Algebras and Representation Theory (2020) 23:693–713
https://doi.org/10.1007/s10468-019-09861-z

The Extension Dimension of Abelian Categories

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Received: 28 February 2018 / Accepted: 23 January 2019 / Published online: 19 February 2019

Abstract

Let $\mathcal{A}$ be an abelian category having enough projective objects and enough injective objects. We prove that if $\mathcal{A}$ admits an additive generating object, then the extension dimension and the weak resolution dimension of $\mathcal{A}$ are identical, and they are at most the representation dimension of $\mathcal{A}$ minus two. By using it, for a right Morita ring Λ, we establish the relation between the extension dimension of the category mod Λ of finitely generated right Λ-modules and the representation dimension as well as the right global dimension of Λ. In particular, we give an upper bound for the extension dimension of mod Λ in terms of the projective dimension of certain class of simple right Λ-modules and the radical layer length of Λ. In addition, we investigate the behavior of the extension dimension under some ring extensions and recollements.

Keywords Extension dimension · Weak resolution dimension · Homological invariants · Radical layer length · Ring extensions · Recollements

1 Introduction

Following the work of Bondal and Van den Bergh [6], Rouquier introduced in [27] the dimension of triangulated categories, which is an invariant that measures how quickly the category can be built from one object. This dimension plays an important role in representation theory. For example, it can be used to compute the representation dimension of artin

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algebras [20, 26]. Let Λ be an artin algebra and mod Λ the category of finitely generated right Λ-modules. Rouquier proved that the dimension of the bounded derived category of mod Λ is at most $\mathrm{LL}(\Lambda) - 1$, where $\mathrm{LL}(\Lambda)$ is the Loewy length of Λ, and this dimension is at most the global dimension gl.dim Λ if Λ is a finite dimensional algebra over a perfect field [27, Proposition 7.37 and Remark 7.26].

As an analogue of the dimension of triangulated categories, the (extension) dimension dim $\mathcal{A}$ of an abelian category $\mathcal{A}$ was introduced by Beligiannis in [3], also see [7]. Let Λ be an artin algebra. Note that the representation dimension of Λ is at most two (that is, Λ is of finite representation type) if and only if dim mod $\Lambda = 0$ [3]. So, like the representation dimension of Λ, the extension dimension dim mod Λ is also an invariant that measures how far Λ is from having finite representation type. It was proved in [3] that dim mod $\Lambda \leq \mathrm{LL}(\Lambda) - 1$, which is a semi-counterpart of the above result of Rouquier. On the other hand, Iyama introduced in [17] the weak resolution dimension of Λ (see also [20]). It is easy to see that the weak resolution dimension of Λ is at most the representation dimension of Λ minus two. Based on these works, in this paper we will study further properties of the extension dimension of abelian categories, especially module categories. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we investigate the relationship between the extension dimension and some other homological invariants. Let $\mathcal{A}$ be an abelian category having enough projective objects and enough injective objects. We prove that if $\mathcal{A}$ admits an additive generating object, then dim $\mathcal{A}$ and the weak resolution dimension of $\mathcal{A}$ are identical, and they are at most the representation dimension of $\mathcal{A}$ minus two. As applications, we get that for a right Morita ring Λ, dim mod $\Lambda \leq$ r.gl.dim Λ (which is the other semi-counterpart of the result of Rouquier) and dim mod Λ is at most the representation dimension of Λ minus two, where r.gl.dim Λ is the right global dimension of Λ; and we also get that dim mod $\Lambda = n - 1$ for the exterior algebra Λ of k^n, where k is a field. In addition, we establish the relation between dim mod Λ and the finitistic dimension of Λ. Finally, we give an upper bound for dim mod Λ in terms of the projective dimension of certain class of simple right Λ-modules and the radical layer length of Λ, such that both gl.dim Λ and $\mathrm{LL}(\Lambda) - 1$ are properly special cases of this upper bound.

In Section 4, we study the behavior of the extension dimension under ring extensions. Let $\Gamma \supseteq \Lambda$ be artin algebras. We prove that dim mod $\Lambda =$ dim mod Γ if $\Gamma \geq \Lambda$ is an excellent extension, and that dim mod $\Lambda \leq$ dim mod $\Gamma + 2$ if $\Gamma \geq \Lambda$ is a left idealized extension. We also prove that if Λ and Γ are separably equivalent artin algebras, then dim mod $\Lambda =$ dim mod Γ.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories and

$$\mathcal{A} \;\underset{\xleftarrow{\;i^!\;}}{\overset{\xleftarrow{\;i^*\;}}{\xrightarrow{\;i_*\;}}}\; \mathcal{B} \;\underset{\xleftarrow{\;j_*\;}}{\overset{\xleftarrow{\;j_!\;}}{\xrightarrow{\;j^*\;}}}\; \mathcal{C}$$

a recollement. In Section 5, we prove that if either $i^!$ or i^* is exact, then $\max\{\dim \mathcal{A}, \dim \mathcal{C}\} \leq \dim \mathcal{B} \leq \dim \mathcal{A} + \dim \mathcal{C} + 1$.

2 Preliminaries

Let $\mathcal{A}$ be an abelian category. The designation *subcategory* will be used for full and additive subcategories of $\mathcal{A}$ which are closed under isomorphisms and the word *functor* will mean an additive functor between additive categories. For a subclass $\mathcal{U}$ of $\mathcal{A}$, we use add $\mathcal{U}$ to denote the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{U}$.

Let $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$ be subcategories of $\mathcal{A}$. Define

$$\mathcal{U}_1 \diamond \mathcal{U}_2 := \text{add}\,\{A \in \mathcal{A} \mid \text{there exists an sequence } 0 \to U_1 \to A \to U_2 \to 0 \text{ in } \mathcal{A} \text{ with } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}.$$

By [7, Proposition 2.2], the operator $\diamond$ is associative, that is, $(\mathcal{U}_1 \diamond \mathcal{U}_2) \diamond \mathcal{U}_3 = \mathcal{U}_1 \diamond (\mathcal{U}_2 \diamond \mathcal{U}_3)$. The category $\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n$ can be inductively described as follows

$$\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n := \text{add}\,\{A \in \mathcal{A} \mid \text{there exists an sequence } 0 \to U \to A \to V \to 0$$

$$\text{in } \mathcal{A} \text{ with } U \in \mathcal{U}_1 \text{ and } V \in \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n\}.$$

For a subclass $\mathcal{U}$ of $\mathcal{A}$, set $\langle \mathcal{U} \rangle_0 = 0$, $\langle \mathcal{U} \rangle_1 = \text{add}\,\mathcal{U}$, $\langle \mathcal{U} \rangle_n = \langle \mathcal{U} \rangle_1 \diamond \langle \mathcal{U} \rangle_{n-1}$ for any $n \geq 2$, and $\langle \mathcal{U} \rangle_\infty = \bigcup_{n \geq 0} \langle \mathcal{U} \rangle_n$ [3]. Note that $\langle \mathcal{U} \rangle_n = \langle \langle \mathcal{U} \rangle_1 \rangle_n$. If T is an object in $\mathcal{A}$ we write $\langle T \rangle_n$ instead of $\langle \{T\} \rangle_n$.

Throughout this paper, by convention, it is assumed that $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

Definition 2.1 [7] For any subcategory $\mathcal{X}$ of $\mathcal{A}$, define

$$\mathbf{size}_{\mathcal{A}} \mathcal{X} := \inf\{n \geq 0 \mid \mathcal{X} \subseteq \langle T \rangle_{n+1} \text{ with } T \in \mathcal{A}\},$$

$$\mathbf{rank}_{\mathcal{A}} \mathcal{X} := \inf\{n \geq 0 \mid \mathcal{X} = \langle T \rangle_{n+1} \text{ with } T \in \mathcal{A}\}.$$

The extension dimension $\dim \mathcal{A}$ of $\mathcal{A}$ is defined to be $\dim \mathcal{A} := \mathbf{rank}_{\mathcal{A}} \mathcal{A}$.

It is easy to see that $\dim \mathcal{A} = \mathbf{rank}_{\mathcal{A}} \mathcal{A} = \mathbf{size}_{\mathcal{A}} \mathcal{A}$. We also have the following easy and useful observations.

Proposition 2.2 *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be subcategories of $\mathcal{A}$ with $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Then we have*

(1) *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are subcategories of $\mathcal{A}$ with $\mathcal{V}_1 \subseteq \mathcal{V}_2$, then $\mathcal{U}_1 \diamond \mathcal{V}_1 \subseteq \mathcal{U}_2 \diamond \mathcal{V}_2$;*
(2) *$\langle \mathcal{U}_1 \rangle_n \subseteq \langle \mathcal{U}_2 \rangle_n$ for any $n \geq 1$;*
(3) *$\langle \mathcal{U}_1 \rangle_n \subseteq \langle \mathcal{U}_1 \rangle_{n+1}$ for any $n \geq 1$;*
(4) $\mathbf{size}_{\mathcal{A}} \mathcal{U}_1 \leq \mathbf{size}_{\mathcal{A}} \mathcal{U}_2$.

For two subcategories $\mathcal{U}, \mathcal{V}$ of $\mathcal{A}$, we set $\mathcal{U} \oplus \mathcal{V} := \{U \oplus V \mid U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$. Note that if $\mathcal{U}$ is closed under finite direct sums, then $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$.

Corollary 2.3 *For any $T_1, T_2 \in \mathcal{A}$ and $m, n \geq 1$, we have*

(1) $\langle T_1 \rangle_m \diamond \langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_{m+n}$;
(2) $\langle T_1 \rangle_m \oplus \langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_{\max\{m,n\}}$.

Proof Since $\langle T_1 \rangle_1 \subseteq \langle T_1 \oplus T_2 \rangle_1$, we have $\langle T_1 \rangle_m \subseteq \langle T_1 \oplus T_2 \rangle_m$ by Proposition 2.2(2). Similarly, $\langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_n$. Thus we have

(1) $\langle T_1 \rangle_m \diamond \langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_m \diamond \langle T_1 \oplus T_2 \rangle_n = \langle T_1 \oplus T_2 \rangle_{m+n}$.
(2) $\langle T_1 \rangle_m \oplus \langle T_2 \rangle_n \subseteq \langle T_1 \oplus T_2 \rangle_m \oplus \langle T_1 \oplus T_2 \rangle_n = \langle T_1 \oplus T_2 \rangle_{\max\{m,n\}}$ by Proposition 2.2(3).

□

We need the following fact.

Lemma 2.4 *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then $F(\langle T\rangle_n) \subseteq \langle F(T)\rangle_n$ for any $T \in \mathcal{A}$ and $n \geq 1$.*

Proof We proceed by induction on n. Let $X \in F(\langle T\rangle_1)$. Then $X = F(Y)$ for some $Y \in \langle T\rangle_1 (= \text{add } T)$. Since $Y \oplus Z \cong T^l$ for some $Z \in \mathcal{A}$ and $l \geq 1$, we have

$$X \oplus F(Z) = F(Y) \oplus F(Z) \cong F(Y \oplus Z) \cong F(T^l) \cong F(T)^l.$$

So $X \in \langle F(T)\rangle_1$ and $F(\langle T\rangle_1) \subseteq \langle F(T)\rangle_1$. The case for $n = 1$ is proved.

Now let $X \in F(\langle T\rangle_n)$ with $n \geq 2$. Then $X = F(Y)$ for some $Y \in \langle T\rangle_n$ and there exists an exact sequence

$$0 \longrightarrow Y_1 \longrightarrow Y \oplus Y' \longrightarrow Y_2 \longrightarrow 0$$

in $\mathcal{A}$ with $Y_1 \in \langle T\rangle_1$, $Y_2 \in \langle T\rangle_{n-1}$ and $Y' \in \langle T\rangle_n$. Since F is exact, we get the following exact sequence

$$0 \longrightarrow F(Y_1) \longrightarrow F(Y) \oplus F(Y') \longrightarrow F(Y_2) \longrightarrow 0.$$

By the induction hypothesis, $F(Y_1) \in F(\langle T\rangle_1) \subseteq \langle F(T)\rangle_1$ and $F(Y_2) \in F(\langle T\rangle_{n-1}) \subseteq \langle F(T)\rangle_{n-1}$. It follows that

$$X = F(Y) \in \langle F(Y_1)\rangle_1 \diamond \langle F(Y_2)\rangle_1 \subseteq \langle F(T)\rangle_1 \diamond \langle F(T)\rangle_{n-1} = \langle F(T)\rangle_n$$

and $F(\langle T\rangle_n) \subseteq \langle F(T)\rangle_n$. □

3 Relations with Some Homological Invariants

In this section, $\mathcal{A}$ is an abelian category.

Definition 3.1 (cf. [17, 20]) Let $M \in \mathcal{A}$. The weak M-resolution dimension of an object X in $\mathcal{A}$, denoted by M-w.resol.dim X, is defined as $\inf\{i \geq 0 \mid$ there exists an exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with all M_j in add $M\}$. The weak M-resolution dimension of $\mathcal{A}$, M-w.resol.dim $\mathcal{A}$, is defined as $\sup\{M\text{-w.resol.dim } X \mid X \in \mathcal{A}\}$. The weak resolution dimension of $\mathcal{A}$ is denoted by w.resol.dim $\mathcal{A}$ and defined as $\inf\{M\text{-w.resol.dim } \mathcal{A} \mid M \in \mathcal{A}\}$.

Let $X \in \mathcal{A}$. Suppose there exists a monomorphism $f : X \longrightarrow E$ in $\mathcal{A}$ such that E is an injective object in $\mathcal{A}$. Then we write $\Omega^{-1}(X) =: \text{Coker} f$ if f is right minimal, i.e. if f is an injective envelope of X. Dually, if $g : P \longrightarrow X$ is a right minimal epimorphism in $\mathcal{A}$ such that P is a projective object in $\mathcal{A}$, then we write $\Omega^1(X) =: \text{Ker} f$. Additionally, define Ω^0 as the identity functor in $\mathcal{A}$. Inductively, for any $n \geq 2$, we write $\Omega^n(X) := \Omega^1(\Omega^{n-1}(X))$ and $\Omega^{-n}(X) := \Omega^{-1}(\Omega^{-(n-1)}(X))$.

Lemma 3.2 [32, Lemma 3.3] *If $\mathcal{A}$ has enough projective objects and enough injective objects, then for any exact sequence*

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$$

in $\mathcal{A}$, we have the following exact sequences

$$0 \longrightarrow \Omega^1(X_3) \longrightarrow X_1 \oplus P \longrightarrow X_2 \longrightarrow 0,$$

$$0 \longrightarrow X_2 \longrightarrow E \oplus X_3 \longrightarrow \Omega^{-1}(X_1) \longrightarrow 0,$$

where P is projective and E is injective in $\mathcal{A}$.

Using Lemma 3.2, we get the following lemma, which is a dual of [7, Lemma 5.8].

Lemma 3.3 *If $\mathcal{A}$ has enough injective objects and*

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0,$$

is an exact sequence in $\mathcal{A}$ with $n \geq 0$, then

$$X \in \langle M_0 \rangle_1 \diamond \langle \Omega^{-1}(M_1) \rangle_1 \diamond \cdots \diamond \langle \Omega^{-n}(M_n) \rangle_1 \subseteq \langle \oplus_{i=0}^{n} \Omega^{-i}(M_i) \rangle_{n+1}.$$

Remark Note that if $\mathcal{A}$ has enough injectives and $X \in \langle Y_1 \rangle_1 \diamond \langle Y_2 \rangle_1$, then $\Omega^{-1}(X) \in \langle \Omega^{-1}(Y_1) \rangle_1 \diamond \langle \Omega^{-1}(Y_2) \rangle_1$. This fact is a sequence of the Horseshoe Lemma and is used to prove Lemma 3.3. This statement and its corresponding dual version will be throughout this paper.

3.1 Representation and Global Dimensions

For a subclass $\mathcal{X}$ of $\mathcal{A}$, recall that a sequence $\mathbb{S}$ in $\mathcal{A}$ is called $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact (resp. $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-exact) if $\mathrm{Hom}_{\mathcal{A}}(X, \mathbb{S})$ (resp. $\mathrm{Hom}_{\mathcal{A}}(\mathbb{S}, X)$) is exact for any $X \in \mathcal{X}$.

Definition 3.4 [2, 8, 26] The representation dimension rep.dim $\mathcal{A}$ of $\mathcal{A}$ is the smallest integer $i \geq 2$ such that there exists $M \in \mathcal{A}$ satisfying the property that for any $X \in \mathcal{A}$,

(1) there exists a $\mathrm{Hom}_{\mathcal{A}}(\mathrm{add}\, M, -)$-exact exact sequence

$$0 \longrightarrow M_{i-2} \longrightarrow M_{i-3} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with all M_j in add M; and

(2) there exists a $\mathrm{Hom}_{\mathcal{A}}(-, \mathrm{add}\, M)$-exact exact sequence

$$0 \longrightarrow X \longrightarrow N_0 \longrightarrow N_1 \longrightarrow \cdots \longrightarrow N_{i-2} \longrightarrow 0$$

in $\mathcal{A}$ with all N_j in add M.

We call $A \in \mathcal{A}$ an additive generating object if add A is a generator for $\mathcal{A}$. It is trivial that if $A \in \mathcal{A}$ is an additive generating object, then all projective objects in $\mathcal{A}$ are in add A.

Theorem 3.5 *Assume that $\mathcal{A}$ admits an additive generating object A. If $\mathcal{A}$ has enough projective objects and enough injective objects, then*

$$\text{w.resol.dim}\, \mathcal{A} = \dim \mathcal{A} \leq \text{rep.dim}\, \mathcal{A} - 2.$$

Proof It is trivial that w.resol.dim $\mathcal{A} \leq$ rep.dim $\mathcal{A} - 2$.

Assume that $\dim \mathcal{A} = n$ and $T \in \mathcal{A}$ such that $\mathcal{A} = \langle T \rangle_{n+1}$. Let $X \in \mathcal{A}$. Then we have an exact sequence

$$0 \longrightarrow X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow 0$$

in $\mathcal{A}$ with $X_1 \in \langle T \rangle_1$ and $X_2 \in \langle T \rangle_n$. Set $M := \oplus_{i=0}^{n} \Omega^i(T) \oplus A$. We will prove M-w.resol.dim $X \leq n$ by induction on n. The case for $n = 0$ is trivial. If $n = 1$, then T-w.resol.dim $X_2 = 0$ and M-w.resol.dim $\Omega^1(X_2) = 0$. By Lemma 3.2, we have an exact sequence

$$0 \longrightarrow \Omega^1(X_2) \longrightarrow X_1 \oplus P \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with P projective. So M-w.resol.dim $X \leq 1$. Now suppose $n \geq 2$. By the induction hypothesis, we have $(\oplus_{i=0}^{n-1}\Omega^i(T) \oplus A)$-w.resol.dim $X_2 \leq n-1$, hence M-w.resol.dim $\Omega^1(X_2) \leq n-1$. It follows that M-w.resol.dim $X \leq n$. Thus we have w.resol.dim $\mathcal{A} \leq n$.

Conversely, assume that w.resol.dim $\mathcal{A} = n$ and $T \in \mathcal{A}$ such that for any $X \in \mathcal{A}$, there exists an exact sequence

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with all M_i in add T. By Lemma 3.3, we have that $X \in \langle \oplus_{i=0}^n \Omega^{-i}(M_i)\rangle_{n+1} \subseteq \langle \oplus_{i=0}^n \Omega^{-i}(T)\rangle_{n+1}$ and $\mathcal{A} \subseteq \langle \oplus_{i=0}^n \Omega^{-i}(T)\rangle_{n+1}$. It follows that $\mathcal{A} = \langle \oplus_{i=0}^n \Omega^{-i}(T)\rangle_{n+1}$. Thus we have dim $\mathcal{A} \leq n$. □

For a ring Λ, we use mod Λ to denote the category of finitely generated right Λ-modules, and we write rep.dim Λ := rep.dim mod Λ if mod Λ is an abelian category. Recall from [11] that a ring Λ is called right Morita if there exist a ring Γ and a Morita duality from mod Λ to mod Γ^{op}. It is known that a ring Λ is right Morita if and only if it is right artinian and there exists a finitely generated injective cogenerator for the category of right Λ-modules [11, p.165]. The class of right Morita rings includes right pure-semisimple rings and artin algebras. For any right noetherian ring Λ, it is clear that w.resol.dim mod $\Lambda \leq$ r.gl.dim Λ, where r.gl.dim Λ is the right global dimension of Λ. So, as an immediate consequence of Theorem 3.5, we have the following

Corollary 3.6 *If Λ is a right Morita ring, then*

$$\text{w.resol.dim mod } \Lambda = \text{dim mod } \Lambda \leq \min\{\text{r.gl.dim } \Lambda, \text{rep.dim } \Lambda - 2\}.$$

Let Λ be an artin algebra. Recall that Λ is called n-Gorenstein if its left and right self-injective dimensions are at most n. Let $\mathcal{P}$ be the subcategory of mod Λ consisting of projective modules. A module $G \in$ mod Λ is called Gorenstein projective if there exists a $\text{Hom}_\Lambda(-, \mathcal{P})$-exact exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in mod Λ with all P_i, P^i in $\mathcal{P}$ such that $G \cong \text{Im}(P_0 \to P^0)$. Recall from [4] that Λ is said to be of finite Cohen-Macaulay type (finite CM-type for short) if there are only finitely many non-isomorphic indecomposable Gorenstein projective modules in mod Λ.

Corollary 3.7 *If Λ is an n-Gorenstein artin algebra of finite CM-type, then* dim mod $\Lambda \leq n$.

Proof Let $M \in$ mod Λ. Because Λ is an n-Gorenstein artin algebra, we have an exact sequence

$$0 \to H_n \to \cdots \to H_1 \to H_0 \to M \to 0$$

in mod Λ with all H_i Gorenstein projective by [12, Theorem 1.4]. Because Λ is of finite CM-type, we may assume that $\{G_1, \cdots, G_n\}$ is the set of non-isomorphic indecomposable Gorenstein projective modules in mod Λ. Set $G := \oplus_{i=0}^n G_i$. Then G-w.resol.dim $M \leq n$ and w.resol.dim mod $\Lambda \leq n$. It follows from Theorem 3.5 that dim mod $\Lambda \leq n$. □

For small dim mod Λ, we have the following

Corollary 3.8 *Let Λ be an artin algebra. Then we have*

(1) [3, Example 1.6(i)] rep.dim $\Lambda \leq 2$ *if and only if* dim mod $\Lambda = 0$;
(2) *if* rep.dim $\Lambda = 3$, *then* dim mod $\Lambda = 1$.

Proof (1) It is trivial by Corollary 3.6.
(2) Let rep.dim $\Lambda = 3$. Then dim mod $\Lambda \geq 1$ by (1); and dim mod $\Lambda \leq$ rep.dim $\Lambda - 2 = 1$ by Corollary 3.6. The assertion follows. □

For a field k and $n \geq 1$, $\wedge(k^n)$ is the exterior algebra of k^n.

Corollary 3.9 dim mod $\wedge(k^n) = n - 1$ *for any* $n \geq 1$.

Proof By [17, Thoerem 4.6], we have w.resol.dim mod $\wedge(k^n) = n - 1$. It follows from Corollary 3.6 that dim mod $\wedge(k^n) = n - 1$. □

3.2 Finitistic Dimension

From Now on, Λ is an Artin Algebra. For a module M in mod Λ, pd$M$ is the projective dimension of M. Set $\mathcal{P}^{<\infty} := \{M \in \text{mod}\,\Lambda \mid \text{pd}M < \infty\}$. Recall that the finitistic dimension fin.dim Λ of Λ is defined as $\sup\{\text{pd}M \mid M \in \mathcal{P}^{<\infty}\}$. It is an unsolved conjecture that fin.dim $\Lambda < \infty$ for every artin algebra Λ. Igusa-Todorov introduced in [16] a powerful function ψ from mod Λ to non-negative integers to study the finiteness of fin.dim Λ. The following lemma gives some useful properties of the Igusa-Todorov function ψ.

Lemma 3.10 [16, Lemma 0.3 and Theorem 0.4]

(1) *For any* $X, Y \in$ mod Λ, $\psi(X) \leq \psi(Y)$ *if* $\langle X\rangle_1 \subseteq \langle Y\rangle_1$;
(2) *if* $0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$ *is an exact sequence in* mod Λ *with* pd$X_3 < \infty$, *then* pd$X_3 \leq \psi(X_1 \oplus X_2) + 1$.

For any subcategory $\mathcal{X}$ of mod Λ and $n \geq 0$, set $\Omega^n(\mathcal{X}) := \{\Omega^n(M) | M \in \mathcal{X}\}$; in particular, $\Omega^0(\mathcal{X}) = \mathcal{X}$.

Proposition 3.11 *The following statements are equivalent.*

(1) fin.dim $\Lambda < \infty$;
(2) *there exists some* $n \geq 0$ *such that* $\mathbf{size}_{\text{mod}\,\Lambda}\Omega^n(\mathcal{P}^{<\infty}) \leq 1$.

Proof (1) ⇒ (2) If fin.dim $\Lambda = m < \infty$, then $\Omega^m(\mathcal{P}^{<\infty}) \subseteq \langle \Lambda\rangle_1$ and $\mathbf{size}_{\text{mod}\,\Lambda}\Omega^m(\mathcal{P}^{<\infty}) = 0$.
(2) ⇒ (1) Let $\mathbf{size}_{\text{mod}\,\Lambda}\Omega^n(\mathcal{P}^{<\infty}) \leq 1$ with $n \geq 0$. Then $\Omega^n(\mathcal{P}^{<\infty}) \subseteq \langle T\rangle_2$ for some $T \in$ mod Λ. Let $X \in \mathcal{P}^{<\infty}$. Then there exists an exact sequence

$$0 \longrightarrow T_1 \longrightarrow \Omega^n(X) \longrightarrow T_2 \longrightarrow 0$$

in mod Λ with $T_1, T_2 \in \langle T \rangle_1$. By Lemma 3.2, we obtain the following exact sequence

$$0 \longrightarrow \Omega^1(T_2) \longrightarrow T_1 \oplus P \longrightarrow \Omega^n(X) \longrightarrow 0$$

with $P \in \langle \Lambda \rangle_1$. Then we have

$$\begin{aligned} \mathrm{pd}X &\leq \mathrm{pd}\Omega^n(X) + n \\ &\leq \psi(\Omega^1(T_2) \oplus T_1 \oplus P) + 1 + n (\text{by Lemma 3.10(2)}) \\ &\leq \psi(\Omega^1(T) \oplus T \oplus \Lambda) + 1 + n, (\text{by Lemma 3.10(1)}) \end{aligned}$$

which implies fin.dim $\Lambda \leq \psi(\Omega^1(T) \oplus T \oplus \Lambda) + 1 + n$.

□

By Proposition 3.11, we have the following

Corollary 3.12 *If* dim mod $\Lambda \leq 1$, *then* fin.dim $\Lambda < \infty$.

3.3 Igusa-Todorov Algebras

Definition 3.13 ([28] and [14, Lemma 3.6]) For an integer $n \geq 0$, Λ is called (n-)Igusa-Todorov if there exists $V \in$ mod Λ such that for any $M \in$ mod Λ, there exists an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \Omega^n(M) \oplus P \longrightarrow 0$$

in mod Λ with $V_1, V_0 \in$ add V and P projective; equivalently, there exists a module $V \in$ mod Λ such that for any $M \in$ mod Λ, there exists an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \Omega^n(M) \longrightarrow 0$$

in mod Λ with $V_1, V_0 \in$ add V.

The class of Igusa-Todorov algebras includes algebras with representation dimension at most 3, algebras with radical cube zero, monomial algebras, left serial algebras and syzygy finite algebras [28].

Theorem 3.14 *For any $n \geq 0$, the following statements are equivalent.*

(1) *Λ is n-Igusa-Todorov;*
(2) $\mathbf{size}_{\mathrm{mod}\,\Lambda}\Omega^n(\mathrm{mod}\,\Lambda) \leq 1$.

Proof (1) ⇒ (2) Let Λ be n-Igusa-Todorov and $X \in \Omega^n(\mathrm{mod}\,\Lambda)$. Then there exists $V \in$ mod Λ such that the following sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow X \longrightarrow 0,$$

in mod Λ with $V_1, V_0 \in$ add V is exact. By Lemma 3.3, Proposition 2.2(1) and Corollary 2.3(1), we have

$$X \in \langle V_0 \rangle_1 \diamond \langle \Omega^{-1}(V_1) \rangle_1 \subseteq \langle V \rangle_1 \diamond \langle \Omega^{-1}(V) \rangle_1 \subseteq \langle V \oplus \Omega^{-1}(V) \rangle_2.$$

And then $\mathbf{size}_{\mathrm{mod}\,\Lambda}\Omega^n(\mathrm{mod}\,\Lambda) \leq 1$ by Definition 2.1.

(2) ⇒ (1) Let $\mathbf{size}_{\text{mod}\,\Lambda}\,\Omega^n(\text{mod}\,\Lambda) \leq 1$ and $X \in \text{mod}\,\Lambda$. Then there exists $T \in \text{mod}\,\Lambda$ such that the following sequence

$$0 \longrightarrow T_1 \longrightarrow \Omega^n(X) \longrightarrow T_2 \longrightarrow 0,$$

in mod Λ with $T_1, T_2 \in \langle T \rangle_1$ is exact. By Lemma 3.2, we obtain the following exact sequence

$$0 \longrightarrow \Omega^1(T_2) \longrightarrow T_1 \oplus P \longrightarrow \Omega^n(X) \longrightarrow 0$$

in mod Λ with P projective. Since both $\Omega^1(T_2)$ and $T_1 \oplus P$ are in add $(\Omega^1(T) \oplus T \oplus \Lambda)$, we have that Λ is n-Igusa-Todorov.

□

The first assertion in the following proposition means that dim mod Λ is an invariant for measuring how far Λ is from being 0-Igusa-Todorov.

Proposition 3.15
(1) *Λ is 0-Igusa-Todorov if and only if* dim mod $\Lambda \leq 1$*;*
(2) *if Λ is n-Igusa-Todorov, then* dim mod $\Lambda \leq n + 1$.

Proof (1) It is trivial by Theorem 3.14.
(2) Let Λ be n-Igusa-Todorov and $X \in \Omega^n(\text{mod}\,\Lambda)$. Then there exists $V \in \text{mod}\,\Lambda$ such that the following sequence

$$0 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in mod Λ with $V_2, V_1 \in$ add V and all P_i projective. Thus w.resol.dim mod $\Lambda \leq n+1$, and therefore dim mod $\Lambda \leq n + 1$ by Theorem 3.5.

□

Moreover, we have the following

Corollary 3.16 dim mod $\Lambda \leq 2$ *if Λ is in one class of the following algebras.*

(1) *monomial algebras;*
(2) *left serial algebras;*
(3) $\text{rad}^{2n+1}\Lambda = 0$ *and* $\Lambda/\text{rad}^n\Lambda$ *is representation finite;*
(4) *2-syzygy finite algebras.*

Proof By [28, Corollaries 2.6, 3.5 and Proposition 2.5], these four classes of algebras are 1-Igusa-Todorov. So the assertions follow from Proposition 3.15. □

3.4 $t_{\mathcal{S}}$-Radical Layer Length

We recall some notions from [15]. Let $\mathcal{C}$ be a length-category, that is, $\mathcal{C}$ is an abelian, skeletally small category and every object of $\mathcal{C}$ has a finite composition series. We denote by $\text{End}_{\mathbb{Z}}(\mathcal{C})$ the category of all additive functors from $\mathcal{C}$ to $\mathcal{C}$, and denote by rad the Jacobson radical lying in $\text{End}_{\mathbb{Z}}(\mathcal{C})$. Let $\alpha, \beta \in \text{End}_{\mathbb{Z}}(\mathcal{C})$ and α be a subfunctor of β, we have the quotient functor $\beta/\alpha \in \text{End}_{\mathbb{Z}}(\mathcal{C})$ which is defined as follows.

(1) $(\beta/\alpha)(M) := \beta(M)/\alpha(M)$ for any $M \in \mathcal{C}$; and
(2) $(\beta/\alpha)(f)$ is the induced quotient morphism: for any $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \alpha(M) & \longrightarrow & \beta(M) & \longrightarrow & \beta(M)/\alpha(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha(f)} & & \downarrow{\scriptstyle \beta(f)} & & \downarrow{\scriptstyle (\beta/\alpha)(f)} & & \\ 0 & \longrightarrow & \alpha(N) & \longrightarrow & \beta(N) & \longrightarrow & \beta(N)/\alpha(N) & \longrightarrow & 0. \end{array}$$

For any $\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, set the α-radical functor $F_\alpha := \mathrm{rad} \circ \alpha$. We define the following two classes

$$\mathcal{F}_\alpha := \{M \in \mathcal{C} \mid \alpha(M) = 0\}, \quad \mathcal{T}_\alpha = \{M \in \mathcal{C} \mid \alpha(M) \cong M\}.$$

Definition 3.17 [15, Definition 3.1] For any $\alpha, \beta \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, the (α, β)-layer length of $M \in \mathcal{C}$, denoted by $\ell\ell_\alpha^\beta(M)$, is defined as $\ell\ell_\alpha^\beta(M) = \inf\{i \geq 0 \mid \alpha \circ \beta^i(M) = 0\}$. Moreover, $\ell\ell_\alpha^\beta$ goes from $\mathcal{C}$ to $\mathbb{N} \cup \{+\infty\}$. And the α-radical layer length $\ell\ell^\alpha := \ell\ell_\alpha^{F_\alpha}$.

Lemma 3.18 [35, Lemma 2.6] *Let $\alpha, \beta \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$. For any $M \in \mathcal{C}$, if $\ell\ell_\alpha^\beta(M) = n$, then $\ell\ell_\alpha^\beta(M) = \ell\ell_\alpha^\beta(\beta^i(M)) + i$ for any $0 \leq i \leq n$; in particular, if $\ell\ell^\alpha(M) = n$, then $\ell\ell^\alpha(F_\alpha^n(M)) = 0$.*

Recall that a torsion pair (or torsion theory) for $\mathcal{C}$ is a pair of classes $(\mathcal{T}, \mathcal{F})$ of objects in $\mathcal{C}$ satisfying the following conditions.

(1) $\mathrm{Hom}_{\mathcal{C}}(M, N) = 0$ for any $M \in \mathcal{T}$ and $N \in \mathcal{F}$;
(2) an object $X \in \mathcal{C}$ is in $\mathcal{T}$ if $\mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{F}} = 0$;
(3) an object $Y \in \mathcal{C}$ is in $\mathcal{F}$ if $\mathrm{Hom}_{\mathcal{C}}(-, Y)|_{\mathcal{T}} = 0$.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair for $\mathcal{C}$. Recall that $t := \mathrm{Trace}_{\mathcal{T}}$ is the so called torsion radical attached to $(\mathcal{T}, \mathcal{F})$. Then $t(M) := \Sigma\{\mathrm{Im} f \mid f \in \mathrm{Hom}_{\mathcal{C}}(T, M)$ with $T \in \mathcal{T}\}$ is the largest subobject of M lying in $\mathcal{T}$.

For a subfunctor $\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ of the identity functor $1_{\mathcal{C}}$ of $\mathcal{C}$, we write $q_\alpha := 1_{\mathcal{C}}/\alpha$. The functor q_α lies in $\mathrm{End}_{\mathbb{Z}}(\mathcal{C})$. In this section, Λ is an artin algebra. Then mod Λ is a length-category. We use rad Λ to denote the Jacobson radical of Λ. For a module M in mod Λ, we use top M to denote the top of M. Set $\mathrm{pd}M = -1$ if $M = 0$. For a subclass $\mathcal{B}$ of mod Λ, the projective dimension $\mathrm{pd}\mathcal{B}$ of $\mathcal{B}$ is defined as

$$\mathrm{pd}\mathcal{B} = \begin{cases} \sup\{\mathrm{pd}M \mid M \in \mathcal{B}\}, & \text{if } \mathcal{B} \neq \emptyset; \\ -1, & \text{if } \mathcal{B} = \emptyset. \end{cases}$$

We use $\mathcal{S}^{<\infty}$ to denote the set of the simple modules in mod Λ with finite projective dimension.

From now on, assume that $\mathcal{S}$ is a subset of $\mathcal{S}^{<\infty}$ and $\mathcal{S}'$ is the set of all the others simple modules in mod Λ. We write $\mathfrak{F}(\mathcal{S}) := \{M \in \text{mod } \Lambda \mid$ there exists a finite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that each quotient M_i/M_{i-1} is isomorphic to some module in $\mathcal{S}\}$. By [15, Lemma 5.7 and Proposition 5.9], we have that $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}))$ is a torsion pair, where

$$\mathcal{T}_{\mathcal{S}} = \{M \in \text{mod } \Lambda \mid \text{top } M \in \text{add } \mathcal{S}'\}.$$

We denote the torsion radical $t_{\mathcal{S}} = \mathrm{Trace}_{\mathcal{T}_{\mathcal{S}}}$. Then $t_{\mathcal{S}}(M) \in \mathcal{T}_{\mathcal{S}}$ and $q_{t_{\mathcal{S}}}(M) \in \mathfrak{F}(\mathcal{S})$ for any $M \in \mathrm{mod}\ \Lambda$. By [15, Proposition 5.3], we have

$$\mathfrak{F}(\mathcal{S}) = \{M \in \mathrm{mod}\ \Lambda \mid t_{\mathcal{S}}(M) = 0\},$$

$$\mathcal{T}_{\mathcal{S}} = \{M \in \mathrm{mod}\ \Lambda \mid t_{\mathcal{S}}(M) = M\}.$$

Theorem 3.19 *Let $\mathcal{S}$ be a subset of the set $\mathcal{S}^{<\infty}$ of all pairwise non-isomorphism simple Λ-modules with finite projective dimension. Then* $\dim \mathrm{mod}\ \Lambda \leq \mathrm{pd}\ \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda)$.

Proof Let $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = n$ and $\mathrm{pd}\mathcal{S} = \alpha$.

If $n = 0$, that is, $t_{\mathcal{S}}(\Lambda) = 0$, then $\Lambda \in \mathfrak{F}(\mathcal{S})$, which implies that $\mathcal{S}$ is the set of all simple modules. Thus $\mathcal{S} = \mathcal{S}^{<\infty}$ and $\mathrm{gl.dim}\ \Lambda = \alpha$. So the assertion follows from Corollary 3.6.

Now let $n \geq 1$ and $M \in \mathrm{mod}\ \Lambda$. Consider the following exact sequence

$$0 \longrightarrow \Omega^{\alpha+2}(M) \longrightarrow L_{\alpha+1} \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

in $\mathrm{mod}\ \Lambda$ with all L_i projective. By Lemma 3.3, we have

$$\begin{aligned} M \in &\ \langle\ L_0\rangle_1 \diamond \langle\Omega^{-1}(L_1)\rangle_1 \diamond \cdots \diamond \langle\Omega^{-\alpha-1}(L_{\alpha+1})\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+2}(M))\rangle_1 \\ \subseteq &\ \langle\ \oplus_{i=0}^{-\alpha-1}\Omega^i(\Lambda)\rangle_{\alpha+2} \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+2}(M))\rangle_1. \end{aligned}$$

We have the following exact sequences

$$\begin{aligned} 0 \to t_{\mathcal{S}}(M) \ &\to\ M \to q_{t_{\mathcal{S}}}(M) \to 0, \\ 0 \to t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M))) \ &\to\ \Omega^1(t_{\mathcal{S}}(M)) \to q_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to 0, \\ 0 \to F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \ &\to\ t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M))) \to \mathrm{top}\, t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M))) \to 0, \\ 0 \to t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \ &\to\ F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to q_{t_{\mathcal{S}}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to 0, \\ 0 \to F^2_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \ &\to\ t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to \mathrm{top}\, t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to 0, \\ &\cdots\cdots\cdots\cdots \\ 0 \to t_{\mathcal{S}}F^{n-2}_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \ &\to\ F^{n-2}_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to q_{t_{\mathcal{S}}}F^{n-2}_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to 0, \\ 0 \to F^{n-1}_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \ &\to\ t_{\mathcal{S}}F^{n-2}_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to \mathrm{top}\, t_{\mathcal{S}}F^{n-2}_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \to 0. \end{aligned}$$

By [15, Lemma 6.3], we have $\ell\ell^{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))) \leq \ell\ell^{t_{\mathcal{S}}}(\Lambda) - 1 = n - 1$. It follows from Lemma 3.18 that $\ell\ell^{t_{\mathcal{S}}}(F^{n-1}_{t_{\mathcal{S}}}\Omega^1(t_{\mathcal{S}}(M))) = 0$, that is, $t_{\mathcal{S}}(F^{n-1}_{t_{\mathcal{S}}}\Omega^1(t_{\mathcal{S}}(M))) = 0$. Then by [15, Proposition 5.3], we have $\mathrm{pd}F^{n-1}_{t_{\mathcal{S}}}\Omega^1(t_{\mathcal{S}}(M)) \leq \alpha$.

We have the following

$$\Omega^{\alpha+2}(M) \cong \Omega^{\alpha+2}(t_{\mathcal{S}}(M)),$$

$$\Omega^{\alpha+2}(t_{\mathcal{S}}(M)) = \Omega^{\alpha+1}(\Omega^1(t_{\mathcal{S}}(M))) \cong \Omega^{\alpha+1}(t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M)))),$$

$$\begin{aligned} 0 \ \to\ &\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))) \to \Omega^{\alpha+1}(t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M)))) \oplus P_1 \\ \to\ &\Omega^{\alpha+1}(\mathrm{top}\, t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M)))) \to 0,\ \text{(exact)} \end{aligned}$$

$$\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))) \cong \Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))),$$

$$\begin{aligned} 0 \ \to\ &\Omega^{\alpha+1}(F^2_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))) \to \Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))) \oplus P_2 \\ \to\ &\Omega^{\alpha+1}(\mathrm{top}\, t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))) \to 0,\ \text{(exact)} \end{aligned}$$

$$\cdots\cdots\cdots\cdots$$

$$\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M)))) \cong \Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M)))),$$

$$\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M)))) \oplus P_{n-1} \cong \Omega^{\alpha+1}(\operatorname{top} t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M)))),$$

where all P_i are projective in mod Λ; we also have the following

$$\begin{aligned}\Omega^{-\alpha-2}(\Omega^{\alpha+2}(M)) &\cong \Omega^{-\alpha-2}(\Omega^{\alpha+2}(t_{\mathcal{S}}(M))) = \Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Omega^1(t_{\mathcal{S}}(M))))\\ &\cong \Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M))))),\end{aligned}$$

$$0 \to \Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))))) \to \Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M)))))\oplus\Omega^{-\alpha-2}(P_1)\oplus E_1$$

$$\to \Omega^{-\alpha-2}(\Omega^{\alpha+1}(\operatorname{top} t_{\mathcal{S}}(\Omega^1(t_{\mathcal{S}}(M))))) \to 0, \ \text{(exact)}$$

$$\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))))) \cong \Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))))),$$

$$0 \to \Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}^2(\Omega^1(t_{\mathcal{S}}(M))))) \to \Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M)))))\oplus\Omega^{-\alpha-2}(P_2)\oplus E_2$$

$$\to \Omega^{-\alpha-2}(\Omega^{\alpha+1}(\operatorname{top} t_{\mathcal{S}}F_{t_{\mathcal{S}}}(\Omega^1(t_{\mathcal{S}}(M))))) \to 0, \ \text{(exact)}$$

$$\cdots\cdots\cdots\cdots$$

$$\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M))))) \cong \Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M))))),$$

$$\Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M)))))\oplus\Omega^{-\alpha-2}(P_{n-1}) \cong \Omega^{-\alpha-2}(\Omega^{\alpha+1}(\operatorname{top} t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}(\Omega^1(t_{\mathcal{S}}(M))))),$$

where all E_i are injective in mod Λ. So

$$\begin{aligned}
&\Omega^{-\alpha-2}(\Omega^{\alpha+2}(M))\\
\cong\ &\Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}\Omega^1(t_{\mathcal{S}}(M))))\\
\in\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\operatorname{top} t_{\mathcal{S}}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1\\
\subseteq\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1\\
=\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1\\
\subseteq\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}^2\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1\\
&\vdots\\
\subseteq\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}^{n-2}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \underbrace{\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \cdots \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1}_{n-2}\\
=\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(F_{t_{\mathcal{S}}}^{n-2}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
\subseteq\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}\Omega^1(t_{\mathcal{S}}(M))) \oplus P_{n-1})\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
=\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\operatorname{top} t_{\mathcal{S}}F_{t_{\mathcal{S}}}^{n-2}\Omega^1(t_{\mathcal{S}}(M))))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
\subseteq\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_1 \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-2}\\
=\ &\langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-1},
\end{aligned}$$

and hence

$$\begin{aligned}
M \in \langle\ &\oplus_{i=0}^{-\alpha-1}\Omega^i(\Lambda)\rangle_{\alpha+2} \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+2}(M))\rangle_1\\
&\subseteq \langle\oplus_{i=0}^{-\alpha-1}\Omega^i(\Lambda)\rangle_{\alpha+2} \diamond \langle\Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{n-1}\\
&\subseteq \langle(\oplus_{i=0}^{-\alpha-1}\Omega^i(\Lambda)) \oplus \Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{\alpha+1+n}. \quad \text{(by Corollary 2.3(1))}
\end{aligned}$$

It follows that

$$\operatorname{mod} \Lambda = \langle (\oplus_{i=0}^{-\alpha-1} \Omega^i(\Lambda)) \oplus \Omega^{-\alpha-2}(\Omega^{\alpha+1}(\Lambda/\operatorname{rad}\Lambda))\rangle_{\alpha+1+n}$$

and $\dim \Lambda \leq \alpha + n$. □

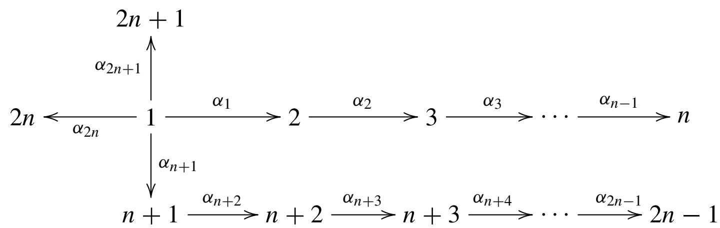
As an application of Theorem 3.19, we have the following

Corollary 3.20
(1) [3, Example 1.6(ii)] $\dim \operatorname{mod} \Lambda \leq LL(\Lambda) - 1$*;*
(2) *(cf. Corollary 3.6 and* [17, 4.5.1(3)]*)* $\dim \operatorname{mod} \Lambda \leq \operatorname{gl.dim} \Lambda$.

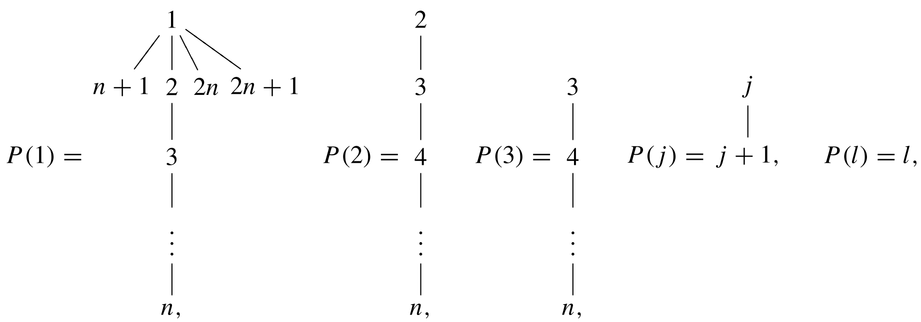
Proof (1) Let $\mathcal{S} = \emptyset$. Then $\operatorname{pd}\mathcal{S} = -1$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (\operatorname{mod} \Lambda, 0)$. By [15, Propposition 5.9(a)], we have $t_{\mathcal{S}}(\Lambda) = \Lambda$ and $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \operatorname{LL}(\Lambda)$. It follows from Theorem 3.19 that $\dim \operatorname{mod} \Lambda \leq \operatorname{LL}(\Lambda) - 1$.
(2) Let $\mathcal{S} = \mathcal{S}^{<\infty} = \{\text{all simple modules in } \operatorname{mod} \Lambda\}$. Then $\operatorname{pd}\mathcal{S} = \operatorname{gl.dim} \Lambda$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (0, \operatorname{mod} \Lambda)$. By [15, Propposition 5.3], we have $t_{\mathcal{S}}(\Lambda) = 0$ and $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = 0$. It follows from Theorem 3.19 that $\dim \operatorname{mod} \Lambda \leq \operatorname{gl.dim} \Lambda$. □

By choosing some suitable $\mathcal{S}$ and applying Theorem 3.19, we may obtain more precise upper bounds for $\dim \operatorname{mod} \Lambda$ than that in Corollary 3.20.

Example 3.21 Consider the bound quiver algebra $\Lambda = kQ/I$, where k is a field and Q is given by



and I is generated by $\{\alpha_i\alpha_{i+1} \mid n+1 \leq i \leq 2n-2\}$ with $n \geq 5$. Then the indecomposable projective Λ-modules are



where $n + 1 \leq j \leq 2n - 2$, $2n - 1 \leq l \leq 2n + 1$ and $P(i + 1) = \operatorname{rad} P(i)$ for any $2 \leq i \leq n - 1$.

We have

$$\mathrm{pd}S(i)=\begin{cases} n-1, & \text{if } i=1;\\ 1, & \text{if } 2\le i\le n-1;\\ 0, & \text{if } i=n,2n,2n+1;\\ 2n-1-i, & \text{if } n+1\le i\le 2n-1.\end{cases}$$

So $\mathcal{S}^{<\infty}=\{\text{all simple modules in mod }\Lambda\}$. Let $\mathcal{S}:=\{S(i)\mid 2\le i\le n\}(\subseteq\mathcal{S}^{<\infty})$ and $\mathcal{S}'$ be all the others simple modules in mod Λ. Then $\mathrm{pd}\mathcal{S}=1$ and $\mathcal{S}'=\{S(i)\mid i=1 \text{ or } n+1\le i\le 2n+1\}$. Because $\Lambda=\oplus_{i=1}^{2n+1}P(i)$, we have

$$\ell\ell^{t_{\mathcal{S}}}(\Lambda)=\max\{\ell\ell^{t_{\mathcal{S}}}(P(i))\mid 1\le i\le 2n+1\}$$

by [15, Lemma 3.4(a)].

In order to compute $\ell\ell^{t_{\mathcal{S}}}(P(1))$, we need to find the least non-negative integer i such that $t_{\mathcal{S}}F^{i}_{t_{\mathcal{S}}}(P(1))=0$. Since top $P(1)=S(1)\in\mathrm{add}\,\mathcal{S}'$, we have $t_{\mathcal{S}}(P(1))=P(1)$ by [15, Proposition 5.9(a)]. Thus

$$F_{t_{\mathcal{S}}}(P(1))=\mathrm{rad}\,t_{\mathcal{S}}(P(1))=\mathrm{rad}(P(1))=S(n+1)\oplus P(2)\oplus S(2n)\oplus S(2n+1).$$

Since top $S(n+1)=S(n+1)\in\mathrm{add}\,\mathcal{S}'$, we have $t_{\mathcal{S}}(S(n+1))=S(n+1)$ by [15, Proposition 5.9(a)]. Similarly, $t_{\mathcal{S}}(S(2n))=S(2n)$ and $t_{\mathcal{S}}(S(2n+1))=S(2n+1)$. Since $P(2)\in\mathfrak{F}(\mathcal{S})$, we have $t_{\mathcal{S}}(P(2))=0$ by [15, Proposition 5.3]. So

$$t_{\mathcal{S}}F_{t_{\mathcal{S}}}(P(1))=t_{\mathcal{S}}(S(n+1)\oplus P(2)\oplus S(2n)\oplus S(2n+1))=S(n+1)\oplus S(2n)\oplus S(2n+1).$$

It follows that

$$F^{2}_{t_{\mathcal{S}}}(P(1))=\mathrm{rad}\,t_{\mathcal{S}}F_{t_{\mathcal{S}}}(P(1))=\mathrm{rad}(S(n+1)\oplus S(2n)\oplus S(2n+1))=0$$

and $t_{\mathcal{S}}F^{2}_{t_{\mathcal{S}}}(P(1))=0$, which implies $\ell\ell^{t_{\mathcal{S}}}(P(1))=2$. Similarly, we have

$$\ell\ell^{t_{\mathcal{S}}}(P(i))=\begin{cases} 0, & \text{if } 2\le i\le n;\\ 2, & \text{if } n+1\le i\le 2n-2;\\ 1, & \text{if } 2n-1\le i\le 2n+1.\end{cases}$$

Consequently, we conclude that $\ell\ell^{t_{\mathcal{S}}}(\Lambda)=\max\{\ell\ell^{t_{\mathcal{S}}}(P(i))\mid 1\le i\le 2n+1\}=2$.

(1) Because $\mathrm{LL}(\Lambda)=n$ and gl.dim $\Lambda=n-1$, we have

$$\dim\mathrm{mod}\,\Lambda\le\min\{\mathrm{gl.dim}\,\Lambda,\mathrm{LL}(\Lambda)-1\}=n-1$$

by Corollary 3.20.

(2) By Theorem 3.19, we have

$$\dim\mathrm{mod}\,\Lambda\le\mathrm{pd}\mathcal{S}+\ell\ell^{t_{\mathcal{S}}}(\Lambda)=1+2=3.$$

The upper bound here is better than that in (1) since $n\ge 5$.

4 Ring Extensions

Let Λ be a subring of a ring Γ such that Λ and Γ have the same identity. Then A is called a ring extension of Λ, and denoted by $\Gamma\ge\Lambda$.

Definition 4.1 A ring extension $\Gamma\ge\Lambda$ is called

(1) [13] a weak excellent extension if

(1.1) Γ is Λ-projective [21]; that is, for a submodule N_Γ of M_Γ, if N_Λ is a direct summand of M_Λ, denoted by $N_\Lambda \mid M_\Lambda$, then $N_\Gamma \mid M_\Gamma$;

(1.2) Γ is a finite extension of Λ; that is, there exists a finite set $\{\gamma_1, \cdots, \gamma_n\}$ in Γ such that $\Gamma = \sum_{i=1}^{n} \gamma_i \Lambda$;

(1.3) Γ_Λ is flat and ${}_\Lambda\Gamma$ is projective;

(2) [5, 21] an excellent extension if it is a weak excellent extension and Γ_Λ and ${}_\Lambda\Gamma$ are free with a common basis $\{\gamma_1, \cdots, \gamma_n\}$, such that $\Lambda\gamma_i = \gamma_i\Lambda$ for any $1 \leq i \leq n$.

(3) [29] a left idealized extension if rad Λ is a left ideal of Γ.

We have the following

Theorem 4.2 *Let $\Gamma \supseteq \Lambda$ be artin algebras. Then we have*

(1) $\dim \operatorname{mod} \Lambda \geq \dim \operatorname{mod} \Gamma$ *if* $\Gamma \geq \Lambda$ *is a weak excellent extension, and* $\dim \operatorname{mod} \Lambda = \dim \operatorname{mod} \Gamma$ *if* $\Gamma \geq \Lambda$ *is an excellent extension;*

(2) $\dim \operatorname{mod} \Lambda \leq \dim \operatorname{mod} \Gamma + 2$ *if* $\Gamma \geq \Lambda$ *is a left idealized extension.*

Proof (1) Let $\Gamma \geq \Lambda$ be a weak excellent extension and $\dim \operatorname{mod} \Lambda = n$ and $T \in \operatorname{mod} \Lambda$ such that $\operatorname{mod} \Lambda = \langle T \rangle_{n+1}$. Let $X \in \operatorname{mod} \Gamma \subseteq \operatorname{mod} \Lambda$. Since ${}_\Lambda\Gamma$ is projective, $-\otimes_\Lambda \Gamma$ is exact. So we have $X \otimes_\Lambda \Gamma \in \langle (T \otimes_\Lambda \Gamma)_\Gamma \rangle_{n+1}$ by Lemma 2.4. Since $X_\Gamma \mid (X \otimes_\Lambda \Gamma)_\Gamma$ by [34, Lemma 1.1], we have $X_\Gamma \in \langle (T \otimes_\Lambda \Gamma)_\Gamma \rangle_{n+1}$. Thus $\operatorname{mod} \Gamma = \langle (T \otimes_\Lambda \Gamma)_\Gamma \rangle_{n+1}$ and $\dim \operatorname{mod} \Gamma \leq n$.

Now let $\Gamma \geq \Lambda$ be an excellent extension and $\dim \operatorname{mod} \Gamma = n$ and $S \in \operatorname{mod} \Gamma \subseteq \operatorname{mod} \Lambda$ such that $\operatorname{mod} \Gamma = \langle S \rangle_{n+1}$. Let $X_\Lambda \in \operatorname{mod} \Lambda$. Then there exists an exact sequence

$$0 \longrightarrow X_1 \longrightarrow X \otimes_\Lambda \Gamma \longrightarrow X_2 \longrightarrow 0$$

in $\operatorname{mod} \Gamma$ with $X_1 \in \langle S_\Gamma \rangle_1$ and $X_2 \in \langle S_\Gamma \rangle_n$. Note that it is also an exact sequence in $\operatorname{mod} \Lambda$. So $(X \otimes_\Lambda \Gamma)_\Lambda \in \langle S_\Lambda \rangle_{n+1}$. Since $X_\Lambda \mid (X \otimes_\Lambda \Gamma)_\Lambda$, we have $X_\Lambda \in \langle S_\Lambda \rangle_{n+1}$. Thus $\operatorname{mod} \Lambda = \langle S_\Lambda \rangle_{n+1}$ and $\dim \operatorname{mod} \Lambda \leq n$.

(2) Let $\dim \operatorname{mod} \Gamma = n$. Then w.resol.dim $\operatorname{mod} \Gamma = n$ by Theorem 3.5. Let $X \in \operatorname{mod} \Lambda$. Since $\Omega_\Lambda^2(X)$ can be viewed as an Γ-module by [30, Lemma 0.2], there exists $V \in \operatorname{mod} \Gamma \subseteq \operatorname{mod} \Lambda$ such that there is an exact sequence

$$0 \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \Omega_\Lambda^2(X) \longrightarrow 0$$

in $\operatorname{mod} \Gamma$ with all V_i in add V_Γ. It is also an exact sequence in $\operatorname{mod} \Lambda$. So $(V_\Lambda \oplus \Lambda)$-w.resol.dim $\operatorname{mod} \Lambda \leq n+2$ and w.resol.dim $\operatorname{mod} \Lambda \leq n+2$. Thus $\dim \operatorname{mod} \Lambda \leq n+2$ by Theorem 3.5. □

In the following, we list some examples of (weak) excellent extensions, in which Theorem 4.2(1) may be applied.

Example 4.3 [5, 14, 21, 33]

(1) For a ring Λ, $M_n(\Lambda)$ (the matrix ring of Λ of degree n) is an excellent extension of Λ.

(2) Let Λ be a ring and G a finite group. If $|G|^{-1} \in \Lambda$, then the skew group ring $\Lambda * G$ is an excellent extension of Λ.
(3) Let Λ be a finite-dimensional algebra over a field k, and let F be a finite separable field extension of k. Then $\Lambda \otimes_k F$ is an excellent extension of Λ.
(4) Let k be a field, and let G be a group and H a normal subgroup of G. If $[G : H]$ is finite and is not zero in k, then kG is an excellent extension of kH.
(5) Let k be a field of charactertistic p, and let G a finite group and H a normal subgroup of G. If H contains a Sylow p-subgroup of G, then kG is an excellent extension of kH.
(6) Let k be a field and G a finite group. If G acts on k (as field automorphisms) with kernel H. Then the skew group ring $k * G$ is an excellent extension of the group ring kH, and the center $Z(kH)$ of kH is an excellent extension of the center $Z(k * G)$ of $k * G$.
(7) Let H be a finite-dimensional semisimple Hopf algebra over a field k and Λ a twisted H-module algebra. Then for any cocycle $\sigma \in \mathrm{Hom}_k(H \otimes H, \Lambda)$, the crossed product algebra $\Lambda\#_\sigma H$ is a weak excellent extension of Λ, but not an excellent extension of Λ in general.
(8) Recall from [25] that a ring Λ is called a right S-ring if any flat module in mod Λ is projective. The class of right S-rings includes semiperfect rings, commutative semilocal rings, subrings of right noetherian rings, subrings of right S-rings, right Ore domains, right nonsingular ring of finite right Goldie dimension, endomorphism rings of right artinian modules and rings with right Krull dimension [9, 25]. Let $\Gamma \geq \Lambda$ be an excellent extension with Λ a right S-ring. If Γ has two ideals I and J such that $\Lambda \cap I = 0$ and $\Gamma = I \oplus J$, then the canonical embedding $\Lambda \hookrightarrow \Gamma/I$ is a weak excellent extension; and it is not an excellent extension if J_Λ is not free.

We recall from [19] the separable equivalence of artin algebras, which includes the derived equivalence of self-injective algebras, Morita equivalence and stable equivalence (of Morita type) [19, 22].

Definition 4.4 [19] Two artin algebras Λ and Γ are called separably equivalent if there exist ${}_\Gamma M_\Lambda$ and ${}_\Lambda N_\Gamma$ such that

(1) M and N are both finitely generated projective as one sided modules;
(2) $M \otimes_\Lambda N \cong \Gamma \oplus X$ as a (Γ, Γ)-bimodule for some ${}_\Gamma X_\Gamma$;
(3) $N \otimes_\Gamma M \cong \Lambda \oplus Y$ as a (Λ, Λ)-bimodule for some ${}_\Lambda Y_\Lambda$.

We have the following

Theorem 4.5 *Let Λ and Γ be artin algebras. If they are separably equivalent, then* $\dim \operatorname{mod} \Lambda = \dim \operatorname{mod} \Gamma$.

Proof Let M and N be as in Definition 4.4. Let $\dim \operatorname{mod} \Gamma = n$. Then there exists $T_\Gamma \in \operatorname{mod} \Gamma$ such that $\operatorname{mod} \Gamma = \langle T_\Gamma \rangle_{n+1}$. Let $L_\Lambda \in \operatorname{mod} \Lambda$. Then $L \otimes_\Lambda N_\Gamma \in \operatorname{mod} \Gamma = \langle T_\Gamma \rangle_{n+1}$. Since ${}_\Gamma M$ is projective in Γ-mod , we have that the functor $- \otimes_\Gamma M : \operatorname{mod} \Gamma \longrightarrow \operatorname{mod} \Lambda$

is exact. By Lemma 2.4, we have $(L \otimes_\Lambda N) \otimes_\Gamma M \in \langle T \otimes_\Gamma M_\Lambda \rangle_{n+1}$. By Definition 4.4(3), there exists a (Λ, Λ)-bimodule Y such that

$$\begin{aligned} L \oplus (L \otimes_\Lambda Y) &\cong (L \otimes_\Lambda \Lambda) \oplus (L \otimes_\Lambda Y) \\ &\cong L \otimes_\Lambda (\Lambda \oplus Y) \\ &\cong L \otimes_\Lambda (N \otimes_\Gamma M) \\ &\cong (L \otimes_\Lambda N) \otimes_\Gamma M \\ &\in \langle T \otimes_\Gamma M_\Lambda \rangle_{n+1}, \end{aligned}$$

and so $L_\Lambda \in \langle T \otimes_\Gamma M_\Lambda \rangle_{n+1}$. It follows that $\operatorname{mod} \Lambda = \langle T \otimes_\Gamma M_\Lambda \rangle_{n+1}$ and $\dim \operatorname{mod} \Lambda \leq n = \dim \operatorname{mod} \Gamma$. Symmetrically, we have $\dim \operatorname{mod} \Gamma \leq \dim \operatorname{mod} \Lambda$. □

As a consequence of Theorem 4.5, we have the following

Corollary 4.6 *Let* Λ, Γ *and* Δ *be finite dimensional algebras over a field* k. *If* Λ *is separably equivalent to* Γ, *then* $\dim \operatorname{mod} \Lambda \otimes_k \Delta = \dim \operatorname{mod} \Gamma \otimes_k \Delta$.

Proof If Λ is separably equivalent to Γ, then $\Lambda \otimes_k \Delta$ is separably equivalent to $\Gamma \otimes_k \Delta$ by [22, p.227, Proposition]. The assertion follows from Theorem 4.5. □

5 Recollements

We recall the notion of recollements of abelian categories.

Definition 5.1 [10] A recollement, denoted by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, of abelian categories is a diagram

$$\mathcal{A} \underset{\overset{\longleftarrow i^! \longrightarrow}{}}{\overset{\overset{\longleftarrow i^* \longrightarrow}{}}{\xrightarrow{\quad i_* \quad}}} \mathcal{B} \underset{\overset{\longleftarrow j_* \longrightarrow}{}}{\overset{\overset{\longleftarrow j_! \longrightarrow}{}}{\xrightarrow{\quad j^* \quad}}} \mathcal{C}$$

of abelian categories and additive functors such that

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs;
(2) i_*, $j_!$ and j_* are fully faithful;
(3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

We list some properties of recollements of abelian categories (see [10, 23, 24]), which will be useful later.

Lemma 5.2 *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *be a recollement of abelian categories. Then we have*

(1) $i^* j_! = 0 = i^! j_*$;
(2) *the functors* i_* *and* j^* *are exact,* $i^!$ *and* j_* *are left exact, and* i^* *and* $j_!$ *are right exact;*
(3) *the functors* i^*, $i^!$ *and* j^* *are dense;*
(4) *all the natural transformations* $i^* i_* \longrightarrow 1_\mathcal{A}$, $1_\mathcal{A} \longrightarrow i^! i_*$, $1_\mathcal{C} \longrightarrow j^* j_!$ *and* $j^* j_* \longrightarrow 1_\mathcal{C}$ *are natural isomorphisms;*
(5) *for any object* $B \in \mathcal{B}$,

    (a) *if* i^* *is exact, there is an exact sequence*

$$0 \longrightarrow j_! j^*(B) \overset{\epsilon_B}{\longrightarrow} B \longrightarrow i_* i^*(B) \longrightarrow 0$$

(b) *if* $i^!$ *is exact, there is an exact sequence*

$$0 \longrightarrow i_*i^!(B) \longrightarrow B\eta_B \overset{-}{\longrightarrow} j_*j^*(B) \longrightarrow 0 .$$

Lemma 5.3 *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *be a recollement of abelian categories. Then we have*

(1) *If* i^* *is exact, then* $j_!$ *is exact;*
(2) *If* $i^!$ *ie exact, then* j_* *is exact.*

Proof (1) Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be an exact sequence in $\mathcal{C}$. Since $j_!$ is right exact by Lemma 5.2(2), we get an exact sequence

$$0 \longrightarrow C \longrightarrow j_!(X) \longrightarrow j_!(Y) \longrightarrow j_!(Z) \longrightarrow 0 \tag{5.1}$$

in $\mathcal{B}$. Notice that j^* is exact and $j^*j_! \cong 1_{\mathcal{C}}$ by Lemma 5.2(2)(4), so $j^*(C) = 0$. Since Im $i_* = \mathrm{Ker}\, j^*$, there exists $C' \in \mathcal{A}$ such that $C \cong i_*(C')$. Since i^* is exact and $i^*j_! = 0$ by Lemma 5.2(2)(1), applying the functor i^* to the exact sequence (5.1) yields $i^*(C) = 0$. It follow that $C' \cong i^*i_*(C') \cong i^*(C) = 0$ and $C = 0$. Thus $j_!$ is exact.

(2) It is dual to (1). □

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of additive categories. Recall from [31] that F is called quasi-dense if for any $D \in \mathcal{D}$, there exists $C \in \mathcal{C}$ such that D is isomorphic to a direct summand of $F(C)$. Obviously, any dense functor is quasi-dense.

Lemma 5.4 *Let* $F : \mathcal{A} \to \mathcal{B}$ *be an exact functor of abelian categories, and let* $\mathcal{A}_1$ *and* $\mathcal{B}_1$ *be subcategories of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively. If the restriction functor* $F : \mathcal{A}_1 \to \mathcal{B}_1$ *is quasi-dense, then* $\mathbf{size}_{\mathcal{A}}\mathcal{A}_1 \geq \mathbf{size}_{\mathcal{B}}\mathcal{B}_1$*; in particular,* $\dim \mathcal{A} \geq \dim \mathcal{B}$.

Proof Suppose $\mathbf{size}_{\mathcal{A}}\mathcal{A}_1 = n$, that is, $\mathcal{A}_1 \subseteq \langle T \rangle_{n+1}$ for some $T \in \mathcal{A}$. Let $X \in \mathcal{B}_1$. Since F is quasi-dense, we have $X \oplus X_1 \cong F(Y)$ for some $Y \in \mathcal{A}_1$ and $X_1 \in \mathcal{B}_1$. It follows from Lemma 2.4 that $X \oplus X_1 \in F(\mathcal{A}_1) \subseteq F(\langle T \rangle_{n+1}) \subseteq \langle F(T) \rangle_{n+1}$. So $X \in \langle F(T) \rangle_{n+1}$ and $\mathcal{B}_1 \subseteq \langle F(T) \rangle_{n+1}$, which implies $\mathbf{size}_{\mathcal{B}}\mathcal{B}_1 \leq n$. □

Let Λ be an artin algebra and e an idempotent of Λ. Then $(\mathrm{mod}\, \Lambda/e\Lambda e, \mathrm{mod}\, \Lambda, \mathrm{mod}\, e\Lambda e)$ is a recollement by [23, Example 2.7]. So $\dim \mathrm{mod}\, \Lambda \geq \dim \mathrm{mod}\, e\Lambda e$ by Lemma 5.4.

Theorem 5.5 *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *be a recollement of abelian categories. If either* $i^!$ *or* i^* *is exact, then*

$$\max\{\dim \mathcal{A}, \dim \mathcal{C}\} \leq \dim \mathcal{B} \leq \dim \mathcal{A} + \dim \mathcal{C} + 1.$$

Proof Let $i^!$ be exact. Since $i^!$ and j^* are exact and dense Lemma 5.2(2)(3), it follows from Lemma 5.4 that $\max\{\dim A, \dim C\} \leq \dim B$.

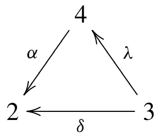
Let $\dim \mathcal{A} = n$ and $\dim \mathcal{C} = m$. Then there exist $X \in \mathcal{A}$ and $Y \in \mathcal{C}$ such that $\mathcal{A} = \langle X \rangle_{n+1}$ and $\mathcal{C} = \langle Y \rangle_{m+1}$. Let $M \in \mathcal{B}$. Since $i^!$ is exact by assumption, we have an exact sequence

$$0 \longrightarrow i_* i^!(M) \longrightarrow M \longrightarrow j_* j^*(M) \longrightarrow 0$$

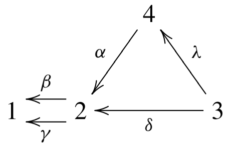
in $\mathcal{B}$. Note that i_* and j_* are exact by Lemmas 5.2(2) and 5.3(2). Since $i^!(M) \in \mathcal{A} = \langle X \rangle_{n+1}$ and $j^*(M) \in \mathcal{C} = \langle Y \rangle_{m+1}$, we have $i_* i^!(M) \in \langle i_*(X) \rangle_{n+1}$ and $j_* j^*(M) \in \langle j_*(Y) \rangle_{m+1}$ by Lemma 2.4. Thus $M \in \langle i_* X \rangle_{n+1} \diamond \langle j_* Y \rangle_{m+1} \subseteq \langle i_* X \oplus j_* Y \rangle_{n+m+2}$ by Corollary 2.3(1), and therefore $\dim \mathcal{B} \leq n + m + 1$.

For the case that i^* is exact, the argument is similar. □

Let $\Lambda, \Lambda', \Lambda''$ be artin algebras and $(\operatorname{mod} \Lambda', \operatorname{mod} \Lambda, \operatorname{mod} \Lambda'')$ be a recollement. If $\dim \operatorname{mod} \Lambda = 0$, then $\dim \operatorname{mod} \Lambda' = 0 = \dim \operatorname{mod} \Lambda''$; that is, Λ is of finite representation type implies that so are Λ' and Λ'' [23]. Conversely, if $\dim \operatorname{mod} \Lambda' = 0 = \dim \operatorname{mod} \Lambda''$, then $\dim \operatorname{mod} \Lambda = 0$ does not hold true in general. For example, let Λ' be the finite dimensional algebra given by the quiver $\cdot$ (a unique vertex without arrows) and Λ'' the finite dimensional algebra given by the quiver



with relation $\lambda\alpha = 0$. Then both Λ' and Λ'' are of finite representation type, and so $\dim \operatorname{mod} \Lambda' = 0 = \dim \operatorname{mod} \Lambda''$ by [3, Example 1.6(i)] (see Corollary 3.8(1)). Define the triangular matrix algebra $\Lambda := \begin{pmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{pmatrix}$, where $M \cong \Lambda' \oplus \Lambda'$, the right Λ''-module structure on M is induced by the unique algebra surjective homomorphism $\Lambda'' \xrightarrow{\phi} \Lambda'$ satisfying $\phi(e_2) = e_1$, $\phi(e_3) = 0$ and $\phi(e_4) = 0$. Then Λ is the finite dimensional algebra given by the quiver



with relations $\delta\gamma = \delta\beta = \lambda\alpha = \alpha\beta = \alpha\gamma = 0$. By [23, Example 2.12], we have that

$$\operatorname{mod} \Lambda' \;\underset{\overset{\longleftarrow i^! \longrightarrow}{}}{\overset{\overset{\longleftarrow i^* \longrightarrow}{}}{\xrightarrow{\;\; i_* \;\;}}}\; \operatorname{mod} \Lambda \;\underset{\overset{\longleftarrow j_* \longrightarrow}{}}{\overset{\overset{\longleftarrow j_! \longrightarrow}{}}{\xrightarrow{\;\; j^* \;\;}}}\; \operatorname{mod} \Lambda''$$

is a recollement, where

$$i^*\left(\begin{pmatrix} X \\ Y \end{pmatrix}_f\right) = \operatorname{Coker} f, \quad i_*(X) = \begin{pmatrix} X \\ 0 \end{pmatrix}, \quad i^!\left(\begin{pmatrix} X \\ Y \end{pmatrix}_f\right) = X,$$
$$j_!(Y) = \begin{pmatrix} Y \\ Y \end{pmatrix}_1, \quad j^*\left(\begin{pmatrix} X \\ Y \end{pmatrix}_f\right) = Y, \quad j_*(Y) = \begin{pmatrix} 0 \\ Y \end{pmatrix}.$$

Because $i^!$ is exact by [18, Lemma 3.2(a)], $\dim \operatorname{mod} \Lambda \leq 1$ by Theorem 5.5. Notice that Λ is of infinite representation type and $\operatorname{rep.dim} \Lambda = 3$ by [1, Example 5.9], so $\dim \operatorname{mod} \Lambda = 1$ by Corollary 3.8(2).

Acknowledgements This work was partially supported by NSFC (No. 11571164), a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions, Postgraduate Research and Practice Innovation Program of Jiangsu Province (Grant No. KYCX17_0019). The authors thank the referee for very useful and detailed suggestions.

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