

Auslander-type Conditions and Weakly Gorenstein Algebras ¹

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Abstract

Let R be an Artin algebra. Under certain Auslander-type conditions, we give some equivalent characterizations of (weakly) Gorenstein algebras in terms of the properties of Gorenstein projective modules and modules satisfying Auslander-type conditions. As applications, we provide some support for several homological conjectures. In particular, we prove that if R is left quasi Auslander, then R is Gorenstein if and only if it is (left and) right weakly Gorenstein; and that if R satisfies the Auslander condition, then R is Gorenstein if and only if it is left or right weakly Gorenstein. This is a reduction of an Auslander–Reiten’s conjecture, which states that R is Gorenstein if R satisfies the Auslander condition.

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1 Introduction

A left and right Noetherian ring is called *Iwanaga–Gorenstein* (*Gorenstein* for short) if its left and right self-injective dimensions are finite. The fundamental theorem in [6] states that a commutative Noetherian ring R is Gorenstein if and only if the flat dimension of the i -th term in a minimal injective coresolution of R as an R -module is at most $i - 1$ for any $i \geq 1$. In the non-commutative case, Auslander proved that the latter condition is left-right symmetric [9, Theorem 3.7]; in this case, R is said to satisfy the *Auslander condition*. Thus, the above result in [6] can be restated as follows: A commutative Noetherian ring is Gorenstein if and only if it satisfies the Auslander condition. Based on it, Auslander and Reiten [2] conjectured that an Artin algebra satisfying the Auslander condition is Gorenstein. We call this conjecture **ARC** for short. It is situated between the well known Nakayama conjecture and the generalized Nakayama conjecture [2, p.2]. All these conjectures remain still open.

As a generalization of the notion of the Auslander condition, Huang and Iyama [16] introduced the notion of Auslander-type conditions of rings as follows. For any $m \geq 0$, a left and right Noetherian ring is said to be $G_\infty(m)$ if for any finitely generated left R -module M and $i \geq 1$, it holds that $\text{Ext}_{R^{op}}^{0 \leq j \leq i-1}(X, R) = 0$ for any right R -submodule X of $\text{Ext}_R^{i+m}(M, R)$; equivalently, if the flat dimension of the i -th term in a minimal injective coresolution of R_R is at most $i + m - 1$ for any $i \geq 1$ [16, p.99]. Non-commutative rings satisfying Auslander-type conditions are analogues of commutative Gorenstein rings. Such rings play a crucial role in homological algebra, representation theory of algebras and non-commutative algebraic geometry, see [2, 3, 8, 9, 11, 12, 16, 18, 19, 21, 24, 26] and references therein. Recently, we introduced modules satisfying Auslander-type condition $G_\infty(m)$ for any $m \geq 0$ [15], see Definition 2.3 below.

As a generalization of the notion of Gorenstein algebras, Ringel and Zhang [23] introduced that of weakly Gorenstein algebras. Marczinzik [20] posed the following question: Is a left

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weakly Gorenstein Artin algebra also right weakly Gorenstein? For the sake of convenience, we state this question as the following conjecture.

Weakly-Gorenstein Symmetry Conjecture (WGSC for short): An Artin algebra is left weakly Gorenstein if and only if it is right weakly Gorenstein.

It is related to the following famous conjecture.

Gorenstein Symmetry Conjecture (GSC for short): For an Artin algebra, its left self-injective dimension is finite if and only if so is its right self-injective dimension.

Note that for a left and right Noetherian ring, its left and right self-injective dimensions coincide if both of them are finite [28, Lemma A]. Thus, an equivalent version of **GSC** is that for an Artin algebra, its left and right self-injective dimensions coincide.

It was proved that **WGSC** implies **GSC** [23, p.33], and that **GSC** holds true for Artin algebras satisfying the Auslander condition [2, Corollary 5.5(b)]. We proved that an Artin algebra satisfying the Auslander condition is Gorenstein if and only if the subcategory of finitely generated modules satisfying the Auslander condition is contravariantly finite [15, Theorem 5.8]. The aim of this paper is to give some equivalent characterizations of (weakly) Gorenstein algebras under certain Auslander-type conditions, and then provide some support for these conjectures mentioned above.

The paper is organized as follows. In Section 2, we give some terminology and preliminary results. Let R be an arbitrary ring. We use $\mathcal{GP}(\text{Mod } R)$ to denote the category of Gorenstein projective left R -modules. For any $m \geq 0$, we use $\mathcal{GP}(\text{Mod } R)^{\leq m}$ to denote the category of left R -modules with Gorenstein projective dimension at most m , and use $\mathcal{G}_\infty(m)$ to denote the category of left R -modules being $G_\infty(m)$.

In Section 3, R is an arbitrary ring. We prove that any module in $\mathcal{G}_\infty(m)$ is isomorphic to a kernel (resp. a cokernel) of a homomorphism from a module with finite flat dimension to certain syzygy module, and as a consequence we get that if a left R -module M satisfies the Auslander condition (that is, $M \in \mathcal{G}_\infty(0)$), then M is an ∞ -flat syzygy module, and the converse holds true if ${}_R R$ satisfies the Auslander condition (Theorem 3.3). For any $m, s \geq 0$, we prove that $\mathcal{GP}(\text{Mod } R) = \mathcal{G}_\infty(m)$ if and only if $\mathcal{GP}(\text{Mod } R)^{\leq s} = \mathcal{G}_\infty(m+s)$ (Proposition 3.5). We also prove that if R is a Gorenstein ring, then any module in $\mathcal{G}_\infty(m)$ has Gorenstein projective dimension at most m (Theorem 3.6).

In Section 4, R is an Artin algebra. We get some equivalent characterizations for ${}_R R \in \mathcal{G}_\infty(m)$ and R being Gorenstein as follows. The case for $m = 0$ in the following result except the statement (2) has been obtained in [27, Corollary 3.5], which is the Gorenstein version of [15, Theorem 5.9].

Theorem 1.1. (Theorem 4.6) *Let $m \geq 0$. Then the following statements are equivalent.*

- (1) ${}_R R \in \mathcal{G}_\infty(m)$ and R is Gorenstein.
 - (2) ${}_R R \in \mathcal{G}_\infty(m)$ and the left self-injective dimension of R is finite.
 - (3) $\mathcal{GP}(\text{Mod } R) \subseteq \mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m}$.
 - (4) $\mathcal{GP}(\text{Mod } R)^{\leq s} \subseteq \mathcal{G}_\infty(m+s) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m+s}$ for any $s \geq 0$.
- (i)_f *The finitely generated version of (i) with $i = 3, 4$.*

Under certain Auslander-type conditions, we get some equivalent characterizations of (weakly) Gorenstein algebras.

Theorem 1.2. (Theorem 4.9) *If ${}_R R \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{\text{op}}$ with $m, m' \geq 0$, then the following statements are equivalent.*

- (1) R is Gorenstein.

- (2) R is left and right weakly Gorenstein.
- (3) The left self-injective dimension of R is finite.
- (4) R is left weakly Gorenstein.
- (5) $\mathcal{GP}(\text{Mod } R)$ coincides with the left orthogonal category of projective left R -modules.
- (i)^{op} The opposite version of (i) with $3 \leq i \leq 5$.

Furthermore, we consider algebras satisfying small Auslander-type conditions. We prove that if R is left quasi Auslander (that is, ${}_R R \in \mathcal{G}_\infty(1)$), then R is Gorenstein if and only if the left or right self-injective dimension of R is finite, and if and only if R is (left and) right weakly Gorenstein (Theorem 4.10). Moreover, we get some equivalent characterizations of Auslander–Gorenstein algebras (Theorem 4.11), which yields that if R satisfies the Auslander condition (that is, ${}_R R \in \mathcal{G}_\infty(0)$), then R is Gorenstein if and only if R is left or right weakly Gorenstein (Corollary 4.12).

Consequently, we conclude that

- (1) Over an Artin algebra R satisfying ${}_R R \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{\text{op}}$ with $m, m' \geq 0$, both **WGSC** and **GSC** hold true (Theorem 1.2).
- (2) Over a left quasi Auslander Artin algebra, **GSC** holds true, but we do not know whether **WGSC** holds true or not (Theorem 4.10).
- (3) Assume that an Artin algebra R satisfies the Auslander condition (equivalently, ${}_R R \in \mathcal{G}_\infty(0)$ and $R_R \in \mathcal{G}_\infty(0)^{\text{op}}$). Then both **WGSC** and **GSC** hold true for R by putting $m = m' = 0$ in Theorem 1.2. Note that **GSC** holds true for an Artin algebra R satisfying the Auslander condition has been obtained in [2, Corollary 5.5(b)]. Moreover, we have that R is Gorenstein if and only if it is left or right weakly Gorenstein (Corollary 4.12). This is a reduction of **ARC**, since Gorenstein algebras are left and right weakly Gorenstein, but the converse does not hold true in general [20, 22, 23].

2 Preliminaries

Throughout this paper, all rings are associative rings with unit and all modules are unital. For a ring R , we use $\text{Mod } R$ to denote the category of left R -modules, and use $\text{mod } R$ to denote the category of finitely generated left R -modules. For a module $M \in \text{Mod } R$, we use $\text{pd}_R M$, $\text{fd}_R M$ and $\text{id}_R M$ to denote the projective, flat and injective dimensions of M , respectively.

Let R be a ring. We write $(-)^* = \text{Hom}(-, R)$. Let $M \in \text{Mod } R$ and let $\sigma_M : M \rightarrow M^{**}$ via $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$ be the canonical evaluation homomorphism. Recall that M is called *torsionless* if σ_M is a monomorphism, and is called *reflexive* if σ_M is an isomorphism. Let

$$\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow E^0(M) \rightarrow E^1(M) \rightarrow \cdots \rightarrow E^i(M) \rightarrow \cdots$$

be a projective resolution and a minimal injective coresolution of M , respectively. For any $n \geq 1$, $\Omega^n(M) := \text{Im}(P_n \rightarrow P_{n-1})$ and $\Omega^{-n}(M) := \text{Im}(E^{n-1}(M) \rightarrow E^n(M))$ are called the *n-syzygy* and *n-cosyzygy* of M , respectively. In particular, $\Omega^0(M) = M$. Note that the *n-syzygy* of M is defined up to projective summands. We write

$$\Omega^n(\text{Mod } R) := \{M \in \text{Mod } R \mid M \text{ is an } n\text{-syzygy module}\} \text{ for any } n \geq 1,$$

$$\Omega^\infty(\text{Mod } R) := \bigcap_{n \geq 1} \Omega^n(\text{Mod } R) \quad \text{and} \quad \Omega^\infty(\text{mod } R) := \Omega^\infty(\text{Mod } R) \cap \text{mod } R.$$

For a subcategory \mathcal{X} of $\text{Mod } R$, we write

$${}^{\perp}\mathcal{X} := \{M \in \text{Mod } R \mid \text{Ext}_R^{\geq 1}(M, X) = 0 \text{ for any } X \in \mathcal{X}\},$$

and write ${}^{\perp}X := {}^{\perp}\mathcal{X}$ if $\mathcal{X} = \{X\}$.

Let R be a left and right Noetherian ring and $M \in \text{mod } R$, and let

$$P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$$

be a projective presentation of M in $\text{mod } R$. Recall from [1] that $\text{Tr } M := \text{Coker } f^*$ is called the *transpose* of M . Note that the transpose of M is defined up to projective summands [1, p.51]. A module $M \in \text{mod } R$ is called ∞ -*torsionfree* if $\text{Tr } M \in {}^{\perp}R_R \cap \text{mod } R^{op}$. We write

$$\mathcal{T}(\text{mod } R) := \{M \in \text{mod } R \mid M \text{ is } \infty\text{-torsionfree}\}.$$

By [1, Theorem 2.17], we have $\mathcal{T}(\text{mod } R) \subseteq \Omega^{\infty}(\text{mod } R)$.

Definition 2.1. ([1]). Let R be a left and right Noetherian ring. A module $M \in \text{mod } R$ is said to *have Gorenstein dimension zero* if

$$\text{Ext}_R^{\geq 1}(M, R) = 0 = \text{Ext}_{R^{op}}^{\geq 1}(\text{Tr } M, R);$$

equivalently, if M is reflexive and

$$\text{Ext}_R^{\geq 1}(M, R) = 0 = \text{Ext}_{R^{op}}^{\geq 1}(M^*, R).$$

Let R be a ring. We write $\mathcal{P}(\text{Mod } R) := \{\text{projective left } R\text{-modules}\}$. Recall from [7] that a module $M \in \text{Mod } R$ is called *Gorenstein projective* if there exists an exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

in $\text{Mod } R$ with all P_i, P^i in $\mathcal{P}(\text{Mod } R)$, such that it remains exact after applying the functor $\text{Hom}_R(-, P)$ for any $P \in \mathcal{P}(\text{Mod } R)$ and $M \cong \text{Im}(P_0 \rightarrow P^0)$. We write

$\mathcal{GP}(\text{Mod } R) := \{\text{Gorenstein projective left } R\text{-modules}\}$ and $\mathcal{GP}(\text{mod } R) := \mathcal{GP}(\text{Mod } R) \cap \text{mod } R$.

It is well known that over a left and right noetherian ring, a finitely generated module has Gorenstein dimension zero if and only if it is Gorenstein projective [4, 7], and thus

$$\mathcal{GP}(\text{mod } R) = ({}^{\perp}R_R \cap \text{mod } R) \cap \mathcal{T}(\text{mod } R).$$

Now, finitely generated modules having Gorenstein dimension zero over left and right noetherian rings are usually referred to as Gorenstein projective modules.

For any $M \in \text{mod } R$ (resp. $\text{mod } R^{op}$), it is well known that M and $\text{Tr } \text{Tr } M$ are projectively equivalent. So we have the following observation.

Lemma 2.2. *Let R be a left and right Noetherian ring. Then a module $M \in \text{mod } R$ (resp. $\text{mod } R^{op}$) is Gorenstein projective if and only if so is $\text{Tr } M$.*

Recall from [9] that a left and right Noetherian ring R is said to satisfy the *Auslander condition* if $\text{fd}_R E^i({}_R R) \leq i$ for any $i \geq 0$. As a generalization of rings satisfying the Auslander condition, Huang and Iyama [16] introduced the notion of rings satisfying Auslander-type conditions, which was extended to that of modules satisfying Auslander-type conditions as follows.

Definition 2.3. ([15]) Let R be a ring and let $m \geq 0$. A module $M \in \text{Mod } R$ is said to be $G_\infty(m)$ if $\text{fd}_R E^i(M) \leq i + m$ for any $i \geq 0$. In particular, M is said to satisfy the *Auslander condition* if it is $G_\infty(0)$.

Let R be a left and right Noetherian ring. Then ${}_R R$ is $G_\infty(m)$ if and only if the ring R is $G_\infty(m)^{op}$ in the sense of [16] (cf. Introduction). Notice that the notion of the Auslander condition is left-right symmetric [9, Theorem 3.7], so the ring R satisfies the Auslander condition if and only if both ${}_R R$ and R_R satisfy the Auslander condition. However, in general, the notion of R being $G_\infty(m)$ is not left-right symmetric when $m \geq 1$ [3, 16]. It should be pointed out that modules satisfying Auslander-type conditions are ubiquitous. For example, if R is a left and right Noetherian ring and $\text{id}_{R^{op}} R \leq m$, then any module in $\text{Mod } R$ is $G_\infty(m)$ [15, Example 4.2(3)]. For more examples of modules satisfying Auslander-type conditions, the reader is referred to [15, Example 4.2].

Let \mathcal{X} be a subcategory of $\text{Mod } R$ and $M \in \text{Mod } R$. The \mathcal{X} -projective dimension $\mathcal{X}\text{-pd}_R M$ of M is defined as $\inf\{n \mid \text{there exists an exact sequence}$

$$0 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in $\text{Mod } R$ with all X_i in $\mathcal{X}\}$. If no such an integer exists, then set $\mathcal{X}\text{-pd}_R M = \infty$. For any $s \geq 0$, we write

$$\mathcal{X}^{\leq s} := \{M \in \text{Mod } R \mid \mathcal{X}\text{-pd}_R M \leq s\}.$$

When $\mathcal{X} = \mathcal{GP}(\text{Mod } R)$ or $\mathcal{GP}(\text{mod } R)$, the \mathcal{X} -projective dimension of M is exactly the Gorenstein projective dimension $\text{G-pd}_R M$ of M .

3 Syzygy modules and Gorenstein projective dimension

In this section, R is an arbitrary ring. For any $m \geq 0$, we write

$$\mathcal{G}_\infty(m) := \{M \in \text{Mod } R \mid M \text{ is } G_\infty(m)\}.$$

Then we have the following inclusion chain

$$\mathcal{G}_\infty(0) \subseteq \mathcal{G}_\infty(1) \subseteq \cdots \subseteq \mathcal{G}_\infty(m) \subseteq \cdots .$$

Lemma 3.1. *If R is a left Noetherian ring and ${}_R R \in \mathcal{G}_\infty(m)$, then any flat module in $\text{Mod } R$ is in $\mathcal{G}_\infty(m)$.*

Proof. It follows from [15, Corollary 3.2]. □

The following lemma is used frequently in the sequel.

Lemma 3.2. *Let*

$$0 \rightarrow M \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^i \rightarrow \cdots \tag{3.1}$$

be an exact sequence in $\text{Mod } R$ and let $m \geq 0$. If $X^i \in \mathcal{G}_\infty(m)$ for any $i \geq 0$, then $M \in \mathcal{G}_\infty(m)$. In particular, the subcategory $\mathcal{G}_\infty(m)$ is closed under kernels of epimorphisms.

Proof. By the exact sequence (3.1) and [13, Corollary 3.9(1)], we get the following exact sequence

$$0 \rightarrow M \rightarrow E^0(X^0) \rightarrow E^1(X^0) \oplus E^0(X^1) \rightarrow \cdots \rightarrow \bigoplus_{i=0}^n E^{n-i}(X^i) \rightarrow \cdots .$$

For the reader's convenience, we give an outline of the construction of this exact sequence, which is dual to that in the proof of [13, Theorem 3.6]. Put $M^i := \text{Im}(X^{i-1} \rightarrow X^i)$ for any $i \geq 1$. Let n be an arbitrary positive integer. We have an exact sequence

$$0 \rightarrow M^n \rightarrow E^0(X^n). \quad (3.2)$$

From (3.2) and the exact sequence $0 \rightarrow X^{n-1} \rightarrow E^0(X^{n-1}) \rightarrow E^1(X^{n-1})$, we obtain the following exact sequence

$$0 \rightarrow M^{n-1} \rightarrow E^0(X^{n-1}) \rightarrow E^1(X^{n-1}) \oplus E^0(X^n). \quad (3.3)$$

Then from (3.3) and the exact sequence $0 \rightarrow X^{n-2} \rightarrow E^0(X^{n-2}) \rightarrow E^1(X^{n-2}) \rightarrow E^2(X^{n-2})$, we obtain the following exact sequence

$$0 \rightarrow M^{n-2} \rightarrow E^0(X^{n-2}) \rightarrow E^1(X^{n-2}) \oplus E^0(X^{n-1}) \rightarrow E^2(X^{n-2}) \oplus E^1(X^{n-1}) \oplus E^0(X^n).$$

Continuing this process, we obtain the following exact sequence

$$0 \rightarrow M \rightarrow E^0(X^0) \rightarrow E^1(X^0) \oplus E^0(X^1) \rightarrow \dots \rightarrow \bigoplus_{i=0}^n E^{n-i}(X^i).$$

Then the desired exact sequence is obtained because of the arbitrariness of n . Since $X^i \in \mathcal{G}_\infty(m)$, we have $\text{fd}_R E^j(X^i) \leq j + m$ for any $i, j \geq 0$. So $\text{fd}_R \bigoplus_{i=0}^n E^{n-i}(X^i) \leq n + m$ for any $n \geq 0$, and thus $M \in \mathcal{G}_\infty(m)$. \square

For any $n \geq 1$, we write $\Omega_{\mathcal{F}}^n(\text{Mod } R) := \{M \in \text{Mod } R \mid \text{there exists an exact sequence}$

$$0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^{n-1}$$

in $\text{Mod } R$ with all F^i flat $\}$, and write $\Omega_{\mathcal{F}}^\infty(\text{Mod } R) := \bigcap_{n \geq 1} \Omega_{\mathcal{F}}^n(\text{Mod } R)$.

The first assertion in the following result shows that any module in $\mathcal{G}_\infty(m)$ is isomorphic to a kernel (resp. a cokernel) of a homomorphism from a module with finite flat dimension to certain syzygy module.

Theorem 3.3. *It holds that*

- (1) *Let $M \in \mathcal{G}_\infty(m)$ with $m \geq 0$. Then for any $n \geq 1$, there exists an exact sequence*

$$0 \rightarrow G_0 \rightarrow X_0 \rightarrow G_1 \rightarrow X_1 \rightarrow 0$$

in $\text{Mod } R$ with $M \cong \text{Im}(X_0 \rightarrow G_1)$ such that the following conditions are satisfied.

- (a) $\text{fd}_R G_0 \leq m - 1$ and $\text{fd}_R G_1 \leq m$.
(b) $X_0 \in \Omega_{\mathcal{F}}^n(\text{Mod } R)$ and $X_1 \in \Omega^{n-1}(\text{Mod } R)$.
(2) $\mathcal{G}_\infty(0) \subseteq \Omega_{\mathcal{F}}^\infty(\text{Mod } R)$ with equality if R is a left Noetherian ring and ${}_R R \in \mathcal{G}_\infty(0)$.

Proof. (1) Let $M \in \mathcal{G}_\infty(m)$ and $n \geq 1$. We have the following two exact and commutative

diagrams:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \dashrightarrow & K & \dashrightarrow & K_0 & \dashrightarrow & K_1 & \dashrightarrow \cdots \dashrightarrow K_{n-1} & \dashrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & X & \dashrightarrow & P_0 & \dashrightarrow & P_1 & \dashrightarrow \cdots \dashrightarrow P_{n-1} & \dashrightarrow & \Omega^{-n}(M) & \dashrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & M & \longrightarrow & E^0(M) & \longrightarrow & E^1(M) & \longrightarrow \cdots \longrightarrow E^{n-1}(M) & \longrightarrow & \Omega^{-n}(M) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\
& 0 & & 0 & & 0 & & 0 & & & &
\end{array}$$

and

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 & \dashrightarrow & K & \dashrightarrow & K_0 & \dashrightarrow & K'_1 & \dashrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \dashrightarrow & X & \dashrightarrow & P_0 & \dashrightarrow & X_1 & \dashrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M & \longrightarrow & E^0(M) & \longrightarrow & \Omega^{-1}(M) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 & & 0 &
\end{array}$$

with all P_i projective in $\text{Mod } R$ and $X_1 := \text{Im}(P_0 \rightarrow P_1) \in \Omega^{n-1}(\text{Mod } R)$. Since $M \in \mathcal{G}_\infty(m)$, we have $\text{fd}_R E^i(M) \leq i + m$ for any $i \geq 0$, and thus $\text{fd}_R K_i \leq i + m - 1$ for any $1 \leq i \leq n - 1$. It follows from the upper row in the first diagram that $\text{fd}_R K'_1 \leq m$.

Consider the following pull-back diagram (Diagram (3.1)):

$$\begin{array}{ccccccc}
& 0 & & 0 \\
& \downarrow & & \downarrow \\
& K'_1 & = & = & = & = & K'_1 \\
& \downarrow & & \downarrow \\
0 & \dashrightarrow & M & \dashrightarrow & G_1 & \dashrightarrow & X_1 & \dashrightarrow & 0 \\
& \parallel & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M & \longrightarrow & E^0(M) & \longrightarrow & \Omega^{-1}(M) & \longrightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & & 0 &
\end{array}$$

From the middle column, we obtain $\text{fd}_R G_1 \leq m$, and there exists an exact sequence

$$0 \rightarrow G_0 \rightarrow F \rightarrow G_1 \rightarrow 0$$

in $\text{Mod } R$ with F flat and $\text{fd}_R G_0 \leq m - 1$. Consider the following pull-back diagram (Diagram (3.2)):

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \\
& & G_0 & = & = & G_0 & \\
& & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & X_0 & \dashrightarrow & F & \dashrightarrow & X_1 \dashrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
& & \vdots & & \vdots & & \parallel \\
0 & \longrightarrow & M & \longrightarrow & G_1 & \longrightarrow & X_1 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

From the middle row, we obtain $X_0 \in \Omega_{\mathcal{F}}^n(\text{Mod } R)$. Now splicing the middle row in Diagram (3.1) and the leftmost column in Diagram (3.2) we get the desired exact sequence.

(2) To prove $\mathcal{G}_\infty(0) \subseteq \Omega_{\mathcal{F}}^\infty(\text{Mod } R)$, it suffices to prove that if $M \in \mathcal{G}_\infty(0)$, then $M \in \Omega_{\mathcal{F}}^n(\text{Mod } R)$ for any $n \geq 1$. Let $M \in \mathcal{G}_\infty(0)$ and $n \geq 1$. By (1), there exists an exact sequence

$$0 \rightarrow M \rightarrow F \rightarrow G_1 \rightarrow 0$$

in $\text{Mod } R$ with F flat and $G_1 \in \Omega_{\mathcal{F}}^{n-1}(\text{Mod } R)$, and so $M \in \Omega_{\mathcal{F}}^n(\text{Mod } R)$.

Now assume that R is a left Noetherian ring and ${}_R R \in \mathcal{G}_\infty(0)$. Then any flat module in $\text{Mod } R$ is in $\mathcal{G}_\infty(0)$ by Lemma 3.1, and thus $\Omega_{\mathcal{F}}^\infty(\text{Mod } R) \subseteq \mathcal{G}_\infty(0)$ by Lemma 3.2. \square

We need the following lemma.

Lemma 3.4. *For any $m, s \geq 0$, we have*

$$\mathcal{G}_\infty(m)^{\leq s} \subseteq \mathcal{G}_\infty(m + s)$$

with equality if $\mathcal{P}(\text{Mod } R) \subseteq \mathcal{G}_\infty(0)$.

Proof. Let $M \in \mathcal{G}_\infty(m)^{\leq s}$ and

$$0 \rightarrow X_s \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

be an exact sequence in $\text{Mod } R$ with all X_i in $\mathcal{G}_\infty(m)$. According to [13, Corollary 3.5], we get the following two exact sequences

$$0 \rightarrow M \rightarrow E \rightarrow \oplus_{i=0}^s E^{i+1}(X_i) \rightarrow \oplus_{i=0}^s E^{i+2}(X_i) \rightarrow \oplus_{i=0}^s E^{i+3}(X_i) \rightarrow \cdots, \quad (3.2)$$

$$0 \rightarrow E^s(X_0) \rightarrow E^{s-1}(X_0) \oplus E^s(X_1) \rightarrow \cdots \rightarrow \oplus_{i=1}^s E^{i-1}(X_i) \rightarrow \oplus_{i=0}^s E^i(X_i) \rightarrow E \rightarrow 0. \quad (3.3)$$

Since all X_i are in $\mathcal{G}_\infty(m)$, we have $\text{fd}_R E^j(X_i) \leq j + m$ for any $j \geq 0$ and $0 \leq i \leq s$. Thus $\text{fd}_R \oplus_{i=0}^s E^{i+j}(X_i) \leq j + m + s$ for any $j \geq 1$. By (3.3), we have that E is a direct summand of $\oplus_{i=0}^s E^i(X_i)$ and $\text{fd}_R E \leq m + s$. Therefore we obtain $M \in \mathcal{G}_\infty(m + s)$ by (3.2).

Now suppose $\mathcal{P}(\text{Mod } R) \subseteq \mathcal{G}_\infty(0)$. We will prove $\mathcal{G}_\infty(m + s) \subseteq \mathcal{G}_\infty(m)^{\leq s}$ by induction on s . The case for $s = 0$ follows trivially. Suppose $s \geq 1$ and $M \in \mathcal{G}_\infty(m + s)$. Let

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$$

be an exact sequence in $\text{Mod } R$ with P projective. Since $P \in \mathcal{G}_\infty(0)$, it follows from [15, Proposition 4.12] that $K \in \mathcal{G}_\infty(m + s - 1)$, and hence $\mathcal{G}_\infty(m)\text{-pd } K \leq s - 1$ by the induction hypothesis. This implies $\mathcal{G}_\infty(m)\text{-pd}_R M \leq s$ and $M \in \mathcal{G}_\infty(m)^{\leq s}$. \square

By Lemma 3.4, we obtain the following result.

Proposition 3.5. *If $\mathcal{P}(\text{Mod } R) \subseteq \mathcal{G}_\infty(0)$, then it holds that*

- (1) $\mathcal{GP}(\text{Mod } R) = \mathcal{G}_\infty(m)$ if and only if $\mathcal{GP}(\text{Mod } R)^{\leq s} = \mathcal{G}_\infty(m + s)$ for any $s \geq 0$.
- (2) If R is a left and right Noetherian ring, then $\mathcal{GP}(\text{mod } R) = \mathcal{G}_\infty(m) \cap \text{mod } R$ if and only if $\mathcal{GP}(\text{mod } R)^{\leq s} = \mathcal{G}_\infty(m + s) \cap \text{mod } R$ for any $s \geq 0$.

About the condition $\mathcal{P}(\text{Mod } R) \subseteq \mathcal{G}_\infty(0)$ in Proposition 3.5, we remark that if R is a left Noetherian ring, then this condition is satisfied if and only if ${}_R R$ satisfied the Auslander condition by [15, Theorem 4.9], and that if R is an Artin algebra, then $\mathcal{P}(\text{Mod } R) = \mathcal{G}_\infty(0)$ if and only if R is Auslander-regular (that is, the algebra R satisfies Auslander condition and the global dimension of R is finite) [15, Theorem 5.9].

Theorem 3.6. *It holds that*

- (1) If R is a Gorenstein ring, then $\mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m}$ for any $m \geq 0$.
- (2) If R is a left Noetherian ring and $\text{id}_R R < \infty$, then $\mathcal{GP}(\text{Mod } R) = \Omega^\infty(\text{Mod } R)$.

Proof. (1) Let R be a Gorenstein ring with $\text{id}_R R = \text{id}_{R^{op}} R \leq n$, and let $M \in \mathcal{G}_\infty(m)$. Then $\text{G-pd}_R M \leq n$ by [7, Theorem 12.3.1]. It suffices to prove $\text{G-pd}_R M \leq m$. The case for $n \leq m$ is trivial. Now suppose $n > m$ and $t := n - m$. Consider the following exact sequence

$$0 \rightarrow M \rightarrow E^0(M) \rightarrow E^1(M) \rightarrow \cdots \rightarrow E^{t-1}(M) \rightarrow K^t \rightarrow 0,$$

where $K^t := \text{Im}(E^{t-1}(M) \rightarrow E^t(M))$. By [7, Theorem 12.3.1] again, we have $\text{G-pd}_R K^t \leq n (= t + m)$. Since $M \in \mathcal{G}_\infty(m)$, we have $\text{pd}_R E^i(M) \leq i + m$ for any $0 \leq i \leq t - 1$. Then it is easy to get $\text{G-pd}_R M \leq m$ by [14, Theorem 3.2 and Remark 4.4(3)(a)].

(2) It suffices to prove $\Omega^\infty(\text{Mod } R) \subseteq \mathcal{GP}(\text{Mod } R)$. If R is a left Noetherian ring and $\text{id}_R R < \infty$, then $\text{id}_R P < \infty$ for any $P \in \mathcal{P}(\text{Mod } R)$ by [5, Theorem 1.1]. Assume that $M \in \Omega^\infty(\text{Mod } R)$ and

$$0 \rightarrow M \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots \rightarrow P^i \rightarrow \cdots$$

is an exact sequence in $\text{Mod } R$ with all P^i in $\mathcal{P}(\text{Mod } R)$. It is easy to see that the kernel of each homomorphism in the above exact sequence is in ${}^\perp \mathcal{P}(\text{Mod } R)$ by dimension shifting. Thus $M \in \mathcal{GP}(\text{Mod } R)$ and $\Omega^\infty(\text{Mod } R) \subseteq \mathcal{GP}(\text{Mod } R)$. \square

4 (Weakly) Gorenstein algebras

In this section, R is an Artin algebra. Under certain Auslander-type conditions, we will give some equivalent characterizations for $\text{id}_R R < \infty$ as well as for (weakly) Gorenstein algebras. As applications, we give some partial answers to some related homological conjectures.

4.1 Auslander-type conditions

For any $M \in \text{Mod } R$ and $m \geq 0$, we write

$${}^{\perp_{\geq m+1}} M := \{A \in \text{Mod } R \mid \text{Ext}_R^{\geq m+1}(A, M) = 0\}.$$

Lemma 4.1. *Let $M \in \text{mod } R$ such that $\Omega^\infty(\text{mod } R) \subseteq {}^{\perp_{\geq m+1}} M \cap \text{mod } R$ for some $m \geq 0$. If there exists some $n \geq 0$ such that $\text{pd}_R E^i(M) \leq n$ for any $i \geq n + m + 1$, then $\text{id}_R M \leq n + m$.*

Proof. Let $M \in \text{mod } R$. Set $K^i := \text{Im}(E^{i-1}(M) \rightarrow E^i(M))$ for any $i \geq 1$. Since $\text{pd}_R E^i(M) \leq n$ for any $i \geq n + m + 1$, by the horseshoe lemma we obtain the following exact and commutative diagram

$$\begin{array}{cccccccc}
& & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_n^{n+m+1} & \longrightarrow & P_n^{n+m+1} & \longrightarrow & P_n^{n+m+2} & \longrightarrow & \dots & \longrightarrow & P_n^{n+m+i} & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-1}^{n+m+1} & \longrightarrow & P_{n-1}^{n+m+2} & \longrightarrow & \dots & \longrightarrow & P_{n-1}^{n+m+i} & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1^{n+m+1} & \longrightarrow & P_1^{n+m+2} & \longrightarrow & \dots & \longrightarrow & P_1^{n+m+i} & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0^{n+m+1} & \longrightarrow & P_0^{n+m+2} & \longrightarrow & \dots & \longrightarrow & P_0^{n+m+i} & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^{n+m+1} & \longrightarrow & E^{n+m+1}(M) & \longrightarrow & E^{n+m+2}(M) & \longrightarrow & \dots & \longrightarrow & E^{n+m+i}(M) & \longrightarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

in $\text{mod } R$ with all P_j and P_j^t projective. Then $K_n^{n+m+1} \in \Omega^\infty(\text{mod } R)$, and thus $K_n^{n+m+1} \in {}^{\perp_{\geq m+1}} M \cap \text{mod } R$ by assumption. It follows from the leftmost column in the above diagram that $K^{n+m+1} \in {}^{\perp_{\geq n+m+1}} M \cap \text{mod } R$. Now applying the functor $\text{Hom}_R(K^{n+m+1}, -)$ to the exact sequence

$$0 \rightarrow M \rightarrow E^0(M) \rightarrow E^1(M) \rightarrow \dots \rightarrow E^{n+m-1}(M) \rightarrow K^{n+m} \rightarrow 0$$

yields $\text{Ext}_R^1(K^{n+m+1}, K^{n+m}) = 0$. It implies that the exact sequence

$$0 \rightarrow K^{n+m} \rightarrow E^{n+m}(M) \rightarrow K^{n+m+1} \rightarrow 0$$

splits and K^{n+m} is a direct summand of $E^{n+m}(M)$. Thus K^{n+m} is injective and $\text{id}_R M \leq n + m$. \square

Remark 4.2. The same argument as above essentially proves the following result: Let R be an arbitrary ring (not necessarily an Artin algebra) and let $M \in \text{Mod } R$ such that $\Omega^\infty(\text{Mod } R) \subseteq {}^{\perp_{\geq m+1}} M$ for some $m \geq 0$. If there exists some $n \geq 0$ such that $\text{pd}_R E^i(M) \leq n$ for any $i \geq n + m + 1$, then $\text{id}_R M \leq n + m$.

Recall from [23] that an Artin algebra R is called *left weakly Gorenstein* if $\mathcal{GP}(\text{mod } R) = {}^{\perp}_R R \cap \text{mod } R$. Symmetrically, the notion of *right weakly Gorenstein algebras* is defined.

Proposition 4.3.

- (1) *Assume that there exists some $n, m \geq 0$ such that $\text{pd}_R E^i({}_R R) \leq n$ for any $i \geq n + m + 1$. If $\Omega^\infty(\text{mod } R) \subseteq {}^{\perp_{\geq m+1}} {}_R R \cap \text{mod } R$, then $\text{id}_R R \leq n + m$.*

- (2) Assume that there exists some $n \geq 0$ such that $\text{pd}_R E^i({}_R R) \leq n$ for any $i \geq n+1$. If R is right weakly Gorenstein and $\Omega^\infty(\text{mod } R) = \mathcal{T}(\text{mod } R)$, then $\text{id}_R R \leq n$.

Proof. (1) Putting $M = {}_R R$ in Lemma 4.1, the assertion follows.

(2) Let $M \in \Omega^\infty(\text{mod } R)$. Then $M \in \mathcal{T}(\text{mod } R)$ by assumption, and so $\text{Tr } M \in {}^\perp_R R \cap \text{mod } R^{op}$. Since R is right weakly Gorenstein by assumption, we have $\text{Tr } M \in {}^\perp_R R \cap \text{mod } R^{op} = \mathcal{GP}(\text{mod } R^{op})$. Thus $M \in \mathcal{GP}(\text{mod } R) \subseteq {}^\perp_R R \cap \text{mod } R$ by Lemma 2.2. This shows $\Omega^\infty(\text{mod } R) \subseteq {}^\perp_R R \cap \text{mod } R$, and then the assertion follows from (1). \square

The following lemma shows that all modules satisfying certain Auslander-type condition over an Artin algebra satisfy the condition about projective dimension in Lemma 4.1.

Lemma 4.4. *If $M \in \mathcal{G}_\infty(m)$ (resp. $N \in \mathcal{G}_\infty(m)^{op}$) with $m \geq 0$, then there exists some $n \geq 0$ such that $\text{pd}_R E^i(M)$ (resp. $\text{pd}_{R^{op}} E^i(N)$) $\leq n$ for any $i \geq 0$.*

Proof. Since R is an Artin algebra, there exist only finitely many non-isomorphic indecomposable injective left R -modules. Let $M \in \mathcal{G}_\infty(m)$. Without loss of generalization, suppose that $\{E^0, \dots, E^t\}$ is the complete set of non-isomorphic indecomposable injective left modules that occur as direct summands of all $E^i(M)$. Then there exists some $n \geq 0$ such that $\text{pd}_R E^i \leq n$ for any $1 \leq i \leq t$, and thus $\text{pd}_R E^i(M) \leq n$ for any $i \geq 0$. Symmetrically, if $N \in \mathcal{G}_\infty(m)^{op}$, then there exists some $n \geq 0$ such that $\text{pd}_{R^{op}} E^i(N) \leq n$ for any $i \geq 0$. \square

As a consequence, we obtain the following result.

Proposition 4.5. *If $\mathcal{GP}(\text{mod } R) = \mathcal{G}_\infty(m) \cap \text{mod } R$ for some $m \geq 0$, then $\text{id}_R R < \infty$.*

Proof. Since ${}_R R \in \mathcal{GP}(\text{mod } R)$, we have ${}_R R \in \mathcal{G}_\infty(m)$ by assumption. It follows from Lemma 4.4 that $\text{pd}_R E^i({}_R R) \leq n$ for any $i \geq 0$. Since any projective module in $\text{mod } R$ is in $\mathcal{G}_\infty(m)$, we have

$$\begin{aligned} \Omega^\infty(\text{mod } R) &\subseteq \mathcal{G}_\infty(m) \cap \text{mod } R \quad (\text{by Lemma 3.2}) \\ &= \mathcal{GP}(\text{mod } R) \quad (\text{by assumption}) \\ &\subseteq {}^\perp_R R \cap \text{mod } R. \end{aligned}$$

Thus $\text{id}_R R \leq n$ by Proposition 4.3(1). \square

We are now in a position to prove the following result, in which assertions (5) and (6) are finitely generated versions of (3) and (4), respectively.

Theorem 4.6. *For any $m \geq 0$, the following statements are equivalent.*

- (1) ${}_R R \in \mathcal{G}_\infty(m)$ and R is Gorenstein.
- (2) ${}_R R \in \mathcal{G}_\infty(m)$ and $\text{id}_R R < \infty$.
- (3) $\mathcal{GP}(\text{Mod } R) \subseteq \mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m}$.
- (4) $\mathcal{GP}(\text{Mod } R)^{\leq s} \subseteq \mathcal{G}_\infty(m+s) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m+s}$ for any $s \geq 0$.
- (5) $\mathcal{GP}(\text{mod } R) \subseteq \mathcal{G}_\infty(m) \cap \text{mod } R \subseteq \mathcal{GP}(\text{mod } R)^{\leq m}$.
- (6) $\mathcal{GP}(\text{mod } R)^{\leq s} \subseteq \mathcal{G}_\infty(m+s) \cap \text{mod } R \subseteq \mathcal{GP}(\text{mod } R)^{\leq m+s}$ for any $s \geq 0$.

Proof. The implications (1) \implies (2), (4) \implies (3) \implies (5) and (4) \implies (6) \implies (5) are trivial. By the symmetric version of [11, Corollary 3], we get (2) \implies (1).

(1) \implies (3) Since R is Gorenstein by (1), we have $\mathcal{G}_\infty(m) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m}$ by Theorem 3.6(1). On the other hand, since ${}_R R \in \mathcal{G}_\infty(m)$ by (1), we have $\mathcal{P}(\text{Mod } R) \subseteq \mathcal{G}_\infty(m)$ by Lemma 3.1, and thus

$$\mathcal{GP}(\text{Mod } R) \subseteq \Omega^\infty(\text{Mod } R) \subseteq \mathcal{G}_\infty(m)$$

by Lemma 3.2.

(5) \implies (2) Since any projective module in $\text{mod } R$ is in $\mathcal{G}_\infty(m) \cap \text{mod } R$ by (5), we have

$$\Omega^\infty(\text{mod } R) \subseteq \mathcal{G}_\infty(m) \cap \text{mod } R \subseteq \mathcal{GP}(\text{mod } R)^{\leq m} \subseteq {}^{\perp_{\geq m+1}} R \cap \text{mod } R$$

by Lemma 3.2 and (5). Since ${}_R R \in \mathcal{G}_\infty(m) \cap \text{mod } R$, there exists some $n \geq 0$ such that $\text{pd}_R E^i({}_R R) \leq n$ for any $i \geq 0$ by Lemma 4.4, and thus $\text{id}_R R \leq n + m$ by Proposition 4.3(1).

(1) + (3) \implies (4) Let $s \geq 0$. Since $\mathcal{GP}(\text{Mod } R) \subseteq \mathcal{G}_\infty(m)$ by (3), we have

$$\mathcal{GP}(\text{Mod } R)^{\leq s} \subseteq \mathcal{G}_\infty(m)^{\leq s} \subseteq \mathcal{G}_\infty(m + s)$$

by Lemma 3.4. Since R is Gorenstein by (1), we have $\mathcal{G}_\infty(m + s) \subseteq \mathcal{GP}(\text{Mod } R)^{\leq m+s}$ by Theorem 3.6(1). \square

We need the following result.

Proposition 4.7. *If $\text{id}_R R < \infty$, then R is right weakly Gorenstein. The converse holds true if one of the following conditions is satisfied.*

- (1) ${}_R R \in \mathcal{G}_\infty(1)$.
- (2) ${}_R R \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{op}$ for some $m, m' \geq 0$.

Proof. The former assertion follows from the symmetric versions of [17, Lemma 3.4] and [23, Theorem 1.2].

Conversely, since ${}_R R \in \mathcal{G}_\infty(1)$ or ${}_R R \in \mathcal{G}_\infty(m)$ with $m \geq 0$ by assumption, it follows from Lemma 4.4 that there exists some $n \geq 0$ such that $\text{pd}_R E^i({}_R R) \leq n$ for any $i \geq 0$. When ${}_R R \in \mathcal{G}_\infty(1)$, we have $\Omega^\infty(\text{mod } R) = \mathcal{T}(\text{mod } R)$ by [3, Proposition 1.6(a)] and the symmetric version of [3, Theorem 0.1]; when ${}_R R \in \mathcal{G}_\infty(m)$ with $m \geq 0$, that is, the algebra R is $\mathcal{G}_\infty(m)$, we also have $\Omega^\infty(\text{mod } R) = \mathcal{T}(\text{mod } R)$ by [16, Theorem 3.4]. Thus $\text{id}_R R \leq n$ by Proposition 4.3(2). \square

The following corollary was proved in [23, p.33], we give it a shorter proof.

Corollary 4.8. *WGSC implies GSC.*

Proof. Suppose that **WGSC** holds true. If $\text{id}_R R = n < \infty$, then R is right weakly Gorenstein by Proposition 4.7, and hence is left weakly Gorenstein. It follows that any n -syzygy module in $\text{mod } R$ is in ${}^{\perp} R \cap \text{mod } R = \mathcal{GP}(\text{mod } R)$. So $\text{G-pd}_R M \leq n$ for any $M \in \text{mod } R$, and hence R is n -Gorenstein (that is, $\text{id}_R R = \text{id}_{R^{op}} R \leq n$) by [7, Theorem 12.3.1]. Symmetrically, we have that if $\text{id}_{R^{op}} R = n < \infty$, then R is n -Gorenstein. Thus **GSC** holds true. \square

The following result shows that the Gorensteinness and weakly Gorensteinness of an Artin algebra are equivalent under certain Auslander-type conditions. It also shows that both **GSC** and **WGSC** hold true for an Artin algebra R such that ${}_R R$ and R_R satisfy certain Auslander-type conditions.

Theorem 4.9. *If ${}_R R \in \mathcal{G}_\infty(m)$ and $R_R \in \mathcal{G}_\infty(m')^{op}$ with $m, m' \geq 0$, then the following statements are equivalent.*

- (1) R is Gorenstein.
- (2) R is left and right weakly Gorenstein.
- (3) $\text{id}_R R < \infty$.
- (4) R is left weakly Gorenstein.
- (5) $\mathcal{GP}(\text{Mod } R) = {}^{\perp} \mathcal{P}(\text{Mod } R)$.

$(i)^{op}$ *Opposite version of (i) with $3 \leq i \leq 5$.*

Proof. It is trivial that (5) \implies (4) and (2) \implies (4). By Proposition 4.7 and its symmetric version, we have (1) \implies (2) and (3) \iff (4)^{op}. By Theorem 4.6 and its symmetric version, we have (1) \iff (3) \iff (3)^{op}. By [7, Corollary 11.5.3], we have (1) \implies (5).

By symmetry, the proof is finished. \square

4.2 Small Auslander-type conditions

Recall from [12] that R is called *left quasi Auslander* if ${}_R R \in \mathcal{G}_\infty(1)$. Compare the following result with Theorem 4.9.

Theorem 4.10. *Let R be a left quasi Auslander algebra. Then the following statements are equivalent.*

- (1) R is Gorenstein.
- (2) $\text{id}_R R < \infty$.
- (3) $\text{id}_{R^{op}} R < \infty$.
- (4) R is left and right weakly Gorenstein.
- (5) R is right weakly Gorenstein.
- (6) $\mathcal{GP}(\text{Mod } R^{op}) = {}^\perp \mathcal{P}(\text{Mod } R^{op})$.

Proof. It is trivial that (4) \implies (5) and (6) \implies (5).

By Proposition 4.7 and its symmetric version, we have (1) \iff (4). By [11, Corollary 4], we have (1) \iff (2) \iff (3). By Proposition 4.7(1), we have (2) \iff (5). By [7, Corollary 11.5.3], we have (1) \implies (6). \square

Theorem 4.10 means that over a left quasi Auslander Artin algebra, **GSC** holds true, but we do not know whether **WGSC** holds true or not.

Recall that R is called *Auslander–Gorenstein* if R satisfies the Auslander condition and R is Gorenstein. In the following result, assertions (5)–(7) are finitely generated versions of (2)–(4) respectively.

Theorem 4.11. *The following statements are equivalent.*

- (1) R is Auslander–Gorenstein.
 - (2) R satisfies the Auslander condition and $\mathcal{GP}(\text{Mod } R) = {}^\perp \mathcal{P}(\text{Mod } R)$.
 - (3) $\mathcal{GP}(\text{Mod } R) = \mathcal{G}_\infty(0)$.
 - (4) $\mathcal{GP}(\text{Mod } R)^{\leq s} = \mathcal{G}_\infty(s)$ for any $s \geq 0$.
 - (5) R satisfies the Auslander condition and R is left weakly Gorenstein.
 - (6) $\mathcal{GP}(\text{mod } R) = \mathcal{G}_\infty(0) \cap \text{mod } R$.
 - (7) $\mathcal{GP}(\text{mod } R)^{\leq s} = \mathcal{G}_\infty(s) \cap \text{mod } R$ for any $s \geq 0$.
- $(i)^{op}$ *Opposite version of (i) with $2 \leq i \leq 7$.*

Proof. The implications (2) \implies (5), (3) \implies (6) and (4) \implies (7) are trivial.

Note that R satisfies the Auslander condition if and only if ${}_R R \in \mathcal{G}_\infty(0)$ and $R_R \in \mathcal{G}_\infty(0)^{op}$, and if and only if ${}_R R \in \mathcal{G}_\infty(0)$ or $R_R \in \mathcal{G}_\infty(0)^{op}$. The implications (3) \iff (4) and (6) \iff (7) follow from Proposition 3.5(1)(2) respectively. The implication (6) \implies (1) follows from Proposition 4.5 and [2, Corollary 5.5(b)]. The implications (1) \iff (3) and (1) \iff (2) \iff (5) follow from Theorems 4.6 and 4.9, respectively.

By symmetry, the proof is finished. \square

Let M be an R -module. An injective coresolution

$$0 \rightarrow M \rightarrow E^0 \xrightarrow{\delta^1} E^1 \xrightarrow{\delta^2} \dots \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} \dots$$

is called *ultimately closed* if there exists some n such that $\text{Im } \delta^n = \bigoplus W_j$ with each W_j isomorphic to a direct summand of some $\text{Im } \delta_{i_j}$ with $i_j < n$. It is clear that a left R -module M has an ultimately closed injective coresolution if $\text{id}_R M < \infty$. An algebra R is said to be of *ultimately closed type* if the minimal injective coresolution of any finitely generated left R -module is ultimately closed [25]. The class of algebras of ultimately closed type includes: (1) Artin algebras with finite global dimension; (2) Artin algebras with radical square zero; (3) Representation-finite algebras; (4) Artin algebras R with Loewy length m such that R/J^{m-1} is representation-finite, where J is the Jacobson radical of R [25, p.110].

Recall from [22] that R is called *torsionless-finite* if there exists only finitely many isomorphism classes of indecomposable torsionless modules in $\text{mod } R$. We claim that any torsionless-finite algebra is of ultimately closed type. Let R be a torsionless-finite algebra. It follows from [22, Corollary 2.2] that R^{op} is also a torsionless-finite algebra and there exists only finitely many isomorphism classes of indecomposable torsionless modules in $\text{mod } R^{op}$. Using the usual duality between $\text{mod } R$ and $\text{mod } R^{op}$ yields that there exists only finitely many isomorphism classes of indecomposable 1-cosyzygy modules in $\text{mod } R$. Thus R is of ultimately closed type. The claim is proved. The class of torsionless-finite algebras includes: (1) Artin algebras R with $R/\text{soc}(R_R)$ representation-finite, where $\text{soc}(R_R)$ is the socle of R_R ; (2) Minimal representation-infinite algebras; (3) Artin algebras stably equivalent to hereditary algebras; (4) Left or right glued algebras; and (5) Special biserial algebras without indecomposable projective-injective modules [22, Section 5].

Note that algebras R such that ${}_R R$ has an ultimately closed injective coresolution (particularly, algebras R of ultimately closed type) are right weakly Gorenstein algebras by the symmetric versions of [17, Theorem 2.4] and [23, Theorem 1.2]. However, such algebras are not Gorenstein in general, thus the following result can be regarded as a reduction of **ARC**.

Corollary 4.12. *If R satisfies the Auslander condition, then the following statements are equivalent.*

- (1) R is Gorenstein.
- (2) R is left or right weakly Gorenstein.
- (3) R is left and right weakly Gorenstein.
- (4) $\mathcal{GP}(\text{mod } R) = \mathcal{T}(\text{mod } R)$.
- (5) $\mathcal{GP}(\text{mod } R) = \mathcal{T}(\text{mod } R) = {}^\perp {}_R R \cap \text{mod } R$.

Proof. Since R satisfies the Auslander condition, we have

$$\mathcal{G}_\infty(0) \cap \text{mod } R = \Omega^\infty(\text{mod } R) = \mathcal{T}(\text{mod } R)$$

by [15, Lemma 5.7]. Now the assertion follows from Theorem 4.11. \square

As indicated above, if ${}_R R$ (resp. R_R) has an ultimately closed injective coresolution, then R is a right (resp. left) weakly Gorenstein algebra. As a consequence of Corollary 4.12, we obtain the following result.

Corollary 4.13. ***ARC** holds true for the following classes of algebras R .*

- (1) ${}_R R$ or R_R has an ultimately closed injective coresolution.
- (2) Algebras of ultimately closed type.

In the following, we give an alternative proof of Corollary 4.13, which is independent of Corollary 4.12. Recall that the *finitistic dimension* $\text{fin.dim } R$ of R is defined as

$$\text{fin.dim } R := \sup\{\text{pd}_R M \mid M \in \text{mod } R \text{ with } \text{pd}_R M < \infty\}.$$

Assume that R satisfies the Auslander condition. Then R^{op} also satisfies the Auslander condition by [9, Theorem 3.7]. It follows from [2, Corollary 5.5(b)] that $\text{id}_R R < \infty$ if and only if $\text{id}_{R^{op}} R < \infty$, and hence $\text{id}_R R = \text{id}_{R^{op}} R$ by [28, Lemma A]. Then

$$\text{fin.dim } R = \text{id}_R R = \text{id}_{R^{op}} R = \text{fin.dim } R^{op} \tag{4.1}$$

by [16, Corollary 5.3(1)]. When R_R has an ultimately closed injective coresolution, it is known from [10, p.2983] that this injective coresolution has a strongly redundant image in the sense of [10]. Then applying [10, Theorem 3] yields $\text{fin.dim } R^{op} < \infty$. Symmetrically, when ${}_R R$ has an ultimately closed injective coresolution (particularly, when R is of ultimately closed type), we have $\text{fin.dim } R < \infty$. So, in both cases, we have $\text{id}_R R = \text{id}_{R^{op}} R < \infty$ by (4.1), that is, R is Gorenstein. Consequently we conclude that **ARC** holds true for R .

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