mathcal{T} \text{ with } \mathcal{U} \text{ resolving, } \mathcal{U}^\wedge = \mathcal{T}\}.$$

Then $\mathfrak{B} = \mathfrak{C} = \mathfrak{D}$ and there is a one-to-one correspondence between $\mathfrak{A}$ and $\mathfrak{B}$.

Proof. By Theorem 4.22, it suffices to show $\mathfrak{B} \subseteq \mathfrak{C}$. Now the assertion follows from Proposition 4.20. □

Example 4.24. The pair $(\mathcal{T}, \mathcal{I}(\xi))$ is a trivial cotorsion pair in $\mathcal{T}$. Obviously, $\mathcal{T}$ is resolving. So, by Theorem 4.23, $(\mathcal{T}, \mathcal{I}(\xi))$ is a left AB context and is a left Frobenius pair in $\mathcal{T}$.

Example 4.25. Assume that $\mathcal{P}(\xi)$ is a generating subcategory and $\mathcal{I}(\xi)$ is a cogenerating subcategory of $\mathcal{T}$. By [21, Theorem 4.6],

$$\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\, T \mid T \in \mathcal{T}\} = \sup\{\xi\text{-}\mathcal{G}\mathrm{id}\, T \mid T \in \mathcal{T}\}.$$

The common value of the quantities in the above equality is called the global ξ-Gorenstein dimension of $\mathcal{T}$. In the sense of Asadollahi and Salarian, $\mathcal{T}$ is called a ξ-Gorenstein triangulated category (see [3, Definition 4.6]).

Assume that $\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\, T \mid T \in \mathcal{T}\} < \infty$. By [3, Theorem 4.13], $(\mathcal{P}(\xi)^{\wedge}, \mathcal{GI}(\xi))$ is a cotorsion pair in $\mathcal{T}$. One have the following facts:

(1) $\mathcal{P}(\xi)^{\wedge}$ and $\mathcal{GI}(\xi)$ are closed under direct summands.

(2) By [17, Remark 3.5], $\mathcal{P}(\xi)^{\wedge}$ $(\supseteq \mathcal{P}(\xi))$ is a resolving subcategory in $\mathcal{T}$, that is, $\mathcal{P}(\xi)^{\wedge}$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms.

(3) The assertion that $\mathcal{GI}(\xi)$ is a ξ-cogenerator of $\mathcal{P}(\xi)^{\wedge}$ is obvious. By Corollary 3.11, we see that $\mathcal{GI}(\xi)$ is coresolving and $\xi xt^{i}_{\xi}(\mathcal{P}(\xi)^{\wedge}, \mathcal{GI}(\xi)) = 0$ for every $i \geq 1$, so $\mathcal{GI}(\xi)$ is $\mathcal{P}(\xi)^{\wedge}$-injective.

Hence, $(\mathcal{P}(\xi)^{\wedge}, \mathcal{GI}(\xi))$ is a left Frobenius pair in $\mathcal{T}$.

Notice that $\mathcal{GI}(\xi)^{\wedge} = \mathcal{GI}(\xi)$, so $(\mathcal{P}(\xi)^{\wedge}, \mathcal{GI}(\xi))$ is a left weak AB context in $\mathcal{T}$ by Theorem 4.22. One can see that $\mathfrak{C} \subsetneq \mathfrak{B}$.

Example 4.26. According to Example 4.7(1), $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a left Frobenius pair in $\mathcal{T}$. By Theorem 4.22 we know that $(\mathcal{GP}(\xi), \mathcal{P}(\xi)^{\wedge})$ is a left weak AB context in $\mathcal{T}$. In addition, if $\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}\, T \mid T \in \mathcal{T}\} < \infty$, then by Theorem 4.23, $(\mathcal{GP}(\xi), \mathcal{P}(\xi)^{\wedge})$ is a left AB context and is a cotorsion pair in $\mathcal{T}$.

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Rerefences

[1] L. Angeleri-Hügel, O. Mendoza, Homological dimensions in cotorsion pairs, *Illinois J. Math.* **53** (2009) 251–263.

[2] J. Asadollahi, S. Salarian, Gorenstein objects in triangulated categories, *J. Algebra* **281** (2004) 264–286.

[3] J. Asadollahi, S. Salarian, Tate cohomology and Gorensteinness for triangulated categories, *J. Algebra* **299** (2006) 480–502.

[4] M. Auslander, R.O. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, *Mem. Soc. Math. France* **38** (1989) 5–37.

[5] M. Auslander, S.O. Smalø, Preprojective modules over Artin algebras, *J. Algebra* **66** (1) (1980) 61–122.

[6] V. Becerril, O. Mendoza, M.A. Pérez, V. Santiago, Frobenius pairs in abelian categories, *J. Homotopy Relat. Struct.* **14** (2019) 1–50.

[7] A. Beligiannis, Relative homological algebra and purity in triangulated categories, *J. Algebra* **227** (2000) 268–361.
[8] D. Bennis, J.R. Garcia Rozas, L.X. Mao, L. Oyonarte, On proper and exact relative homological dimensions, *Algebra Colloq.* **27** (3) (2020) 621–642.
[9] P. Eklof, J. Trlifaj, How to make Ext vanish, *Bull. London Math. Soc.* **33** (2001) 41–51.
[10] E. Enochs, O.M.G. Jenda, *Relative Homological Algebra*, De Gruyter Exp. Math., 30, Walter de Gruyter & Co., Berlin, 2000.
[11] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, De Gruyter Exp. Math., 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2006.
[12] M. Hashimoto, *Auslander-Buchweitz Approximations of Equivariant Modules*, London Math. Soc. Lecture Note Ser., 282, Cambridge Univ. Press, Cambridge, 2000.
[13] J. He, P.Y. Zhou, On the relation between n-cotorsion pairs and $(n+1)$-cluster tilting subcategories, *J. Algebra Appl.* **21** (1) (2022) Paper No. 2250011, 12 pp.
[14] J.S. Hu, D.D. Zhang, P.Y. Zhou, Proper classes and Gorensteinness in extriangulated categories, *J. Algebra* **551** (2020) 23–60.
[15] Y.G. Hu, H.L. Yao, X.R. Fu, Tilting objects in triangulated categories, *Comm. Algebra* **48** (1) (2020) 410–429.
[16] M. Huerta, O. Mendoza, M.A. Pérez, n-Cotorsion pairs, *J. Pure Appl. Algebra* **225** (5) (2021) Paper No. 106556, 34 pp.
[17] X. Ma, T.W. Zhao, Resolving resolution dimensions in triangulated categories, *Open Math.* **19** (1) (2021) 121–143.
[18] X. Ma, T.W. Zhao, Z.Y. Huang, Resolving subcategories of triangulated categories and relative homological dimension, *Acta Math. Sin.* (Engl. Ser.) **33** (11) (2017) 1513–1535.
[19] H. Nakaoka, Y. Palu, Extriangulated categories, Hovey twin cotorsion pairs and model structures, *Cah. Topol. Géom. Différ. Catég.* **60** (2019) 117–193.
[20] A. Neeman, *Triangulated Categories*, Annals Math. Studies, 148, Princeton Univ. Press, Princeton, NJ, 2001.
[21] W. Ren, Z.K. Liu, Gorenstein homological dimensions for triangulated categories, *J. Algebra* **410** (2014) 258–276.
[22] X.Y. Yang, Notes on proper class of triangles, *Acta Math. Sin.* (Engl. Ser.) **29** (11) (2013) 2137–2154.
[23] X.Y. Yang, Model structures on triangulated categories, *Glasgow Math. J.* **57** (2) (2015) 263–284.
[24] X.Y. Yang, Z.C. Wang, Proper resolutions and Gorensteinness in triangulated categories, *Rocky Mountain J. Math.* **47** (3) (2017) 1013–1053.