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EXTENSION CLOSURE OF RELATIVE *k*-TORSIONFREE MODULES

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In this article, we study the extension closure of the category of modules consisting of relative k-torsionfree modules. Some previous results related to the extension closure of k-torsionfree modules are extended and strengthened.

Key Words: Extension closed; ω-k-torsionfree modules; (Strong) grade.

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1. INTRODUCTION AND MAIN RESULTS

Let Λ be a ring. We use $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$) to denote the category of finitely generated left Λ-modules (resp. right Λ-modules). We always assume that Λ and Γ are Artinian algebras and ${}_\Lambda\omega_\Gamma$ is a faithfully balanced self-orthogonal bimodule, that is, ${}_\Lambda\omega_\Gamma$ satisfies the following conditions: (1) ${}_\Lambda\omega$ is in $\operatorname{mod}\Lambda$ and ω_Γ is in $\operatorname{mod}\Gamma^{op}$; (2) the natural maps $\Gamma \to \operatorname{End}({}_\Lambda\omega)^{op}$ and $\Lambda \to \operatorname{End}(\omega_\Gamma)$ are isomorphisms; (3) $\operatorname{Ext}^i_\Lambda({}_\Lambda\omega, {}_\Lambda\omega) = 0$ and $\operatorname{Ext}^i_\Gamma(\omega_\Gamma, \omega_\Gamma) = 0$ for any $i \geq 1$. We use $\operatorname{add}_\Lambda\omega$ (resp. $\operatorname{add}\omega_\Gamma$) to denote the subcategory of $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{op}$) consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda\omega$ (resp. ω_Γ).

Suppose that $A \in \operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{op}$). Then, we call $\operatorname{Hom}_\Lambda({}_\Lambda A, {}_\Lambda\omega_\Gamma)$ (resp. $\operatorname{Hom}_\Gamma(A_\Gamma, {}_\Lambda\omega_\Gamma)$) the dual module of A with respect to ω, and denote these modules by A^ω. For a homomorphism f between the Λ-modules (resp. Γ^{op}-modules), we put $f^\omega = \operatorname{Hom}(f, {}_\Lambda\omega_\Gamma)$. Let $\sigma_A : A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$, for any $x \in A$ and $f \in A^\omega$, be the canonical evaluation homomorphism. Then, we call A *ω-torsionless* (resp. *ω-reflexive*) if σ_A is a monomorphism (resp. an isomorphism).

Now let $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a minimal projective resolution of A. Then we have an exact sequence $0 \to A^\omega \to P_0^\omega \xrightarrow{f^\omega} P_1^\omega \to \operatorname{Coker} f^\omega \to 0$. We call $\operatorname{Coker} f^\omega$ the transpose (with respect to ${}_\Lambda\omega_\Gamma$) of A, and denote it by $\operatorname{Tr}_\omega A$. For a positive integer k, a module A in $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{op}$) is called *$\omega$-$k$-torsionfree* if $\operatorname{Ext}^i_\Gamma(\operatorname{Tr}_\omega A, \omega)$ (resp. $\operatorname{Ext}^i_\Lambda(\operatorname{Tr}_\omega A, \omega)$) $= 0$ for any $1 \leq i \leq k$. A is called *ω-k-syzygy* if there is an exact sequence $0 \to A \to X_0 \to X_1 \to \cdots \xrightarrow{f_{k-1}} X_{k-1}$ with all X_i in $\operatorname{add}_\Lambda\omega$

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(resp. $\mathrm{add}\,\omega_\Gamma$). At this moment, $\mathrm{Coker}\, f_{k-1}$ is called a *ω-k-cosyzygy module*. We remark that a module is ω-torsionless (resp. ω-reflexive) if and only if it is ω-1-torsionfree (resp. ω-2-torsionfree) (see Huang, 2001).

Put ${}_\Lambda\omega_\Gamma = {}_\Lambda\Lambda_\Lambda$. Then, in this case, the notions of *ω-k-torsionfree* modules and ω-k-syzygy modules are just the k-torsionfree modules and k-syzygy modules, respectively (see Auslander and Bridger, 1969 for the definitions of k-torsionfree modules and k-syzygy modules). We use $\mathcal{T}^k_\omega(\Gamma^{op})$ (resp. $\Omega^k_\omega(\Gamma^{op})$) to denote the full subcategory of $\mathrm{mod}\,\Gamma^{op}$ consisting of ω-k-torsionfree modules (resp. ω-k-syzygy modules). By Huang (2001, Theorem 1), we have $\mathcal{T}^k_\omega(\Gamma^{op}) \subseteq \Omega^k_\omega(\Gamma^{op})$.

Recall that a full subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$) is said to be *extension closed* if the middle term B of any short sequence $0 \to A \to B \to C \to 0$ is in $\mathcal{X}$ provided that the end terms A and C are in $\mathcal{X}$. The extension closure of the subcategory consisting of k-syzygy modules and the subcategory consisting of k-torsionfree modules have been studied in terms of flat dimensions and grade of modules by Auslander and Reiten (1996) and by the author Huang (1999). In Huang (2003) we have further studied the extension closure of $\Omega^k_\omega(\Gamma^{op})$. Motivated by the above results, we will deal with the extension closure of $\mathcal{T}^k_\omega(\Gamma^{op})$.

Let $A \in \mathrm{mod}\,\Lambda$. If there exists an exact sequence $\cdots \to \omega_n \to \cdots \to \omega_1 \to \omega_0 \to A \to 0$ in $\mathrm{mod}\,\Lambda$ with each $\omega_i \in \mathrm{add}\,{}_\Lambda\omega$ for any $i \geq 0$, then we define ω-resol.$\dim_\Lambda(A) = \inf\{n \mid$ there exists an exact sequence $0 \to \omega_n \to \cdots \to \omega_1 \to \omega_0 \to A \to 0$ in $\mathrm{mod}\,\Lambda$ with each $\omega_i \in \mathrm{add}_\Lambda\omega$ for any $0 \leq i \leq n\}$. We set ω-resol.$\dim_\Lambda(A)$ infinity if there does not exist such an integer (see Auslander and Buchweitz, 1989).

Suppose that

$$0 \to {}_\Lambda\omega \to E_0 \to E_1 \cdots \to E_i \to \cdots$$

is a minimal injective resolution of ${}_\Lambda\omega$ and k is a positive integer. One of the main results in this article is the following theorem.

Theorem 1.1. *If* ω-resol.$\dim_\Lambda\big(\bigoplus_{i=0}^{k-1} E_i\big) \leq k$, *then* $\mathcal{T}^k_\omega(\Gamma^{op})$ *is extension closed.*

Let $A \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$) and i a non-negative integer. Then, the grade of A with respect to ω, written as $\mathrm{grade}_\omega A$, is said to be greater than or equal to i if $\mathrm{Ext}^j_\Lambda(A, \omega) = 0$ (resp. $\mathrm{Ext}^j_\Gamma(A, \omega) = 0$), for any $0 \leq j < i$. We also say that the strong grade of A with respect to ω, written as $\mathrm{s.grade}_\omega A$, is greater than or equal to i if $\mathrm{grade}_\omega B \geq i$ for all submodules B of A. The following result characterizes the assumption of Theorem 1.1 in terms of strong grade of modules.

Theorem 1.2. *Let m be an integer with $m \geq -k$. Then* ω-resol.$\dim_\Lambda\big(\bigoplus_{i=0}^{k-1} E_i\big) \leq k+m$ *if and only if* $\mathrm{s.grade}_\omega \mathrm{Ext}^{k+m+1}_\Gamma(N, \omega) \geq k$ *for any* $N \in \mathrm{mod}\,\Gamma^{op}$.

In Section 2, some lemmas we will give that will be useful in the rest of this article. The proofs of Theorems 1.1–1.2 will be given and some corollaries will be listed in Section 3. In particular, as a corollary of Theorem 1.2, we get that ${}_\Lambda\Lambda$ has dominant dimension greater than or equal to k if and only if $\mathrm{s.grade}_\Lambda \mathrm{Ext}^1_\Lambda(N, \Lambda) \geq k$ for any $N \in \mathrm{mod}\,\Lambda^{op}$ (Corollary 3.10).

2. SOME LEMMAS

In this section we give some lemmas which will be useful in proving the main results.

For a Λ-module (resp. Γ^{op}-module) X, we use $l.\mathrm{id}_\Lambda(X)$ (resp. $r.\mathrm{id}_\Gamma(X)$) and $l.\mathrm{fd}_\Lambda(X)$ (resp. $r.\mathrm{fd}_\Gamma(X)$) to denote the left (resp. right) injective dimension and flat dimension of X, respectively. We first make the following observation.

Lemma 2.1. *Let E be Λ-injective (resp. Γ^{op}-injective) and n a non-negative integer. Then $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E))$ (resp. $r.\mathrm{fd}_\Lambda(\mathrm{Hom}_\Gamma({}_\Lambda\omega_\Gamma, E))) \leq n$ if and only if $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{n+1}_\Gamma(A, \omega), E)$ (resp. $\mathrm{Hom}_\Gamma(\mathrm{Ext}^{n+1}_\Lambda(A, \omega), E)) = 0$ for any $A \in \mathrm{mod}\,\Gamma^{op}$ (resp. $\mathrm{mod}\,\Lambda$).*

Proof. The proof is trivial by Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3). □

Lemma 2.2. (1) $r.\mathrm{id}_\Gamma(\omega) = \sup\{l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E)) \mid {}_\Lambda E$ is injective$\}$.

(2) *Let E be injective in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$). Then $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E))$ (resp. $r.\mathrm{fd}_\Lambda(\mathrm{Hom}_\Gamma({}_\Lambda\omega_\Gamma, E))) \leq n$ if and only if ω-resol.dim$_\Lambda(E)$ (resp. ω-resol.dim$_\Gamma(E)) \leq n$.*

Proof. (1) By Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3.), we have

$$\mathrm{Tor}^\Gamma_i(A, \mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E)) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}^i_\Gamma(A, \omega), E) \qquad (\dagger)$$

for any $A \in \mathrm{mod}\,\Gamma^{op}$ and ${}_\Lambda E$ injective and $i \geq 1$.

If $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E)) \leq n(<\infty)$ for any injective module ${}_\Lambda E$, then (†) induces $\mathrm{Hom}_\Lambda(\mathrm{Ext}^{n+1}_\Gamma(A, \omega), E) \cong \mathrm{Tor}^\Gamma_{n+1}(A, \mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E)) = 0$. Now taking ${}_\Lambda E$ as an injective cogenerator in $\mathrm{mod}\,\Lambda$, we see that $\mathrm{Ext}^{n+1}_\Gamma(A, \omega) = 0$ and $r.\mathrm{id}_\Gamma(\omega) \leq n$.

Conversely, if $r.\mathrm{id}_\Gamma(\omega) \leq n(<\infty)$, then $\mathrm{Ext}^{n+1}_\Gamma(A, \omega) = 0$ for any $A \in \mathrm{mod}\,\Gamma^{op}$ and $\mathrm{Tor}^\Gamma_{n+1}(A, \mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E)) = 0$ for any injective module ${}_\Lambda E$ by the isomorphism (†). This implies that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E)) \leq n$.

(2) See Huang (2005, Lemma 2.7). □

Remark. Lemma 2.2(1) is a generalization of Enochs and Jenda (2000, Proposition 9.1.6).

Lemma 2.3 (Huang, 2005, Proposition 3.4). *The following statements are equivalent*:

(1) ω-resol.dim$_\Lambda(E_0) \leq 1$;
(2) *σ_X is an essential monomorphism for any ω-torsionless module X in* $\mathrm{mod}\,\Lambda$;
(3) *$f^{\omega\omega}$ is a monomorphism for any monomorphism $f: X \to Y$ in $\mathrm{mod}\,\Lambda$ with Y ω-torsionless*;
(4) $\mathrm{grade}_\omega \mathrm{Ext}^1_\Lambda(X, \omega) \geq 1$ *for any X in* $\mathrm{mod}\,\Lambda$.

Lemma 2.4 (Huang, 2006, Theorem 17.5.4). *The following statements are equivalent*:

(1) s.grade$_\omega$Ext$_\Gamma^{i+1}(N,\omega) \geq i$ *for any* $N \in \operatorname{mod}\Gamma^{op}$ *and* $1 \leq i \leq k$;
(2) grade$_\omega$Ext$_\Lambda^{i}(M,\omega) \geq i$ *for any* $M \in \operatorname{mod}\Lambda$ *and* $1 \leq i \leq k$.

3. PROOFS OF THE MAIN RESULTS

In this section, we prove the main results (Theorems 1.1–1.2) of this article. As applications of our main results, we list some corollaries, some of which are known results.

We now prove Theorem 1.1 in several steps.

Lemma 3.1. ω-resol.dim$_\Lambda(E_0) \leq 1$ *if and only if* $\mathcal{T}_\omega^1(\Gamma^{op})$ *is extension closed.*

Proof. Note that $\mathcal{T}_\omega^1(\Gamma^{op}) = \{X \in \operatorname{mod}\Gamma^{op} \mid X$ is ω-torsionless$\}$. Then the conclusion follows from Lemma 2.3 and Huang (2003, Proposition 4.2.). □

Lemma 3.2. *If* ω-resol.dim$_\Lambda\big(\bigoplus_{i=0}^{k-1}E_i\big) \leq k$ *(where* $k \geq 2$*) and* $0 \to A \xrightarrow{f} B \to C \to 0$ *is an exact sequence in* $\operatorname{mod}\Gamma^{op}$ *with* C ω-k-*torsionfree, then*:

(1) grade$_\omega$ Coker $f^\omega \geq k$;
(2) *B is ω-reflexive provided that A is ω-reflexive.*

Proof. (1) Note that C is a ω-k-torsionfree module, that is, $\operatorname{Ext}_\Lambda^i(\operatorname{Tr}_\omega C,\omega) = 0$ for any $1 \leq i \leq k$. Then C is ω-reflexive and $\operatorname{Ext}_\Lambda^i(C^\omega,\omega) = 0$ for any $1 \leq i \leq k-2$.

Let

$$P_{k-1} \to \cdots \to P_1 \to P_0 \to C^\omega \to 0$$

be a minimal projective resolution of C^ω in $\operatorname{mod}\Lambda$. Then we have an induced exact sequence

$$0 \to C \to P_0^\omega \to P_1^\omega \to \cdots \to P_{k-1}^\omega \to H \to 0,$$

where $H = \operatorname{Coker}(P_{k-2}^\omega \to P_{k-1}^\omega)$. Since ω-resol.dim$_\Lambda\big(\bigoplus_{i=0}^{k-1}E_i\big) \leq k$, $l.\operatorname{fd}_\Gamma\big(\operatorname{Hom}_\Lambda\big({}_\Lambda\omega_\Gamma, \bigoplus_{i=0}^{k-1}E_i\big)\big) \leq k$ by Lemma 2.2. We can see, from Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3), that $\operatorname{Hom}_\Lambda\big(\operatorname{Ext}_\Gamma^1(C,\omega), \bigoplus_{i=0}^{k-1}E_i\big) \cong \operatorname{Tor}_1^\Gamma\big(C, \operatorname{Hom}_\Lambda\big({}_\Lambda\omega_\Gamma, \bigoplus_{i=0}^{k-1}E_i\big)\big) \cong \operatorname{Tor}_{k+1}^\Gamma\big(H, \operatorname{Hom}_\Lambda\big({}_\Lambda\omega_\Gamma, \bigoplus_{i=0}^{k-1}E_i\big)\big) = 0$. Now, we have $\operatorname{Hom}_\Lambda\big(\operatorname{Coker}f^\omega, \bigoplus_{i=0}^{k-1}E_i\big) = 0$ since $\operatorname{Coker}f^\omega$ is a submodule of $\operatorname{Ext}_\Gamma^1(C,\omega)$, from which we get our conclusion.

(2) By (1), we have grade$_\omega$Coker $f^\omega \geq 2$. Now, by applying $\operatorname{Hom}_\Lambda(-,\omega)$ to the exact sequence $0 \to C^\omega \to B^\omega \xrightarrow{f^\omega} A^\omega \to \operatorname{Coker}f^\omega \to 0$, we get the exact sequence $0 \to A^{\omega\omega} \xrightarrow{f^{\omega\omega}} B^{\omega\omega} \to C^{\omega\omega}$ and the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma_A} & & \downarrow{\scriptstyle\sigma_B} & & \downarrow{\scriptstyle\sigma_C} & & \\ 0 & \longrightarrow & A^{\omega\omega} & \xrightarrow{f^{\omega\omega}} & B^{\omega\omega} & \longrightarrow & C^{\omega\omega}. & & \end{array}$$

Since σ_A and σ_C are isomorphisms, σ_B is also an isomorphism by the snake lemma, that is, B is ω-reflexive. □

Corollary 3.3.

(1) *Let* $0 \to A \xrightarrow{f} B \to C \to 0$ *be an exact sequence in* $\mathrm{mod}\,\Gamma^{op}$. *If* $\mathrm{grade}_\omega \mathrm{Coker}\, f^\omega \geq 2$, *then B is in* $\mathcal{T}^2_\omega(\Gamma^{op})$ *provided that both A and C are in* $\mathcal{T}^2_\omega(\Gamma^{op})$.

(2) *If* $\omega\text{-resol.dim}_\Lambda\big(E_0 \bigoplus E_1\big) \leq 2$, *then* $\mathcal{T}^2_\omega(\Gamma^{op})$ *is extension closed.*

Proof. (1) By Huang (2001, Lemma 4), $\mathcal{T}^2_\omega(\Gamma^{op}) = \{X \in \mathrm{mod}\,\Gamma^{op} \mid X \text{ is } \omega\text{-reflexive}\}$. Then our conclusion follows from the proof of Lemma 3.2(2).

(2) It is easy by (1) and Lemma 3.2(1). □

Lemma 3.4. *If* $\omega\text{-resol.dim}_\Lambda\big(\bigoplus_{i=0}^{k-1} E_i\big) \leq k$ *(where* $k \geq 3$*), then* $\mathcal{T}^k_\omega(\Gamma^{op})$ *is extension closed.*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in $\mathrm{mod}\,\Gamma^{op}$ with A and C ω-k-torsionfree. Then A and C are ω-reflexive and $\mathrm{Ext}^i_\Lambda(A^\omega, \omega) = 0 = \mathrm{Ext}^i_\Lambda(C^\omega, \omega)$ for any $1 \leq i \leq k-2$. By Lemma 3.2(2), B is ω-reflexive.

Since $0 \to \mathrm{Coker}\, g^\omega \to A^\omega \to \mathrm{Coker}\, f^\omega \to 0$ is exact, and since $\mathrm{grade}_\omega \mathrm{Coker}\, f^\omega \geq k$ by Lemma 3.2(1), we have $\mathrm{Ext}^i_\Lambda(\mathrm{Coker}\, g^\omega, \omega) = 0$ for any $1 \leq i \leq k-2$. Also, from the exact sequence $0 \to C^\omega \xrightarrow{g^\omega} B^\omega \to \mathrm{Coker}\, g^\omega \to 0$, we can easily see that $\mathrm{Ext}^i_\Lambda(B^\omega, \omega) \cong \mathrm{Ext}^i_\Lambda(\mathrm{Coker}\, g^\omega, \omega) = 0$, for any $1 \leq i \leq k-2$. Noting that B is ω-reflexive, we hence have $\mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega B, \omega) = 0$, for any $1 \leq i \leq k$ and so B is ω-k-torsionfree. □

Now Theorem 1.1 follows from Lemma 3.1, Corollary 3.3(2), and Lemma 3.4.

Corollary 3.5. *If* $\omega\text{-resol.dim}_\Lambda(E_i) \leq i+1$ *for any* $0 \leq i \leq k-1$, *then* $\mathcal{T}^i_\omega(\Gamma^{op})$ *is extension closed for any* $1 \leq i \leq k$.

Remark. Theorem 1.1 generalizes Huang (1999, Theorem 2.3). By Lemma 2.2, if $r.\mathrm{id}_\Gamma(\omega) \leq k$, then $\omega\text{-resol.dim}_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k$. So Theorem 1.1 is also a generalization of Huang (2003, Proposition 4.1).

By Huang (1999, Theorem 3.3), the converse of Corollary 3.5 holds when ${}_\Lambda\omega_\Gamma = {}_\Lambda\Lambda_\Lambda$. Assume that $k \leq 2$ and $\mathcal{T}^i_\omega(\Gamma^{op})$ is extension closed for any $1 \leq i \leq k$. Then $\mathrm{grade}_\omega \mathrm{Ext}^i_\Lambda(M, \omega) \geq i$ for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leq i \leq k$ by Huang (2003, Theorem 4.1). So $\mathrm{s.grade}_\omega \mathrm{Ext}^{i+1}_\Gamma(N, \omega) \geq i$ for any $N \in \mathrm{mod}\,\Gamma^{op}$ and $1 \leq i \leq k$ by Lemma 2.4. It follows from Corollary 3.8(2) (see below) that $\omega\text{-resol.dim}_\Lambda(E_i) \leq i+1$ for any $0 \leq i \leq k-1$. This means that the converse of Corollary 3.5 holds when $k \leq 2$. However, we don't know whether this converse holds in general.

We use $\Omega^{-k}_\omega(\Gamma^{op})$ to denote the full subcategory of $\mathrm{mod}\,\Gamma^{op}$ consisting of ω-k-cosyzygy modules. In the following result, we give some equivalent conditions of $\mathcal{T}^i_\omega(\Gamma^{op})$ being extension closed for any $1 \leq i \leq k$, which extends Huang (2003, Theorem 3.3).

Theorem 3.6. *The following statements are equivalent*:

(1) $\text{s.grade}_\omega \text{Ext}_\Gamma^{i+1}(N, \omega) \geq i$ *for any* $N \in \Omega_\omega^{-i}(\Gamma^{op})$ *and* $1 \leq i \leq k$;
(2) $\Omega_\omega^i(\Gamma^{op})$ *is extension closed for any* $1 \leq i \leq k$;
(3) $\Omega_\omega^i(\Gamma^{op})$ *is extension closed and* $\Omega_\omega^i(\Gamma^{op}) = \mathcal{T}_\omega^i(\Gamma^{op})$ *for any* $1 \leq i \leq k$;
(4) $\mathcal{T}_\omega^i(\Gamma^{op})$ *is extension closed for any* $1 \leq i \leq k$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By Huang (2003, Theorem 3.3).

(3) $\Rightarrow$ (4) It is trivial.

(4) $\Rightarrow$ (1) We proceed by using induction on k. The case $k \leq 2$ follows from the above argument.

Now suppose $k \geq 3$. By induction assumption, $\text{s.grade}_\omega \text{Ext}_\Gamma^{i+1}(N, \omega) \geq i$ for any $N \in \Omega_\omega^{-i}(\Gamma^{op})$ and $1 \leq i \leq k-1$. It follows from Huang (2003, Theorem 3.1) that $\Omega_\omega^i(\Gamma^{op}) = \mathcal{T}_\omega^i(\Gamma^{op})$ for any $1 \leq i \leq k$. Then by (4), $\Omega_\omega^i(\Gamma^{op})$ is extension closed for any $1 \leq i \leq k$, that is, the condition (2) holds. We have known (2) $\Leftrightarrow$ (1), so our conclusion follows. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. This proof is similar to that of Huang (1999, Theorem 2.8). For the sake of completeness, we give here the proof. "if-part":

We proceed by using induction on i. Suppose that $\text{s.grade}_\omega \text{Ext}_\Gamma^{k+m+1}(N, \omega) \geq k$, for any $N \in \text{mod}\,\Gamma^{op}$. We first prove that $\omega\text{-resol.dim}_\Lambda(E_0) \leq k+m$. By assumption, we have $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, \omega), \omega) = 0$. We now claim that $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, \omega), E_0) = 0$. Assume to the contrary that there exists $0 \neq f : \text{Ext}_\Gamma^{k+m+1}(N, \omega) \to E_0$ and $\text{Im} f \cap \omega \neq 0$ (since ω is essential in E_0). Hence, there exists a submodule $X(=f^{-1}(\text{Im} f \cap \omega))$ of $\text{Ext}_\Gamma^{k+m+1}(N, \omega)$ such that $\text{Hom}_\Lambda(X, \omega) \neq 0$, which contradicts $\text{s.grade}_\omega \text{Ext}_\Gamma^{k+m+1}(N, \omega) \geq k$. Then, by Lemmas 2.1 and 2.2, we have $\text{l.fd}_\Gamma(\text{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E_0)) \leq k+m$ and $\omega\text{-resol.dim}_\Lambda(E_0) \leq k+m$.

Now suppose that $i \geq 1$. Consider the exact sequence

$$0 \to K_{i-1} \to E_{i-1} \to K_i \to 0,$$

where $K_{i-1} = \text{Ker}(E_{i-1} \to E_i)$ and $K_i = \text{Im}(E_{i-1} \to E_i)$. Then for any $X \subset \text{Ext}_\Gamma^{k+m+1}(N, \omega)$, we have an exact sequence

$$\text{Hom}_\Lambda(X, E_{i-1}) \to \text{Hom}_\Lambda(X, K_i) \to \text{Ext}_\Lambda^1(X, K_{i-1}) \to 0.$$

Since $\text{s.grade}_\omega \text{Ext}_\Gamma^{k+m+1}(N, \omega) \geq k$ and $1 \leq i \leq k-1$, $\text{Ext}_\Lambda^1(X, K_{i-1}) \cong \text{Ext}_\Lambda^i(X, \omega) = 0$. By induction assumption and Lemma 2.1, we have $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, \omega), E_{i-1}) = 0$. Since E_{i-1} is injective, $\text{Hom}_\Lambda(X, E_{i-1}) = 0$. Hence, it follows that $\text{Hom}_\Lambda(X, K_i) = 0$. Observe that E_i is the injective envelope of K_i, by using a similar argument to the case $i = 0$, we can show that $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, \omega), E_i) = 0$. Hence, we have $l.\text{fd}_\Gamma(\text{Hom}_\Lambda({}_\Lambda\omega_\Gamma, E_i)) \leq k+m$ and $\omega\text{-resol.dim}_\Lambda(E_i) \leq k+m$. "only if-part": Suppose that $\omega\text{-resol.dim}_\Lambda\big(\bigoplus_{i=0}^{k-1} E_i\big) \leq k+m$. Then, by Lemmas 2.2 and 2.1, we have $l.\text{fd}_\Gamma\big(\text{Hom}_\Lambda\big({}_\Lambda\omega_\Gamma, \bigoplus_{i=0}^{k-1} E_i\big)\big) \leq k+m$ and $\text{Hom}_\Lambda\big(\text{Ext}_\Gamma^{k+m+1}(N, \omega), \bigoplus_{i=0}^{k-1} E_i\big) = 0$

for any $N \in \operatorname{mod}\Gamma^{op}$. Let X be any submodule of $\operatorname{Ext}_\Gamma^{k+m+1}(N, \omega)$. Then $\operatorname{Hom}_\Lambda\big(X, \bigoplus_{i=0}^{k-1} E_i\big) = 0$. Put $K_0 = \omega$ and $K_i = \operatorname{Im}(E_{i-1} \to E_i)$ for any $1 \le i \le k-1$. Then $\operatorname{Hom}_\Lambda(X, K_i) = 0$ for any $0 \le i \le k-1$. By dimension shifting, $\operatorname{Ext}_\Lambda^{i+1}(X, K_0) \cong \operatorname{Ext}_\Lambda^1(X, K_i)$ and $\operatorname{Ext}_\Lambda^1(X, K_i) \cong \operatorname{Hom}_\Lambda(X, K_{i+1})$ for any $0 \le i \le k-2$. Hence we conclude that $\operatorname{Hom}_\Lambda(X, \omega) = 0 = \operatorname{Ext}_\Lambda^i(X, \omega)$ for any $1 \le i \le k-1$. This completes the proof. □

By putting $m = -1$, then by Theorem 1.2, we have the following Corollary.

Corollary 3.7.

(1) *ω-resol.dim$_\Lambda\big(\bigoplus_{i=0}^{k-1} E_i\big) \le k-1$ if and only if s.grade$_\omega \operatorname{Ext}_\Gamma^k(N, \omega) \ge k$ for any $N \in \operatorname{mod}\Gamma^{op}$.*
(2) *ω-resol.dim$_\Lambda(E_i) \le i$ for any $0 \le i \le k-1$ if and only if s.grade$_\omega \operatorname{Ext}_\Gamma^i(N, \omega) \ge i$ for any $N \in \operatorname{mod}\Gamma^{op}$ and $1 \le i \le k$.*

Putting $m = 0$, then by Theorem 1.2, we have the following result, in which the first assertion gives a characterization of the assumption in Theorem 1.1 and the second one is just Auslander and Reiten (1996, Proposition 2.2) when ${}_\Lambda\omega_\Gamma = {}_\Lambda\Lambda_\Lambda$. Compare the strong grade condition of modules in the second assertion with that in Theorem 3.6(1).

Corollary 3.8.

(1) *ω-resol.dim$_\Lambda\big(\bigoplus_{i=0}^{k-1} E_i\big) \le k$ if and only if s.grade$_\omega \operatorname{Ext}_\Gamma^{k+1}(N, \omega) \ge k$ for any $N \in \operatorname{mod}\Gamma^{op}$.*
(2) *ω-resol.dim$_\Lambda(E_i) \le i+1$ for any $0 \le i \le k-1$ if and only if s.grade$_\omega \operatorname{Ext}_\Gamma^{i+1}(N, \omega) \ge i$ for any $N \in \operatorname{mod}\Gamma^{op}$ and $1 \le i \le k$. In this case, $\mathcal{T}_\omega^i(\Gamma^{op}) = \Omega_\omega^i(\Gamma^{op})$ for any $1 \le i \le k$.*

Proof. Our assertion follows from Theorem 1.2 and Huang (2003, Theorem 3.3). □

Putting $m = -k$, then by Theorem 1.2, we have the following corollary.

Corollary 3.9. *Each E_i is in add$_\Lambda\omega$ for any $0 \le i \le k-1$ if and only if s.grade$_\omega \operatorname{Ext}_\Gamma^1(N, \omega) \ge k$ for any $N \in \operatorname{mod}\Gamma^{op}$.*

Recall that a module M in $\operatorname{mod}\Lambda$ is said to have *dominant dimension* greater than or equal to k, written dom.dim$({}_\Lambda M) \ge k$, if each of the first k terms in a minimal injective resolution of M is Λ-projective (see Tachikawa, 1973). By Corollary 3.9, we get a characterization of ${}_\Lambda\Lambda$ having dominant dimension greater than or equal to k as follows.

Corollary 3.10. dom.dim$({}_\Lambda\Lambda) \ge k$ *if and only if* s.grade$_\Lambda \operatorname{Ext}_\Lambda^1(N, \Lambda) \ge k$ *for any* $N \in \operatorname{mod}\Lambda^{op}$.

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