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HOMOLOGICAL INVARIANTS UNDER FROBENIUS EXTENSIONS

BY

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Abstract. Let A/S be a Frobenius extension of artin algebras such that S is commutative and A is an S-algebra. We prove that if (C, T) is a tilting pair of right S-modules, then $(C \otimes_S A, T \otimes_S A)$ is a tilting pair of right A-modules; conversely, if (C, T) is a tilting pair of right A-modules, then (C, T) is also a tilting pair of right S-modules. We also prove that the so-called (l, n)-condition and certain classes of algebras are preserved under right-split or separable Frobenius extensions. Finally, we prove that the validity of some homological conjectures is preserved under (separable) Frobenius extensions.

1. Introduction. As a generalization of Frobenius algebras, Frobenius extensions were introduced by Kasch [23], and then studied by Nakayama and Tsuzuku [30] and Morita [28]. As a generalization of separable algebras, separable extensions were introduced by Hirata and Sugano [16], who made a thorough study of these in connection with Galois theory for noncommutative rings and generalizations of Azumaya algebras. Separable extensions are closely related to Frobenius extensions. A ring extension that is both a separable extension and a Frobenius extension is called a *separable Frobenius extension*. In addition, if the base ring is commutative, then a Frobenius extension is both left-split and right-split [8, III.4.8, Lemma 2].

Many algebraists have studied the invariant properties of artin algebras under (separable) Frobenius extensions, such as projectivity, injectivity, Gorensteinness, (Gorenstein) homological dimension, representation dimension, tilting theory and homological conjectures; see [13, 18, 32, 33, 42, 43, 44] and references therein. In particular, Zhang [42] proposed a question: Is the validity of some homological conjectures preserved under excellent extensions? Fu, Xu and Zhao [13] showed that the validity of the Gorenstein symmetric conjecture is preserved under Frobenius extensions, which generalized a result in [42].

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In this paper, we focus on connecting (separable) Frobenius extensions with tilting theory, homological conjectures and certain algebraic structures. The outline of this article is as follows.

In Section 2, we give some notations and preliminary results.

For a ring R, we use $\operatorname{mod} R$ to denote the category of finitely generated right R-modules. Let A and S be artin algebras. In Section 3, let A/S be a Frobenius extension such that S is commutative and A is an S-algebra. We prove that if (C, T) is a tilting pair in $\operatorname{mod} S$, then $(C \otimes_S A, T \otimes_S A)$ is a tilting pair in $\operatorname{mod} A$; conversely, if (C, T) is a tilting pair in $\operatorname{mod} A$, then (C_S, T_S) is a tilting pair in $\operatorname{mod} S$ (Theorem 3.7). In particular, we obtain tilting modules under Frobenius extensions, which is then used to study the Wakamatsu tilting conjecture and tilted algebras.

In Section 4, we prove that the so-called (l, n)-condition is preserved under right-split Frobenius extensions (Theorem 4.5). As applications, we find that the dominant dimension is invariant under right-split Frobenius extensions (Corollary 4.6). In addition, we show that the quasi-tilted algebra (respectively, tilted algebra) is preserved under (separable) Frobenius extensions when the base ring is commutative; see Theorem 4.13 (respectively, Theorem 4.14).

In Section 5, some homological conjectures are studied, such as the (strong) Nakayama conjecture, the finitistic dimension conjecture, the Auslander–Reiten conjecture, and others. We prove that the validity of these homological conjectures is preserved under (separable) Frobenius extensions when the base ring is commutative; see Corollary 5.1 and Theorems 5.2–5.6.

In Section 6, we give some examples to illustrate the results obtained.

2. Preliminaries. Recall that if S is a subring of a ring A such that S and A have the same identity, then A is called a *ring extension* of S, denoted by A/S. Let A/S be a ring extension and let $S \overset{l}{\hookrightarrow} A$ be the inclusion of rings. Then there exists a *restriction functor* $\operatorname{Res} : \operatorname{mod} A \to \operatorname{mod} S$ which sends $M_A \mapsto M_S$, given by $m \cdot s := m \cdot l(s)$. In the opposite direction, there exist two natural functors:

(1) $\mathbb{T} = - \otimes_S A_A : \operatorname{mod} S \to \operatorname{mod} A$, given by $M_S \mapsto M \otimes {}_S A_A$.
(2) $\mathbb{H} = \operatorname{Hom}_S({}_A A_S, -) : \operatorname{mod} S \to \operatorname{mod} A$, given by $M_S \mapsto \operatorname{Hom}_S({}_A A_S, M_S)$.

It is easy to check that both $(\mathbb{T}, \operatorname{Res})$ and $(\operatorname{Res}, \mathbb{H})$ are adjoint pairs.

Definition 2.1 (see [22, Definition 1.1 and Theorem 1.2]). A ring extension A/S is a *Frobenius extension* if one of the following equivalent conditions holds:

(1) The functors $\mathbb{T}$ and $\mathbb{H}$ are naturally equivalent.
(2) ${}_S A_A \cong \operatorname{Hom}_S({}_A A_S, {}_S S_S)$ and A_S is finitely generated projective.

(3) ${}_AA_S \cong \mathrm{Hom}_{S^{\mathrm{op}}}({}_SA_A, {}_SS_S)$ and ${}_SA$ is finitely generated projective.
(4) There exist an S-S-homomorphism $\tau : A \to S$ and elements $x_i, y_i \in A$ such that $\sum_i x_i\tau(y_ia) = a$ and $\sum_i \tau(ax_i)y_i = a$ for any $a \in A$.

Remark 2.2. If A/S is a Frobenius extension, then both $(\mathbb{T}, \mathrm{Res})$ and $(\mathrm{Res}, \mathbb{T})$ are adjoint pairs. So $\mathbb{T}$ and Res are exact functors, and hence they preserve projectives and injectives.

Definition 2.3. Let A/S be a ring extension.

(1) ([22, Definition 2.12]) A/S is called a *separable extension* if
$$\mu : A \otimes_S A \to A, \quad a \otimes b \mapsto ab,$$
is a split epimorphism of A-A-bimodules.
(2) ([39, p. 35]) A/S is called *right-split* (respectively, *left-split*) if the inclusion map $S \hookrightarrow A$ is a split monomorphism of right (respectively, left) S-modules. Moreover, A/S is called *split* if it is both left-split and right-split.

Many examples of Frobenius extensions can be found in [9, 12, 18, 21, 22, 32, 34, 35, 40, 41]. If a ring extension A/S is both a Frobenius extension and a separable extension, then it is called a *separable Frobenius extension*. If a ring extension A/S is both a Frobenius extension and a right-split extension, then it is called a *right-split Frobenius extension*. In particular, when S is a commutative ring, if A/S is an excellent extension of rings, then it is separable [31]; if A/S is a Frobenius extension of rings, then it is split [8, III.4.8, Lemma 2].

Let A/S be a ring extension and M_A an A-module. Then M_S is a right S-module. There exists a natural surjective map $\pi : M \otimes_S A \to M$ given by $m \otimes a \mapsto ma$ for any $m \in M$ and $a \in A$. It is easy to check that π is split as a homomorphism of S-modules. However, π is not split as a homomorphism of A-modules in general; see Example 6.1(1) below.

The following lemma is a characterization of separable extensions.

Lemma 2.4 ([32, Lemma 2.9]). *Let A/S be a ring extension. Then the following statements are equivalent:*

(1) *A/S is a separable extension.*
(2) *For any A-A-bimodule M, $M \otimes_S A \to M$ is a split epimorphism of A-A-bimodules.*
(3) *There exists an element $e \in A \otimes_S A$ such that $\mu(e) = 1_A$ and $ae = ea$ for any $a \in A$.*

For any $M \in \mathrm{mod}\, A$, we use $\mathrm{add}\, M$ to denote the subcategory of $\mathrm{mod}\, A$ consisting of all direct summands of finite direct sums of M. For two right A-modules M and N, we use $M_A \mid N_A$ to denote that M_A is a direct summand of N_A.

LEMMA 2.5. *Assume that S is commutative and A is an S-algebra. Then* $\operatorname{add}(M \otimes_S A_A) \subseteq \operatorname{add} M_A$ *for any* $M \in \operatorname{mod} A$. *Moreover, if A/S is separable, then* $\operatorname{add} M_A = \operatorname{add}(M \otimes_S A_A)$.

Proof. By [35, Lemma 3], we have ${}_AA \otimes_S A_A \in \operatorname{add} {}_AA_A$. Then

$$M_S \otimes_S A_A \cong (M \otimes_A A) \otimes_S A_A \cong M \otimes_A (A \otimes_S A)_A \in \operatorname{add} M_A.$$

Note that $M_A \mid M \otimes_S A_A$ by Lemma 2.4, so $\operatorname{add} M_A = \operatorname{add}(M \otimes_S A_A)$. ∎

3. Tilting modules. In this section we assume that all rings are artin algebras, all modules are finitely generated right modules unless stated otherwise, and A/S is a Frobenius extension of artin algebras. Let $M \in \operatorname{mod} A$. We say that M is *selforthogonal* if $\operatorname{Ext}_A^{\geq 1}(M, M) = 0$. We use $\operatorname{pd}_A M$ (respectively, $\operatorname{id}_A M$, $\operatorname{fd}_A M$) to denote the projective (respectively, injective, flat) dimensions of M.

DEFINITION 3.1 ([26]). A module $T \in \operatorname{mod} A$ is called *n-tilting* if it satisfies the following conditions:

(T1) $\operatorname{pd}_A T \leq n$.
(T2) T is selforthogonal.
(T3) There exists an exact sequence

$$0 \to A_A \to T_0 \to T_1 \to \cdots \to T_{n-1} \to T_n \to 0$$

in $\operatorname{mod} A$ with all T_i in $\operatorname{add} T$.

Note that a 1-tilting module is called *classical tilting* [7, 15]; in this case, (T3) is equivalent to $|T| = |A|$, where $|T|$ denotes the number of pairwise nonisomorphic indecomposable direct summands of T in $\operatorname{mod} A$. Moreover, if T is an n-tilting A-module, then $|T| = |A|$ by [26, Theorem 1.19].

To construct tilting modules, Miyashita [27] introduced the notion of tilting pairs.

DEFINITION 3.2 ([27]). A pair (C, T) of modules in $\operatorname{mod} A$ is called *tilting* if it satisfies the following conditions:

(1) C and T are selforthogonal.
(2) There exist exact sequences

$$0 \to C \to T_0 \to T_1 \to \cdots \to T_{m-1} \to T_m \to 0,$$
$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to T \to 0$$

in $\operatorname{mod} A$ with $m, n \geq 0$, all T_i in $\operatorname{add} T$ and all C_i in $\operatorname{add} C$.

Let (C, T) be a tilting pair as in Definition 3.2. Then $m = n$ (see [27]); in this case, we call (C, T) an *n-tilting pair*. We also say that T is *C-tilting* or C is *T-cotilting*. If $C = A$, then C-tilting modules are exactly tilting modules [27].

Remark 3.3. Let (C, T) be a tilting pair and (C', T') be a pair in $\operatorname{mod} A$. If $\operatorname{add} C' = \operatorname{add} C$ and $\operatorname{add} T' = \operatorname{add} T$, then (C', T') is also a tilting pair by [38, Lemma 2.1].

As a generalization of tilting modules, we recall the notion of Wakamatsu tilting modules.

Definition 3.4 ([25, 37]). A module $T \in \operatorname{mod} A$ is called *Wakamatsu tilting* if T is selforthogonal, and there exists an exact sequence

$$0 \to A_A \to T_0 \to T_1 \to \cdots \to T_i \to \cdots$$

in $\operatorname{mod} A$ with all T_i in $\operatorname{add} T$ and such that after applying the functor $\operatorname{Hom}_A(-, T)$ the sequence is still exact.

For convenience, we give the following result.

Lemma 3.5. *Let $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} S$. Then for any $i \geq 0$,*

$$\operatorname{Ext}^i_S(M_S, N_S) \cong \operatorname{Ext}^i_A(M_A, N \otimes_S A_A),$$
$$\operatorname{Ext}^i_S(M_S, S_S) \cong \operatorname{Ext}^i_A(M_A, A_A).$$

Proof. By the adjoint isomorphism, for any $i \geq 0$, we have

$$\begin{aligned} \operatorname{Ext}^i_S(M_S, N_S) &\cong \operatorname{Ext}^i_S(M \otimes_A A_S, N_S) \\ &\cong \operatorname{Ext}^i_A(M_A, \operatorname{Hom}_S({}_A A_S, N_S)) \\ &\cong \operatorname{Ext}^i_A(M_A, N \otimes_S A_A). \end{aligned}$$

The last isomorphism is obvious. ∎

In the rest of this section, we always assume that S is commutative and A is an S-algebra.

Lemma 3.6.

(1) *If $M, N \in \operatorname{mod} S$ with $\operatorname{Ext}^{\geq 0}_S(M, N) = 0$, then*

$$\operatorname{Ext}^{\geq 0}_A(M \otimes_S A, N \otimes_S A) = 0.$$

In particular, if M is a selforthogonal S-module, then $M \otimes_S A_A$ is a selforthogonal A-module.

(2) *Let $M, N \in \operatorname{mod} A$ be such that $\operatorname{Ext}^{\geq 0}_A(M, N) = 0$. Then*

$$\operatorname{Ext}^{\geq 0}_S(M_S, N_S) = 0.$$

In particular, if M is a selforthogonal A-module, then M_S is a selforthogonal S-module.

Proof. (1) By the adjoint isomorphism, we have

$$\begin{aligned} \operatorname{Ext}^i_A(M \otimes_S A_A, N \otimes_S A_A) &\cong \operatorname{Ext}^i_S(M_S, \operatorname{Hom}_A({}_S A_A, N \otimes_S A_A)) \\ &\cong \operatorname{Ext}^i_S(M_S, N \otimes_S A_S) \end{aligned}$$

for any $i \geq 0$. Noticing that $N \otimes_S A_S \in \operatorname{add} N_S$, it follows that

$$\operatorname{Ext}_A^i(M \otimes_S A_A, N \otimes_S A_A) = 0$$

from the assumption that $\operatorname{Ext}_S^i(M, N) = 0$.

(2) Let

$$\mathbf{P}^\bullet := \cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to 0$$

be the deleted complex of projective resolution of M_A. For any $i \geq 0$, we have

$$\begin{aligned}
\operatorname{Ext}_S^i(M_S, N_S) &\cong \operatorname{Ext}_A^i(M_A, N \otimes_S A_A) \quad \text{(by Lemma 3.5)} \\
&\cong H^i(\operatorname{Hom}_A(\mathbf{P}^\bullet, N \otimes_S A)) \\
&\cong H^i(\operatorname{Hom}_A(\mathbf{P}^\bullet, N) \otimes_S A) \\
&\cong H^i(\operatorname{Hom}_A(\mathbf{P}^\bullet, N)) \otimes_S A \\
&\cong \operatorname{Ext}_A^i(M, N) \otimes_S A \\
&= 0 \quad \text{(by assumption).} \ \blacksquare
\end{aligned}$$

THEOREM 3.7. *For any $n \geq 0$, the following hold:*

(1) *If (C, T) is an n-tilting pair in* $\operatorname{mod} S$, *then $(C \otimes_S A_A, T \otimes_S A_A)$ is an n-tilting pair in* $\operatorname{mod} A$.
(2) *If (C, T) is an n-tilting pair in* $\operatorname{mod} A$, *then (C_S, T_S) is an n-tilting pair in* $\operatorname{mod} S$.

Proof. (1) By Lemma 3.6(1), $C \otimes_S A_A$ and $T \otimes_S A_A$ are selforthogonal A-modules.

By assumption, there exist two exact sequences

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to T \to 0$$

and

$$0 \to C \to T_0 \to T_1 \to \cdots \to T_{n-1} \to T_n \to 0$$

in $\operatorname{mod} S$ with $C_i \in \operatorname{add} C$ and $T_i \in \operatorname{add} T$ for any $0 \leq i \leq n$. Since ${}_S A$ is projective, we get the exact sequences

$$\begin{aligned}
0 \to C_n \otimes_S A_A \to C_{n-1} \otimes_S A_A \to \cdots \to C_1 \otimes_S A_A& \\
\to C_0 \otimes_S A_A \to T \otimes_S A_A \to 0&
\end{aligned}$$

and

$$\begin{aligned}
0 \to C \otimes_S A_A \to T_0 \otimes_S A_A \to T_1 \otimes_S A_A \to \cdots& \\
\to T_{n-1} \otimes_S A_A \to T_n \otimes_S A_A \to 0&
\end{aligned}$$

in $\operatorname{mod} A$ with $C_i \otimes_S A_A \in \operatorname{add}(C \otimes_S A_A)$ and $T_i \otimes_S A_A \in \operatorname{add}(T \otimes_S A_A)$ for any $0 \leq i \leq n$. Thus $(C \otimes_S A_A, T \otimes_S A_A)$ is a tilting pair in $\operatorname{mod} A$.

(2) By Lemma 3.6(2), C_S and T_S are selforthogonal S-modules. By assumption, there exist two exact sequences

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to T \to 0$$

and

$$0 \to C \to T_0 \to T_1 \to \cdots \to T_{n-1} \to T_n \to 0$$

in $\operatorname{mod} A$ with $C_i \in \operatorname{add} C$ and $T_i \in \operatorname{add} T$ for any $0 \leq i \leq n$, which are also exact in $\operatorname{mod} S$ with $C_i \in \operatorname{add} C_S$ and $T_i \in \operatorname{add} T_S$ for any $0 \leq i \leq n$.. Thus (C_S, T_S) is a tilting pair in $\operatorname{mod} S$. ∎

Corollary 3.8. *For any $n \geq 0$, the following hold:*

(1) *If T is an n-tilting S-module, then $T \otimes_S A_A$ is an n-tilting A-module.*
(2) *If T is an n-tilting A-module, then T_S is an n-tilting S-module.*

Proof. (1) By assumption, (S_S, T_S) is an n-tilting pair in $\operatorname{mod} S$. It follows from Theorem 3.7(1) that $(S \otimes_S A_A \cong A_A, T \otimes_S A_A)$ is an n-tilting pair in $\operatorname{mod} A$, and hence $T \otimes_S A_A$ is an n-tilting A-module.

(2) By assumption, (A_A, T_A) is an n-tilting pair in $\operatorname{mod} A$. It follows from Theorem 3.7(2) that (A_S, T_S) is an n-tilting pair in $\operatorname{mod} S$. Note that $S_S \mid A_S$ and A_S is projective, so $\operatorname{add} S_S = \operatorname{add} A_S$. Then (S_S, T_S) is an n-tilting pair in $\operatorname{mod} S$ by Remark 3.3, and hence T_S is an n-tilting S-module. ∎

If S is not commutative, then assertion (2) in Corollary 3.8 may not be true in general; see Example 6.2 below.

Lemma 3.9.

(1) *If $T \in \operatorname{mod} A$ is n-tilting, then $\operatorname{add} T_A = \operatorname{add} T \otimes_S A_A$.*
(2) *If $T \in \operatorname{mod} S$ is n-tilting, then $\operatorname{add} T_S = \operatorname{add} T \otimes_S A_S$.*

Proof. (1) By Lemma 2.5, we have $\operatorname{add}(T \otimes_S A_A) \subseteq \operatorname{add} T_A$. It follows from Corollary 3.8 that T_S is an n-tilting S-module and $T \otimes_S A_A$ is an n-tilting A-module. By [26, Theorem 1.19], we have $|T \otimes_S A_A| = |A| = |T_A|$, and thus $\operatorname{add} T_A = \operatorname{add}(T \otimes_S A_A)$.

(2) It follows from Corollary 3.8 that $T \otimes_S A_A$ is an n-tilting A-module and $T \otimes_S A_S$ is an n-tilting S-module. So $|T \otimes_S A_S| = |S| = |T_S|$ by [26, Theorem 1.19]. Since ${}_S A$ is projective, we have $T \otimes_S A_S \in \operatorname{add} T_S$, and thus $\operatorname{add} T_S = \operatorname{add}(T \otimes_S A_S)$. ∎

For a ring R, we say that two modules M and N in $\operatorname{mod} R$ are *add-isomorphic* if $\operatorname{add} M = \operatorname{add} N$. We use n-tilt R to denote the class of n-tilting modules in $\operatorname{mod} R$ up to add-isomorphism.

Theorem 3.10. *For any $n \geq 0$, we have a bijection*

$$n\text{-tilt}\, A \underset{\Psi}{\overset{\Phi}{\leftrightarrows}} n\text{-tilt}\, S$$

given by $\Phi(T_S) = T \otimes_S A_A$ *and* $\Psi(T'_A) = T'_S$ *for any* $T_S \in n\text{-}\operatorname{tilt} S$ *and any* $T'_A \in n\text{-}\operatorname{tilt} A$.

Proof. By Corollary 3.8, it suffices to show $\Psi\Phi = \mathrm{Id}$ and $\Phi\Psi = \mathrm{Id}$.

For any $T_S \in n\text{-}\operatorname{tilt} S$, we have $\Psi\Phi(T_S) = \Psi(T \otimes_S A_A) = T \otimes_S A_S$. It follows from Lemma 3.9 that $\operatorname{add}(T \otimes_S A_S) = \operatorname{add} T_S$, and thus $\Psi\Phi = \mathrm{Id}$. On the other hand, for any $T'_A \in n\text{-}\operatorname{tilt} A$, we have $\Phi\Psi(T'_A) = \Phi(T'_S) = T' \otimes_S A_A$. It follows from Lemma 3.9 that $\operatorname{add}(T' \otimes_S A_A) = \operatorname{add} T'_A$, and thus $\Phi\Psi = \mathrm{Id}$. ∎

Fu, Xu and Zhao [13, Proposition 3.4] showed that if T_S is a Wakamatsu tilting S-module, then $T \otimes_S A_A$ is a Wakamatsu tilting A-module. Conversely, we have the following result.

PROPOSITION 3.11. *If* T_A *is a Wakamatsu tilting* A*-module, then* T_S *is a Wakamatsu tilting* S*-module.*

Proof. By assumption, T_A is a selforthogonal A-module and there exists an exact sequence

$$0 \to A_A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \to \cdots \xrightarrow{f_i} T_i \to \cdots$$

with $T_i \in \operatorname{add} T$ for any $i \geq 0$ such that after applying $\operatorname{Hom}_A(-, T)$ the sequence is still exact. Set $K_i := \operatorname{Coker} f_i$. Then $\operatorname{Ext}^1_A(K_i, T) = 0$. On the other hand,

$$0 \to A_S \to T_0 \to T_1 \to \cdots \to T_i \to \cdots$$

is also exact in $\operatorname{mod} S$. By Lemma 3.6(2), T_S is a selforthogonal S-module and $\operatorname{Ext}^1_S((K_i)_S, T_S) = 0$. Since A_S is projective, we have $\operatorname{Ext}^1_S(A_S, T_S) = 0$. Since $S_S \mid A_S$, we get from [38, Lemma 2.1] an exact sequence

$$0 \to S_S \xrightarrow{f'_0} T'_0 \xrightarrow{f'_1} T'_1 \to \cdots \xrightarrow{f'_i} T'_i \to \cdots \tag{3.1}$$

in $\operatorname{mod} S$ with $T'_i \in \operatorname{add} T_S$ and $\operatorname{Ext}^1_S((\operatorname{Coker} f'_i)_S, T_S) = 0$ for any $i \geq 0$. Then after applying the functor $\operatorname{Hom}_S(-, T_S)$ the sequence (3.1) is still exact. Thus T_S is a Wakamatsu tilting S-module. ∎

4. Certain classes of algebras. Let A be a two-sided noetherian ring. Recall that A is called *Iwanaga–Gorenstein* (*Gorenstein* for short) if $\operatorname{id}_A A = \operatorname{id}_{A^{\mathrm{op}}} A < \infty$.

DEFINITION 4.1 ([4, 19]). Let A be a two-sided noetherian ring.

(1) For any $l, n \geq 1$, A is said to satisfy the (l, n)-*condition* if in the minimal injective coresolution

$$0 \to A_A \to I^0 \to I^1 \to \cdots \tag{4.1}$$

of A_A, we have $\operatorname{fd}_A I^i < l$ for any $0 \leq i < n$.

(2) For any $k \geq 1$, A is called *Auslander k-Gorenstein* if A satisfies the (l, l)-condition for any $0 < l \leq k$. If A is Auslander k-Gorenstein for all k, then A is said to satisfy the *Auslander condition.*
(3) A is called *Auslander–Gorenstein* if it satisfies the Auslander condition and is Gorenstein.
(4) A is called *Auslander–regular* if it satisfies the Auslander condition and its global dimension $\mathrm{gl.dim}\, A$ is finite.

Definition 4.2.

(1) ([17]) Let A be a two-sided Noetherian ring. The *flat-dominant dimension* $\mathrm{fd.dom.dim}\, A$ of A is defined as

$$\mathrm{fd.dom.dim}\, A := \sup\{n \mid A \text{ satisfies the } (1, n)\text{-condition}\}.$$

If no such integer exists, then we set $\mathrm{fd.dom.dim}\, A = \infty$.
(2) ([2, 36]) Let A be an artin algebra. The *dominant dimension* $\mathrm{dom.dim}\, A$ of A is defined as

$$\mathrm{dom.dim}\, A := \sup\{n \mid A \text{ satisfies the } (1, n)\text{-condition}\}.$$

If no such integer exists, then we set $\mathrm{dom.dim}\, A = \infty$.

If A is an artin algebra, then $\mathrm{fd.dom.dim}\, A = \mathrm{dom.dim}\, A$.

Definition 4.3 ([20]). For any $n \geq 1$, A is called *n-Auslander* if $\mathrm{gl.dim}\, A \leq n + 1 \leq \mathrm{dom.dim}\, A$. In particular, 1-Auslander algebras are exactly classical *Auslander algebras.*

The following result seems to be well-known.

Lemma 4.4. *Let A/S be a Frobenius extension between noetherian rings. For any $M \in \mathrm{mod}\, A$ and $N \in \mathrm{mod}\, S$, the following hold:*

(1) $\mathrm{pd}_S\, M \leq \mathrm{pd}_A\, M$, $\mathrm{id}_S\, M \leq \mathrm{id}_A\, M$, *and* $\mathrm{fd}_S\, M \leq \mathrm{fd}_A\, M$.
(2) $\mathrm{pd}_A\, N \otimes_S A \leq \mathrm{pd}_S\, N$, $\mathrm{id}_A\, N \otimes_S A \leq \mathrm{id}_S\, N$, *and* $\mathrm{fd}_A\, N \otimes_S A \leq \mathrm{fd}_S\, N$.
(3) *If A/S is separable, then* $\mathrm{pd}_S\, M = \mathrm{pd}_A\, M$.
(4) *If S is commutative, then* $\mathrm{pd}_A\, N \otimes_S A = \mathrm{pd}_S\, N$.

Theorem 4.5. *Let A/S be a Frobenius extension between noetherian rings. If S satisfies the (l, n)-condition, then so does A. The converse holds true if A/S is right-split.*

Proof. Assume that S satisfies the (l, n)-condition. Let

$$0 \to S_S \to I^0 \to I^1 \to \cdots$$

be a minimal injective coresolution of S_S with $\mathrm{fd}_S\, I^i < l$ for any $0 \leq i < n$. Since ${}_S A$ is projective, we have the exact sequence

$$0 \to S \otimes_S A_A \cong A_A \to I^0 \otimes_S A_A \to I^1 \otimes_S A_A \to \cdots$$

of right A-modules with all $I^i \otimes_S A_A$ injective. One can take a minimal injective coresolution

$$0 \to A_A \to J^0 \to J^1 \to \cdots$$

of A_A, where J^i is a direct summand of $I^i \otimes_S A_A$ for any $i \geq 0$. So $\mathrm{fd}_A J^i \leq \mathrm{fd}_A I^i \otimes_S A_A \leq \mathrm{fd}_S I^i < l$ for any $0 \leq i < n$ by Lemma 4.4, and thus A satisfies the (l, n)-condition.

Conversely, assume that A/S is right-split and A satisfies the (l, n)-condition. Let

$$0 \to A_A \to I^0 \to I^1 \to \cdots$$

be a minimal injective coresolution of A_A with $\mathrm{fd}\, I^i < l$ for any $0 \leq i < n$, which is also an exact sequence of right S-modules. Noticing that $S_S \mid A_S$, one can take a minimal injective coresolution

$$0 \to S_S \to J^0 \to J^1 \to \cdots$$

of S_S, where J^i is a direct summand of I^i for any $i \geq 0$. So $\mathrm{fd}_S J^i \leq \mathrm{fd}_S I^i \leq \mathrm{fd}_A I^i < l$ for any $0 \leq i < n$ by Lemma 4.4, and thus S satisfies the (l, n)-condition. ∎

The following result is an immediate consequence of Theorem 4.5, which has been obtained in [39]. Note that there are examples of Frobenius extensions A/S such that $\mathrm{dom.dim}\, A > \mathrm{dom.dim}\, S$ [39, Remark 2.5(2)].

Corollary 4.6 (cf. [39, p. 35]). *Let A/S be a Frobenius extension.*

(1) *If A and S are noetherian rings, then* $\mathrm{fd.dom.dim}\, A \geq \mathrm{fd.dom.dim}\, S$. *Furthermore, if A/S is right-split, then* $\mathrm{fd.dom.dim}\, S = \mathrm{fd.dom.dim}\, A$.
(2) *If A and S are artin algebras, then* $\mathrm{dom.dim}\, A \geq \mathrm{dom.dim}\, S$. *Furthermore, if A/S is right-split, then* $\mathrm{dom.dim}\, S = \mathrm{dom.dim}\, A$.

The following result provides a partial answer to [45, Section 5, Question].

Corollary 4.7. *Let A/S be a Frobenius extension between noetherian rings. For any $k \geq 1$, if S is Auslander k-Gorenstein, then so is A. The converse holds true if A/S is right-split.*

The following lemma is easy.

Lemma 4.8. *Let A/S be a Frobenius extension between noetherian rings.*

(1) $\mathrm{id}_A A \leq \mathrm{id}_S S$ *and* $\mathrm{id}_{A^{\mathrm{op}}} A \leq \mathrm{id}_{S^{\mathrm{op}}} S$.
(2) *If A/S is right-split (respectively, left-split), then* $\mathrm{id}_S S = \mathrm{id}_A A$ *(respectively,* $\mathrm{id}_{S^{\mathrm{op}}} S = \mathrm{id}_{A^{\mathrm{op}}} A$*).*

In particular, if S is commutative, then $\mathrm{id}_S S = \mathrm{id}_A A$ and $\mathrm{id}_{S^{\mathrm{op}}} S = \mathrm{id}_{A^{\mathrm{op}}} A$; moreover, A is selfinjective (respectively, Gorenstein) if and only if so is S [13, Corollaries 3.11, 3.12]. By Theorem 4.5 and Lemma 4.8, we obtain the following result.

COROLLARY 4.9. *Let A/S be a Frobenius extension between noetherian rings. If S is Auslander–Gorenstein, then so is A. The converse holds true if A/S is split.*

The following is a consequence of Theorem 4.5 and [13, Corollary 4.13].

COROLLARY 4.10. *Let A/S be a Frobenius extension between noetherian rings. Assume that S is commutative. If A is an Auslander-regular algebra* (*respectively, n-Auslander algebra*), *then so is S. The converse holds true if A/S is separable.*

In the rest of this section, A is an artin algebra.

DEFINITION 4.11 ([15]). The algebra A is called *tilted* if there exists a hereditary algebra H and a 1-tilting H-module such that $A = \operatorname{End}_H T$.

Let A be a tilted algebra. Following [1, Lemma VIII.3.2], we know that $\operatorname{gl.dim} A \leq 2$ and $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$ for any finitely generated indecomposable A-module. As a generalization of tilted algebra, Happel, Reiten and Smalø introduced the notion of quasi-tilted algebras.

DEFINITION 4.12 ([10, 14]). The algebra A is called *quasi-tilted* if it satisfies the following conditions:

(1) $\operatorname{gl.dim} A \leq 2$.
(2) For any finitely generated indecomposable A-module X, either $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$.

THEOREM 4.13. *Assume that A/S is a Frobenius extension and S is commutative. If A is a quasi-tilted algebra, then so is S. The converse holds true if A/S is separable.*

Proof. (1) Assume that A is a quasi-tilted algebra. Then $\operatorname{gl.dim} S \leq \operatorname{gl.dim} A \leq 2$ by [13, Corollary 4.13]. Since S is commutative, A/S is split by [8, III.4.8, Lemma 2], and thus ${}_S S_S \in \operatorname{add} {}_S A_S$. Let $M \in \operatorname{mod} S$ be indecomposable. Then $M_S \cong M \otimes_S S_S \mid M \otimes_S A_S$. Notice that $M \otimes_S A_A \in \operatorname{mod} A$, so there exists an indecomposable module $N \in \operatorname{mod} A$ such that $M_S \mid N_S$. If $\operatorname{pd}_A N \leq 1$, then $\operatorname{pd}_S M \leq \operatorname{pd}_S N \leq \operatorname{pd}_A N \leq 1$ by Lemma 4.4. If $\operatorname{pd}_A N = 2$, then $\operatorname{id}_A N \leq 1$ by assumption. So $\operatorname{id}_S N \leq \operatorname{id}_A N \leq 1$ by Lemma 4.4, and hence $\operatorname{id}_S M \leq 1$. Thus S is quasi-tilted.

Conversely, assume that A/S is separable and S is a quasi-tilted algebra. By [13, Corollary 4.13], we have $\operatorname{gl.dim} A = \operatorname{gl.dim} S \leq 2$. Let $M \in \operatorname{mod} A$ be indecomposable. Then M is also a right S-module. Notice that $M_A \mid M \otimes_S A_A$ by Lemma 2.4, so there exists an indecomposable module $N \in \operatorname{mod} S$ such that $M_A \mid N \otimes_S A_A$. If $\operatorname{pd}_S N \leq 1$, then $\operatorname{pd}_A N \otimes_S A_A \leq \operatorname{pd}_S N \leq 1$ by Lemma 4.4, and so $\operatorname{pd}_A M \leq 1$. If $\operatorname{pd}_S N = 2$, then $\operatorname{id}_S N \leq 1$ by assumption. So $\operatorname{id}_A N \otimes_S A_A \leq \operatorname{id}_S N \leq 1$ by Lemma 4.4, and hence $\operatorname{id}_A M \leq 1$. Thus A is quasi-tilted. ∎

Let $T \in \operatorname{mod} A$ be 1-tilting. It is well known that $(\mathcal{T}, \mathcal{F})$ is a torsion pair, where

$$\mathcal{T} = \operatorname{Gen} T := \{M \in \operatorname{mod} A \mid \text{there exists an exact sequence } \widetilde{T} \to M \to 0 \text{ with } \widetilde{T} \in \operatorname{add} T\}$$

and

$$\mathcal{F} := \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\}.$$

By [6], A is tilted if and only if there exists a 1-tilting module $T \in \operatorname{mod} A$ such that $\operatorname{Hom}_A(M, T) = 0$ for any $M \in \operatorname{add}(\mathcal{T} \setminus T)$. For brevity, we call this equivalent condition of tilted algebras the *TA-condition*.

THEOREM 4.14. *Assume that A/S is a Frobenius extension such that S is commutative and A is an S-algebra. If A/S is separable, then A is a tilted algebra if and only if so is S.*

Proof. If A is tilted, then there exists a 1-tilting A-module T satisfying the TA-condition. By Corollary 3.8, T_S is 1-tilting. Let $M_S \in \operatorname{add}(\operatorname{Gen} T_S \setminus T_S)$. Then

$$M_S \otimes_S A_A \in \operatorname{add}(\operatorname{Gen}(T \otimes_S A) \setminus T \otimes_S A) = \operatorname{add}(\operatorname{Gen} T_A \setminus T_A)$$

by Lemma 2.5. On the other hand, we have

$$\begin{aligned} \operatorname{Hom}_S(M_S, T_S) &\cong \operatorname{Hom}_S(M_S, \operatorname{Hom}_A({}_SA_A, T_A)) \\ &\cong \operatorname{Hom}_A(M \otimes_S A_A, T_A). \end{aligned}$$

By assumption, $\operatorname{Hom}_A(M \otimes_S A_A, T_A) = 0$, and so $\operatorname{Hom}_S(M_S, T_S) = 0$. Thus S is tilted.

Conversely, if S is tilted, then there exists a 1-tilting S-module T satisfying the TA-condition. By Corollary 3.8, $T \otimes_S A_A$ is a 1-tilting A-module. We claim that

$$\text{if } M_A \in \operatorname{add}(\operatorname{Gen}(T \otimes_S A) \setminus T \otimes_S A), \text{ then } M_S \in \operatorname{add}(\operatorname{Gen} T_S \setminus T_S).$$

In fact, suppose $M_S = T'_S \oplus M'_S$ with $0 \neq T'_S \in \operatorname{add} T_S$ and $\operatorname{add} M'_S \cap \operatorname{add} T_S = \{0\}$. Note that $M_A \mid M \otimes_S A_A = (T'_S \otimes_S A_A) \oplus (M'_S \otimes_S A_A)$. By assumption, we have $(\operatorname{add} T'_S \otimes_S A_A) \cap \operatorname{add} M_A = \{0\}$, so $M_A \mid M'_S \otimes_S A_A$, and hence $M_S \mid M'_S \otimes_S A_S \in \operatorname{add} M'_S$. It follows that $T'_S = 0$, which is a contradiction. Thus $M_S \in \operatorname{add}(\operatorname{Gen} T_S \setminus T_S)$. The claim is proved.

Since $\operatorname{Hom}_S(M_S, T_S) = 0$ by assumption, we have

$$\begin{aligned} \operatorname{Hom}_A(M_A, T \otimes_S A_A) &\cong \operatorname{Hom}_S(\operatorname{Hom}_A({}_SA_A, M_A), T_S) \\ &\cong \operatorname{Hom}_S(M_S, T_S) = 0. \end{aligned}$$

Thus A is tilted. ∎

5. Homological conjectures. The following homological conjectures are important in the representation theory of artin algebras [4, 5, 24].

Let A be an artin algebra.

Finitistic Dimension Conjecture (FDC).

$$\mathrm{fin.dim}\, A := \sup \{\mathrm{pd}_A M \mid M \in \mathrm{mod}\, A \text{ with } \mathrm{pd}_A M < \infty\} < \infty.$$

Nakayama Conjecture (NC). If $\mathrm{dom.dim}\, A = \infty$, then A is selfinjective.

Strong Nakayama Conjecture (SNC). For any module $M \in \mathrm{mod}\, A$, if $\mathrm{Ext}_A^{\geq 0}(M, A) = 0$, then $M = 0$.

Auslander–Reiten Conjecture (ARC). For any module $M \in \mathrm{mod}\, A$, if $\mathrm{Ext}_A^{\geq 0}(M, M) = 0 = \mathrm{Ext}_A^{\geq 0}(M, A)$, then M is projective.

Auslander and Reiten [4] raised the following conjecture, but they did not name it. To avoid confusion, we name it the Auslander–Gorenstein Conjecture according to its meaning.

Auslander–Gorenstein Conjecture (AGC). If A satisfies the Auslander condition, then A is Gorenstein.

Recall that a module $M \in \mathrm{mod}\, A$ is said to have *Gorenstein dimension zero* [3] (or be *Gorenstein projective* [11]) if the following conditions are satisfied: (1) M is reflexive; (2) $\mathrm{Ext}_A^{\geq 1}(M, A) = 0 = \mathrm{Ext}_{A^{\mathrm{op}}}^{\geq 1}(\mathrm{Hom}_A(M, A), A)$.

Gorenstein Projective Conjecture (GPC). If M is a Gorenstein projective A-module such that $\mathrm{Ext}_A^i(M, M) = 0$ for any $i \geq 1$, then M is projective.

Wakamatsu Tilting Conjecture (WTC). If T is a Wakamatsu tilting A-module with $\mathrm{pd}_A T < \infty$, then T is tilting.

Gorenstein Symmetric Conjecture (GSC). $\mathrm{id}_A A < \infty$ if and only if $\mathrm{id}_{A^{\mathrm{op}}} A < \infty$; equivalently, $\mathrm{id}_A A = \mathrm{id}_{A^{\mathrm{op}}} A$.

In this section, assume that A/S is a Frobenius extension of artin algebras. We will study the invariance of some homological conjectures under Frobenius extensions. The following result shows that NC is preserved under right-split Frobenius extensions, where the sufficiency has been obtained in [39, p. 35].

Corollary 5.1. *If A/S is right-split, then A satisfies NC if and only if so does S.*

Proof. Assume that A satisfies NC and $\mathrm{dom.dim}\, S = \infty$. Then $\mathrm{dom.dim}\, A = \infty$ by Corollary 4.6, and so A is selfinjective by assumption. By Lemma 4.8, S is selfinjective. Thus S satisfies NC.

Conversely, assume that S satisfies NC and $\mathrm{dom.dim}\, A = \infty$. Then $\mathrm{dom.dim}\, S = \infty$ by Corollary 4.6, and so S is selfinjective by assumption. By Lemma 4.8, A is selfinjective. Thus A satisfies NC. ∎

THEOREM 5.2. *If either A/S is split or S is commutative, then A satisfies GSC if and only if so does S.*

Proof. By Lemma 4.8, A satisfies GSC if and only if $\mathrm{id}_A A = \mathrm{id}_{A^{\mathrm{op}}} A$, if and only if $\mathrm{id}_S S = \mathrm{id}_{S^{\mathrm{op}}} S$, and if and only if S satisfies GSC. ∎

THEOREM 5.3. *Assume that A/S is right-split. If S satisfies AGC, then so does A. The converse holds true if A/S is split.*

Proof. Assume that S satisfies AGC. If A satisfies the Auslander condition, then so does S by Theorem 4.5, and so S is Gorenstein by assumption. By Lemma 4.8, A is Gorenstein. Thus A satisfies AGC.

Now assume that A/S is split and A satisfies AGC. If S satisfies the Auslander condition, then so does A by Theorem 4.5, and so A is Gorenstein by assumption. By Lemma 4.8, S is Gorenstein. Thus S satisfies AGC. ∎

THEOREM 5.4.

(1) *Assume that A/S is separable. If S satisfies FDC, then so does A.*
(2) *Assume that S is commutative. If A satisfies FDC, then so does S.*

Proof. (1) Assume that S satisfies FDC and $\mathrm{fin.dim}\, S = n < \infty$. Let $M \in \mathrm{mod}\, A$ with $\mathrm{pd}_A M < \infty$. Then $\mathrm{pd}_S M \le \mathrm{pd}_A M < \infty$ by Lemma 4.4, and hence $\mathrm{pd}_S M \le n$. By Lemma 4.4, we have $\mathrm{pd}_A M = \mathrm{pd}_S M \le n$. Thus $\mathrm{fin.dim}\, A \le n$ and A satisfies FDC.

(2) Assume that A satisfies FDC and $\mathrm{fin.dim}\, A = n < \infty$. Let $M \in \mathrm{mod}\, S$ with $\mathrm{pd}_S M < \infty$. Then $\mathrm{pd}_A M \otimes_S A_A \le \mathrm{pd}_S M < \infty$ by Lemma 4.4, and hence $\mathrm{pd}_A M \otimes_S A_A \le n$. It follows from Lemma 4.4 that $\mathrm{pd}_S M = \mathrm{pd}_A M \otimes_S A_A \le n$. Thus $\mathrm{fin.dim}\, S \le n$ and S satisfies FDC. ∎

THEOREM 5.5.

(1) *Assume that A/S is separable. If S satisfies SNC, then so does A.*
(2) *Assume that S is commutative. If A satisfies SNC, then so does S.*

Proof. (1) Assume that S satisfies SNC. Let $M \in \mathrm{mod}\, A$ with $\mathrm{Ext}_A^{\geq 0}(M, A) = 0$. Then by Lemma 3.5, we have

$$\mathrm{Ext}_S^i(M_S, S_S) \cong \mathrm{Ext}_A^i(M_A, A_A) = 0 \tag{5.1}$$

for any $i \ge 0$. It follows that $M_S = 0$ and $M \otimes_S A_A = 0$. Since $M_A \mid M \otimes_S A_A$ by Lemma 2.4, we find that $M_A = 0$ and A satisfies SNC.

(2) Assume that A satisfies SNC. Let $M \in \mathrm{mod}\, S$ with $\mathrm{Ext}_S^{\geq 0}(M, S) = 0$. Then

$$\begin{aligned} \mathrm{Ext}_A^i(M \otimes_S A_A, A_A) &\cong \mathrm{Ext}_S^i(M_S, \mathrm{Hom}_A({}_S A_A, A_A)) \\ &\cong \mathrm{Ext}_S^i(M_S, A_S) \end{aligned} \tag{5.2}$$

for any $i \geq 0$. Note that $A_S \in \operatorname{add} S_S$, so $\operatorname{Ext}^i_A(M \otimes_S A_A, A_A) = 0$. It follows that $M \otimes_S A_A = 0$, and so $M \otimes_S A_S = 0$ as a right S-module. Notice that $M_S \mid M \otimes_S A_S$, thus $M_S = 0$ and S satisfies SNC. ∎

THEOREM 5.6. *Assume that S is commutative and A is an S-algebra. If A satisfies ARC (respectively, GPC, WTC), then so does S. The converse holds true if A/S is separable.*

Proof. **(ARC)** Assume that A satisfies ARC. Let $M \in \operatorname{mod} S$ with

$$\operatorname{Ext}^{\geq 1}_S(M, M) = 0 = \operatorname{Ext}^{\geq 1}_S(M, S).$$

Then $\operatorname{Ext}^i_A(M \otimes_S A_A, M \otimes_S A_A) = 0$ by Lemma 3.6(1), and consequently $\operatorname{Ext}^i_A(M \otimes_S A_A, A_A) = 0$ by (5.2). It follows that $M \otimes_S A_A$ is projective. So $M \otimes_S A_S$ is a projective right S-module, and hence M_S is a projective right S-module because $M_S \mid M \otimes_S A_S$. Thus S satisfies ARC.

Conversely, assume that A/S is separable and S satisfies ARC. Let M be an A-module such that $\operatorname{Ext}^{\geq 1}_A(M, M) = 0 = \operatorname{Ext}^{\geq 1}_A(M, A)$. Then $\operatorname{Ext}^i_S(M_S, M_S) = 0$ by Lemma 3.6(2), and hence $\operatorname{Ext}^i_S(M_S, S_S) = 0$ by (5.1). It follows that M_S is projective. So $M \otimes_S A_A$ is a projective right A-module. Note that $M_A \mid M \otimes_S A_A$ by Lemma 2.4, so M_A is projective, and therefore A satisfies ARC.

(GPC) Assume that A satisfies GPC. Let $M \in \operatorname{mod} S$ be Gorenstein projective with $\operatorname{Ext}^{\geq 1}_S(M, M) = 0$. Then $M \otimes_S A_A$ is Gorenstein projective with $\operatorname{Ext}^{\geq 1}_A(M \otimes_S A_A, M \otimes_S A_A) = 0$ by [32, Lemma 2.3] and Lemma 3.6(1). It follows that $M \otimes_S A_A$ is projective, and so $M \otimes_S A_S$ is a projective S-module. Notice that $M_S \mid M \otimes_S A_S$, thus M_S is projective and S satisfies GPC.

Conversely, assume that A/S is separable and S satisfies GPC. Let $M \in \operatorname{mod} A$ be Gorenstein projective with $\operatorname{Ext}^{\geq 1}_A(M, M) = 0$. Then M_S is Gorenstein projective with $\operatorname{Ext}^i_S(M_S, M_S) = 0$ by [43, Theorem 3.2] (or [32, Lemma 2.2]) and Lemma 3.6(2). It follows that M_S is projective, and hence $M \otimes_S A_A$ is a projective right A-module. Notice that $M_A \mid M \otimes_S A_A$ by Lemma 2.4, thus M_A is projective and A satisfies GPC.

(WTC) Assume that A satisfies WTC. Let $T \in \operatorname{mod} S$ be Wakamatsu tilting with $\operatorname{pd}_S T < \infty$. Then $T \otimes_S A_A \in \operatorname{mod} A$ is Wakamatsu tilting and $\operatorname{pd}_A T \otimes_S A_A = \operatorname{pd}_S T < \infty$ by [13, Proposition 3.4] and Lemma 4.4. Thus $T \otimes_S A_A$ is a tilting right A-module. It follows from Corollary 3.8 that $T \otimes_S A_S$ is a tilting right S-module. Notice that $T_S \mid T \otimes_S A_S \in \operatorname{add} T_S$, so $\operatorname{add} T_S = \operatorname{add}(T \otimes_S A_S)$. Thus T_S is a tilting right S-module and S satisfies WTC.

Conversely, assume that A/S is separable and S satisfies WTC. Let $T \in \operatorname{mod} A$ be Wakamatsu tilting with $\operatorname{pd}_A T < \infty$. Then T_S is Wakamatsu tilting and $\operatorname{pd}_S T = \operatorname{pd}_A T < \infty$ by Proposition 3.11 and Lemma 4.4. Thus

T_S is tilting. It follows from Corollary 3.8 that $T \otimes_S A_A$ is a tilting right A-module. Notice that $\operatorname{add} T_A = \operatorname{add}(T \otimes_S A_A)$ by Lemma 2.5, thus T_A is tilting and A satisfies WTC. ■

6. Examples. Now we give some examples to explain the results obtained. In this section, we assume that k is an algebraically closed field.

EXAMPLE 6.1. Let S be a finite-dimensional k-algebra given by the quiver

$$1 \xrightarrow{\beta} 2.$$

Set $A := S[x]/(x^2)$. Then A is a finite-dimensional k-algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$$

with relations $\alpha^2 = 0 = \gamma^2$ and $\alpha\beta = \beta\gamma$. By [32, Lemma 3.1], A/S is a Frobenius extension. The Auslander–Reiten quivers of S and A are, respectively,

$$\begin{tikzcd} & \substack{1\\2} \arrow[dr] & \\ 2 \arrow[ur] & & 1 \end{tikzcd}$$

and

$$\begin{tikzcd}
 & 1 \arrow[dr] & & \substack{1\\2} \arrow[dr] & & 2 \arrow[dr] & \\
\substack{12\\2} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\12\\2} \arrow[r] & \substack{1\\12} \arrow[ur] \arrow[dr] & & \substack{12\\2} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\12\\2} \arrow[r] & \cdots \\
 & 2 \arrow[ur] \arrow[dr] & & \substack{12\\12} \arrow[ur] \arrow[dr] & & 1 \arrow[ur] & \\
 & & \substack{2\\2} \arrow[ur] & & \substack{1\\1} \arrow[ur] & &
\end{tikzcd}$$

(1) Take an A-module $M = \substack{12\\12}$. Then $M \otimes_S A = \substack{1\\12\\2} \oplus \substack{2\\2} \oplus \substack{1\\1}$. It is trivial that M is not a direct summand of $M \otimes_S A$ as A-modules. Thus the natural surjective map $\pi : M \otimes_S A \to M$ given by $m \otimes a \mapsto ma$ for any $m \in M$ and $a \in A$ is not a split epimorphism of A-modules.

(2) Take a minimal injective coresolution

$$0 \to S_S \to \substack{1\\2} \oplus \substack{1\\2} \to 1 \to 0$$

of S_S. Since S satisfies the $(2,2)$-condition, so does A by Theorem 4.5. In fact, A_A has a minimal injective coresolution

$$0 \to A_A \to \substack{1\\12\\2} \oplus \substack{1\\12\\2} \to \substack{1\\1} \to 0.$$

On the other hand, S also satisfies the $(1,1)$-condition, hence so does A by Theorem 4.5. Clearly, $\text{dom.dim}\, S = 1 = \text{dom.dim}\, A$ since A/S is right-split. This also follows from [39, Remark 2.5].

EXAMPLE 6.2. Let $A = M_4(k)$ be a finite-dimensional k-algebra, and let S be the subalgebra generated by

$$e_1 := E_{11} + E_{44}, \quad e_2 := E_{22} + E_{33}, E_{21}, E_{31}, E_{41}, E_{42}, E_{43},$$

where all E_{ij} are primitive orthogonal idempotents for any $0 \leq i, j \leq 4$. By [29, Example 7.1], A/S is a Frobenius extension. Obviously, S is not commutative. Notice that A_A is a tilting A-module and $A_S = (e_1S)^{\oplus 4}$, so $|A_S| = 1 < 2 = |S|$, and thus A_S is not a tilting S-module. This shows that the condition that S is commutative in Corollary 3.8(2) is necessary.

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