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Selforthogonal Modules with Finite Injective Dimension III

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Abstract Let R be a left Noetherian ring, S a right Noetherian ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. We give some equivalent conditions that the injective dimension of U_S is finite implies that of ${}_RU$ is also finite. As an application, under the assumption that the injective dimensions of ${}_RU$ and U_S are finite, we construct a hereditary and complete cotorsion theory by some subcategories associated with ${}_RU$.

Keywords Generalized tilting modules · Injective dimension · $\mathcal{X}$-resolution dimension · U-torsionless property

Mathematics Subject Classifications (2000) 16E10 · 16E30

1 Introduction

Let R be a ring. We use mod R (resp. mod R^{op}) to denote the category of finitely generated left (resp. right) R-modules.

We define $\mathrm{gen}^*({}_RR) = \{X \in \mathrm{mod}\, R \mid$ there exists an exact sequence $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to X \to 0$ in mod R with P_i projective for any $i \geq 0\}$ (see [15]). It

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is clear that $\mathrm{gen}^*({}_RR) = \mathrm{mod}\, R$ if R is a left Noetherian ring. For a module ${}_RU \in \mathrm{mod}\, R$, we use ${}^{\perp}_RU$ to denote the full subcategory of mod R consisting of the modules C with $\mathrm{Ext}^i_R({}_RC, {}_RU) = 0$ for any $i \geq 1$. ${}_RU$ is called *selforthogonal* if ${}_RU \in {}^{\perp}_RU$.

Definition 1.1 [15] A selforthogonal module ${}_RU \in \mathrm{gen}^*({}_RR)$ is called a *generalized tilting module* (sometimes it is also called a *Wakamatsu tilting module*, see [11]) if there exists an exact sequence:

$$0 \to {}_RR \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

such that: (1) $U_i \in \mathrm{add}_R\, U$ for any $i \geq 0$, where $\mathrm{add}_R\, U$ denotes the full subcategory of mod R consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_RU$, and (2) after applying the functor $\mathrm{Hom}_R(\ , U)$ the sequence is still exact.

Let R and S be any rings. Recall that a bimodule ${}_RU_S$ is called *faithfully balanced* if $R = \mathrm{End}(U_S)$ and $S = \mathrm{End}({}_RU)$. By [15, Corollary 3.2], we have that ${}_RU_S$ is faithfully balanced and selforthogonal with ${}_RU \in \mathrm{gen}^*({}_RR)$ and $U_S \in \mathrm{gen}^*(S_S)$ if and only if ${}_RU$ is generalized tilting with $S = \mathrm{End}({}_RU)$ if and only if U_S is generalized tilting with $R = \mathrm{End}(U_S)$. In particular, it is trivial that ${}_RR$ is a generalized tilting module. Let R be an Artinian algebra. Then a (co)tilting module in mod R is generalized tilting (see [11]). On the other hand, the direct sum of all representatives of indecomposable injective R-modules which appear in the minimal injective resolution of ${}_RR$ as direct summands of some term is a generalized tilting module (see [14]).

Auslander and Reiten pointed out in [3, p.150] that it would be interesting to know that if one of the left and right self-injective dimensions of an Artinian algebra is finite implies the other is also finite. In [3, Prop. 6.10–6.12] they studied this question by using the finiteness of the finitistic dimension and the contravariant finiteness of some subcategories of mod R. When R and S are left and right Artinian rings, we gave in [8, Theorem] a partial answer to this Auslander and Reiten's question in term of the grade of modules with respect to a faithfully balanced and selforthogonal bimodule ${}_RU_S$, which implies that the injective dimension of ${}_RU$ is at most one if and only if that of U_S is also at most one. On the other hand, in [9, Theorem 2.7] we generalized [14, Theorem] to Noetherian rings, and proved that if R is a left Noetherian ring, S is a right Noetherian ring and ${}_RU$ is a generalized tilting module with $S = \mathrm{End}({}_RU)$, then the injective dimensions of ${}_RU$ and U_S are identical provided that both of them are finite.

In this paper, we will give some equivalent conditions that the injective dimension of U_S is finite implies that of ${}_RU$ is also finite. The main result is the following

Theorem 1.2 *Let R be a left Noetherian ring, S a right Noetherian ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. If the injective dimension of U_S is finite, then the following statements are equivalent for a non-negative integer n.*

(1) *The injective dimension of ${}_RU$ is at most n.*
(2) ${}^{\perp}_RU$-resol.dim$_R(M) \leq n$ *for any* $M \in \mathrm{mod}\, R$.
(3) *For any $M \in \mathrm{mod}\, R$, there exists an exact sequence:*

$$0 \to K \to E \to M \to 0$$

with $\mathrm{add}_R\, U$-resol.dim$_R(K) \leq n-1$ *and* $E \in {}^{\perp}_RU$.

(4) *For any* $M \in \operatorname{mod} R$*, there exists an exact sequence:*

$$0 \to M \to K' \to E' \to 0$$

with $\operatorname{add}_R U\text{-resol.dim}_R(K') \leq n$ *and* $E' \in {}_R^{\perp}U$.

This theorem extends [1, Theorem 1.8], [3, Prop. 6.11 and 6.12] and [12, Prop. 3.2(a)], see Section 3 for the details.

In Section 2, we introduce the notion of the U-torsionless property and deduce some basic facts about it for the later use. In Section 3, we give the proof of Theorem 1.2. Actually, we will prove this theorem in a more general situation. As an application of Theorem 1.2, under the assumption that the injective dimensions of ${}_RU$ and U_S are finite, we prove in Section 4 that (the subcategory of $\operatorname{mod} R$ consisting the modules M with $\operatorname{add}_R U\text{-resol.dim}_R(M) < \infty$, ${}_R^{\perp}U$) forms a hereditary and complete cotorsion theory.

2 The U-torsionless Property

In this section we introduce the notion of the U-torsionless property and give it some characterizations. The following result is [15, Corollary 3.2].

Proposition 2.1 *For a bimodule* ${}_RU_S$*, the following statements are equivalent.*

(1) ${}_RU$ *is a generalized tilting module with* $S = \operatorname{End}({}_RU)$.
(2) U_S *is a generalized tilting module with* $R = \operatorname{End}(U_S)$.
(3) ${}_RU_S$ *is a faithfully balanced and selforthogonal bimodule with* ${}_RU \in \operatorname{gen}^*({}_RR)$ *and* $U_S \in \operatorname{gen}^*(S_S)$.

In the rest of this paper, we always assume that R is a left Noetherian ring, S is a right Noetherian ring and ${}_RU$ is a generalized tilting module with $S = \operatorname{End}({}_RU)$. All modules considered are finitely generated.

Let A be in $\operatorname{mod} R$ (resp. $\operatorname{mod} S^{op}$). We use $\operatorname{l.id}_R(A)$ (resp. $\operatorname{r.id}_S(A)$) to denote the injective dimension of ${}_RA$ (resp. A_S). We call $\operatorname{Hom}_R({}_RA, {}_RU_S)$ (resp. $\operatorname{Hom}_S(A_S, {}_RU_S)$) the *dual module* of A with respect to ${}_RU_S$, and denote either of these modules by A^*. For a homomorphism f between R-modules (resp. S^{op}-modules), we put $f^* = \operatorname{Hom}(f, {}_RU_S)$. Let $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. A is called *U-torsionless* (resp. *U-reflexive*) if σ_A is a monomorphism (resp. an isomorphism). Under the assumption of ${}_RU$ being generalized tilting with $S = \operatorname{End}({}_RU)$ (more generally, ${}_RU_S$ being faithfully balanced), it is easy to see that any projective module in $\operatorname{mod} R$ (resp. $\operatorname{mod} S^{op}$) is U-reflexive.

Definition 2.2 Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} R$. $\mathcal{X}$ is said to have the *U-torsionless property* (resp. the *U-reflexive property*) if every module in $\mathcal{X}$ is U-torsionless (resp. U-reflexive).

Recall from [2] that $M \in \operatorname{mod} R$ is said to have *generalized Gorenstein dimension zero* (with respect to ${}_RU_S$), denoted by $\text{G-dim}_U(M) = 0$, if the following conditions are satisfied: (1) M is U-reflexive, and (2) $M \in {}_R^{\perp}U$ and $M^* \in {}^{\perp}U_S$. We use $\mathcal{G}_U$ to denote the full subcategory of mod R consisting of the modules with generalized Gorenstein dimension zero.

We give in the following result some characterizations of the U-torsionless property.

Proposition 2.3 *The following statements are equivalent.*

(1) ${}_R^{\perp}U$ *has the U-torsionless property.*
(2) ${}_R^{\perp}U$ *has the U-reflexive property.*
(3) ${}_R^{\perp}U = \mathcal{G}_U$.

Proof The implications of (3) ⇒ (2) ⇒ (1) are trivial. Now we prove (1) ⇒ (3). Assume that (1) holds true. Let $M \in {}_R^{\perp}U$ and $Q \xrightarrow{g} M^* \to 0$ an exact sequence in mod S^{op} with Q projective, and let f be the composition: $M \xrightarrow{\sigma_M} M^{**} \xrightarrow{g^*} Q^*$ with $Q^* \in \mathrm{add}_R U$, that is, $f = g^*\sigma_M$. Notice that M is U-torsionless by (1), so σ_M and f are monomorphisms. Putting $K = \mathrm{Coker}\, f$, then it is not difficult to verify that $\mathrm{Ext}_R^1(K, U) = 0$ and we get an exact sequence: $0 \to K^* \to Q^{**} \xrightarrow{f^*} M^* \to 0$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & Q^* & \xrightarrow{\pi} & K & \longrightarrow & 0 \\ & & \downarrow{\sigma_M} & & \downarrow{\sigma_{Q^*}} & & \downarrow{\sigma_K} & & \\ 0 & \longrightarrow & M^{**} & \xrightarrow{f^{**}} & Q^{***} & \xrightarrow{\pi^{**}} & K^{**} & & \end{array}$$

Since $M \in {}_R^{\perp}U$, so is K. Then by (1), we have that K is U-torsionless and σ_K is a monomorphism. Since σ_{Q^*} is an isomorphism, σ_M is also an isomorphism and M is U-reflexive. Similarly, we get that K is also U-reflexive. Hence π^{**} is an epimorphism and $\mathrm{Ext}_S^1(M^*, U) = 0$. Similarly, we get that $\mathrm{Ext}_S^1(K^*, U) = 0$ and so $\mathrm{Ext}_S^2(M^*, U) = 0$. Continuing this process, we get that $\mathrm{Ext}_S^i(M^*, U) = 0$ for any $i \geq 1$ and $M \in \mathcal{G}_U$. □

Let $N \in \mathrm{mod}\, S^{op}$ and let

$$0 \to N \xrightarrow{\delta_0} I_0 \xrightarrow{\delta_1} I_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} I_i \xrightarrow{\delta_{i+1}} \cdots$$

be an injective resolution of N. Recall from [4] that such an injective resolution is called *ultimately closed* if there exists a positive integer k such that $\mathrm{Im}\, \delta_k = \bigoplus_{j=0}^m W_j$, where each W_j is a direct summand of $\mathrm{Im}\, \delta_{i_j}$ with $i_j < k$. It is clear that if $\mathrm{r.id}_S(U) < \infty$, then the minimal injective resolution of U_S is ultimately closed. Let S be a semi-primary ring with $J(S)^{m+1} = 0$, where $J(S)$ is the Jacobson radical of S and m is a non-negative integer. If there is only a finite number of finitely generated indecomposable right $S/J(S)^m$-modules up to isomorphism, then any module in mod S^{op} has an ultimately closed injective resolution (see [13, p.110]). By [10, Theorem 2.4], if U_S has an ultimately closed injective resolution, then ${}_R^{\perp}U$ has the U-reflexive property.

Let $M \in \operatorname{mod} R$ and n be a non-negative integer. Recall from [2] that M is said to have *generalized Gorenstein dimension* at most n (with respect to ${}_RU_S$), denoted by $\text{G-dim}_U(M) \leq n$, if there exists an exact sequence $0 \to M_n \to \cdots \to M_1 \to M_0 \to M \to 0$ in $\operatorname{mod} R$ with $M_i \in \mathcal{G}_U$ for any $0 \leq i \leq n$.

Lemma 2.4

(1) *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence in* $\operatorname{mod} R$ *with* $C \in \mathcal{G}_U$. *Then* $A \in \mathcal{G}_U$ *if and only if* $B \in \mathcal{G}_U$.
(2) *For a module* $M \in \operatorname{mod} R$ *and a non-negative integer n,* $\text{G-dim}_U(M) \leq n$ *if and only if* $\Omega^n(M) \in \mathcal{G}_U$*, where* $\Omega^n(M)$ *is the n-th syzygy of M.*

Proof See [7, Lemma 5.8 and Theorem 5.9] respectively. The arguments there remain valid here, so we omit the proof. □

We use $\mathcal{X}_U$ to denote the full subcategory of ${}^{\perp}_{R}U$ consisting of the modules C such that there exists an exact sequence $0 \to C \to U_0 \xrightarrow{d_0} U_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} U_i \xrightarrow{d_i} \cdots$ in $\operatorname{mod} R$ with $U_i \in \operatorname{add}_R U$ and $\operatorname{Im} d_i \in {}^{\perp}_{R}U$ for any $i \geq 0$ (see [3]). As an application of Proposition 2.3, we have the following result, which is a generalization of [2, Proposition 4.3].

Proposition 2.5

(1) ${}^{\perp}_{R}U \supseteq \mathcal{G}_U = \mathcal{X}_U$;
(2) *If* ${}^{\perp}_{R}U$ *has the U-torsionless property, then* ${}^{\perp}_{R}U = \mathcal{G}_U = \mathcal{X}_U$.

Proof

(1) It suffices to prove $\mathcal{G}_U = \mathcal{X}_U$. Let $M \in \mathcal{G}_U$ and

$$\cdots \xrightarrow{f_i} Q_i \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} Q_1 \xrightarrow{f_0} Q_0 \to M^* \to 0$$

be an exact sequence in $\operatorname{mod} S^{op}$ with Q_i projective for any $i \geq 0$. Then M is U-reflexive and $M^* \in {}^{\perp}U_S$. So we get an exact sequence:

$$0 \to M \to Q_0^* \xrightarrow{f_0^*} Q_1^* \xrightarrow{f_1^*} \cdots \xrightarrow{f_{i-1}^*} Q_i^* \xrightarrow{f_i^*} \cdots$$

in $\operatorname{mod} R$ with $Q_i^* \in \operatorname{add}_R U$ for any $i \geq 0$. It is clear that $\text{G-dim}_U(M^*) = 0$, so $\text{G-dim}_U(\operatorname{Im} f_i) = 0$ for any $i \geq 0$ by Lemma 2.4(2). Thus $\operatorname{Im} f_i^* \cong (\operatorname{Im} f_i)^*$ and $\text{G-dim}_U(\operatorname{Im} f_i^*) = 0$ for any $i \geq 0$. Therefore we conclude that $M \in \mathcal{X}_U$ and $\mathcal{G}_U \subseteq \mathcal{X}_U$.
Conversely, let $M \in \mathcal{X}_U$ and

$$0 \to M \to U_0 \xrightarrow{d_0} U_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} U_i \xrightarrow{d_i} \cdots$$

be an exact sequence in mod R with $U_i \in \mathrm{add}_R U$ and $\mathrm{Im}\, d_i \in {}_R^{\perp}U$ for any $i \geq 0$. By using an argument similar to that in the proof of Proposition 2.3, we get that M is U-reflexive. In addition, we have an exact sequence:

$$\cdots \xrightarrow{d_i^*} U_i^* \xrightarrow{d_{i-1}^*} \cdots \xrightarrow{d_1^*} U_1^* \xrightarrow{d_0^*} U_0^* \to M^* \to 0.$$

Note that U_i is U-reflexive for any $i \geq 0$. So, if the functor $\mathrm{Hom}_S(\ , U)$ is applied, then the induced sequence:

$$0 \to M^{**} \to U_0^{**} \xrightarrow{d_0^{**}} U_1^{**} \xrightarrow{d_1^{**}} \cdots \xrightarrow{d_{i-1}^{**}} U_i^{**} \xrightarrow{d_i^{**}} \cdots$$

is also exact, which implies that $M^* \in {}^{\perp}U_S$. Then $M \in \mathcal{G}_U$ and $\mathcal{X}_U \subseteq \mathcal{G}_U$.

(2) follows from (1) and Proposition 2.3. □

3 Proof of Theorem 1.2

In this Section, we give the proof of Theorem 1.2. In fact, we will prove it in a more general situation.

Let $\mathcal{X}$ be a full subcategory of mod R and A a module in mod R. If there exists an exact sequence $\cdots \to X_n \to \cdots \to X_1 \to X_0 \to A \to 0$ in mod R with $X_i \in \mathcal{X}$ for any $i \geq 0$, then we define the *$\mathcal{X}$-resolution dimension* of A, denoted by $\mathcal{X}$-resol.dim$_R(A)$, as $\inf\{n \mid$ there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to A \to 0$ in mod R with $X_i \in \mathcal{X}$ for any $0 \leq i \leq n\}$. We set $\mathcal{X}$-resol.dim$_R(A)$ equal to infinity if no such integer exists (see [1]).

Proposition 3.1 *For a non-negative integer n,* $\mathrm{l.id}_R(U) \leq n$ *if and only if* ${}_R^{\perp}U$-resol.dim$_R(M) \leq n$ *for any* $M \in$ mod R.

Proof The case $n = 0$ is trivial. Now suppose $n \geq 1$.

Assume that $\mathrm{l.id}_R(U) \leq n$. Let $M \in$ mod R and

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be an exact sequence in mod R with P_i projective for any $0 \leq i \leq n-1$. Then $\mathrm{Ext}_R^j(K, U) \cong \mathrm{Ext}_R^{n+j}(M, U) = 0$ for any $j \geq 1$. So $K \in {}_R^{\perp}U$ and ${}_R^{\perp}U$-resol.dim$_R(M) \leq n$.

Conversely, let M be any module in mod R. Then ${}_R^{\perp}U$-resol.dim$_R(M) \leq n$ and there exists an exact sequence:

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

in mod R with X_i in ${}_R^{\perp}U$ for any $0 \leq i \leq n$. So $\mathrm{Ext}_R^{n+j}(M, U) \cong \mathrm{Ext}_R^j(X_n, U) = 0$ for any $j \geq 1$ and hence $\mathrm{l.id}_R(U) \leq n$. □

The following result generalizes [3, Prop. 6.11].

Corollary 3.2 *Let R be a left Artinian ring. If ${}_R^{\perp}U$ has the U-torsionless property, then the following statements are equivalent.*

(1) $\mathrm{l.id}_R(U) < \infty$.
(2) ${}_R^{\perp}U$-resol.dim$_R(M) < \infty$ *for any* $M \in$ mod R.

Proof (1) $\Rightarrow$ (2) follows from Proposition 3.1.

(2) $\Rightarrow$ (1) Since R is a left Artinian ring, there exist only finitely many simple R-modules up to isomorphism, say $S_1, \cdots, S_t$. By assumption, ${}^{\perp}_{R}U$-resol.dim$_R(S_j) < \infty$ for any $1 \leq j \leq t$. Putting $n = \max\left\{{}^{\perp}_{R}U\text{-resol.dim}_R(S_j) \mid 1 \leq j \leq t\right\}$, then $\mathrm{Ext}_R^{n+i}(S_j, U) = 0$ for any $i \geq 1$ and $1 \leq j \leq t$.

Let

$$0 \to {}_RU \xrightarrow{\alpha_0} E_0 \xrightarrow{\alpha_1} E_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} E_{n-1} \xrightarrow{\alpha_n} \cdots$$

be the minimal injective resolution of ${}_RU$. Then

$$\mathrm{Ext}_R^i(S_j, \mathrm{Im}\,\alpha_n) \cong \mathrm{Ext}_R^{n+i}(S_j, U) = 0$$

for any $i \geq 1$ and $1 \leq j \leq t$. So $\mathrm{Im}\,\alpha_n$ is injective and $\mathrm{l.id}_R(U) \leq n$. □

Lemma 3.3 *For a module $M \in \mathrm{mod}\, R$, $\mathrm{G\text{-}dim}_U(M) \leq 1$ if and only if there exists an exact sequence:*

$$0 \to K \to E \to M \to 0,$$

with $K \in \mathrm{add}_R U$ and $E \in \mathcal{G}_U$.

Proof The sufficiency is trivial. So it suffices to prove the necessity. Suppose G-dim$_U(M) \leq 1$. By Lemma 2.4(2), there exists an exact sequence:

$$0 \to E_1 \to P \to M \to 0$$

in mod R with P projective and $E_1 \in \mathcal{G}_U$. By Proposition 2.5, $E_1 \in \mathcal{X}_U$. So there exists an exact sequence:

$$0 \to E_1 \to K \to E_2 \to 0$$

with $K \in \mathrm{add}_R U$ and $E_1 \in \mathcal{X}_U (= \mathcal{G}_U)$. Consider the push-out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E_1 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & E_2 & = & E_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the middle column in the above diagram, it is easy to get that $E \in \mathcal{G}_U$. So the middle row is as desired. □

Proposition 3.4 *Let n be a non-negative integer. For a module $M \in \operatorname{mod} R$, G-$\dim_U(M) \leq n$ if and only if there exists an exact sequence:*

$$0 \to K \to E \to M \to 0$$

with $\operatorname{add}_R U$-resol.$\dim_R(K) \leq n-1$ *and* $E \in \mathcal{G}_U$.

Proof The sufficiency is trivial. In the following we will prove the necessity by using induction on n. The case $n \leq 1$ follows from Lemma 3.3. Now suppose $n \geq 2$. We have an exact sequence:

$$0 \to M' \to P \to M \to 0$$

in mod R with P projective and G-$\dim_U(M') \leq n-1$. By the induction hypothesis, we have an exact sequence:

$$0 \to K' \to E' \to M' \to 0$$

with $\operatorname{add}_R U$-resol.$\dim_R(K') \leq n-2$ and $E' \in \mathcal{G}_U$. Since $\mathcal{G}_U = \mathcal{X}_U$ by Proposition 2.5, $E' \in \mathcal{X}_U$. So there exists an exact sequence:

$$0 \to E' \to U_0 \to E'' \to 0$$

with $U_0 \in \operatorname{add}_R U$ and $E'' \in \mathcal{X}_U (= \mathcal{G}_U)$. First, consider the push-out diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & E' & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & U_0 & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E'' & = & E'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle row, we have that $\mathrm{add}_R\, U\text{-resol.dim}_R(K) \leq n-1$. Next, consider the following push-out diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & E'' & = & E'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

From the middle column in the above diagram, it is easy to get that $E \in \mathcal{G}_U$. So the middle row is as desired. □

Proposition 3.5 is interesting in its own right.

Proposition 3.5 *For any non-negative integer n, the following statements are equivalent:*

(1) *For any* $M \in \mathrm{mod}\, R$*, there exists an exact sequence:*

$$0 \to K \to E \to M \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K) \leq n-1$ *and* $E \in \mathcal{G}_U$*.*

(2) *For any* $M \in \mathrm{mod}\, R$*, there exists an exact sequence:*

$$0 \to M \to K' \to E' \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K') \leq n$ *and* $E' \in \mathcal{G}_U$*.*

Proof (1) $\Rightarrow$ (2) For $M \in \mathrm{mod}\, R$, there exists an exact sequence:

$$0 \to K \to E \to M \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K) \leq n-1$ and $E \in \mathcal{G}_U$. Since $\mathcal{G}_U = \mathcal{X}_U$ by Proposition 2.5, $E \in \mathcal{X}_U$. So there exists an exact sequence:

$$0 \to E \to V_0 \to E' \to 0$$

with $V_0 \in \operatorname{add}_R U$ and $E' \in \mathcal{X}_U(= \mathcal{G}_U)$. Consider the push-out diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & V_0 & \longrightarrow & K' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E' & = & E' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From the middle row in the above diagram, we have that $\operatorname{add}_R U$-resol.dim$_R$ $(K') \leq n$. So the third column $0 \to M \to K' \to E' \to 0$ is as desired.
(2) $\Rightarrow$ (1) Let M be in mod R. Then by (2), there exists an exact sequence:

$$0 \to M \to K' \to E' \to 0$$

with $\operatorname{add}_R U$-resol.dim$_R(K') \leq n$ and $E' \in \mathcal{G}_U$. So there exists an exact sequence:

$$0 \to K \to U_0 \to K' \to 0$$

with $U_0 \in \operatorname{add}_R U$ and $\operatorname{add}_R U$-resol.dim$_R(K) \leq n - 1$. Consider the pull-back diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E & \longrightarrow & U_0 & \longrightarrow & E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & K' & \longrightarrow & E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since both U_0 and E' are in $\mathcal{G}_U$, it follows from Lemma 2.4(1) and the exactness of the middle row in the above diagram that E is also in $\mathcal{G}_U$. So the first column $0 \to K \to E \to M \to 0$ is as desired. □

Proposition 3.6 *If ${}^{\perp}_{R}U$ has the U-torsionless property, then the following statements are equivalent, for any non-negative integer n.*

(1) $\mathrm{l.id}_R(U) \leq n$.

(2) *For any $M \in \mathrm{mod}\, R$, there exists an exact sequence:*

$$0 \to K \to E \to M \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K) \leq n-1$ *and* $E \in \mathcal{G}_U$.

(3) *For any $M \in \mathrm{mod}\, R$, there exists an exact sequence:*

$$0 \to M \to K' \to E' \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K') \leq n$ *and* $E' \in \mathcal{G}_U$.

Proof (1) ⇔ (2) follows from Propositions 2.5(2), 3.1 and 3.4. (2) ⇔ (3) follows from Proposition 3.5. □

Remark 3.7 Note that the implication of (2) ⇒ (1) in Proposition 3.6 still holds without the assumption that ${}^{\perp}_{R}U$ has the U-torsionless property. In fact, condition (2) implies that $\text{G-dim}_U(M) \leq n$ and ${}^{\perp}_{R}U\text{-resol.dim}_R(M) \leq n$ for any $M \in \mathrm{mod}\, R$. So $\mathrm{l.id}_R(U) \leq n$ by Proposition 3.1. In addition, by Proposition 3.5, (2) ⇔ (3) is still true without an assumption as above. So the only place where the assumption that ${}^{\perp}_{R}U$ has the U-torsionless property in Proposition 3.6 is used is in showing (1) ⇒ (2).

Summarizing the result of this section, we obtain the following

Theorem 3.8 *If ${}^{\perp}_{R}U$ has the U-torsionless property, then the following statements are equivalent for any non-negative integer n.*

(1) $\mathrm{l.id}_R(U) \leq n$.

(2) ${}^{\perp}_{R}U\text{-resol.dim}_R(M) \leq n$ *for any* $M \in \mathrm{mod}\, R$.

(3) *For any $M \in \mathrm{mod}\, R$, there exists an exact sequence:*

$$0 \to K \to E \to M \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K) \leq n-1$ *and* $E \in {}^{\perp}_{R}U$.

(4) *For any $M \in \mathrm{mod}\, R$, there exists an exact sequence:*

$$0 \to M \to K' \to E' \to 0$$

with $\mathrm{add}_R\, U\text{-resol.dim}_R(K') \leq n$ *and* $E' \in {}^{\perp}_{R}U$.

Proof (1) ⇔ (2) follows from Proposition 3.1; (1) ⇔ (3) ⇔ (4) follow from Propositions 3.6 and 2.5. □

We have pointed out in Section 2 that if U_S has an ultimately closed injective resolution, especially, if $\mathrm{r.id}_S(U) < \infty$, then ${}^{\perp}_{R}U$ has the U-torsionless property. So Theorem 1.2 is a special case of Theorem 3.8.

Assume that $\mathcal{C} \supset \mathcal{D}$ are full subcategories of mod R and $C \in \mathcal{C}$, $D \in \mathcal{D}$. The homomorphism $D \to C$ is said to be a *$\mathcal{D}$-precover* of C if $\mathrm{Hom}_R(X, D) \to \mathrm{Hom}_R(X, C) \to 0$ is exact for any $X \in \mathcal{D}$. The subcategory $\mathcal{D}$ is said to be *precovering* in $\mathcal{C}$ if every module in $\mathcal{C}$ has a $\mathcal{D}$-precover. A *$\mathcal{D}$-precover* f of C is called *special* if it is an epimorphism and $\mathrm{Ext}^1_R(X, \mathrm{Ker}\, f) = 0$ for any $X \in \mathcal{D}$. Dually, the homomorphism $C \to D$ is said to be a *$\mathcal{D}$-preenvelope* of C if $\mathrm{Hom}_R(D, X) \to \mathrm{Hom}_R(C, X) \to 0$ is exact for any $X \in \mathcal{D}$. The subcategory $\mathcal{D}$ is said to be *preenveloping* in $\mathcal{C}$ if every module in $\mathcal{C}$ has a $\mathcal{D}$-preenvelope. A *$\mathcal{D}$-preenvelope* f of C is called *special* if it is a monomorphism and $\mathrm{Ext}^1_R(\mathrm{Coker}\, f, X) = 0$ for any $X \in \mathcal{D}$ (see [5]).

It is not difficult to verify that the exact sequences in Theorem 3.8 (or Theorem 1.2) (3) and (4) are a special ${}^{\perp}_{R}U$-precover and a special $\widehat{\mathrm{add}_R U}^n$-preenvelope of M, respectively, where $\widehat{\mathrm{add}_R U}^n$ denotes the full subcategory of mod R consisting of the modules with $\mathrm{add}_R U$-resolution dimension at most n. In addition, if ${}^{\perp}_{R}U$ has the U-torsionless property, then ${}^{\perp}_{R}U = \mathcal{G}_U = \mathcal{X}_U$ by Proposition 2.5. So Theorem 1.2 extends [1, Theorem 1.8], [3, Prop. 6.11 and 6.12] and [12, Prop. 3.2(a)].

4 Cotorsion Theory

We use $\widehat{\mathrm{add}_R U}$ to denote the full subcategory of mod R consisting of the modules with finite $\mathrm{add}_R U$-resolution dimension. In this section, as an application of Theorem 3.8, we prove that if both $\mathrm{l.id}_R(U)$ and $\mathrm{r.id}_S(U)$ are finite, then $(\widehat{\mathrm{add}_R U}, {}^{\perp}_{R}U)$ forms a hereditary and complete cotorsion theory.

Proposition 4.1 *The following statements are equivalent.*

(1) *For any $M \in$ mod R, there exists an exact sequence:*

$$0 \to K \to E \to M \to 0$$

with $K \in \widehat{\mathrm{add}_R U}$ and $E \in \mathcal{G}_U$.

(2) *For any $M \in$ mod R, there exists an exact sequence:*

$$0 \to M \to K' \to E' \to 0$$

with $K' \in \widehat{\mathrm{add}_R U}$ and $E' \in \mathcal{G}_U$.

If both $\mathrm{l.id}_R(U)$ and $\mathrm{r.id}_S(U)$ are finite, then the above equivalent conditions hold true.

Proof We omit the proof of (1) ⇔ (2), as it is similar to the proof of Proposition 3.5. The last assertion follows from Theorem 3.8. □

In the following, we assume that both $\mathrm{l.id}_R(U)$ and $\mathrm{r.id}_S(U)$ are finite. In this case, we have ${}^{\perp}_{R}U = \mathcal{G}_U = \mathcal{X}_U$, by Proposition 2.5(2).

Let $\mathcal{X}$ be a subcategory of mod R. We use ${}^{\perp_1}\mathcal{X}$ (resp. $\mathcal{X}^{\perp_1}$) to denote the subcategory $\{Y \in \text{mod } R \mid \mathrm{Ext}^1_R(Y, \mathcal{X}) = 0 \ (\text{resp. } \mathrm{Ext}^1_R(\mathcal{X}, Y) = 0)\}$, and use ${}^{\perp}\mathcal{X}$ (resp. $\mathcal{X}^{\perp}$) to denote the subcategory $\{Y \in \text{mod } R \mid \mathrm{Ext}^i_R(Y, \mathcal{X}) = 0 \ (\text{resp. } \mathrm{Ext}^i_R(\mathcal{X}, Y) = 0) \text{ for any } i \geq 1\}$.

Corollary 4.2

(1) $\widehat{\mathrm{add}_R U} \bigcap {}_R^{\perp}U = \mathrm{add}_R U$.
(2) ${}^{\perp_1}(\widehat{\mathrm{add}_R U}) = {}^{\perp}(\widehat{\mathrm{add}_R U}) = {}_R^{\perp}U$.
(3) $({}_R^{\perp}U)^{\perp_1} = ({}_R^{\perp}U)^{\perp} = \widehat{\mathrm{add}_R U}$.

Proof

(1) is a straightforward verification.
(2) It is easy to see that ${}^{\perp_1}(\widehat{\mathrm{add}_R U}) \supseteq {}^{\perp}(\widehat{\mathrm{add}_R U}) \supseteq {}_R^{\perp}U$. Now assume that $M \in {}^{\perp_1}(\widehat{\mathrm{add}_R U})$. By Proposition 4.1(1), there exists an exact sequence $0 \to K \to E \to M \to 0$ with $K \in \widehat{\mathrm{add}_R U}$ and $E \in {}_R^{\perp}U$. Then this exact sequence splits and $M \in {}_R^{\perp}U$. So we have ${}^{\perp_1}(\widehat{\mathrm{add}_R U}) \subseteq {}_R^{\perp}U$.
(3) It is easy to see that $({}_R^{\perp}U)^{\perp_1} \supseteq ({}_R^{\perp}U)^{\perp} \supseteq \widehat{\mathrm{add}_R U}$. Assume that $M \in ({}_R^{\perp}U)^{\perp_1}$. By Proposition 4.1(2), there exists an exact sequence $0 \to M \to K' \to E' \to 0$ with $K' \in \widehat{\mathrm{add}_R U}$ and $E' \in {}_R^{\perp}U$. Then this exact sequence splits and M is isomorphic to a direct summand of $K'(\in \widehat{\mathrm{add}_R U})$. Since $\widehat{\mathrm{add}_R U}$ is closed under direct summands by [1], $M \in \widehat{\mathrm{add}_R U}$ and $({}_R^{\perp}U)^{\perp_1} \subseteq \widehat{\mathrm{add}_R U}$. □

Recall that a pair $(\mathcal{C}, \mathcal{D})$ of subcategories of mod R is called a *cotorsion theory* if $\mathcal{C} = {}^{\perp_1}\mathcal{D}$ and $\mathcal{D} = \mathcal{C}^{\perp_1}$. The class $\mathcal{W}$=$\mathcal{C} \bigcap \mathcal{D}$ is called the *kernel* of the cotorsion theory. A cotorsion theory $(\mathcal{C}, \mathcal{D})$ is called *complete* if $\mathcal{C}$ is precovering and $\mathcal{D}$ is preenveloping in mod R, and called *hereditary* if $\mathrm{Ext}_R^i(C, D) = 0$ for any $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $i \geq 1$ (see [5, 6]).

Theorem 4.3 $\left(\widehat{\mathrm{add}_R U}, {}_R^{\perp}U\right)\left(= \left(\widehat{\mathrm{add}_R U}, {}^{\perp}(\widehat{\mathrm{add}_R U})\right) = \left(\left({}_R^{\perp}U\right)^{\perp}, {}_R^{\perp}U\right)\right)$ *is a hereditary and complete cotorsion theory with kernel* $\mathrm{add}_R U$.

Proof By Corollary 4.2, we have that $\left(\widehat{\mathrm{add}_R U}, {}_R^{\perp}U\right)\left(= \left(\widehat{\mathrm{add}_R U}, {}^{\perp}(\widehat{\mathrm{add}_R U})\right) = \left(\left({}_R^{\perp}U\right)^{\perp}, {}_R^{\perp}U\right)\right)$ is a hereditary cotorsion theory with kernel $\mathrm{add}_R U$.

On the other hand, it is easy to verify that the exact sequence in Proposition 4.1(1) is a special ${}_R^{\perp}U$-precover of M, and that in Proposition 4.1(2) is a special $\widehat{\mathrm{add}_R U}$-preenvelope of M. So the above cotorsion theory is complete. □

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