COMMUNICATIONS IN ALGEBRA, 29(10), 4811–4821 (2001)

CODIMENSION AND REGULARITY OVER COHERENT SEMILOCAL RINGS

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ABSTRACT

In this paper, the results on codimension and regularity over noetherian local rings and coherent local rings are extended to coherent semilocal rings and some useful examples of coherent semilocal rings are constructed.

1991 Mathematics Subject Classification: 13H99, 16E10.

INTRODUCTION

Throughout this paper it is assumed that all rings are commutative coherent rings with identity and all modules are unitary. Our aim in this paper is to extend the results on codimension and regularity which were studied in [7] and [8] for noetherian local rings and coherent local rings to coherent semilocal rings.

Let R be a ring and M an R-module. Recall that M is called finitely presented if there is a finitely generated R-module P and a finitely generated submodule N of P such that $P/N \simeq M$. R is called a coherent ring if every finitely generated ideal of R is finitely presented. R is called a regular ring if every finitely generated ideal of R has finite projective dimension ([5]). In case R is a noetherian ring, the notion of regularity given here coincides with that in [7].

In § 2, we study the codimension of modules, and show that if R is a regular coherent semilocal ring and every maximal ideal of R is finitely generated, then $m\text{-}codim_R(R) = Codim_{R_m}(R_m) = w.gl.dim\, R_m$ for every maximal ideal m of R, and $Codim_R(R) = w.gl.dim\, R$.

§ 3 deals with the regularity of coherent semilocal rings. In [7] Theorem 60 and Theorem 69, Kaplansky proved that a noetherian local ring R is regular if and only if the unique maximal ideal of R is generated by a regular R-sequence. In [8], we proved that a coherent local ring R whose maximal ideal m is finitely generated is regular if and only if m is generated by a regular R-sequence (see [8] Theorem 2.6). The main result of this section is that if R is an indecomposable coherent semilocal ring whose every maximal ideal is finitely generated, then R is regular if and only if every maximal ideal of R is generated by a regular R-sequence if and only if every maximal ideal m which satisfies $w.gl.dim\, R = w.gl.dim\, R_m$ is generated by a regular R-sequence if and only if there exists a maximal ideal m of R such that $w.gl.dim\, R = w.gl.dim\, R_m$ and m is generated by a regular R-sequence.

In § 4 we provide an example of a non-noetherian indecomposable coherent semilocal ring R with exactly t maximal ideals and with weak global dimension equal to s, for any natural numbers t and s.

In this paper, we use J, $Max(R)$, $gl.dim\, R$, $w.gl.dim\, R$, $pd_R(M)$, $fd_R(M)$, $id_R(M)$, $Codim_R(M)$, $FP\text{-}id_R(M)$, ${}_R\mathbf{M}$, $f.g.\text{-}{}_R\mathbf{M}$, $f.r.\text{-}{}_R\mathbf{M}$ for the Jacobson radical, the maximal spectrum, global dimension, weak global dimension of R, projective dimension, flat dimension, injective dimension, codimension, FP-injective dimension of R-module M, the category of R-modules, the category of finitely generated R-modules and the category of finitely presented R-modules, respectively.

1 PRELIMINARIES

Definition 1. *Let R be a ring and I an ideal of R, $M \in {}_R\mathbf{M}$. A finite sequence $\alpha_1, \ldots, \alpha_n \in I$ is called a regular M-sequence if α_i is not a zero divisor for the module $M/(\alpha_1, \ldots, \alpha_{i-1})M$, $i = 1, \ldots, n$, and $M \neq (\alpha_1, \ldots, \alpha_n)M$. We define $I\text{-}codim_R(M) = \sup\{t \mid \alpha_1, \ldots, \alpha_t$ is a regular M-sequence in $I\}$ and*

$Codim_R(M) = \sup\{I\text{-}codim_R(M) \mid I \text{ is an ideal of } R\}$ *which are called the codimension of M in I and codimension of M respectively. Since every ideal of R must be contained in a maximal ideal of R, we have* $Codim_R(M) = \sup\{m\text{-}codim_R(M) \mid m \in Max(R)\}$. *Particularly* $Codim_R(R) = \sup\{m\text{-}codim_R(R) \mid m \in Max(R)\}$.

Definition 2. *Let R be a ring,* $M \in {}_R\mathbf{M}$. *The FP-injective dimension of M, denoted by* $FP\text{-}id_R(M)$, *is equal to the least integer* $n \geq 0$ *for which* $Ext_R^{n+1}(P, M) = 0$ *for every finitely presented R-module P. If no such n exists set* $FP\text{-}id_R(M) = \infty$.

In this section, we shall give some lemmas which will be used later.

Lemma 1.1 ([5] Corollary 2.5.5). *Let R be a coherent ring,* $M \in f.r._R\mathbf{M}$. *Then* $pd_R(M) = fd_R(M)$.

Lemma 1.2 ([6] Theorem 5). *If R is a coherent semilocal ring, then*

$$w.gl.dim\, R = fd_R(R/J) = id_R(R/J) = FP\text{-}id_R(R/J).$$

Lemma 1.3. *Let R be a ring,* $m \in Max(R)$, $M \in {}_R M$. *If* $\alpha_1, \ldots, \alpha_n \in m$ *is a regular M-sequence, then the sequence* $\alpha_1, \ldots, \alpha_n$, *considered as elements in* R_m, *is a regular* M_m*-sequence.*

Proof. See the proof of Proposition 1.2 in [1]. □

Lemma 1.4. *Let R be a coherent local ring,* $M \in f.r._R\mathbf{M}$. *Then* $Codim_R(M) \leq w.gl.dim\, R$.

Proof. Let m be the maximal ideal of R and let $\alpha_1, \ldots, \alpha_s \in m$ be any regular M-sequence in m. By [8] Lemma 2.3 we have $pd_R(M/(\alpha_1, \ldots, \alpha_s)M) = pd_R(M) + s$. Since M is finitely presented, so is $M/(\alpha_1, \ldots, \alpha_s)M$. Therefore $s \leq pd_R(M/(\alpha_1, \ldots, \alpha_s)M) = fd_R(M/\ (\alpha_1, \ldots, \alpha_s)M) \leq w.gl.dim\, R$. Thus we have $Codim_R(M) \leq w.gl.dim\, R$. □

Corollary 1.5. *Let R be a coherent ring,* $M \in f.r._R\mathbf{M}$. *Then* $Codim_R(M) \leq w.gl.dim\, R$.

Proof. By Lemma 1.3 and Lemma 1.4, we have $Codim_R(M) = \sup\{m\text{-}codim_R (M) \mid m \in Max(R)\} \leq \sup\{Codim_{R_m}(M_m) \mid m \in Max(R)\} \leq \sup\{w.gl.dim\, R_m \mid m \in Max(R)\} = w.gl.dim\, R$. □

Lemma 1.6. *Let R be a semilocal domain but not a field.*

(1) *If $m_1 \in Max(R)$ and m_1 is finitely generated, then there exists an irreducible element α of R in m_1;*

(2) *If $m_1 \in Max(R)$ and $\alpha \in m_1$ is an irreducible element of R, then $R/(\alpha)$ is a local ring with unique maximal ideal $\bar{m}_1 = m_1/\alpha R$.*

(3) *If $m \in Max(R)$ is finitely generated and $\alpha \in m$ is an irreducible element of R, then α, considered as an element in R_m, can extended to a minimal set of generators of the unique maximal ideal mR_m of R_m.*

Proof. (1) We Assume that $Max(R) = \{m_1, m_2, \ldots, m_t\}$ and $m_i \neq m_j$, $\forall i \neq j$. We only need to prove that there exists $\alpha \in m_1$ such that $\alpha \notin m_i m_j$, $\forall 1 \leq i,j \leq t$.

Since $m_1, m_2, \ldots, m_t$ are all different maximal ideals of R, $m_1 m_i = m_1 \cap m_i \subsetneq m_1$, $\forall 2 \leq i \leq t$.

Consider the natural homomorphism $\phi: R \to R_{m_1}$, $\varphi(r) = \frac{r}{1}$. Since R is a domain, φ is injective. If $m_1^2 = m_1$, then $(m_1 R_{m_1})^2 = m_1^2 R_{m_1} = m_1 R_{m_1}$. Since m_1 is finitely generated, so is $m_1 R_{m_1}$. It follows from Nakayama's Lemma that $m_1 R_{m_1} = 0$, so $m_1 = 0$ and R is a field, which is a contradiction. Therefore $m_1^2 \subsetneq m_1$ and every $m_1 m_i (1 \leq i \leq t)$ is a proper submodule of m_1 as an R-module and $\bigcup_{i=1}^{t} m_1 m_i \subsetneq m_1$. Thus there exists $\alpha \in m_1$ such that $\alpha \notin m_1 m_i$, $\forall 1 \leq i \leq t$. If $\alpha \in m_i m_j$, $i, j \neq 1$, then $\alpha = \alpha_i \alpha_j$, $\alpha_i \in m_i$, $\alpha_j \in m_j$ and $\alpha_i, \alpha_j \notin m_1$. So $\alpha = \alpha_i \alpha_j \notin m_1$, which is a contradiction. Therefore $\alpha \notin m_i m_j$, $\forall 1 \leq i,j \leq t$.

(2) It is easy to see that $\mathrm{Max}(R/\alpha R) = \{m/\alpha R \mid m \in Max(R),\ \alpha \in m\}$. Since α is an irreducible element of R, $\alpha \notin m_1 m_i = m_1 \bigcap m_i$, $i = 2, \ldots, t$. So $\alpha \notin m_i$, $i = 2, \ldots, t$, and hence $\mathrm{Max}(R/\alpha R) = \{m_1/\alpha R\}$ and $R/\alpha R$ is a local ring.

(3) Since α is an irreducible element of R, α is also an irreducible element of R_m and $\alpha \notin (mR_m)^2$. From [3] § 8.3 Exercise 1 we know that α can be extended to a minimal set of generators of mR_m. □

Lemma 1.7 ([3] P. 299, Theorem 2). *If R is a semilocal ring, then R has an indecomposable decomposition $R = R_1 \oplus \cdots \oplus R_n$ such that every $R_i (1 \leq i \leq n)$ is an indecomposable semilocal ring.*

Let R be a ring and $R = R_1 \oplus \cdots \oplus R_n$ an indecomposable decomposition of R. Then every maximal ideal of R has the form

$$m = R_1 \oplus \cdots \oplus R_{i-1} \oplus m_i \oplus R_{i+1} \oplus \cdots \oplus R_n \tag{1}$$

where $m_i \in Max(R_i)$, and we have a natural ring isomorphism $R_m \simeq R_{i_{m_i}}$. □

Lemma 1.8. *Let R be a coherent local ring with maximal ideal m. If m is finitely generated, then R is regular if and only if $w.gl.dim\, R < \infty$ if and only if* *m is generated by a regular R-sequence. In addition, if m is generated by a regular R-sequence with q elements, then* $Codim_R(R) = w.gl.dim\, R = q$.

Proof. See [8] Theorem 2.1, Theorem 2.6 and Corollary 2.7. □

Lemma 1.9. *Let R be a regular coherent local ring with maximal ideal m. If m is finitely generated and* $\alpha, \alpha_2, \ldots, \alpha_n$ *is a minimal set of generators of m, then*

$$w.gl.dim\, R/\alpha R = w.gl.dim\, R - 1 < \infty.$$

In this case $R/\alpha R$ is also a regular coherent local ring.

Proof. By Lemma 1.8, we have w.gl.dim $R < \infty$.

Set $\bar{R} = R/\alpha R$, $\bar{m} = m/\alpha R$, $A = \alpha R$, $B = \alpha m + \alpha_2 R + \cdots + \alpha_q R$. It is easy to verify that

$$m = A + B, \quad A \cap B = \alpha m.$$

So

$$m/\alpha m = (A + B)/\alpha m \simeq (A/\alpha m) \oplus (B/\alpha m)$$

and hence

$$\bar{m} = m/\alpha R \simeq (m/\alpha m)/(\alpha R/\alpha m) = (m/\alpha m)/(A/\alpha m) \simeq B/\alpha m,$$

that is, $\bar{m}$ is isomorphic to a direct summand of $m/\alpha m$ as an R-module, and so as an $\bar{R}$-module. Therefore $pd_{\bar{R}}(\bar{m}) \leq pd_{\bar{R}}(m/\alpha m)$.

It follows from [11] Corollary 5 that R is a GCD domain, then α is not a zero divisor on m and R, and thus by [5] Theorem 3.1.2 we have that $pd_{\bar{R}}(\bar{m}) \leq pd_{\bar{R}}(m/\alpha m) = pd_{R/\alpha R}(m/\alpha m) = pd_R(m) < \infty$ and $pd_{\bar{R}}(\bar{R}/\bar{m}) \leq pd_{\bar{R}}(\bar{m}) + 1 < \infty$. By Lemma 1.2

$$w.gl.dim\, R/\alpha R = w.gl.dim\, \bar{R} = fd_{\bar{R}}(\bar{R}/\bar{m}) < \infty.$$

Therefore by [5] Corollary 3.1.4 we have $w.gl.dim\, R/\alpha R = w.gl.dim\, R - 1$. □

2 CODIMENSION

Theorem 2.1. *Let R be a regular indecomposable coherent semilocal ring. If every maximal ideal of R is finitely generated, then*

(1) $m\text{-}codim_R(R) = Codim_{R_m}(R_m) = w.gl.dim\, R_m,\ \forall m \in Max(R)$.
(2) $Codim_R(R) = w.gl.dim\, R$.

Proof. (1) Suppose m is a maximal ideal of R. Since R is a coherent ring and m is finitely generated, it follows from [5] Theorem 2.4.2 that R_m is a coherent local ring with maximal ideal mR_m and mR_m is finitely generated, and $w.gl.dim\,R_m \leq w.gl.dim\,R < \infty$. By Lemma 1.8, R_m is regular and mR_m is generated by a t elements regular R_m-sequence and $Codim_{R_m}(R_m) = w.gl.dim\,R_m = t < \infty$.

We will prove $m\text{-}codim_R(R) = t$ by induction on t.

If $t = 0$, then R_m is a field and $mR_m = 0$. Since R is a regular indecomposable coherent semilocal ring, it follows from [11] Corollary 5 that R is a GCD domain and thus $m = 0$ and $m\text{-}codim_R(R) = 0$.

Now suppose $t > 0$. If $m = 0$, then R is a field and $t = w.gl.dim\,R_m = 0$, which is a contradiction. So $m \neq 0$. By Lemma 1.6, there exists an irreducible element α of R in m such that $R/\alpha R$ is a local ring with maximal ideal $m/\alpha R$ and α, considered as an element in R_m, can be extended to a minimal set of generators of mR_m. By Lemma 1.9 we have $w.gl.dim\,R_m/\alpha R_m = w.gl.dim\,R_m - 1 = t - 1$.

We denote $R/\alpha R$ by $\bar{R}$, then $\bar{R}$ is also coherent. Since $\bar{R}$ is a local ring with maximal ideal $\bar{m} = m/\alpha R$, $\bar{R}_{\bar{m}} \simeq \bar{R}$, and since it is easy to verify that $\bar{R}_{\bar{m}} \simeq R_m/\alpha R_m$,

$$\begin{aligned} w.gl.dim\,\bar{R} &= w.gl.dim\,\bar{R}_{\bar{m}} = w.gl.dim\,R_m/\alpha R_m \\ &= w.gl.dim\,R_m - 1 = t - 1 < \infty. \end{aligned}$$

By induction hypothesis, we have $\bar{m}\text{-}codim_{\bar{R}}(\bar{R}) = t - 1$. In addition, $m\text{-}codim_R(R) = \bar{m}\text{-}codim_{\bar{R}}(\bar{R}) + 1$, so $m\text{-}codim_R(R) = t$.

The statement (2) is an immediate consequence of (1). □

Corollary 2.2. *Let R be a regular coherent semilocal ring. If every maximal ideal of R is finitely generated, then*

(1) $m\text{-}codim_R(R) = Codim_{R_m}(R_m) = w.gl.dim\,R_m,\ \forall m \in Max(R)$.
(2) $Codim_R(R) = w.gl.dim\,R$.

Proof. By Lemma 1.7, R has an indecomposable decomposition: $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, where every R_i $(1 \leq i \leq n)$ is a regular indecomposable coherent semilocal ring. Now, $\forall m \in Max(R)$ we assume

$$m = m_1 \oplus R_2 \oplus \cdots \oplus R_n, \quad m_1 \in Max(R_1).$$

It is easy to verify that $m\text{-}codim_R(R) = m_1\text{-}codim_{R_1}(R_1)$. Then by Theorem 2.1 we have

$$m\text{-}codim_R(R) = m_1\text{-}codim_{R_1}(R_1) = w.gl.dim\,R_{1_{m_2}} = w.gl.dim\,R_m.$$

The statement (2) is an immediate consequence of (1). □

3 REGULARITY OF COHERENT SEMILOCAL RINGS

Theorem 3.1. *Let R be a coherent semilocal ring. If every maximal ideal of R is finitely generated, then the following statements are equivalent:*

(1) R *is regular*;
(2) $w.gl.dim\, R < \infty$;
(3) R_m *is regular*, $\forall m \in Max(R)$;
(4) $pd_R(R/m) < \infty,\ \forall m \in Max(R)$;
(5) $pd_R(R/J) < \infty$;
(6) $id_R(R/m) < \infty$, $\forall m \in Max(R)$;
(7) $id_R(R/J) < \infty$;
(8) $FP\text{-}id_R(R/m) < \infty$, $\forall m \in Max(R)$;
(9) $FP\text{-}id_R(R/J) < \infty$.

Proof. Assume that $m_1, \ldots, m_t$ are all maximal ideals of R. Then $R/J \simeq \oplus_{i=1}^{t} R/m_i$ and $w.gl.dim\, R = \sup\{w.gl.dim\, R_{m_i} | 1 \le i \le t\} = \sup\{fd_R(R/m_i) | 1 \le i \le t\}$.

By Lemma 1.2 we have $w.gl.dim\, R = fd_R(R/J) = id_R(R/J) = FP\text{-}id_R(R/J)$. Since every m_i $(1 \le i \le t)$ is finitely generated, $J = \bigcap_{i=1}^{t} m_i = \prod_{i=1}^{t} m_i$ is also finitely generated. Thus by Lemma 1.1 $fd_R(R/m_i) = pd_R(R/m_i)$, for any $1 \le i \le t$ and therefore $fd_R(R/J) = pd_R(R/J)$. Now we only need to prove (2) $\Rightarrow$ (1). The other implications are trivial by above argument. By [5] Corollary 1.3.9 we have $w.gl.dim\, R = \{pd_R(R/I) \mid I$ is a finitely generated ideal of $R| \le |pd_R(I) + 1 \mid I$ is a finitely generated ideal of $R\}$. Since $w.gl.dim\, R < \infty$, $pd_R(I) < \infty$ for every finitely generated ideal I of R. Thus R is regular. □

Theorem 3.2. *Let R be an indecomposable coherent semilocal ring. If every maximal ideal of R is finitely generated, then the following statements are equivalent:*

(1) *R is regular;*
(2) *Every maximal ideal m of R is generated by a regular R-sequence with q elements, where* $q = w.gl.dim\, R_m$;
(3) *Every maximal ideal m of R satisfying* $w.gl.dim\, R_m = w.gl.dim\, R$ *is generated by a regular R-sequence.*
(4) *There is a maximal ideal m of R such that* $w.gl.dim\, R = w.gl.dim\, R_m$ *and m is generated by a regular R-sequence.*

Proof. The implications of (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1) Let $m \in Max(R)$ with $w.gl.dim\, R = w.gl.dim\, R_m$ and let m be generated by a regular R-sequence $\alpha_1, \ldots, \alpha_t$. Then by Lemma 1.3, $\alpha_1, \ldots, \alpha_t$,

considered as elements in mR_m, is a regular R_m-sequence in mR_m and $mR_m = (\alpha_1, \ldots, \alpha_t)R_m$, and thus by Lemma 1.8 we have that $w.gl.dim\, R_m = Codim_{R_m}(R_m) = t$. So $w.gl.dim\, R = w.gl.dim\, R_m < \infty$ and hence R is regular by Theorem 3.1.

(1) $\Rightarrow$ (2) Suppose m is any maximal ideal of R and $w.gl.dim\, R_m = q$.

If $q = 0$, the conclusion is trivial. Now suppose $q > 0$. It follows from [11] Corollary 5 that R is a GCD domain. By Lemma 1.6, there exists an irreducible element α_1 of R in m such that $R/\alpha_1 R$ (denoted by $\bar{R}$) is a local ring with unique maximal ideal $\bar{m} = m/\alpha_1 R$. Similar to the proof of Theorem 2.1, we can prove $w.gl.dim \bar{R} = w.gl.dim\, R_m - 1 = q - 1 < \infty$. So $\bar{R}$ is a regular coherent local ring whose unique maximal ideal $\bar{m}$ is also finitely generated. It follows from Lemma 1.8 that $\bar{m}$ is generated by a regular $\bar{R}$-sequence: $\bar{\alpha}_2, \ldots, \bar{\alpha}_q$. Thus $m = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ and $\alpha_1, \alpha_2, \ldots, \alpha_q$ is a regular R-sequence. □

Corollary 3.3. *Let R be a coherent semilocal ring and let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_t$ be an indecomposable decomposition of R. If every maximal ideal of R is finitely generated and $w.gl.dim\, R > 0$, then the following statements are equivalent:*

(1) R is regular;
(2) For every maximal ideal $m = R_1 \oplus \cdots \oplus R_{i-1} \oplus m_i \oplus R_{i+1} \oplus \cdots \oplus R_t$, $m_i \in Max(R_i)$, of R, if $m_i \neq 0$, then m is generated by a q elements regular R-sequence, where $q = w.gl.dim\, R_m = w.gl.dim\, R_{i_{m_i}}$.
(3) Every maximal ideal m of R which satisfies $w.gl.dim\, R = w.gl.dim\, R_m$ is generated by a regular R-sequence.
(4) There is a maximal ideal m of R such that $w.gl.dim\, R_m = w.gl.dim\, R$ and m is generated by a regular R-sequence.

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) It is similar to the proof of (4) $\Rightarrow$ (1) in the proof of Theorem 3.2.

(1) $\Rightarrow$ (2) For convenience sake, we assume that $m = m_1 \oplus R_2 \oplus \cdots \oplus R_t, m_1 \in Max(R_1), m_1 \neq 0$. Since $w.gl.dim\, R > 0$, such an m must exist. Since $w.gl.dim\, R = \sup\{w.gl.dim\, R_i | 1 \leq i \leq t\}$, $w.gl.dim\, R_1 \leq w.gl.dim\, R < \infty$, and thus R_1 is a regular indecomposable coherent semilocal ring. Therefore by Theorem 3.2 the maximal ideal m_1 of R is generated by a q elements regular R_1-sequence $\alpha'_1, \ldots, \alpha'_q$. Set $\alpha_i = \alpha'_i + e_2 + \cdots + e_t$, $1 \leq i \leq q$, where $e_i (2 \leq i \leq t)$ is the identity of R_i. It is easy to show that $\alpha_1, \ldots, \alpha_q$ is a regular R-sequence and $m = (\alpha_1, \ldots, \alpha_q)$. So m is generated by a q elements R-sequence and $q = w.gl.dim\, R_{1_{m_1}} = w.gl.dim\, R_m$. □

Remark. When R satisfies the conditions of Corollary 3.3 and R is regular, we can not obtain that every maximal ideal of R is generated by a regular R-sequence. For example, let (R_1, m_1) be a noetherian local ring and $0 < gl.dim\, R_1 < \infty$. Set $R = R_1 \oplus F$, F is a field. It is easy to see that R satisfies the conditions of Corollary 3.3 and $0 < w.gl.dim\, R < \infty$. But the maximal ideal $m = R_1 \oplus 0 \neq 0$ can not be generated by a regular R-sequence. The Jacobson radical of R is $J = m_1 \oplus 0 \neq 0$ and J also can not be generated by a regular R-sequence.

4 SOME EXAMPLES

In this section, we construct some interesting examples of non-noetherian regular indecomposable coherent semilocal rings.

Proposition 4.1. *Let t, n be any two natural numbers. Then there exists a non-noetherian indecomposable coherent semilocal ring R such that: (1) R has exactly t maximal ideals; (2) $w.gl.dim\, R = n+1$, $gl.dim\, R = n+2$; (3) Every maximal ideal of R is not finitely generated.*

Proof. Let K be a field with characteristic zero. Set $A = \bigcup_{n \geq 1} K[[x^{\frac{1}{n}}]]$. It follows from [8] Example 2 that A is a non-noetherian coherent local ring with unique maximal ideal $m = \{f(x) \in A \mid f(0) = 0\}$ which is not finitely generated and $w.gl.dim\, A = 1$, $gl.dim\, A = 2$.

Let $\Lambda = A[t_1, t_2, \ldots, t_n]$ be the polynomial ring over A on the indeterminates $t_1, t_2, \ldots, t_n$. Set

$$m_i = m + (t_1 + i - 1)\Lambda + t_2\Lambda + \cdots + t_n\Lambda, \quad 1 \leq i \leq t.$$

It is easy to verify that $m_1, \ldots, m_t$ are different maximal ideals of Λ. Set $S = \Lambda - \bigcup_{i=1}^{t} m_i = \{s \in \Lambda \mid s \notin \bigcup_{i=1}^{t} m_i\}$. It is clear that S is a multiplicatively closed subset of Λ and $1 \in S$, we denote by Λ_S the localization ring of Λ at S. We shall prove that Λ_S satisfies all conditions of Proposition 4.1.

First, by [11] Corollary 5, A is a domain. So Λ and Λ_S are domains and hence Λ_S is indecomposable.

Since A is a coherent ring of global dimension two, it follows from [5] Theorem 7.3.14 that Λ is a coherent ring and thus from [5] Theorem 2.4.2 we know that Λ_S is also a coherent ring.

It is easy to verify that $m_1\Lambda_S, \ldots, m_t\Lambda_S$ are all different maximal ideals of Λ_S and every $m_i\Lambda_S (1 \leq i \leq t)$ is not finitely generated.

By [5] Hilbert Syzygies Theorem 1.3.17, $w.gl.dim\, \Lambda = w.gl.dim\ A + n = n + 1$ and $gl.dim\, \Lambda = gl.dim A + n = n + 2$.

It follows from [5] Theorem 1.3.13 that $w.gl.dim\, \Lambda_S \leq w.gl.dim\, \Lambda = n + 1$ and $gl.dim\, \Lambda_S \leq gl.dim\, \Lambda = n + 2$. Let $0 \neq \alpha \in m$, we may easily show that $\alpha, t_1, t_2, \ldots, t_n$ is a regular Λ-sequence in m_1. By Lemma 1.3, $\alpha, t_1, \ldots, t_n$, considered as elements in Λ_{m_1}, is a regular R_{m_1}-sequence and thus by Theorem 2.1 we have $w.gl.dim\,\Lambda_{m_1} = Codim\,\Lambda_{m_1} \geq n+1$. Set $S_1 = \Lambda - m_1$, then $S \subset S_1$. It follows from [2] § 7.2 Proposition 7.4 that $\Lambda_{m_1} = \Lambda_{S_1} \simeq (\Lambda_S)_{S_1/S}$. Thus by [5] Theorem 1.3.13 we have

$$w.gl.dim\,\Lambda_S \geq w.gl.dim\,\Lambda_{m_1} \geq n+1 \quad \text{and} \quad gl.dim\,\Lambda_S \geq gl.dim\,\Lambda_{m_1} \geq n+1.$$

Therefore $w.gl.dim\,\Lambda_S = n+1$ and $n+1 \leq gl.dim\,\Lambda_{m_1} \leq gl.dim\,\Lambda_S \leq n+2$.

If $gl.dim\,\Lambda_{m_1} = n+1$, then $gl.dim\,\Lambda_{m_1} = w.gl.dim\,\Lambda_{m_1} = n+1 < \infty$ and so by [5] Theorem 6.2.15 the maximal ideal $m_1\Lambda_{m_1}$ is finitely generated, which is a contradiction. Hence $gl.dim\,\Lambda_{m_1} = n+2$. In addition, $gl.dim\,\Lambda_{m_1} \leq gl.dim\,\Lambda_S \leq n+2$. Thus $gl.dim\,\Lambda_S = n+2$. □

Proposition 4.2. *Let t, n be any two natural numbers. Then there exists a non-noetherian indecomposable coherent semilocal ring R such that:*

(1) R has exactly t maximal ideals;
(2) $w.gl.dim\,R = gl.dim\,R = n+2$;
(3) Every maximal ideal of R is finitely generated.

Proof. Let (A, m) be an umbrella ring (see [9] for the definition) with characteristic zero (For example, let $\Lambda = \mathbb{Z}[x]$, $m = (2, x) \in Max(\Lambda)$, $T = \Lambda_m$, K the quotient field of T. It follows from [9] that $A = K[[t)) = \{f(t) \in K[[t]] \mid f(0) \in T\}$ is an umbrella ring and it is easy to see that the characteristic of A is zero).

By [8] Example 3, we know that A is a coherent local domain and $w.gl.dim\,A = gl.dim\,A = 2$ and m can be generated by a two elements regular A-sequence $\{\alpha_1, \alpha_2\}$ and there is a non-finitely generated prime ideal of A. So A is a non-noetherian super regular coherent local ring.

Set

$$\Lambda = A[t_1, t_2, \ldots, t_n],$$
$$m_i = (\alpha_1, \alpha_2, t_1 + i - 1, t_2, \ldots, t_n), \quad 1 \leq i \leq t.$$

Since Char $A = 0$, it is easy to prove that $m_1, \ldots, m_t$ are different maximal ideals of Λ. Set

$$S = \Lambda - \bigcup_{i=1}^{t} m_i = \left\{ s \in \Lambda | s \notin \bigcup_{i=1}^{t} m_i \right\}.$$

We can prove that Λ_S satisfies the conditions of Proposition 4.2. The proof here is similar to that of Proposition 4.1, we omit it. □

ACKNOWLEDGMENT

Supported by the National Science Foundation of China (19771046).

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Received October 1999
Revised June 2000