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Auslander-type conditions and weakly Gorenstein algebras

Zhaoyong Huang

School of Mathematics, Nanjing University, Nanjing, Jiangsu Province, P.R. China

Correspondence
Zhaoyong Huang, School of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P.R. China.
Email: huangzy@nju.edu.cn

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Abstract

Let R be an Artin algebra. Under certain Auslander-type conditions, we give some equivalent characterizations of (weakly) Gorenstein algebras in terms of the properties of Gorenstein projective modules and modules satisfying Auslander-type conditions. As applications, we provide some support for several homological conjectures. In particular, we prove that if R is left quasi-Auslander, then R is Gorenstein if and only if it is (left and) right weakly Gorenstein; and that if R satisfies the Auslander condition, then R is Gorenstein if and only if it is left or right weakly Gorenstein. This is a reduction of an Auslander–Reiten's conjecture, which states that R is Gorenstein if R satisfies the Auslander condition.

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1 | INTRODUCTION

A left and right Noetherian ring is called *Iwanaga–Gorenstein* (*Gorenstein* for short) if its left and right self-injective dimensions are finite. The fundamental theorem in [6] states that a commutative Noetherian ring R is Gorenstein if and only if the flat dimension of the ith term in a minimal injective coresolution of R as an R-module is at most $i-1$ for any $i \geqslant 1$. In the noncommutative case, Auslander proved that the latter condition is left–right symmetric [9, Theorem 3.7]; in this case, R is said to satisfy the *Auslander condition*. Thus, the above result in [6] can be restated as follows: A commutative Noetherian ring is Gorenstein if and only if it satisfies the Auslander condition. Based on it, Auslander and Reiten [2] conjectured that an Artin algebra satisfying the Auslander condition is Gorenstein. We call this conjecture *ARC* for short. It is situated between the well-known Nakayama conjecture and the generalized Nakayama conjecture [2, p.2]. All these conjectures remain still open.

As a generalization of the notion of the Auslander condition, Huang and Iyama [16] introduced the notion of Auslander-type conditions of rings as follows. For any $m \geqslant 0$, a left and right Noetherian ring is said to be $G_\infty(m)$ if for any finitely generated left R-module M and $i \geqslant 1$, it holds that $\mathrm{Ext}_{R^{op}}^{0\leqslant j\leqslant i-1}(X,R)=0$ for any right R-submodule X of $\mathrm{Ext}_R^{i+m}(M,R)$; equivalently, if the flat dimension of the ith term in a minimal injective coresolution of R_R is at most $i+m-1$ for any $i \geqslant 1$ [16, p. 99]. Noncommutative rings satisfying Auslander-type conditions are analogues of commutative Gorenstein rings. Such rings play a crucial role in homological algebra, representation theory of algebras, and noncommutative algebraic geometry, see [2, 3, 8, 9, 11, 12, 16, 18, 19, 21, 24, 26] and references therein. Recently, we introduced modules satisfying Auslander-type condition $G_\infty(m)$ for any $m \geqslant 0$ [15], see Definition 2.3 below.

As a generalization of the notion of Gorenstein algebras, Ringel and Zhang [23] introduced that of weakly Gorenstein algebras. Marczinzik [20] posed the following question: Is a left weakly Gorenstein Artin algebra also right weakly Gorenstein? For the sake of convenience, we state this question as the following conjecture.

Weakly-Gorenstein symmetry conjecture (**WGSC**): An Artin algebra is left weakly Gorenstein if and only if it is right weakly Gorenstein.

It is related to the following famous conjecture.

Gorenstein symmetry conjecture (**GSC**): For an Artin algebra, its left self-injective dimension is finite if and only if so is its right self-injective dimension.

Note that for a left and right Noetherian ring, its left and right self-injective dimensions coincide if both of them are finite [28, Lemma A]. Thus, an equivalent version of *GSC* is that for an Artin algebra, its left and right self-injective dimensions coincide.

It was proved that *WGSC* implies *GSC* [23, p. 33], and that *GSC* holds true for Artin algebras satisfying the Auslander condition [2, Corollary 5.5(b)]. We proved that an Artin algebra satisfying the Auslander condition is Gorenstein if and only if the subcategory of finitely generated modules satisfying the Auslander condition is contravariantly finite [15, Theorem 5.8]. The aim of this paper is to give some equivalent characterizations of (weakly) Gorenstein algebras under certain Auslander-type conditions, and then provide some support for these conjectures mentioned above.

The paper is organized as follows. In Section 2, we give some terminology and preliminary results. Let R be an arbitrary ring. We use $\mathcal{GP}(\mathrm{Mod}\,R)$ to denote the category of Gorenstein projective left R-modules. For any $m \geqslant 0$, we use $\mathcal{GP}(\mathrm{Mod}\,R)^{\leqslant m}$ to denote the category of left R-modules with Gorenstein projective dimension at most m, and use $\mathcal{G}_\infty(m)$ to denote the category of left R-modules being $G_\infty(m)$.

In Section 3, R is an arbitrary ring. We prove that any module in $\mathcal{G}_\infty(m)$ is isomorphic to a kernel (resp. a cokernel) of a homomorphism from a module with finite flat dimension to certain syzygy module, and as a consequence, we get that if a left R-module M satisfies the Auslander condition (i.e., $M \in \mathcal{G}_\infty(0)$), then M is an ∞-flat syzygy module, and the converse holds true if ${}_RR$ satisfies the Auslander condition (Theorem 3.3). For any $m, s \geqslant 0$, we prove that $\mathcal{GP}(\mathrm{Mod}\,R)=\mathcal{G}_\infty(m)$ if and only if $\mathcal{GP}(\mathrm{Mod}\,R)^{\leqslant s}=\mathcal{G}_\infty(m+s)$ (Proposition 3.5). We also prove that if R is a Gorenstein ring, then any module in $\mathcal{G}_\infty(m)$ has Gorenstein projective dimension at most m (Theorem 3.6).

In Section 4, R is an Artin algebra. We get some equivalent characterizations for ${}_RR \in \mathcal{G}_\infty(m)$ and R being Gorenstein as follows. The case for $m=0$ in the following result except the statement (2) has been obtained in [27, Corollary 3.5], which is the Gorenstein version of [15, Theorem 5.9].

Theorem 1.1 (Theorem 4.6). *Let $m \geqslant 0$. Then, the following statements are equivalent.*

(1) ${}_RR \in G_\infty(m)$ *and R is Gorenstein.*
(2) ${}_RR \in G_\infty(m)$ *and the left self-injective dimension of R is finite.*
(3) $\mathcal{GP}(\text{Mod}\, R) \subseteq \mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\text{Mod}\, R)^{\leqslant m}$.
(4) $\mathcal{GP}(\text{Mod}\, R)^{\leqslant s} \subseteq \mathcal{G}_\infty(m+s) \subseteq \mathcal{GP}(\text{Mod}\, R)^{\leqslant m+s}$ *for any* $s \geqslant 0$.
$(i)_f$ *The finitely generated version of* (i) *with* $i = 3, 4$.

Under certain Auslander-type conditions, we get some equivalent characterizations of (weakly) Gorenstein algebras.

Theorem 1.2 (Theorem 4.9). *If* ${}_RR \in \mathcal{G}_\infty(m)$ *and* $R_R \in \mathcal{G}_\infty(m')^{op}$ *with* $m, m' \geqslant 0$, *then the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) *R is left and right weakly Gorenstein.*
(3) *The left self-injective dimension of R is finite.*
(4) *R is left weakly Gorenstein.*
(5) $\mathcal{GP}(\text{Mod}\, R)$ *coincides with the left orthogonal category of projective left R-modules.*
$(i)^{op}$ *The opposite version of* (i) *with* $3 \leqslant i \leqslant 5$.

Furthermore, we consider algebras satisfying small Auslander-type conditions. We prove that if R is left quasi-Auslander (i.e., ${}_RR \in \mathcal{G}_\infty(1)$), then R is Gorenstein if and only if the left or right self-injective dimension of R is finite, and if and only if R is (left and) right weakly Gorenstein (Theorem 4.10). Moreover, we get some equivalent characterizations of Auslander–Gorenstein algebras (Theorem 4.11), which yields that if R satisfies the Auslander condition (i.e., ${}_RR \in \mathcal{G}_\infty(0)$), then R is Gorenstein if and only if R is left or right weakly Gorenstein (Corollary 4.12).

Consequently, we conclude that

(1) Over an Artin algebra R satisfying ${}_RR \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{op}$ with $m, m' \geqslant 0$, both *WGSC* and *GSC* hold true (Theorem 1.2).
(2) Over a left quasi-Auslander Artin algebra, *GSC* holds true, but we do not know whether *WGSC* holds true or not (Theorem 4.10).
(3) Assume that an Artin algebra R satisfies the Auslander condition (equivalently, ${}_RR \in \mathcal{G}_\infty(0)$ and $R_R \in \mathcal{G}_\infty(0)^{op}$). Then, both *WGSC* and *GSC* hold true for R by putting $m = m' = 0$ in Theorem 1.2. Note that *GSC* holds true for an Artin algebra R satisfying the Auslander condition has been obtained in [2, Corollary 5.5(b)]. Moreover, we have that R is Gorenstein if and only if it is left or right weakly Gorenstein (Corollary 4.12). This is a reduction of *ARC*, since Gorenstein algebras are left and right weakly Gorenstein, but the converse does not hold true in general [20, 22, 23].

2 | PRELIMINARIES

Throughout this paper, all rings are associative rings with unit and all modules are unital. For a ring R, we use $\text{Mod}\, R$ to denote the category of left R-modules, and use $\text{mod}\, R$ to denote the category of finitely generated left R-modules. For a module $M \in \text{Mod}\, R$, we use $\text{pd}_R M$, $\text{fd}_R M$, and $\text{id}_R M$ to denote the projective, flat, and injective dimensions of M, respectively.

Let R be a ring. We write $(-)^* := \mathrm{Hom}(-, R)$. Let $M \in \mathrm{Mod}\, R$ and let $\sigma_M : M \to M^{**}$ via $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$ be the canonical evaluation homomorphism. Recall that M is called *torsionless* if σ_M is a monomorphism, and is called *reflexive* if σ_M is an isomorphism. Let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

and

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to \cdots$$

be a projective resolution and a minimal injective coresolution of M, respectively. For any $n \geqslant 1$, $\Omega^n(M) := \mathrm{Im}(P_n \to P_{n-1})$ and $\Omega^{-n}(M) := \mathrm{Im}(E^{n-1}(M) \to E^n(M))$ are called the *n-syzygy* and *n-cosyzygy* of M, respectively. In particular, $\Omega^0(M) = M$. Note that the n-syzygy of M is defined up to projective summands. We write

$$\Omega^n(\mathrm{Mod}\, R) := \{M \in \mathrm{Mod}\, R \mid M \text{ is an } n\text{-syzygy module}\} \text{ for any } n \geqslant 1,$$

$$\Omega^\infty(\mathrm{Mod}\, R) := \cap_{n\geqslant 1}\Omega^n(\mathrm{Mod}\, R) \;\text{ and }\; \Omega^\infty(\mathrm{mod}\, R) := \Omega^\infty(\mathrm{Mod}\, R) \cap \mathrm{mod}\, R.$$

For a subcategory $\mathcal{X}$ of $\mathrm{Mod}\, R$, we write

$${}^\perp\mathcal{X} := \{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{\geqslant 1}(M, X) = 0 \text{ for any } X \in \mathcal{X}\},$$

and write ${}^\perp X := {}^\perp\mathcal{X}$ if $\mathcal{X} = \{X\}$.

Let R be a left and right Noetherian ring and $M \in \mathrm{mod}\, R$, and let

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

be a projective presentation of M in $\mathrm{mod}\, R$. Recall from [1] that $\mathrm{Tr}\, M := \mathrm{Coker}\, f^*$ is called the *transpose* of M. Note that the transpose of M is defined up to projective summands [1, p.51]. A module $M \in \mathrm{mod}\, R$ is called *∞-torsionfree* if $\mathrm{Tr}\, M \in {}^\perp R_R \cap \mathrm{mod}\, R^{op}$. We write

$$\mathcal{T}(\mathrm{mod}\, R) := \{M \in \mathrm{mod}\, R \mid M \text{ is } \infty\text{-torsionfree}\}.$$

By [1, Theorem 2.17], we have $\mathcal{T}(\mathrm{mod}\, R) \subseteq \Omega^\infty(\mathrm{mod}\, R)$.

Definition 2.1 [1]. Let R be a left and right Noetherian ring. A module $M \in \mathrm{mod}\, R$ is said to *have Gorentein dimension zero* if

$$\mathrm{Ext}_R^{\geqslant 1}(M, R) = 0 = \mathrm{Ext}_{R^{op}}^{\geqslant 1}(\mathrm{Tr}\, M, R);$$

equivalently, if M is reflexive and

$$\mathrm{Ext}_R^{\geqslant 1}(M, R) = 0 = \mathrm{Ext}_{R^{op}}^{\geqslant 1}(M^*, R).$$

Let R be a ring. We write $\mathcal{P}(\operatorname{Mod} R) := \{\text{projective left } R\text{-modules}\}$. Recall from [7] that a module $M \in \operatorname{Mod} R$ is called *Gorenstein projective* if there exists an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\operatorname{Mod} R$ with all P_i, P^i in $\mathcal{P}(\operatorname{Mod} R)$, such that it remains exact after applying the functor $\operatorname{Hom}_R(-, P)$ for any $P \in \mathcal{P}(\operatorname{Mod} R)$ and $M \cong \operatorname{Im}(P_0 \to P^0)$. We write

$$\mathcal{GP}(\operatorname{Mod} R) := \{\text{Gorenstein projective left } R\text{-modules}\} \text{ and } \mathcal{GP}(\operatorname{mod} R) := \mathcal{GP}(\operatorname{Mod} R) \cap \operatorname{mod} R.$$

It is well known that over a left and right noetherian ring, a finitely generated module has Gorenstein dimension zero if and only if it is Gorenstein projective [4, 7], and thus,

$$\mathcal{GP}(\operatorname{mod} R) = ({}^{\perp}{}_R R \cap \operatorname{mod} R) \cap \mathcal{T}(\operatorname{mod} R).$$

Now, finitely generated modules having Gorenstein dimension zero over left and right noetherian rings are usually referred to as Gorenstein projective modules.

For any $M \in \operatorname{mod} R$ (resp. $\operatorname{mod} R^{op}$), it is well known that M and $\operatorname{Tr}\operatorname{Tr} M$ are projectively equivalent. So, we have the following observation.

Lemma 2.2. *Let R be a left and right Noetherian ring. Then, a module $M \in \operatorname{mod} R$ (resp. $\operatorname{mod} R^{op}$) is Gorenstein projective if and only if so is $\operatorname{Tr} M$.*

Recall from [9] that a left and right Noetherian ring R is said to satisfy the *Auslander condition* if $\operatorname{fd}_R E^i({}_R R) \leqslant i$ for any $i \geqslant 0$. As a generalization of rings satisfying the Auslander condition, Huang and Iyama [16] introduced the notion of rings satisfying Auslander-type conditions, which was extended to that of modules satisfying Auslander-type conditions as follows.

Definition 2.3 [15]. Let R be a ring and let $m \geqslant 0$. A module $M \in \operatorname{Mod} R$ is said to be $G_\infty(m)$ if $\operatorname{fd}_R E^i(M) \leqslant i + m$ for any $i \geqslant 0$. In particular, M is said to satisfy the *Auslander condition* if it is $G_\infty(0)$.

Let R be a left and right Noetherian ring. Then, ${}_R R$ is $G_\infty(m)$ if and only if the ring R is $G_\infty(m)^{op}$ in the sense of [16] (cf. Introduction). Notice that the notion of the Auslander condition is left–right symmetric [9, Theorem 3.7], so the ring R satisfies the Auslander condition if and only if both ${}_R R$ and R_R satisfy the Auslander condition. However, in general, the notion of R being $G_\infty(m)$ is not left–right symmetric when $m \geqslant 1$ [3, 16]. It should be pointed out that modules satisfying Auslander-type conditions are ubiquitous. For example, if R is a left and right Noetherian ring and $\operatorname{id}_{R^{op}} R \leqslant m$, then any module in $\operatorname{Mod} R$ is $G_\infty(m)$ [15, Example 4.2(3)]. For more examples of modules satisfying Auslander-type conditions, the reader is referred to [15, Example 4.2].

Let $\mathcal{X}$ be a subcategory of $\operatorname{Mod} R$ and $M \in \operatorname{Mod} R$. The *$\mathcal{X}$-projective dimension* $\mathcal{X}\text{-}\operatorname{pd}_R M$ of M is defined as $\inf\{n \mid$ there exists an exact sequence

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all X_i in $\mathcal{X}\}$. If no such an integer exists, then set $\mathcal{X}\text{-}\operatorname{pd}_R M = \infty$. For any $s \geqslant 0$, we write

$$\mathcal{X}^{\leqslant s} := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-}\operatorname{pd}_R M \leqslant s\}.$$

When $\mathcal{X} = \mathcal{GP}(\mathrm{Mod}\, R)$ or $\mathcal{GP}(\mathrm{mod}\, R)$, the $\mathcal{X}$-projective dimension of M is exactly the Gorenstein projective dimension $\text{G-pd}_R\, M$ of M.

3 | SYZYGY MODULES AND GORENSTEIN PROJECTIVE DIMENSION

In this section, R is an arbitrary ring. For any $m \geqslant 0$, we write

$$\mathcal{G}_\infty(m) := \{M \in \mathrm{Mod}\, R \mid M \text{ is } G_\infty(m)\}.$$

Then, we have the following inclusion chain:

$$\mathcal{G}_\infty(0) \subseteq \mathcal{G}_\infty(1) \subseteq \cdots \subseteq \mathcal{G}_\infty(m) \subseteq \cdots.$$

Lemma 3.1. *If R is a left Noetherian ring and ${}_RR \in \mathcal{G}_\infty(m)$, then any flat module in* Mod R *is in $\mathcal{G}_\infty(m)$.*

Proof. It follows from [15, Corollary 3.2]. □

The following lemma is used frequently in the sequel.

Lemma 3.2. *Let*

$$0 \to M \to X^0 \to X^1 \to \cdots \to X^i \to \cdots \tag{3.1}$$

be an exact sequence in Mod R *and let $m \geqslant 0$. If $X^i \in \mathcal{G}_\infty(m)$ for any $i \geqslant 0$, then $M \in \mathcal{G}_\infty(m)$. In particular, the subcategory $\mathcal{G}_\infty(m)$ is closed under kernels of epimorphisms.*

Proof. By the exact sequence (3.1) and [13, Corollary 3.9(1)], we get the following exact sequence:

$$0 \to M \to E^0(X^0) \to E^1(X^0) \oplus E^0(X^1) \to \cdots \to \oplus_{i=0}^{n} E^{n-i}(X^i) \to \cdots.$$

For the reader's convenience, we give an outline of the construction of this exact sequence, which is dual to that in the proof of [13, Theorem 3.6]. Put $M^i := \mathrm{Im}(X^{i-1} \to X^i)$ for any $i \geqslant 1$. Let n be an arbitrary positive integer. We have an exact sequence

$$0 \to M^n \to E^0(X^n). \tag{3.2}$$

From (3.2) and the exact sequence $0 \to X^{n-1} \to E^0(X^{n-1}) \to E^1(X^{n-1})$, we obtain the following exact sequence:

$$0 \to M^{n-1} \to E^0(X^{n-1}) \to E^1(X^{n-1}) \oplus E^0(X^n). \tag{3.3}$$

Then, from (3.3) and the exact sequence $0 \to X^{n-2} \to E^0(X^{n-2}) \to E^1(X^{n-2}) \to E^2(X^{n-2})$, we obtain the following exact sequence:

$$0 \to M^{n-2} \to E^0(X^{n-2}) \to E^1(X^{n-2}) \oplus E^0(X^{n-1}) \to E^2(X^{n-2}) \oplus E^1(X^{n-1}) \oplus E^0(X^n).$$

Continuing this process, we obtain the following exact sequence:

$$0 \to M \to E^0(X^0) \to E^1(X^0) \oplus E^0(X^1) \to \cdots \to \oplus_{i=0}^{n} E^{n-i}(X^i).$$

Then, the desired exact sequence is obtained because of the arbitrariness of n. Since $X^i \in G_\infty(m)$, we have $\mathrm{fd}_R\, E^j(X^i) \leqslant j + m$ for any $i, j \geqslant 0$. So, $\mathrm{fd}_R \oplus_{i=0}^{n} E^{n-i}(X^i) \leqslant n + m$ for any $n \geqslant 0$, and thus $M \in G_\infty(m)$. □

For any $n \geqslant 1$, we write $\Omega^n_{\mathcal{F}}(\mathrm{Mod}\, R) := \{M \in \mathrm{Mod}\, R \mid$ there exists an exact sequence

$$0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1}$$

in $\mathrm{Mod}\, R$ with all F^i flat$\}$, and write $\Omega^\infty_{\mathcal{F}}(\mathrm{Mod}\, R) := \cap_{n \geqslant 1} \Omega^n_{\mathcal{F}}(\mathrm{Mod}\, R)$.

The first assertion in the following result shows that any module in $\mathcal{G}_\infty(m)$ is isomorphic to a kernel (resp. a cokernel) of a homomorphism from a module with finite flat dimension to certain syzygy module.

Theorem 3.3. *It holds that*

(1) *Let $M \in \mathcal{G}_\infty(m)$ with $m \geqslant 0$. Then, for any $n \geqslant 1$, there exists an exact sequence*

$$0 \to G_0 \to X_0 \to G_1 \to X_1 \to 0$$

in $\mathrm{Mod}\, R$ with $M \cong \mathrm{Im}(X_0 \to G_1)$ such that the following conditions are satisfied.
  (a) $\mathrm{fd}_R\, G_0 \leqslant m - 1$ *and* $\mathrm{fd}_R\, G_1 \leqslant m$.
  (b) $X_0 \in \Omega^n_{\mathcal{F}}(\mathrm{Mod}\, R)$ *and* $X_1 \in \Omega^{n-1}(\mathrm{Mod}\, R)$.

(2) $\mathcal{G}_\infty(0) \subseteq \Omega^\infty_{\mathcal{F}}(\mathrm{Mod}\, R)$ *with equality if R is a left Noetherian ring and ${}_RR \in \mathcal{G}_\infty(0)$.*

Proof.

(1) Let $M \in \mathcal{G}_\infty(m)$ and $n \geqslant 1$. We have the following two exact and commutative diagrams:

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \dashrightarrow & K & \dashrightarrow & K_0 & \dashrightarrow & K_1 & \dashrightarrow & \cdots & \dashrightarrow & K_{n-1} & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \dashrightarrow & X & \dashrightarrow & P_0 & \dashrightarrow & P_1 & \dashrightarrow & \cdots & \dashrightarrow & P_{n-1} & \dashrightarrow & \Omega^{-n}(M) \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \| \\
0 & \longrightarrow & M & \longrightarrow & E^0(M) & \longrightarrow & E^1(M) & \longrightarrow & \cdots & \longrightarrow & E^{n-1}(M) & \longrightarrow & \Omega^{-n}(M) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}$$

and

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K & \dashrightarrow & K_0 & \dashrightarrow & K_1' & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & X & \dashrightarrow & P_0 & \dashrightarrow & X_1 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E^0(M) & \longrightarrow & \Omega^{-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with all P_i projective in $\operatorname{Mod} R$ and $X_1 := \operatorname{Im}(P_0 \to P_1) \in \Omega^{n-1}(\operatorname{Mod} R)$. Since $M \in \mathcal{G}_\infty(m)$, we have $\operatorname{fd}_R E^i(M) \leqslant i + m$ for any $i \geqslant 0$, and thus $\operatorname{fd}_R K_i \leqslant i + m - 1$ for any $1 \leqslant i \leqslant n - 1$. It follows from the upper row in the first diagram that $\operatorname{fd}_R K_1' \leqslant m$.

Consider the following pull-back diagram (Diagram (3.1)):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_1' & = & K_1' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & M & \dashrightarrow & G_1 & \dashrightarrow & X_1 & \dashrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E^0(M) & \longrightarrow & \Omega^{-1}(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

From the middle column, we obtain $\operatorname{fd}_R G_1 \leqslant m$, and there exists an exact sequence

$$0 \to G_0 \to F \to G_1 \to 0$$

in Mod R with F flat and $\mathrm{fd}_R\, G_0 \leqslant m-1$. Consider the following pull-back diagram (Diagram (3.2)):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_0 & = & G_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & X_0 & \dashrightarrow & F & \dashrightarrow & X_1 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

From the middle row, we obtain $X_0 \in \Omega^n_{\mathcal{F}}(\mathrm{Mod}\, R)$. Now splicing the middle row in Diagram (3.1) (i.e., the bottom row in Diagram (3.2)) and the leftmost column in Diagram (3.2), we get the desired exact sequence.

(2) To prove $\mathcal{G}_\infty(0) \subseteq \Omega^\infty_{\mathcal{F}}(\mathrm{Mod}\, R)$, it suffices to prove that if $M \in \mathcal{G}_\infty(0)$, then $M \in \Omega^n_{\mathcal{F}}(\mathrm{Mod}\, R)$ for any $n \geqslant 1$. Let $M \in \mathcal{G}_\infty(0)$ and $n \geqslant 1$. By (1), there exists an exact sequence

$$0 \to M \to F \to G_1 \to 0$$

in Mod R with F flat and $G_1 \in \Omega^{n-1}(\mathrm{Mod}\, R)$, and so $M \in \Omega^n_{\mathcal{F}}(\mathrm{Mod}\, R)$.

Now assume that R is a left Noetherian ring and ${}_RR \in \mathcal{G}_\infty(0)$. Then any flat module in Mod R is in $\mathcal{G}_\infty(0)$ by Lemma 3.1, and thus $\Omega^\infty_{\mathcal{F}}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(0)$ by Lemma 3.2. □

We need the following lemma.

Lemma 3.4. *For any $m, s \geqslant 0$, we have*

$$\mathcal{G}_\infty(m)^{\leqslant s} \subseteq \mathcal{G}_\infty(m+s)$$

with equality if $\mathcal{P}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(0)$.

Proof. Let $M \in \mathcal{G}_\infty(m)^{\leqslant s}$ and

$$0 \to X_s \to \cdots \to X_1 \to X_0 \to M \to 0$$

be an exact sequence in Mod R with all X_i in $\mathcal{G}_\infty(m)$. According to [13, Corollary 3.5], we get the following two exact sequences:

$$0 \to M \to E \to \oplus_{i=0}^s E^{i+1}(X_i) \to \oplus_{i=0}^s E^{i+2}(X_i) \to \oplus_{i=0}^s E^{i+3}(X_i) \to \cdots, \tag{3.2}$$

$$0 \to E^s(X_0) \to E^{s-1}(X_0) \oplus E^s(X_1) \to \cdots \to \oplus_{i=1}^s E^{i-1}(X_i) \to \oplus_{i=0}^s E^i(X_i) \to E \to 0. \tag{3.3}$$

Since all X_i are in $\mathcal{G}_\infty(m)$, we have $\mathrm{fd}_R E^j(X_i) \leqslant j+m$ for any $j \geqslant 0$ and $0 \leqslant i \leqslant s$. Thus, $\mathrm{fd}_R \bigoplus_{i=0}^{s} E^{i+j}(X_i) \leqslant j+m+s$ for any $j \geqslant 1$. By (3.3), we have that E is a direct summand of $\bigoplus_{i=0}^{s} E^i(X_i)$ and $\mathrm{fd}_R E \leqslant m+s$. Therefore, we obtain $M \in \mathcal{G}_\infty(m+s)$ by (3.2).

Now suppose $\mathcal{P}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(0)$. We will prove $\mathcal{G}_\infty(m+s) \subseteq \mathcal{G}_\infty(m)^{\leqslant s}$ by induction on s. The case for $s=0$ follows trivially. Suppose $s \geqslant 1$ and $M \in \mathcal{G}_\infty(m+s)$. Let

$$0 \to K \to P \to M \to 0$$

be an exact sequence in $\mathrm{Mod}\, R$ with P projective. Since $P \in \mathcal{G}_\infty(0)$, it follows from [15, Proposition 4.12] that $K \in \mathcal{G}_\infty(m+s-1)$, and hence $\mathcal{G}_\infty(m)$-pd $K \leqslant s-1$ by the induction hypothesis. This implies $\mathcal{G}_\infty(m)$-$\mathrm{pd}_R\, M \leqslant s$ and $M \in \mathcal{G}_\infty(m)^{\leqslant s}$. □

By Lemma 3.4, we obtain the following result.

Proposition 3.5. *If $\mathcal{P}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(0)$, then it holds that*

(1) *$\mathcal{GP}(\mathrm{Mod}\, R) = \mathcal{G}_\infty(m)$ if and only if $\mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant s} = \mathcal{G}_\infty(m+s)$ for any $s \geqslant 0$.*
(2) *If R is a left and right Noetherian ring, then $\mathcal{GP}(\mathrm{mod}\, R) = \mathcal{G}_\infty(m) \cap \mathrm{mod}\, R$ if and only if $\mathcal{GP}(\mathrm{mod}\, R)^{\leqslant s} = \mathcal{G}_\infty(m+s) \cap \mathrm{mod}\, R$ for any $s \geqslant 0$.*

About the condition $\mathcal{P}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(0)$ in Proposition 3.5, we remark that if R is a left Noetherian ring, then this condition is satisfied if and only if ${}_R R$ satisfied the Auslander condition by [15, Theorem 4.9], and that if R is an Artin algebra, then $\mathcal{P}(\mathrm{Mod}\, R) = \mathcal{G}_\infty(0)$ if and only if R is Auslander-regular (i.e., the algebra R satisfies Auslander condition and the global dimension of R is finite) [15, Theorem 5.9].

Theorem 3.6. *It holds that*

(1) *If R is a Gorenstein ring, then $\mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant m}$ for any $m \geqslant 0$.*
(2) *If R is a left Noetherian ring and $\mathrm{id}_R\, R < \infty$, then $\mathcal{GP}(\mathrm{Mod}\, R) = \Omega^\infty(\mathrm{Mod}\, R)$.*

Proof.

(1) Let R be a Gorenstein ring with $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R \leqslant n$, and let $M \in \mathcal{G}_\infty(m)$. Then G-$\mathrm{pd}_R\, M \leqslant n$ by [7, Theorem 12.3.1]. It suffices to prove G-$\mathrm{pd}_R\, M \leqslant m$. The case for $n \leqslant m$ is trivial. Now suppose $n > m$ and $t := n - m$. Consider the following exact sequence:

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^{t-1}(M) \to K^t \to 0,$$

where $K^t := \mathrm{Im}(E^{t-1}(M) \to E^t(M))$. By [7, Theorem 12.3.1] again, we have G-$\mathrm{pd}_R\, K^t \leqslant n (= t+m)$. Since $M \in \mathcal{G}_\infty(m)$, we have $\mathrm{pd}_R\, E^i(M) \leqslant i+m$ for any $0 \leqslant i \leqslant t-1$. Then, it is easy to get G-$\mathrm{pd}_R\, M \leqslant m$ by [14, Theorem 3.2 and Remark 4.4(3)(a)].
(2) It suffices to prove $\Omega^\infty(\mathrm{Mod}\, R) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)$. If R is a left Noetherian ring and $\mathrm{id}_R\, R < \infty$, then $\mathrm{id}_R\, P < \infty$ for any $P \in \mathcal{P}(\mathrm{Mod}\, R)$ by [5, Theorem 1.1]. Assume that $M \in \Omega^\infty(\mathrm{Mod}\, R)$ and

$$0 \to M \to P^0 \to P^1 \to \cdots \to P^i \to \cdots$$

is an exact sequence in $\mathrm{Mod}\, R$ with all P^i in $\mathcal{P}(\mathrm{Mod}\, R)$. It is easy to see that the kernel of each homomorphism in the above exact sequence is in ${}^{\perp}\mathcal{P}(\mathrm{Mod}\, R)$ by dimension shifting. Thus, $M \in \mathcal{GP}(\mathrm{Mod}\, R)$ and $\Omega^\infty(\mathrm{Mod}\, R) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)$. □

4 | (WEAKLY) GORENSTEIN ALGEBRAS

In this section, R is an Artin algebra. Under certain Auslander-type conditions, we will give some equivalent characterizations for $\mathrm{id}_R R < \infty$ as well as for (weakly) Gorenstein algebras. As applications, we give some partial answers to some related homological conjectures.

4.1 | Auslander-type conditions

For any $M \in \mathrm{Mod}\, R$ and $m \geqslant 0$, we write

$$^{\perp_{\geqslant m+1}}M := \{A \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{\geqslant m+1}(A, M) = 0\}.$$

Lemma 4.1. *Let $M \in \mathrm{mod}\, R$ such that $\Omega^{\infty}(\mathrm{mod}\, R) \subseteq {}^{\perp_{\geqslant m+1}}M \cap \mathrm{mod}\, R$ for some $m \geqslant 0$. If there exists some $n \geqslant 0$ such that $\mathrm{pd}_R E^i(M) \leqslant n$ for any $i \geqslant n+m+1$, then $\mathrm{id}_R M \leqslant n+m$.*

Proof. Let $M \in \mathrm{mod}\, R$. Set $K^i := \mathrm{Im}(E^{i-1}(M) \to E^i(M))$ for any $i \geqslant 1$. Since $\mathrm{pd}_R E^i(M) \leqslant n$ for any $i \geqslant n+m+1$, by the horseshoe lemma, we obtain the following exact and commutative diagram:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & K_n^{n+m+1} & \to & P_n^{n+m+1} & \to & P_n^{n+m+2} & \to & \cdots & \to & P_n^{n+m+i} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & P_{n-1} & \to & P_{n-1}^{n+m+1} & \to & P_{n-1}^{n+m+2} & \to & \cdots & \to & P_{n-1}^{n+m+i} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & P_1 & \to & P_1^{n+m+1} & \to & P_1^{n+m+2} & \to & \cdots & \to & P_1^{n+m+i} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & P_0 & \to & P_0^{n+m+1} & \to & P_0^{n+m+2} & \to & \cdots & \to & P_0^{n+m+i} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & K^{n+m+1} & \to & E^{n+m+1}(M) & \to & E^{n+m+2}(M) & \to & \cdots & \to & E^{n+m+i}(M) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 & & 0 & & 0 & & & & 0
\end{array}$$

in $\mathrm{mod}\, R$ with all P_j and P_j^t projective. Then, $K_n^{n+m+1} \in \Omega^{\infty}(\mathrm{mod}\, R)$, and thus $K_n^{n+m+1} \in {}^{\perp_{\geqslant m+1}}M \cap \mathrm{mod}\, R$ by assumption. It follows from the leftmost column in the above diagram that

$K^{n+m+1} \in {}^{\perp_{\geqslant n+m+1}}M \cap \operatorname{mod} R$. Now applying the functor $\operatorname{Hom}_R(K^{n+m+1}, -)$ to the exact sequence

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^{n+m-1}(M) \to K^{n+m} \to 0$$

yields $\operatorname{Ext}^1_R(K^{n+m+1}, K^{n+m}) = 0$. It implies that the exact sequence

$$0 \to K^{n+m} \to E^{n+m}(M) \to K^{n+m+1} \to 0$$

splits and K^{n+m} is a direct summand of $E^{n+m}(M)$. Thus, K^{n+m} is injective and $\operatorname{id}_R M \leqslant n+m$. □

Remark 4.2. The same argument as above essentially proves the following result: Let R be an arbitrary ring (not necessarily an Artin algebra) and let $M \in \operatorname{Mod} R$ such that $\Omega^\infty(\operatorname{Mod} R) \subseteq {}^{\perp_{\geqslant m+1}}M$ for some $m \geqslant 0$. If there exists some $n \geqslant 0$ such that $\operatorname{pd}_R E^i(M) \leqslant n$ for any $i \geqslant n+m+1$, then $\operatorname{id}_R M \leqslant n+m$.

Recall from [23] that an Artin algebra R is called *left weakly Gorenstein* if $\mathcal{GP}(\operatorname{mod} R) = {}^{\perp}{}_R R \cap \operatorname{mod} R$. Symmetrically, the notion of *right weakly Gorenstein algebras* is defined.

Proposition 4.3.

(1) *Assume that there exists some* $n, m \geqslant 0$ *such that* $\operatorname{pd}_R E^i({}_R R) \leqslant n$ *for any* $i \geqslant n+m+1$. *If* $\Omega^\infty(\operatorname{mod} R) \subseteq {}^{\perp_{\geqslant m+1}}{}_R R \cap \operatorname{mod} R$, *then* $\operatorname{id}_R R \leqslant n+m$.
(2) *Assume that there exists some* $n \geqslant 0$ *such that* $\operatorname{pd}_R E^i({}_R R) \leqslant n$ *for any* $i \geqslant n+1$. *If* R *is right weakly Gorenstein and* $\Omega^\infty(\operatorname{mod} R) = \mathcal{T}(\operatorname{mod} R)$, *then* $\operatorname{id}_R R \leqslant n$.

Proof.

(1) Putting $M = {}_R R$ in Lemma 4.1, the assertion follows.
(2) Let $M \in \Omega^\infty(\operatorname{mod} R)$. Then $M \in \mathcal{T}(\operatorname{mod} R)$ by assumption, and so $\operatorname{Tr} M \in {}^{\perp}R_R \cap \operatorname{mod} R^{op}$. Since R is right weakly Gorenstein by assumption, we have $\operatorname{Tr} M \in {}^{\perp}R_R \cap \operatorname{mod} R^{op} = \mathcal{GP}(\operatorname{mod} R^{op})$. Thus, $M \in \mathcal{GP}(\operatorname{mod} R) \subseteq {}^{\perp}{}_R R \cap \operatorname{mod} R$ by Lemma 2.2. This shows $\Omega^\infty(\operatorname{mod} R) \subseteq {}^{\perp}{}_R R \cap \operatorname{mod} R$, and then the assertion follows from (1). □

The following lemma shows that all modules satisfying certain Auslander-type condition over an Artin algebra satisfy the condition about projective dimension in Lemma 4.1.

Lemma 4.4. *If* $M \in \mathcal{G}_\infty(m)$ *(resp.* $N \in \mathcal{G}_\infty(m)^{op}$*) with* $m \geqslant 0$, *then there exists some* $n \geqslant 0$ *such that* $\operatorname{pd}_R E^i(M)$ *(resp.* $\operatorname{pd}_{R^{op}} E^i(N)$*)* $\leqslant n$ *for any* $i \geqslant 0$.

Proof. Since R is an Artin algebra, there exist only finitely many nonisomorphic indecomposable injective left R-modules. Let $M \in \mathcal{G}_\infty(m)$. Without of generalization, suppose that $\{E^0, \dots, E^t\}$ is the complete set of nonisomorphic indecomposable injective left modules that occur as direct summands of all $E^i(M)$. Then, there exists some $n \geqslant 0$ such that $\operatorname{pd}_R E^i \leqslant n$ for any $1 \leqslant i \leqslant t$, and thus $\operatorname{pd}_R E^i(M) \leqslant n$ for any $i \geqslant 0$. Symmetrically, if $N \in \mathcal{G}_\infty(m)^{op}$, then there exists some $n \geqslant 0$ such that $\operatorname{pd}_{R^{op}} E^i(N) \leqslant n$ for any $i \geqslant 0$. □

As a consequence, we obtain the following result.

Proposition 4.5. *If $\mathcal{GP}(\mathrm{mod}\, R) = \mathcal{G}_\infty(m) \cap \mathrm{mod}\, R$ for some $m \geqslant 0$, then $\mathrm{id}_R\, R < \infty$.*

Proof. Since ${}_RR \in \mathcal{GP}(\mathrm{mod}\, R)$, we have ${}_RR \in \mathcal{G}_\infty(m)$ by assumption, It follows from Lemma 4.4 that $\mathrm{pd}_R\, E^i({}_RR) \leqslant n$ for any $i \geqslant 0$. Since any projective module in $\mathrm{mod}\, R$ is in $\mathcal{G}_\infty(m)$, we have

$$\begin{aligned}\Omega^\infty(\mathrm{mod}\, R) &\subseteq \mathcal{G}_\infty(m) \cap \mathrm{mod}\, R \ \ \text{(by Lemma 3.2)}\\ &= \mathcal{GP}(\mathrm{mod}\, R) \ \ \text{(by assumption)}\\ &\subseteq {}^\perp{}_RR \cap \mathrm{mod}\, R.\end{aligned}$$

Thus, $\mathrm{id}_R\, R \leqslant n$ by Proposition 4.3(1). □

We are now in a position to prove the following result, in which assertions (5) and (6) are finitely generated versions of (3) and (4), respectively.

Theorem 4.6. *For any $m \geqslant 0$, the following statements are equivalent.*

(1) ${}_RR \in \mathcal{G}_\infty(m)$ *and R is Gorenstein.*
(2) ${}_RR \in \mathcal{G}_\infty(m)$ *and* $\mathrm{id}_R\, R < \infty$.
(3) $\mathcal{GP}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant m}$.
(4) $\mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant s} \subseteq \mathcal{G}_\infty(m+s) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant m+s}$ *for any* $s \geqslant 0$.
(5) $\mathcal{GP}(\mathrm{mod}\, R) \subseteq \mathcal{G}_\infty(m) \cap \mathrm{mod}\, R \subseteq \mathcal{GP}(\mathrm{mod}\, R)^{\leqslant m}$.
(6) $\mathcal{GP}(\mathrm{mod}\, R)^{\leqslant s} \subseteq \mathcal{G}_\infty(m+s) \cap \mathrm{mod}\, R \subseteq \mathcal{GP}(\mathrm{mod}\, R)^{\leqslant m+s}$ *for any* $s \geqslant 0$.

Proof. The implications (1) $\Longrightarrow$ (2), (4) $\Longrightarrow$ (3) $\Longrightarrow$ (5) and (4) $\Longrightarrow$ (6) $\Longrightarrow$ (5) are trivial. By the symmetric version of [11, Corollary 3], we get (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (3) Since R is Gorenstein by (1), we have $\mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant m}$ by Theorem 3.6(1). On the other hand, since ${}_RR \in \mathcal{G}_\infty(m)$ by (1), we have $\mathcal{P}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(m)$ by Lemma 3.1, and thus,

$$\mathcal{GP}(\mathrm{Mod}\, R) \subseteq \Omega^\infty(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(m)$$

by Lemma 3.2.

(5) $\Longrightarrow$ (2) Since any projective module in $\mathrm{mod}\, R$ is in $\mathcal{G}_\infty(m) \cap \mathrm{mod}\, R$ by (5), we have

$$\Omega^\infty(\mathrm{mod}\, R) \subseteq \mathcal{G}_\infty(m) \cap \mathrm{mod}\, R \subseteq \mathcal{GP}(\mathrm{mod}\, R)^{\leqslant m} \subseteq {}^{\perp_{\geqslant m+1}}{}_RR \cap \mathrm{mod}\, R$$

by Lemma 3.2 and (5). Since ${}_RR \in \mathcal{G}_\infty(m) \cap \mathrm{mod}\, R$, there exists some $n \geqslant 0$ such that $\mathrm{pd}_R\, E^i({}_RR) \leqslant n$ for any $i \geqslant 0$ by Lemma 4.4, and thus, $\mathrm{id}_R\, R \leqslant n + m$ by Proposition 4.3(1).

(1) + (3) $\Longrightarrow$ (4) Let $s \geqslant 0$. Since $\mathcal{GP}(\mathrm{Mod}\, R) \subseteq \mathcal{G}_\infty(m)$ by (3), we have

$$\mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant s} \subseteq \mathcal{G}_\infty(m)^{\leqslant s} \subseteq \mathcal{G}_\infty(m+s)$$

by Lemma 3.4. Since R is Gorenstein by (1), we have $\mathcal{G}_\infty(m+s) \subseteq \mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant m+s}$ by Theorem 3.6(1). □

We need the following result.

Proposition 4.7. *If $\mathrm{id}_R\, R < \infty$, then R is right weakly Gorenstein. The converse holds true if one of the following conditions is satisfied.*

(1) ${}_RR \in \mathcal{G}_\infty(1)$.
(2) ${}_RR \in \mathcal{G}_\infty(m)$ *and* $R_R \in \mathcal{G}_\infty(m')^{op}$ *for some* $m, m' \geqslant 0$.

Proof. The former assertion follows from the symmetric versions of [17, Lemma 3.4] and [23, Theorem 1.2].

Conversely, since ${}_RR \in \mathcal{G}_\infty(1)$ or ${}_RR \in \mathcal{G}_\infty(m)$ with $m \geqslant 0$ by assumption, it follows from Lemma 4.4 that there exists some $n \geqslant 0$ such that $\mathrm{pd}_R\, E^i({}_RR) \leqslant n$ for any $i \geqslant 0$. When ${}_RR \in \mathcal{G}_\infty(1)$, we have $\Omega^\infty(\mathrm{mod}\, R) = \mathcal{T}(\mathrm{mod}\, R)$ by [3, Proposition 1.6(a)] and the symmetric version of [3, Theorem 0.1]; when $R_R \in \mathcal{G}_\infty(m')^{op}$ with $m' \geqslant 0$, that is, the algebra R is $G_\infty(m')$, we also have $\Omega^\infty(\mathrm{mod}\, R) = \mathcal{T}(\mathrm{mod}\, R)$ by [16, Theorem 3.4]. Thus, $\mathrm{id}_R\, R \leqslant n$ by Proposition 4.3(2). □

The following corollary was proved in [23, p.33], we give it a shorter proof.

Corollary 4.8. *WGSC implies GSC.*

Proof. Suppose that *WGSC* holds true. If $\mathrm{id}_R\, R = n < \infty$, then R is right weakly Gorenstein by Proposition 4.7, and hence is left weakly Gorenstein. It follows that any n-syzygy module in $\mathrm{mod}\, R$ is in ${}^{\perp}{}_RR \cap \mathrm{mod}\, R = \mathcal{GP}(\mathrm{mod}\, R)$. So $\text{G-pd}_R\, M \leqslant n$ for any $M \in \mathrm{mod}\, R$, and hence, R is n-Gorenstein (i.e., $\mathrm{id}_R\, R = \mathrm{id}_{R^{op}}\, R \leqslant n$) by [7, Theorem 12.3.1]. Symmetrically, we have that if $\mathrm{id}_{R^{op}}\, R = n < \infty$, then R is n-Gorenstein. Thus, *GSC* holds true. □

The following result shows that the Gorensteinness and weakly Gorensteinness of an Artin algebra are equivalent under certain Auslander-type conditions. It also shows that both *GSC* and *WGSC* hold true for an Artin algebra R such that ${}_RR$ and R_R satisfy certain Auslander-type conditions.

Theorem 4.9. *If* ${}_RR \in \mathcal{G}_\infty(m)$ *and* $R_R \in \mathcal{G}_\infty(m')^{op}$ *with* $m, m' \geqslant 0$, *then the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) *R is left and right weakly Gorenstein.*
(3) $\mathrm{id}_R\, R < \infty$.
(4) *R is left weakly Gorenstein.*
(5) $\mathcal{GP}(\mathrm{Mod}\, R) = {}^{\perp}\mathcal{P}(\mathrm{Mod}\, R)$.
$(i)^{op}$ *Opposite version of* (i) *with* $3 \leqslant i \leqslant 5$.

Proof. It is trivial that (5) $\Longrightarrow$ (4) and (2) $\Longrightarrow$ (4). By Proposition 4.7 and its symmetric version, we have (1) $\Longrightarrow$ (2) and (3) $\Longleftrightarrow$ $(4)^{op}$. By Theorem 4.6 and its symmetric version, we have (1) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ $(3)^{op}$. By [7, Corollary 11.5.3], we have (1) $\Longrightarrow$ (5).

By symmetry, the proof is finished. □

4.2 | Small Auslander-type conditions

Recall from [12] that R is called *left quasi-Auslander* if ${}_RR \in \mathcal{G}_\infty(1)$. Compare the following result with Theorem 4.9.

Theorem 4.10. *Let R be a left quasi-Auslander algebra. Then, the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) $\mathrm{id}_R R < \infty$.
(3) $\mathrm{id}_{R^{op}} R < \infty$.
(4) *R is left and right weakly Gorenstein.*
(5) *R is right weakly Gorenstein.*
(6) $\mathcal{GP}(\mathrm{Mod}\, R^{op}) = {}^{\perp}\mathcal{P}(\mathrm{Mod}\, R^{op})$.

Proof. It is trivial that (4) $\Longrightarrow$ (5) and (6) $\Longrightarrow$ (5).

By Proposition 4.7 and its symmetric version, we have (1) $\Longleftrightarrow$ (4). By [11, Corollary 4], we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). By Proposition 4.7(1), we have (2) $\Longleftrightarrow$ (5). By [7, Corollary 11.5.3], we have (1) $\Longrightarrow$ (6). □

Theorem 4.10 means that over a left quasi-Auslander Artin algebra, *GSC* holds true, but we do not know whether *WGSC* holds true or not.

Recall that R is called *Auslander–Gorenstein* if R satisfies the Auslander condition and R is Gorenstein. In the following result, assertions (5)–(7) are finitely generated versions of (2)–(4), respectively.

Theorem 4.11. *The following statements are equivalent.*

(1) *R is Auslander–Gorenstein.*
(2) *R satisfies the Auslander condition and* $\mathcal{GP}(\mathrm{Mod}\, R) = {}^{\perp}\mathcal{P}(\mathrm{Mod}\, R)$.
(3) $\mathcal{GP}(\mathrm{Mod}\, R) = \mathcal{G}_{\infty}(0)$.
(4) $\mathcal{GP}(\mathrm{Mod}\, R)^{\leqslant s} = \mathcal{G}_{\infty}(s)$ *for any* $s \geqslant 0$.
(5) *R satisfies the Auslander condition and R is left weakly Gorenstein.*
(6) $\mathcal{GP}(\mathrm{mod}\, R) = \mathcal{G}_{\infty}(0) \cap \mathrm{mod}\, R$.
(7) $\mathcal{GP}(\mathrm{mod}\, R)^{\leqslant s} = \mathcal{G}_{\infty}(s) \cap \mathrm{mod}\, R$ *for any* $s \geqslant 0$.
$(i)^{op}$ *Opposite version of (i) with* $2 \leqslant i \leqslant 7$.

Proof. The implications (2) $\Longrightarrow$ (5), (3) $\Longrightarrow$ (6) and (4) $\Longrightarrow$ (7) are trivial.

Note that R satisfies the Auslander condition if and only if ${}_RR \in \mathcal{G}_{\infty}(0)$ and $R_R \in \mathcal{G}_{\infty}(0)^{op}$, and if and only if ${}_RR \in \mathcal{G}_{\infty}(0)$ or $R_R \in \mathcal{G}_{\infty}(0)^{op}$. The implications (3) $\Longleftrightarrow$ (4) and (6) $\Longleftrightarrow$ (7) follow from Proposition 3.5(1)(2), respectively. The implication (6) $\Longrightarrow$ (1) follows from Proposition 4.5 and [2, Corollary 5.5(b)]. The implications (1) $\Longleftrightarrow$ (3) and (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (5) follow from Theorems 4.6 and 4.9, respectively.

By symmetry, the proof is finished. □

Let M be an R-module. An injective coresolution

$$0 \to M \to E^0 \xrightarrow{\delta^1} E^1 \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} \cdots$$

is called *ultimately closed* if there exists some n such that $\mathrm{Im}\,\delta^n = \oplus W_j$ with each W_j isomorphic to a direct summand of some $\mathrm{Im}\,\delta_{i_j}$ with $i_j < n$. It is clear that a left R-module M has an ultimately closed injective coresolution if $\mathrm{id}_R M < \infty$. An algebra R is said to be of *ultimately closed type* if the minimal injective coresolution of any finitely generated left R-module is ultimately closed

[25]. The class of algebras of ultimately closed type includes: (1) Artin algebras with finite global dimension; (2) Artin algebras with radical square zero; (3) Representation-finite algebras; and (4) Artin algebras R with Loewy length m such that R/J^{m-1} is representation-finite, where J is the Jacobson radical of R [25, p.110].

Recall from [22] that R is called *torsionless-finite* if there exists only finitely many isomorphism classes of indecomposable torsionless modules in $\operatorname{mod} R$. We claim that any torsionless-finite algebra is of ultimately closed type. Let R be a torsionless-finite algebra. It follows from [22, Corollary 2.2] that R^{op} is also a torsionless-finite algebra and there exists only finitely many isomorphism classes of indecomposable torsionless modules in $\operatorname{mod} R^{op}$. Using the usual duality between $\operatorname{mod} R$ and $\operatorname{mod} R^{op}$ yields that there exists only finitely many isomorphism classes of indecomposable 1-cosyzygy modules in $\operatorname{mod} R$. Thus, R is of ultimately closed type. The claim is proved. The class of torsionless-finite algebras includes: (1) Artin algebras R with $R/\operatorname{soc}(R_R)$ representation-finite, where $\operatorname{soc}(R_R)$ is the socle of R_R; (2) Minimal representation-infinite algebras; (3) Artin algebras stably equivalent to hereditary algebras; (4) left or right glued algebras; and (5) special biserial algebras without indecomposable projective-injective modules [22, Section 5].

Note that algebras R such that ${}_RR$ has an ultimately closed injective coresolution (particularly, algebras R of ultimately closed type) are right weakly Gorenstein algebras by the symmetric versions of [17, Theorem 2.4] and [23, Theorem 1.2]. However, such algebras are not Gorenstein in general, thus the following result can be regarded as a reduction of *ARC*.

Corollary 4.12. *If R satisfies the Auslander condition, then the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) *R is left or right weakly Gorenstein.*
(3) *R is left and right weakly Gorenstein.*
(4) $\mathcal{GP}(\operatorname{mod} R) = \mathcal{T}(\operatorname{mod} R)$.
(5) $\mathcal{GP}(\operatorname{mod} R) = \mathcal{T}(\operatorname{mod} R) = {}^{\perp}{}_RR \cap \operatorname{mod} R$.

Proof. Since R satisfies the Auslander condition, we have

$$\mathcal{G}_\infty(0) \cap \operatorname{mod} R = \Omega^\infty(\operatorname{mod} R) = \mathcal{T}(\operatorname{mod} R)$$

by [15, Lemma 5.7]. Now the assertion follows from Theorem 4.11. □

As indicated above, if ${}_RR$ (resp. R_R) has an ultimately closed injective coresolution, then R is a right (resp. left) weakly Gorenstein algebra. As a consequence of Corollary 4.12, we obtain the following result.

Corollary 4.13. *ARC holds true for the following classes of algebras R.*

(1) *${}_RR$ or R_R has an ultimately closed injective coresolution.*
(2) *Algebras of ultimately closed type.*

In the following, we give an alternative proof of Corollary 4.13, which is independent of Corollary 4.12. Recall that the *finitistic dimension* $\operatorname{fin.dim} R$ of R is defined as

$$\operatorname{fin.dim} R := \sup\{\operatorname{pd}_R M \mid M \in \operatorname{mod} R \text{ with } \operatorname{pd}_R M < \infty\}.$$

Assume that R satisfies the Auslander condition. Then, R^{op} also satisfies the Auslander condition by [9, Theorem 3.7]. It follows from [2, Corollary 5.5(b)] that $\mathrm{id}_R R < \infty$ if and only if $\mathrm{id}_{R^{op}} R < \infty$, and hence, $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R$ by [28, Lemma A]. Then,

$$\mathrm{fin.dim}\, R = \mathrm{id}_R R = \mathrm{id}_{R^{op}} R = \mathrm{fin.dim}\, R^{op} \tag{4.1}$$

by [16, Corollary 5.3(1)]. When R_R has an ultimately closed injective coresolution, it is known from [10, p.2983] that this injective coresolution has a strongly redundant image in the sense of [10]. Then applying [10, Theorem 3] yields $\mathrm{fin.dim}\, R^{op} < \infty$. Symmetrically, when $_R R$ has an ultimately closed injective coresolution (particularly, when R is of ultimately closed type), we have $\mathrm{fin.dim}\, R < \infty$. So, in both cases, we have $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R < \infty$ by (4.1), that is, R is Gorenstein. Consequently, we conclude that *ARC* holds true for R.

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ORCID

Zhaoyong Huang https://orcid.org/0000-0002-1621-7373

REFERENCES

1. M. Auslander and M. Bridger, *Stable module theory*, Memoirs Amer. Math. Soc., vol. 94, Amer. Math. Soc., Providence, RI, 1969.
2. M. Auslander and I. Reiten, *k-Gorenstein algebras and syzygy modules*, J. Pure Appl. Algebra **92** (1994), 1–27.
3. M. Auslander and I. Reiten, *Syzygy modules for Noetherian rings*, J. Algebra **183** (1996), 167–185.
4. L. L. Avramov and A. Martsinkovsky, *Absolute, relative and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. **85** (2002), 393–440.
5. H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 18–29.
6. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
7. E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, 2nd ed., de Gruyter Exp. in Math., vol. 30, Walter de Gruyter, Berlin, New York, 2011.
8. K. Erdmann, T. Holm, O. Iyama, and J. Schröer, *Radical embeddings and representation dimension*, Adv. Math. **185** (2004), 159–177.
9. R. M. Fossum, P. A. Griffith, and I. Reiten, *Trivial extensions of abelian categories. Homological algebra of trivial extensions of abelian categories with applications to ring theory*, Lecture Notes in Math., vol. 456, Springer, Berlin-Heidelberg-New York, 1975.
10. K. R. Fuller and Y. Wang, *Redundancy in resolutions and finitistic dimensions of Noetherian rings*, Comm. Algebra **21** (1993), 2983–2994.
11. Z. Y. Huang, *Selforthogonal modules with finite injective dimension II*, J. Algebra **264** (2003), 262–268.
12. Z. Y. Huang, *Syzygy modules for quasi k-Gorenstein rings*, J. Algebra **299** (2006), 21–32.
13. Z. Y. Huang, *Proper resolutions and Gorenstein categories*, J. Algebra **393** (2013), 142–169.
14. Z. Y. Huang, *Homological dimensions relative to preresolving subcategories II*, Forum Math. **34** (2022), 507–530.

15. Z. Y. Huang, *On Auslander-type conditions of modules*, Publ. Res. Inst. Math. Sci. **59** (2023), 57–88.
16. Z. Y. Huang and O. Iyama, *Auslander-type conditions and cotorsion pairs*, J. Algebra **318** (2007), 93–110.
17. Z. Y. Huang and G. H. Tang, *Self-orthogonal modules over coherent rings*, J. Pure Appl. Algebra **161** (2001), 167–176.
18. Y. Iwanaga and H. Sato, *On Auslander's n-Gorenstein rings*, J. Pure Appl. Algebra **106** (1996), 61–76.
19. O. Iyama, *Symmetry and duality on n-Gorenstein rings*, J. Algebra **269** (2003), 528–535.
20. R. Marczinzik, *Gendo-symmetric algebras, dominant dimensions and Gorenstein homological algebra*, arXiv:1608.04212, 2016.
21. J. Miyachi, *Injective resolutions of Noetherian rings and cogenerators*, Proc. Amer. Math. Soc. **128** (2000), 2233–2242.
22. C. M. Ringel, *On the representation dimension of Artin algebras*, Bull. Inst. Math. Acad. Sin. (N.S.) **7** (2012), 33–70.
23. C. M. Ringel and P. Zhang, *Gorenstein-projective modules and semi-Gorenstein-projective modules*, Algebra Number Theory **14** (2020), 1–36.
24. S. P. Smith, *Some finite-dimensional algebras related to elliptic curves*, Representation theory of algebras and related topics (Mexico City, 1994), CMS Conf. Proc., vol. 19, Amer. Math. Soc., Providence, RI, 1996, pp. 315–348.
25. H. Tachikawa, *Quasi-Frobenius rings and generalizations, QF-3 and QF-1 rings*, Lecture Notes in Math., vol. 351, Springer, Berlin-Heidelberg-New York, 1973.
26. T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra **275** (2004), 3–39.
27. J. Wang, Y. X. Li, J. Y. Wu, and J. S. Hu, *Auslander conditions and tilting-like cotorsion pairs*, J. Algebra **635** (2023), 220–234.
28. A. Zaks, *Injective dimension of semiprimary rings*, J. Algebra **13** (1969), 73–86.