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HOMOLOGICAL INVARIANTS RELATED TO SEMIDUALIZING BIMODULES

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Abstract. Let R and S be rings and ${}_RC_S$ a semidualizing bimodule. We show that the supremum of the C-projective dimensions of C-flat left R-modules is less than or equal to that for left R-modules with finite C-projective dimension, and the latter is less than or equal to the supremum of the C-injective dimensions of projective (or flat) left S-modules. We also show that the supremum of the C-projective dimensions of injective left R-modules and that of the C-injective dimensions of projective left S-modules are identical provided that both of them are finite. Finally, we show that the supremum of the C-projective dimensions of C-flat left R-modules (a relative homological invariant) and that of the projective dimensions of flat left S-modules (an absolute homological invariant) coincide.

1. Introduction. The study of semidualizing modules in commutative rings was initiated by Foxby [10] and Golod [12]. Then Holm and White [16] extended it to arbitrary associative rings. Many authors have studied the properties of semidualizing modules and related modules; see for example [10], [12], [15]–[16], [25], [28], [32]–[40] and the references therein. Among various research areas, one basic theme is to extend the "absolute" classical results in homological algebra to the "relative" setting with respect to semidualizing modules. One of the motivations of this paper comes from a classical result due to Jensen [23, Proposition 6], which states that any flat left R-module has finite projective dimension over a ring R with finite left finitistic dimension. Simson [29, Theorem 2.7] extended this result to skeletally small additive categories. Another motivation comes from Emmanouil and Talelli's work [7], in which the relations between the supremum of the projective lengths of injective left R-modules, the supremum of the injective lengths of projective left R-modules, the finitistic dimension and the left self-injective dimension of a ring R were established. We are interested in whether these results have relative counterparts with respect to semidualizing modules. The paper is organized as follows.

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In Section 2, we give some terminology and some preliminary results.

Let R and S be rings and ${}_RC_S$ a semidualizing bimodule. In Section 3, we show that the supremum of the C-projective dimensions of C-flat left R-modules is less than or equal to that for left R-modules with finite C-projective dimension, and the latter is less than or equal to the supremum of the C-injective dimensions of projective (or flat) left S-modules. The former part of this result is a C-version of Jensen's result mentioned above.

In Section 4, we show that the supremum of the C-projective dimensions of injective left R-modules and the supremum of the C-injective dimensions of projective left S-modules are identical provided that both are finite. If S is a right coherent ring, then any C-Gorenstein projective left R-module is C-Gorenstein flat provided that the supremum of the C-projective dimensions of C-flat left R-modules is finite. At the end of this section, we give a negative answer to the following open question posed by White [40]: for a commutative ring R, if M is a left R-module with finite projective dimension, must the projective and C-Gorenstein projective dimensions of M be identical?

In Section 5, we prove that if R is a left noetherian ring, then the direct sum of the first $n+1$ terms in a minimal injective resolution of ${}_RC$ is a Σ-embedding cogenerator for the category of modules with C-projective dimension at most n; and if the supremum of the C-projective dimensions of C-flat left R-modules is at most m, then the direct sum of the first $m+n+1$ terms in a minimal injective resolution of ${}_RC$ is a Σ-embedding cogenerator for the category of modules with C-flat dimension at most n. Finally, we show that the supremum of the C-projective dimensions of C-flat left R-modules (a relative homological invariant) and the supremum of the projective dimensions of flat left S-modules (an absolute homological invariant) coincide.

2. Preliminaries. Throughout this paper, all rings are associative rings with unit. Let R be a ring. We use $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) to denote the category of left (resp. right) R-modules, and use $\operatorname{mod} R$ (resp. $\operatorname{mod} R^{\mathrm{op}}$) to denote the category of finitely presented left (resp. right) R-modules. Let $M \in \operatorname{Mod} R$. We write $\operatorname{Add}_R M$ (resp. $\operatorname{add}_R M$) for the subcategory of $\operatorname{Mod} R$ consisting of all direct summands of direct sums (resp. finite direct sums) of copies of M. We use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots$$

to denote a minimal injective resolution of M.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{Mod} R$. We write

$$\mathcal{X}^{\perp} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(X, M) = 0\},$$

$$^{\perp}\mathcal{X} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(M, X) = 0\}.$$

A sequence

$$\mathbb{M} := \cdots \to M_1 \to M_2 \to M_3 \to \cdots$$

in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(\mathcal{X}, -)$-*exact* (resp. $\operatorname{Hom}_R(-, \mathcal{X})$-*exact*) if $\operatorname{Hom}_R(X, \mathbb{M})$ (resp. $\operatorname{Hom}_R(\mathbb{M}, X)$) is exact for any $X \in \mathcal{X}$. An exact sequence

$$\cdots \to X_1 \to X_0 \to M \to 0$$

(of finite or infinite length) in $\operatorname{Mod} R$ is called an $\mathcal{X}$-*resolution* of M if all X_i are in $\mathcal{X}$. The $\mathcal{X}$-*projective dimension* $\mathcal{X}\text{-}\operatorname{pd}_R M$ of M is defined as $\inf\{n \mid$ there exists an $\mathcal{X}$-resolution

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

of M in $\operatorname{Mod} R\}$. Dually, the notions of an $\mathcal{X}$-*coresolution* and the $\mathcal{X}$-*injective dimension* $\mathcal{X}\text{-}\operatorname{id}_R M$ are defined. In particular, we use $\operatorname{pd}_R M$, $\operatorname{fd}_R M$ and $\operatorname{id}_R M$ to denote the projective, flat and injective dimensions of M respectively.

We first give the following

LEMMA 2.1. *Let $\mathcal{X}$ and $\mathcal{C}$ be full subcategories of* $\operatorname{Mod} R$ *with $\mathcal{C}$ additive.*

(1) *If $\mathcal{X} \cup \mathcal{C} \subseteq \mathcal{C}^{\perp}$ and $\mathcal{C}\text{-}\operatorname{pd}_R X \leq m$ $(< \infty)$ for any $X \in \mathcal{X}$, then for a module $M \in \operatorname{Mod} R$ with $\mathcal{X}\text{-}\operatorname{pd}_R M \leq n$ $(< \infty)$, we have $\mathcal{C}\text{-}\operatorname{pd}_R M \leq m + n$.*

(2) *If $\mathcal{X} \cup \mathcal{C} \subseteq {}^{\perp}\mathcal{C}$ and $\mathcal{C}\text{-}\operatorname{id}_R X \leq m$ $(< \infty)$ for any $X \in \mathcal{X}$, then for a module $M \in \operatorname{Mod} R$ with $\mathcal{X}\text{-}\operatorname{id}_R M \leq n$ $(< \infty)$, we have $\mathcal{C}\text{-}\operatorname{id}_R M \leq m + n$.*

Proof. (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{X}\text{-}\operatorname{pd}_R M \leq n$ and let

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0 \tag{2.1}$$

be an exact sequence in $\operatorname{Mod} R$ with all X_i in $\mathcal{X}$. Because $\mathcal{X} \subseteq \mathcal{C}^{\perp}$ by assumption, the exact sequence (2.1) is $\operatorname{Hom}_R(\mathcal{C}, -)$-exact. Since $\mathcal{C}\text{-}\operatorname{pd}_R X \leq m$ and $\mathcal{C} \subseteq \mathcal{C}^{\perp}$ by assumption, for any $0 \leq i \leq n$ we have a $\operatorname{Hom}_R(\mathcal{C}, -)$-exact exact sequence

$$0 \to C_i^m \to \cdots \to C_i^1 \to C_i^0 \to X_i \to 0$$

in $\operatorname{Mod} R$ with all C_i^j in $\mathcal{C}$. By [17, Corollary 3.7], we get an exact sequence

$$0 \to C_{m+n} \to \cdots \to C_1 \to C_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all C_t being direct sums of some modules in $\{C_i^j\}_{0 \leq i \leq n}^{0 \leq j \leq m}$. Because $\mathcal{C}$ is additive, all C_t are in $\mathcal{C}$ and $\mathcal{C}\text{-}\operatorname{pd}_R M \leq m + n$.

(2) This is dual to (1). ∎

DEFINITION 2.2 ([16]). Let R and S be rings.

(1) An R-S-bimodule ${}_RC_S$ is called *semidualizing* if:

(a1) ${}_RC$ admits a degreewise finite R-projective resolution.

(a2) C_S admits a degreewise finite S-projective resolution.

(b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \mathrm{Hom}_{S^{\mathrm{op}}}(C,C)$ is an isomorphism.

(b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \mathrm{Hom}_R(C,C)$ is an isomorphism.

(c1) $\mathrm{Ext}_R^{\geq 1}(C,C)=0$, that is, ${}_RC$ is *self-orthogonal.*

(c2) $\mathrm{Ext}_{S^{\mathrm{op}}}^{\geq 1}(C,C)=0$, that is, C_S is *self-orthogonal.*

(2) A semidualizing bimodule ${}_RC_S$ is called *faithful* if:

(f1) $M \in \mathrm{Mod}\, R$ and $\mathrm{Hom}_R(C,M)=0$ imply $M=0$.

(f2) $N \in \mathrm{Mod}\, S^{\mathrm{op}}$ and $\mathrm{Hom}_{S^{\mathrm{op}}}(C,N)=0$ imply $N=0$.

Typical examples of semidualizing bimodules include the free module of rank one, dualizing modules over a Cohen–Macaulay local ring and the ordinary Matlis dual bimodule ${}_\Lambda D(\Lambda)_\Lambda$ of ${}_\Lambda\Lambda_\Lambda$ over an artin algebra Λ. Over a commutative ring, all semidualizing modules are faithful [16, Proposition 3.1].

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. For convenience, we write $(-)_* := \mathrm{Hom}(C,-)$, and

$$ {}_RC^{\perp} := \{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{i\geq 1}(C,M)=0\}, $$
$$ {C_S}^{\top} := \{N \in \mathrm{Mod}\, S \mid \mathrm{Tor}^S_{i\geq 1}(C,N)=0\}. $$

Following [16], set

$$ \mathcal{F}_C(R) := \{C\otimes_S F \mid F \text{ is flat in } \mathrm{Mod}\, S\}, $$
$$ \mathcal{P}_C(R) := \{C\otimes_S P \mid P \text{ is projective in } \mathrm{Mod}\, S\}, $$
$$ \mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in } \mathrm{Mod}\, R\}. $$

The modules in $\mathcal{F}_C(R)$, $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called *C-flat*, *C-projective* and *C-injective* respectively. Symmetrically, the classes of $\mathcal{F}_C(S^{\mathrm{op}})$, $\mathcal{P}_C(S^{\mathrm{op}})$ and $\mathcal{I}_C(R^{\mathrm{op}})$ are defined. Set $(-)^+ := \mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$. We have the following

Lemma 2.3.

(1) *If $M \in \mathcal{F}_C(R)$, then $M^+ \in \mathcal{I}_C(R^{\mathrm{op}})$.*

(2) *If S is a right coherent ring and $N \in \mathcal{I}_C(R^{\mathrm{op}})$, then $N^+ \in \mathcal{F}_C(R)$.*

Proof. (1) This follows directly from the adjoint isomorphism theorem.

(2) Let S be a right coherent ring and $N \in \mathcal{I}_C(R^{\mathrm{op}})$. Then there exists an injective module I in $\mathrm{Mod}\, S^{\mathrm{op}}$ such that $N = I_*$. By [11, Lemma 2.16(c)],

$$ C\otimes_S I^+ \cong {I_*}^+ (= N^+). $$

By [9, Theorem 2.2], $I^+ \in \mathrm{Mod}\, S$ is flat. So N^+ $(\cong C\otimes_S I^+) \in \mathcal{F}_C(R)$. ∎

Let $M \in \mathrm{Mod}\, R$ and $N \in \mathrm{Mod}\, S$. Then we have two canonical valuation homomorphisms:

$$ \theta_M : C\otimes_S M_* \to M $$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$, and

$$\mu_N : N \to (C \otimes_S N)_*$$

defined by $\mu_N(x)(c) = c \otimes x$ for any $x \in N$ and $c \in C$.

Definition 2.4 ([16]).

(1) The *Auslander class* $\mathcal{A}_C(S)$ with respect to C consists of all left S-modules N satisfying the following conditions:
  (A1) $N \in {C_S}^{\top}$.
  (A2) $C \otimes_S N \in {}_RC^{\perp}$.
  (A3) μ_N is an isomorphism in $\operatorname{Mod} S$.

(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all left R-modules M satisfying the following conditions:
  (B1) $M \in {}_RC^{\perp}$.
  (B2) $M_* \in {C_S}^{\top}$.
  (B3) θ_M is an isomorphism in $\operatorname{Mod} R$.

For a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$ and $n \geq 0$, we write

$$\mathcal{X}\text{-pd}^{\leq n}(R) := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-pd}_R M \leq n\},$$
$$\mathcal{X}\text{-pd}^{<\infty}(R) := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-pd}_R M < \infty\}.$$

We use $\mathcal{I}(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of all injective modules. The following two lemmas will be used frequently.

Lemma 2.5.

(1) $\mathcal{I}(R) \cup \mathcal{F}_C(R)\text{-pd}^{<\infty}(R) \subseteq \mathcal{B}_C(R) \subseteq {}_RC^{\perp} = \mathcal{P}_C(R)^{\perp}$.
(2) $\mathcal{I}_C(R^{\mathrm{op}}) \subseteq {}^{\perp}\mathcal{I}_C(R^{\mathrm{op}})$ *and* $\mathcal{I}_C(S) \subseteq {}^{\perp}\mathcal{I}_C(S)$.

Proof. (1) By [16, Lemma 4.1 and Corollary 6.1] and [35, Theorem 3.9],

$$\mathcal{I}(R) \cup \mathcal{F}_C(R)\text{-pd}^{<\infty}(R) \subseteq \mathcal{B}_C(R) \subseteq {}_RC^{\perp}.$$

It is well known that $\operatorname{Ext}^n_R(\bigoplus_{i\in I} A_i, M) \cong \prod_{i\in I} \operatorname{Ext}^n_R(A_i, M)$ for any family $\{A_i\}_{i\in I}$ of modules, $M \in \operatorname{Mod} R$ and $n \geq 1$. Because $\mathcal{P}_C(R) = \operatorname{Add}_R C$ by [36, Proposition 3.4(2)], it is easy to get $\mathcal{P}_C(R)^{\perp} = {}_RC^{\perp}$.

(2) This follows from [16, Lemma 4.1 and Theorem 6.4(b)]. ∎

The following result is used frequently below.

Lemma 2.6 ([35, Theorem 3.9] and [36, Theorem 3.5]).

(1) $\operatorname{fd}_S M_* \leq \mathcal{F}_C(R)\text{-pd}_R M$ *for any* $M \in \operatorname{Mod} R$*, with equality holding if* $M \in \mathcal{B}_C(R)$.
(2) $\operatorname{pd}_S M_* \leq \mathcal{P}_C(R)\text{-pd}_R M$ *for any* $M \in \operatorname{Mod} R$*, with equality holding if* $M \in \mathcal{B}_C(R)$.
(3) $\operatorname{id}_R C \otimes_S N \leq \mathcal{I}_C(S)\text{-id}_S N$ *for any* $N \in \operatorname{Mod} S$*, with equality holding if* $N \in \mathcal{A}_C(S)$.

The following notions were introduced by Holm and Jøgensen [15] for commutative rings. We give their non-commutative versions.

Definition 2.7. Let M be in $\operatorname{Mod} R$.

(1) M is called *C-Gorenstein projective* if:

  (i) $\operatorname{Ext}_R^{\geq 1}(M, G) = 0$ for any $G \in \mathcal{P}_C(R)$.

  (ii) There exists a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence

$$\mathbb{G} := 0 \to M \to G^0 \to G^1 \to \cdots$$

  in $\operatorname{Mod} R$ with all G^i in $\mathcal{P}_C(R)$.

(2) M is called *C-Gorenstein flat* if:

  (i) $\operatorname{Tor}_{\geq 1}^R(E, M) = 0$ for any $E \in \mathcal{I}_C(R^{\mathrm{op}})$.

  (ii) There exists an exact sequence

$$\mathbb{Q} := 0 \to M \to Q^0 \to Q^1 \to \cdots$$

  in $\operatorname{Mod} R$ with all Q^i in $\mathcal{F}_C(R)$ such that $E \otimes_R \mathbb{Q}$ is exact for any $E \in \mathcal{I}_C(R^{\mathrm{op}})$.

We use $\mathcal{GP}_C(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of all C-Gorenstein projective modules. If we put ${}_RC_S = {}_RR_R$, then C-Gorenstein projective modules and C-Gorenstein flat modules are the classical *Gorenstein projective modules* and *Gorenstein flat modules* respectively [4, 5, 8, 14].

Lemma 2.8. *For a module $M \in \operatorname{Mod} R$, if $\mathcal{P}_C(R)$-$\operatorname{pd}_R M < \infty$, then $\mathcal{P}_C(R)$-$\operatorname{pd}_R M = \mathcal{GP}_C(R)$-$\operatorname{pd}_R M$.*

Proof. The case of commutative rings has been proved in [40, Proposition 2.16]. The argument there is also valid in our setting. ∎

3. The C-version of a result of Jensen. In this section, we investigate the relationships between some homological invariants related to ${}_RC_S$. We first define the *finitistic C-projective dimension* $\mathrm{F}\mathcal{P}_C$-$\dim R$ of R as

$$\mathrm{F}\mathcal{P}_C\text{-}\dim R := \sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M < \infty\},$$

and the *finitistic C-Gorenstein projective dimension* $\mathrm{F}\mathcal{GP}_C$-$\dim R$ as

$$\mathrm{F}\mathcal{GP}_C\text{-}\dim R := \sup\{\mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M < \infty\}.$$

We write the supremum of the C-projective dimensions of C-flat left R-modules as

$$\operatorname{spclfc} R := \sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M \mid M \in \mathcal{F}_C(R)\}.$$

The following result is a C-version of [23, Proposition 6]. It plays a key role in what follows.

Proposition 3.1. $\operatorname{spclfc} R \leq \mathrm{F}\mathcal{P}_C$-$\dim R$.

Proof. The proof is modified from [23, Proposition 6]. Let $\mathrm{F}\mathcal{P}_C$-$\dim R < \infty$ and $M \in \mathcal{F}_C(R)$. Then $M \cong C \otimes_S F$ for some flat module F in $\mathrm{Mod}\, S$. Now take an exact sequence

$$0 \to B \to F_0 \to F \to 0 \tag{3.1}$$

in $\mathrm{Mod}\, S$ with F_0 free and B flat. Assume that B is generated by $\aleph$ elements, where $\aleph$ is a finite or an infinite cardinal number. We claim that $\mathrm{pd}_S B \le \mathrm{F}\mathcal{P}_C$-$\dim R$.

We proceed by transfinite induction on $\aleph$. If $\aleph \le \aleph_0$, then there exists a pure exact sequence

$$0 \to B \to F' \to F'/B \to 0$$

in $\mathrm{Mod}\, S$ such that F' is a free submodule of F_0 and F' is generated by at most $\aleph_0$ elements. Hence F'/B is a countably related flat module by [22]. Now it follows from [21, Lemma 2] (see also [27, Lemma 1.2]) that $\mathrm{pd}_S F'/B \le 1$. So B is projective and $\mathrm{pd}_S B \le \mathrm{F}\mathcal{P}_C$-$\dim R$. Next from the proof of [23, Proposition 6], we know that there exists a transfinite sequence $(C_\beta)_{\beta<\Omega}$ of pure submodules C_β such that $B = \bigcup_{\beta<\Omega} C_\beta$ with $C_{\beta_1} \subseteq C_{\beta_2}$ for $\beta_1 \le \beta_2$, and each C_β is generated by less than $\aleph$ elements. Then by the induction hypothesis, we have $\mathrm{pd}_S C_\beta \le \mathrm{F}\mathcal{P}_C$-$\dim R$. So $\mathrm{pd}_S B \le \mathrm{F}\mathcal{P}_C$-$\dim R$ by [2, Proposition 3]. The claim is proved.

By the claim and the exact sequence (3.1), we have $\mathrm{pd}_S F < \infty$. Notice that $F \in \mathcal{A}_S(C)$ by [16, Lemma 4.1], so $\mu_F : F \to (C \otimes_S F)_*$ is an isomorphism, and hence $\mathcal{P}_C(R)$-$\mathrm{pd}_R M = \mathcal{P}_C(R)$-$\mathrm{pd}_R(C \otimes_S F) = \mathrm{pd}_S(C \otimes_S F)_* = \mathrm{pd}_S F < \infty$ by Lemma 2.6(2). It follows that $\mathcal{P}_C(R)$-$\mathrm{pd}_R M \le \mathrm{F}\mathcal{P}_C$-$\dim R$. ∎

For a subcategory $\mathcal{X}$ of $\mathrm{Mod}\, R$, following [36] we write

$$\mathrm{id}_R \mathcal{X} := \sup\{\mathrm{id}_R X \mid X \in \mathcal{X}\}.$$

The following result improves [36, Proposition 3.6].

Corollary 3.2.

$$\sup\{\mathcal{P}_C(R)\text{-}\mathrm{pd}_R M \mid M \in \mathrm{Mod}\, R \textit{ with } \mathcal{F}_C(R)\text{-}\mathrm{pd}_R M < \infty\} \le \mathrm{F}\mathcal{P}_C\text{-}\dim R \le \mathrm{id}_R \mathcal{P}_C(R).$$

Proof. If $\mathrm{F}\mathcal{P}_C$-$\dim R < \infty$, then $\mathrm{spclfc}\, R \le \mathrm{F}\mathcal{P}_C$-$\dim R$ by Proposition 3.1. It follows from Lemmas 2.5 and 2.1(1) that $\mathcal{P}_C(R)$-$\mathrm{pd}_R M < \infty$ for any $M \in \mathrm{Mod}\, R$ with $\mathcal{F}_C(R)$-$\mathrm{pd}_R M < \infty$. Thus the first inequality follows.

Let $M \in \mathrm{Mod}\, R$ with $\mathcal{P}_C(R)$-$\mathrm{pd}_R M = n$ $(< \infty)$ and

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to M \to 0$$

be an exact sequence in $\mathrm{Mod}\, R$ with all C_i in $\mathcal{P}_C(R)$ $(= \mathrm{Add}_R C$ by [36, Proposition 3.4(2)]). Then $\mathrm{Ext}^n_R(M, C_n) \ne 0$ and $\mathrm{id}_R C_n \ge n$. So $\mathrm{id}_R \mathcal{P}_C(R) \ge n$ and the second inequality follows. ∎

Motivated by [7, Section 2], we write the supremum of the C-injective dimensions of projective left S-modules as

$$\operatorname{siclp} S := \sup\{\mathcal{I}_C(S)\text{-id}_S\, P \mid P \in \operatorname{Mod} S \text{ is projective}\},$$

and write the supremum of the C-injective dimensions of flat left S-modules as

$$\operatorname{siclf} S := \sup\{\mathcal{I}_C(S)\text{-id}_S\, F \mid F \in \operatorname{Mod} S \text{ is flat}\}.$$

In the special case of those commutative noetherian rings S with $\operatorname{siclp} S \leq n$, it was proved in [33, Theorem 2.6] that these are precisely the rings over which every finitely generated module can be embedded into a module with C-projective dimension at most n.

THEOREM 3.3.

(1) $\operatorname{spclfc} R \leq \mathrm{F}\mathcal{GP}_C\text{-dim}\, R = \mathrm{F}\mathcal{P}_C\text{-dim}\, R \leq \mathrm{id}_R\, \mathcal{P}_C(R) = \operatorname{siclp} S = \operatorname{siclf} S$.
(2) *If R is a left noetherian ring, then* $\mathrm{F}\mathcal{P}_C\text{-dim}\, R \leq \mathrm{id}_R\, C = \operatorname{siclp} S$.

Proof. (1) By Proposition 3.1, Lemma 2.8 and Corollary 3.2, we have $\operatorname{spclfc} R \leq \mathrm{F}\mathcal{P}_C\text{-dim}\, R \leq \mathrm{F}\mathcal{GP}_C\text{-dim}\, R \leq \mathrm{id}_R\, \mathcal{P}_C(R)$.

Now suppose that $\mathrm{F}\mathcal{P}_C\text{-dim}\, R = n\ (< \infty)$ and $M \in \operatorname{Mod} R$ satisfies $\mathcal{GP}_C(R)\text{-pd}_R\, M < \infty$. By [25, Corollary 3.4], there exists $M' \in \operatorname{Mod} R$ such that $\mathcal{P}_C(R)\text{-pd}_R\, M' = \mathcal{GP}_C(R)\text{-pd}_R\, M$. So $\mathcal{GP}_C(R)\text{-pd}_R\, M \leq n$. It follows that $\mathrm{F}\mathcal{GP}_C\text{-dim}\, R \leq \mathrm{F}\mathcal{P}_C\text{-dim}\, R$. The first equality follows.

Assume that $\operatorname{siclp} S = n\ (< \infty)$ and $M\ (\cong C \otimes_S P) \in \mathcal{P}_C(R)$ with P projective in $\operatorname{Mod} S$. Then there exists an exact sequence

$$0 \to P \to {I^0}_* \to {I^1}_* \to \cdots \to {I^n}_* \to 0 \tag{3.2}$$

in $\operatorname{Mod} S$ with all I^i injective in $\operatorname{Mod} R$. By Lemma 2.5(1), all I^i are in $\mathcal{B}_C(R)$. So ${I^i}_* \in {C_S}^{\top}$ and $C \otimes_S {I^i}_* \cong I^i$ for any $0 \leq i \leq n$. Then applying the functor $C \otimes_S -$ to (3.2) yields the exact sequence

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

in $\operatorname{Mod} R$. It follows that $\mathrm{id}_R\, M \leq n$ and $\mathrm{id}_R\, \mathcal{P}_C(R) \leq \operatorname{siclp} S$. By using a dual argument, we get $\operatorname{siclp} S \leq \mathrm{id}_R\, \mathcal{P}_C(R)$. The second equality follows.

Obviously $\operatorname{siclp} S \leq \operatorname{siclf} S$. Now let $\operatorname{siclp} S = n\ (< \infty)$ and suppose $F \in \operatorname{Mod} S$ is flat. Then $\mathrm{F}\mathcal{P}_C\text{-dim}\, R \leq n$ and $\mathcal{P}_C(R)\text{-pd}_R(C \otimes_S F) < \infty$ by the above argument. Let

$$0 \to C_m \to \cdots \to C_1 \to C_0 \to C \otimes_S F \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all C_i in $\mathcal{P}_C(R)$. By Lemma 2.6(3), we have $\mathrm{id}_R\, C_i \leq \operatorname{siclp} S \leq n$. Thus $\mathrm{id}_R(C \otimes_S F) \leq n$. Note that $F \in \mathcal{A}_C(R)$ by [16, Lemma 4.1]. Then $\mathcal{I}_C(S)\text{-id}_S\, F \leq n$ by Lemma 2.6(3) again. This yields $\operatorname{siclf} S \leq n$. Hence we conclude that $\operatorname{siclp} S = \operatorname{siclf} S$.

(2) Let R be a left noetherian ring. Then $\mathrm{id}_R\, \mathcal{P}_C(R) \leq \mathrm{id}_R\, C$ by [3, Theorem 1.1]. Now the first inequality follows from Corollary 3.2.

Since $\operatorname{id}_R C = \mathcal{I}_C(S)\text{-}\operatorname{id}_S S$ by Lemma 2.6(3), we have $\operatorname{id}_R C \leq \operatorname{siclp} S$. Now let $\operatorname{id}_R C = n$ ($< \infty$). Since R is left noetherian, by [3, Theorem 1.1] we have $\operatorname{id}_R G \leq n$ for any $G \in \mathcal{P}_C(R)$. It follows from Lemma 2.6(3) that $\mathcal{I}_C(S)\text{-}\operatorname{id}_S P \leq n$ for any projective module P in $\operatorname{Mod} S$. Thus $\operatorname{siclp} S \leq n$ and $\operatorname{siclp} S \leq \operatorname{id}_R C$. ∎

Note that Theorem 3.3(1) extends [14, Theorem 2.28] and [7, Proposition 2.1]. The inequality in Theorem 3.3(2) can be strict, as illustrated in the following example. We refer to [1] for the notions of quivers and their representations.

Example 3.4. Let R be the bound quiver algebra kQ/J^2, where k is a field, Q is the quiver

$$\circlearrowleft \circ 1 \longrightarrow \circ 2 \longrightarrow \circ 3 \longrightarrow \circ 4,$$

kQ is the path k-algebra of Q, and J is the two-sided ideal of kQ generated by the arrows. If C is the R-R-bimodule R, then $\mathrm{F}\mathcal{P}_C\text{-}\dim R = 0$ [13, Example 1.2], but $\operatorname{id}_R C = \infty$.

4. Some relative homological invariants.

In classical homological algebra, it is known that for any module $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M \leq n$, the nth yoke in every flat resolution of M is flat. As described in the following result, an analogous result holds for C-flat dimension of modules.

Lemma 4.1. *Let ${}_RC_S$ be faithful and $M \in \operatorname{Mod} R$. If there exist exact sequences*

$$0 \to M_n \to \cdots \to C_1 \to C_0 \to M \to 0 \quad \textit{and}$$
$$0 \to D_n \to \cdots \to D_1 \to D_0 \to M \to 0$$

in $\operatorname{Mod} R$ *with all* C_i *and* D_i *in* $\mathcal{F}_C(R)$, *then* $M_n \in \mathcal{F}_C(R)$.

Proof. Applying the functor $(-)^+$ to both the sequences, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M^+ & \longrightarrow & D_0^+ & \longrightarrow & \cdots & \longrightarrow & D_{n-1}^+ & \longrightarrow & D_n^+ & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \\ 0 & \longrightarrow & M^+ & \longrightarrow & C_0^+ & \longrightarrow & \cdots & \longrightarrow & C_{n-1}^+ & \longrightarrow & M_n^+ & \longrightarrow & 0 \end{array}$$

By Lemma 2.3(1), all C_i^+ and D_i^+ lie in $\mathcal{I}_C(R^{\mathrm{op}})$. Then the existence of all f_i follows from Lemma 2.5(2). Now we may view the sequence $(f_0, \ldots, f_{n-1}, f_n)$ as a quasi-isomorphism between the complexes

$$0 \to D_0^+ \to \cdots \to D_{n-1}^+ \to D_n^+ \to 0 \quad \text{and}$$
$$0 \to C_0^+ \to \cdots \to C_{n-1}^+ \to M_n^+ \to 0.$$

We therefore obtain an exact sequence

$$0 \to D_0^+ \to D_1^+ \oplus C_0^+ \to \cdots \to D_n^+ \oplus C_{n-1}^+ \to M_n^+ \to 0.$$

Then $M_n^+ \in \mathcal{I}_C(R^{\mathrm{op}})$ ($\subseteq \mathcal{A}_C(R^{\mathrm{op}})$) and $M_n^+ \otimes_R C \in \operatorname{Mod} S^{\mathrm{op}}$ is injective by Lemma 2.3(1) and [16, Lemma 5.1(c)]. Note that $M_n^+ \otimes_R C \cong \operatorname{Hom}_R(C, M_n)^+$ by [11, Lemma 2.16(c)]. So $\operatorname{Hom}_R(C, M_n) \in \operatorname{Mod} S$ is flat by [11, Corollary 2.18(b)], and hence it is in $\mathcal{A}_C(S)$ by [16, Lemma 4.1]. Then $M_n \in \mathcal{B}_C(R)$ by [34, Lemma 1.7]. It follows from [16, Lemma 5.1(a)] that $M_n \in \mathcal{F}_C(R)$. ∎

The following example shows that the assumption about the faithfulness of ${}_RC_S$ in the above lemma is necessary.

Example 4.2. Let k be an algebraically closed field and let $R = kQ$ be the path k-algebra of dimension 3 of the quiver

$$1\circ \longrightarrow \circ 2.$$

Put $C = I(1) \oplus I(2)$. Then ${}_RC_R$ is a non-faithful semidualizing bimodule and there exist exact sequences

$$0 \to S(2) \to P(1) \to S(1) \to 0 \quad \text{and}$$
$$0 \to I(1) \to I(1)^2 \to S(1) \to 0,$$

where $I(1)$ and $P(1)$ are in $\mathcal{F}_C(R)$, but $S(2)$ is not in $\mathcal{F}_C(R)$.

Motivated by the corresponding notions introduced in [7], in an analogous way we write the supremum of the C-projective dimensions of injective left R-modules as

$$\operatorname{spcli} R := \sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R I \mid I \in \operatorname{Mod} R \text{ is injective}\},$$

and write the supremum of the C-flat dimensions of injective left R-modules as

$$\operatorname{sfcli} R := \sup\{\mathcal{F}_C(R)\text{-}\operatorname{pd}_R I \mid I \in \operatorname{Mod} R \text{ is injective}\}.$$

Next we turn to further investigate the relationship of the aforementioned relative invariants. The following two results extend [7, Proposition 2.2 and Corollary 2.3] respectively.

Theorem 4.3.

(1) *If* $\operatorname{spcli} R < \infty$ *and* $\operatorname{siclp} S < \infty$, *then* $\operatorname{spcli} R = \operatorname{siclp} S$.
(2) *If* ${}_RC_S$ *is faithful, then* $\operatorname{spcli} R \leq \operatorname{sfcli} R + \operatorname{spclfc} R$.

Proof. (1) Let $\operatorname{spcli} R = n$, and let $I \in \operatorname{Mod} R$ be injective with $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R I = n$. Thus there exists an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to I \to 0$$

in $\operatorname{Mod} R$ with all C_i in $\mathcal{P}_C(R)$. Then $\operatorname{Ext}_R^n(I, C_n) \neq 0$, which implies that $\operatorname{id}_R C_n \geq n$. We may assume that $C_n \cong C \otimes_S P$ for some projective module P

in $\operatorname{Mod} S$. Then $\mathcal{I}_C(S)\text{-id}_S P = \operatorname{id}_R C_n \geq n$ by Lemma 2.6(3), implying that $\operatorname{siclp} S \geq n$. With the aid of Lemma 2.6(2), a similar argument gives the converse inequality.

(2) Let $\operatorname{sfcli} R = n$ $(< \infty)$ and $\operatorname{spclfc} R = m$ $(< \infty)$, and let $I \in \operatorname{Mod} R$ be injective. Since $I \in \mathcal{B}_C(R)$, by [35, Theorem 3.9 and Proposition 3.7] there exists an exact sequence

$$0 \to K_n \to C_{n-1} \to \cdots \to C_0 \to I \to 0$$

in $\operatorname{Mod} R$ with all C_i in $\mathcal{P}_C(R)$. Since $\mathcal{F}_C(R)\text{-pd}_R I \leq \operatorname{sfcli} R = n$, it follows from Lemma 4.1 that $K_n \in \mathcal{F}_C(R)$. Since $\operatorname{spclfc} R = m$, we have $\mathcal{P}(R)_C\text{-pd}_R K_n \leq m$ and $\mathcal{P}(R)_C\text{-pd}_R I \leq m + n$. ∎

Corollary 4.4. *Let ${}_RC_S$ be faithful. Then the following statements are equivalent:*

(1) $\operatorname{spcli} R = \operatorname{siclp} S < \infty$.
(2) $\operatorname{sfcli} R < \infty$ *and* $\operatorname{siclp} S < \infty$.

Proof. The implication (1)⇒(2) is trivial.

Let $\operatorname{sfcli} R < \infty$ and $\operatorname{siclp} S < \infty$. Then $\operatorname{spclfc} R < \infty$ by Theorem 3.3(1). So $\operatorname{spcli} R < \infty$ by Theorem 4.3(2). Now the implication (2)⇒(1) follows from Theorem 4.3(1). ∎

In the following result, we give a sufficient condition for a C-Gorenstein projective module to be C-Gorenstein flat.

Proposition 4.5. *Let S be a right coherent ring. If* $\operatorname{spclfc} R < \infty$ *(in particular, if* $\mathrm{F}\mathcal{P}_C\text{-dim}\, R < \infty$*), then any C-Gorenstein projective module in* $\operatorname{Mod} R$ *is C-Gorenstein flat.*

Proof. By Proposition 3.1, we have $\operatorname{spclfc} R \leq \mathrm{F}\mathcal{P}_C\text{-dim}\, R$. Now let S be a right coherent ring and $\operatorname{spclfc} R < \infty$. If $M \in \operatorname{Mod} R$ is C-Gorenstein projective module, then by definition there exists a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence

$$\mathbb{G} := \cdots \to P_1 \to P_0 \to G^0 \to G^1 \to \cdots$$

in $\operatorname{Mod} R$ with all G^i in $\mathcal{P}_C(R)$, P_i projective and $M \cong \Im(P_0 \to G^0)$, such that $\operatorname{Hom}_R(\mathbb{G}, H)$ is exact for any module $H \in \mathcal{P}_C(R)$. By induction on dimension, it is not difficult to show that $\operatorname{Hom}_R(\mathbb{G}, H')$ is exact for any $H' \in \operatorname{Mod} R$ with $\mathcal{P}_C(R)\text{-pd}_R H' < \infty$.

Now let $E \in \mathcal{I}_C(R^{\mathrm{op}})$. Then $E^+ \in \mathcal{F}_C(R)$ by Lemma 2.3(2), and so $\mathcal{P}_C(R)\text{-pd}_R E^+ < \infty$ by assumption. This implies that $\operatorname{Hom}_R(\mathbb{G}, E^+)$ is exact. Thus $E \otimes_R \mathbb{G}$ is exact by the adjoint isomorphism theorem. It follows that M is C-Gorenstein flat. ∎

Recall that the *big finitistic dimension* $\operatorname{FPD} R$ of R is defined as

$$\operatorname{FPD} R := \sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \operatorname{pd}_R M < \infty\}.$$

Following [7], we write the supremum of the projective dimensions of flat left R-modules as

$$\operatorname{splf} R := \sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ is flat}\}.$$

Putting ${}_RC_S = {}_RR_R$ in Proposition 4.5, we immediately get the following result, which is a slight generalization of [14, Proposition 3.4].

Corollary 4.6. *Let R be a right coherent ring. If* $\operatorname{splf} R < \infty$ (*in particular, if* $\operatorname{FPD} R < \infty$), *then any Gorenstein projective module in* $\operatorname{Mod} R$ *is Gorenstein flat.*

We use $\mathcal{P}(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of all projective modules. Recall from [14] that a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$ is called *projectively resolving* if $\mathcal{P}(R) \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under extensions and kernels of epimorphisms.

White asked in [40, Question 2.15]: for a commutative ring R, if $M \in \operatorname{Mod} R$ with $\operatorname{pd}_R M < \infty$, must $\operatorname{pd}_R M$ be equal to $\mathcal{GP}_C(R)$-$\operatorname{pd}_R M$? The following example illustrates that the answer to both this question and its non-commutative version is negative in general. In addition, Holm and White stated in [16, Corollary 6.4] that $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are projectively resolving if ${}_RC_S$ is faithful. The following example also shows that this result is not true.

Example 4.7. (1) Let R be a non-self-injective commutative artinian local ring with maximal ideal m. For example, we can take for R the ring $k[[X,Y]]/(X^2, XY, Y^2)$ with k a field (see [6, p. 15]). Then $C := I^0(R/m)$ is a faithfully semidualizing module and C is C-(Gorenstein) projective. But C is an injective cogenerator for $\operatorname{Mod} R$, so $\operatorname{pd}_R C = \operatorname{id}_R R \neq 0$. We also claim that $R \notin \mathcal{P}_C(R)$. Indeed, otherwise, there exists a projective module P in $\operatorname{Mod} R$ such that $R \cong C \otimes_R P$. It follows that R is injective, a contradiction. Consequently, $\mathcal{P}_C(R)$ is not projectively resolving.

(2) Let R be a Gorenstein artin algebra with $\operatorname{id}_R R = \operatorname{id}_{R^{\text{op}}} R = n \geq 1$. For example, we can take for R the bound quiver algebra kQ/J^2, where k is an algebraically closed field, Q is the quiver

$$\circ 1 \xrightarrow{\alpha_1} \circ 2 \xrightarrow{\alpha_2} \circ 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \circ n+1,$$

kQ is the path k-algebra of Q, and J is the two-sided ideal of kQ generated by the arrows. Put $C := \bigoplus_{i=0}^{n} I^i(R)$. Then by [39, Corollary 3.2], it is easy to see that C is a semidualizing (R,S)-bimodule, where $S = \operatorname{End}_R C$. Because C is an injective cogenerator for $\operatorname{Mod} R$ by [19, Theorem 2], we have $\operatorname{pd}_R C = \operatorname{fd}_R C = \operatorname{id}_{R^{\text{op}}} R = n$ (≥ 1) by [20, Proposition 1]. But C is C-(Gorenstein) projective.

The following result shows that the answer to White's question mentioned above is affirmative under some condition.

PROPOSITION 4.8. *Assume that $\mathcal{P}_C(R)$ is projectively resolving and that $M \in \operatorname{Mod} R$. If $\operatorname{pd}_R M < \infty$, then*

$$\operatorname{pd}_R M = \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M = \mathcal{G}\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M.$$

Proof. By assumption, $\mathcal{P}_C(R)$ is projectively resolving; in particular, $\mathcal{P}(R) \subseteq \mathcal{P}_C(R)$. So we have $\operatorname{pd}_R M \geq \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M$. On the other hand, by Lemma 2.5(1), we have $\mathcal{P}_C(R) \subseteq \mathcal{P}(R)^{\perp} \cap {}^{\perp}\mathcal{P}(R)$. So, if $\operatorname{pd}_R M < \infty$, then $\operatorname{pd}_R M = \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M$ by [18, Theorem 3.10]. It follows from Lemma 2.8 that $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M = \mathcal{G}\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M$. ∎

5. Σ-embedding cogenerators. Recall from [26] that a module $A \in \operatorname{Mod} R$ is called a *Σ-embedding cogenerator* for a subcategory $\mathcal{B}$ of $\operatorname{Mod} R$ if every module in $\mathcal{B}$ admits an injection to a direct sum of copies of A.

THEOREM 5.1. *Let R be a left noetherian ring and $\mathcal{X}$ a subcategory of $\operatorname{Mod} R$ with $\mathcal{X} \subseteq C^{\perp}$, and let $m, n \geq 0$.*

(1) *If $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R X \mid X \in \mathcal{X}\} \leq m$, then $\bigoplus_{t=0}^{m+n} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{X}\text{-}\operatorname{pd}^{\leq n}(R)$.*
(2) *If $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R X \mid X \in \mathcal{X}\} < \infty$, then $\bigoplus_{t\geq 0} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{X}\text{-}\operatorname{pd}^{<\infty}(R)$.*

Proof. (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{X}\text{-}\operatorname{pd}_R M \leq n$ and $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R X \mid X \in \mathcal{X}\} \leq m$. Then by Lemma 2.1(1), we have an exact sequence

$$0 \to C_{m+n} \to \cdots \to C_1 \to C_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all C_t in $\mathcal{P}_C(R)$ $(= \operatorname{Add}_R C)$. Because R is a left noetherian ring, all $I^j(C_t)$ are in $\operatorname{Add}_R I^j(C)$ for any $j \geq 0$. By [26, Corollary 1.3] (cf. [17, Corollary 3.5]), M can be embedded into (a direct summand of) $\bigoplus_{t=0}^{m+n} I^t(C_t)$. So M can be embedded into a direct sum of copies of $\bigoplus_{t=0}^{m+n} I^t(C)$. It follows that $\bigoplus_{t=0}^{m+n} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{X}\text{-}\operatorname{pd}^{\leq n}(R)$.

(2) This is a direct consequence of (1). ∎

Putting $\mathcal{X} = \mathcal{P}_C(R)$ in Theorem 5.1, we have the following result in which the first assertion is a C-version of [26, Theorem 2.2].

COROLLARY 5.2. *Let R be a left noetherian ring, and let $n \geq 0$. Then:*

(1) *$\bigoplus_{t=0}^{n} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{P}_C(R)\text{-}\operatorname{pd}^{\leq n}(R)$.*
(2) *$\bigoplus_{t\geq 0} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{P}_C(R)\text{-}\operatorname{pd}^{<\infty}(R)$.*

As another application of Theorem 5.1, we have the following

COROLLARY 5.3. *Let R be a left noetherian ring, and let $m, n \geq 0$.*

(1) *If $\operatorname{spclfc} R \leq m$ (in particular, if $\mathrm{F}\mathcal{P}_C\text{-}\dim R \leq m$ or $\operatorname{id}_R C \leq m$), then $\bigoplus_{t=0}^{m+n} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{F}_C(R)\text{-}\operatorname{pd}^{\leq n}(R)$.*

(2) *If* $\operatorname{spclfc} R < \infty$ *(in particular, if* $\mathrm{F}\mathcal{P}_C\text{-}\dim R < \infty$ *or* $\mathrm{id}_R C < \infty$*), then* $\bigoplus_{t\geq 0} I^t(C)$ *is a* Σ*-embedding cogenerator for* $\mathcal{F}_C(R)\text{-pd}^{<\infty}(R)$*.*

Proof. By Theorem 3.3, we have

$$\operatorname{spclfc} R \leq \mathrm{F}\mathcal{P}_C\text{-}\dim R \leq \mathrm{id}_R C.$$

Note that $\mathcal{F}_C(R) \subseteq C^{\perp}$ by Lemma 2.5(1). So, if we put $\mathcal{X} = \mathcal{F}_C(R)$ in Theorem 5.1, then the assertions follow. ∎

Putting $C = R$ in Corollary 5.3, we have the following result in which the second assertion generalizes [26, Corollary 2.3].

Corollary 5.4. *Let* R *be a left noetherian ring, and let* $m, n \geq 0$*.*

(1) *If* $\operatorname{splf} R \leq m$ *(in particular, if* $\operatorname{FPD} R \leq m$ *or* $\mathrm{id}_R R \leq m$*), then* $\bigoplus_{t=0}^{m+n} I^t(R)$ *is a* Σ*-embedding cogenerator for the subcategory of* Mod R *consisting of modules with flat dimension at most* n*.*
(2) *If* $\operatorname{splf} R < \infty$ *(in particular, if* $\operatorname{FPD} R < \infty$ *or* $\mathrm{id}_R R < \infty$*), then* $\bigoplus_{t\geq 0} I^t(C)$ *is a* Σ*-embedding cogenerator for the subcategory of* Mod R *consisting of all modules with finite flat dimension.*

In view of Proposition 4.5 and Corollary 5.3, we need more information about (the finiteness of) $\operatorname{spclfc} R$.

Lemma 5.5. *Let* $m, n \geq 0$*.*

(1) *If* $\mathrm{pd}_S N \leq n$ *for any flat module* N *in* Mod S*, then* $\mathcal{P}_C(R)\text{-pd}_R M \leq m + n$ *for any module* $M \in \operatorname{Mod} R$ *with* $\mathcal{F}_C(R)\text{-pd}_R M \leq m$*.*
(2) *If* $\mathrm{pd}_S N \leq n$ *(resp.* $< \infty$*) for any module* $N \in \operatorname{Mod} S$ *with* $\mathrm{fd}_S N < \infty$*, then* $\mathcal{P}_C(R)\text{-pd}_R M \leq n$ *(resp.* $< \infty$*) for any module* $M \in \operatorname{Mod} R$ *with* $\mathcal{F}_C(R)\text{-pd}_R M < \infty$*.*

Proof. (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)\text{-pd}_R M \leq m$. By Lemmas 2.5(1) and 2.6(1), we have $M \in \mathcal{B}_C(R)$ and $\mathrm{fd}_S M_* \leq m$. Then by assumption and dimension shifting, $\mathrm{pd}_S M_* \leq m + n$. So there exists an exact sequence

$$0 \to P_{m+n} \to \cdots \to P_1 \to P_0 \to M_* \to 0 \tag{5.1}$$

in Mod S with all P_i projective. Applying $C \otimes_S -$ to (5.1) yields the exact sequence

$$0 \to C \otimes_S P_{m+n} \to \cdots \to C \otimes_S P_1 \to C \otimes_S P_0 \to C \otimes_S M_* \ (\cong M) \to 0$$

in Mod R with all $C \otimes_S P_i$ in $\mathcal{P}_C(R)$. Hence $\mathcal{P}_C(R)\text{-pd}_R M \leq m + n$.

(2) This has been essentially proved in (1). ∎

Let k be a field, and let S be a right-Noetherian k-algebra for which there exists a left-Noetherian k-algebra R and a dualizing complex ${}_R D_S$. Then $\mathrm{pd}_S N < \infty$ for any module $N \in \operatorname{Mod} S$ with $\mathrm{fd}_S N < \infty$ [24, Theorem]. So $\mathcal{P}_C(R)\text{-pd}_R M < \infty$ for any module $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)\text{-pd}_R M < \infty$ by Lemma 5.5(2).

As a consequence of Lemma 5.5(1), we have the following result, which shows that the relative homological invariant spclfc R coincides with the absolute homological invariant splf S. Compare it with Proposition 3.1.

THEOREM 5.6. *spclfc R = splf S.*

Proof. Putting $m = 0$ in Lemma 5.5(1), it is easy to get spclfc $R \leq$ splf S. Now let spclfc $R = n$ $(< \infty)$ and $N \in \operatorname{Mod} S$ be flat. Then $C \otimes_S N \in \mathcal{F}_C(R)$ and there exists an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to C \otimes_S N \to 0$$

in $\operatorname{Mod} R$ with all C_i in $\mathcal{P}_C(R)$ $(= \operatorname{Add}_R C)$. Applying $\operatorname{Hom}_R(C, -)$ yields an exact sequence

$$0 \to C_{n*} \to \cdots \to C_{1*} \to C_{0*} \to (C \otimes_S N)_* \to 0$$

in $\operatorname{Mod} S$ with all C_{i*} projective. Note that $N \in \mathcal{A}_C(S)$ by [16, Lemma 4.1]. Hence $N \cong (C \otimes_S N)_*$ and $\operatorname{pd}_S N \leq n$. Thus splf $S \leq$ spclfc R. ■

We finish the paper by the following interesting open problem suggested by a referee.

OPEN PROBLEM 5.7. Let $\mathcal{C}$ be a preadditive category and $\operatorname{Mod} \mathcal{C}$ the category of additive contravariant functors from $\mathcal{C}$ to abelian groups. Similar problems on pure projective resolutions, pure projective and pure injective dimensions have been studied in $\operatorname{Mod} \mathcal{C}$ [29]–[31]. It is interesting to study how to define a semidualizing bimodule T in $\operatorname{Mod} \mathcal{C}$ such that the results in this paper still hold true after replacing C (in $\operatorname{Mod} R$) by T (in $\operatorname{Mod} \mathcal{C}$).

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