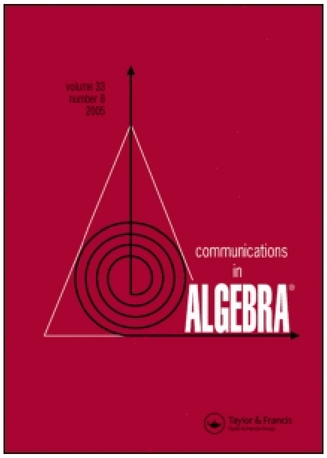

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EXTENSION CLOSURE OF k-TORSIONFREE MODULES

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To the memory of Professor Maurice Auslander

ABSTRACT. Let Λ be a left and right Noetherian ring. For a positive integer k, we give an equivalent condition that flat dimensions of the first k terms in the minimal injective resolution of Λ are less than or equal to k. In this case we show that the subcategory consisting of k-torsionfree modules is extension closed. Moreover we prove that for a Noetherian algebra every subcategory consisting of i-torsionfree modules is extension closed for any $1 \leq i \leq k$ if and only if every subcategory consisting of i-th syzygy modules is extension closed for any $1 \leq i \leq k$. Our results generalize the main results in Auslander and Reiten [4].

Key words. extension closed, k-torsionfree modules, Noetherian rings, flat dimension.

1. INTRODUCTION

Throughout this paper Λ is a left and right Noetherian ring and mod Λ (resp. mod Λ^{op}) is the category of finitely generated left (resp. right) Λ-modules. Let $\mathfrak{X}$ be a full subcategory of mod Λ. $\mathfrak{X}$ is called extension closed, if the middle term B of any short exact sequence $0 \to A \to B \to C \to 0$ is in

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$\mathcal{X}$, provided the end terms A, C are in $\mathcal{X}$. For a left (resp. right) Λ-module A, we use the notation $l.fd_\Lambda(A)$ (resp. $r.fd_\Lambda(A)$) and $l.id_\Lambda(A)$ (resp. $r.id_\Lambda(A)$) to denote left (resp. right) flat dimension and left (resp. right) injective dimension of A respectively, and we put $A^* = Hom_\Lambda(A, \Lambda)$. In addition, we assume that

$$0 \to \Lambda \to I_0 \to I_1 \cdots \to I_i \to \dots$$

is a minimal injective resolution of Λ as a right Λ-module.

Let X be in mod Λ and i a non-negative integer. We denote grade $X \geq i$ if $Ext^j_\Lambda(X, \Lambda) = 0$ for any $0 \leq j < i$. We denote s.grade $X \geq i$ if grade $A \geq i$ for each submodule A of X. Let k be a positive integer, and assume there is an exact sequence $0 \to Y \to P_{k-1} \cdots \to P_0 \to X \to 0$ with the P_i's projective Λ-modules. Then Y is the k-th syzygy module of X. By $\Omega^k(mod\Lambda)$ we denote the full subcategory of mod Λ consisting of k-th syzygy modules and by add $\Omega^k(mod\Lambda)$ we denote the full subcategory of mod Λ consisting of direct summands of k-th syzygy modules. Auslander and Reiten studied when $\Omega^k(mod\Lambda)$ is extension closed in [3] and [4]. This condition is stated in term of flat dimension and grade of Λ-modules by them as follows.

Theorem AR ([4, Theorem 4.7]). *Let Λ be a left and right Noetherian ring and k a positive integer. Then the following conditions are equivalent.*

(a) $\Omega^i(mod\Lambda)$ is extension closed for $1 \leq i \leq k$;

(b) add $\Omega^i(mod\Lambda)$ is extension closed for $1 \leq i \leq k$;

(c) $r.fd_\Lambda(I_i) \leq i+1$ for $0 \leq i \leq k-1$;

(d) s.grade $Ext^{i+1}_\Lambda(Y, \Lambda) \geq i$ for all Y in mod Λ and $1 \leq i \leq k$.

If Λ is a Noetherian algebra, that is, Λ is an algebra over a commutative Noetherian ring R and Λ is a finitely generated R-module, then the following condition is equivalent to the above conditions.

(e) grade $Ext^i_\Lambda(X, \Lambda) \geq i$ for all X in mod Λ^{op} and $1 \leq i \leq k$.

For a positive integer i let $\mathcal{X}_i$ be the full subcategory of mod Λ consisting of i-torsionfree modules (see Definition 2.2). It is not difficult to see that $\mathcal{X}_i \subseteq \Omega^i(mod\Lambda)$. Under the assumption of (c) in the above Theorem, it follows from Auslander and Reiten [4, Theorem 1.7 and Proposition 2.2] that $\mathcal{X}_i = \Omega^i(mod\Lambda)$. We attempt to generalize and develope above Theorem by using the extension closure of $\mathcal{X}_i$. In fact for a positive integer k we will prove that if $r.fd_\Lambda(\bigoplus_{i=0}^{k-1} I_i) \leq k$, then $\mathcal{X}_k$ is extension closed (Theorem 2.3), and a necessary and sufficient condition for $r.fd_\Lambda(\bigoplus_{i=0}^{k-1} I_i) \leq k$ is that s.grade $Ext^{k+1}_\Lambda(M, \Lambda) \geq k$ for any M in mod Λ (Theorem 2.8). In section 3 we will prove that if Λ is a Noetherian algebra, then $\mathcal{X}_i$ is extension closed for any $1 \leq i \leq k$ if and only if $\Omega^i(mod\Lambda)$ is extension closed for any $1 \leq i \leq k$. In this case $\mathcal{X}_i = \Omega^i(mod\Lambda)$ for any $1 \leq i \leq k$ (Theorem 3.1).

2. Flat Dimension and Extension Closure

From this section we assume that all modules are finitely generated and k is a positive integer.

Definition 2.1. A left (resp. right) Λ-module M is called a left (resp. right) W^k-module if $Ext^i_\Lambda(M,\Lambda)=0$ for $1\leq i\leq k$.

Let M be a W^k-module. If $P_{k+1}\to\cdots\to P_1\to P_0\to M\to 0$ is a projective resolution of M, we have an exact sequence of the form $0\to M^*\to P_0^*\to P_1^*\cdots\to P_{k+1}^*$.

We also recall a definition from Auslander and Reiten [4].

Definition 2.2. Let M be in mod Λ. M is said to be a k-torsionfree module if TrM is a W^k-module, where TrM is the transpose of M.

Remark. For any M in mod Λ, from the exact sequence $0\to Ext^1_\Lambda(TrM,\Lambda)\to M\overset{\sigma_M}{\to} M^{**}\to Ext^2_\Lambda(TrM,\Lambda)\to 0$ (see Auslander [1, Proposition 6.3]) we know that M is 1-torsionfree if and only if M is torsionless, and M is 2-torsionfree if and only if M is reflexive. In addition, If $k\geq 2$ and r.gl.dim Λ (right global dimension of Λ) $=k-2$, then M is k-torsionfree if and only if M is projective.

Theorem 2.3. *If $r.fd_\Lambda(\bigoplus_{i=0}^{k-1} I_i)\leq k$, then $\mathfrak{X}_k$ is extension closed.*

Proof. Let $0\to A\overset{f}{\to} B\to C\to 0$ be an exact sequence of mod Λ with A, C k-torsionfree. We want to prove that B is also k-torsionfree.

We first prove that grade Coker $f^*\geq k$. If $k=1$, then by [4, Proposition 2.2] s.grade $Ext^1_\Lambda(C,\Lambda)\geq 1$ because of $fd_\Lambda(I_0)\leq 1$. But $Coker f^*$ is a submodule of $Ext^1_\Lambda(C,\Lambda)$, so grade Coker $f^*\geq 1$. If $k=2$, then C is 2-torsionfree, that is, C is reflexive. Suppose $P_1\to P_0\to C^*\to 0$ is a projective resolution of C^*. Then we get an exact sequence of the form

$$0\to C\cong C^{**}\to P_0^*\to P_1^*\to H\to 0 \tag{1}$$

where $H=Coker(P_0^*\to P_1^*)$. Now suppose $k\geq 3$. Then A^*, C^* are W^{k-2}-modules. Consider an exact sequence of the form

$$P_{k-1}\to\cdots\to P_1\to P_0\to C^*\to 0$$

where P_i' s are projective modules. We have an exact sequence

$$0\to C\cong C^{**}\to P_0^*\to P_1^*\to\cdots\to P_{k-1}^*\to H\to 0 \tag{2}$$

where $H=Coker(P_{k-2}^*\to P_{k-1}^*)$. ¿From the above exact sequences (1) and (2) for $k\geq 2$ and from [5, Chapter VI, Proposition 5.3] it follows that $Hom_\Lambda(Ext^1_\Lambda(C,\Lambda),\bigoplus_{i=0}^{k-1} I_i)\cong Tor_1^\Lambda(\bigoplus_{i=0}^{k-1} I_i, C)\cong Tor_{k+1}^\Lambda(\bigoplus_{i=0}^{k-1} I_i, H)=0$. We then have $Hom_\Lambda(Coker f^*,\bigoplus_{i=0}^{k-1} I_i)=0$ since $Coker f^*$ is a submodule of $Ext^1_\Lambda(C,\Lambda)$ and $\bigoplus_{i=0}^{k-1} I_i$ is injective. It is easy to see that $Ext^i_\Lambda(Coker f^*,\Lambda)=0$ for $0\leq i\leq k-1$.

Now we consider the following exact commutative diagram with last two split rows;

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_0 \bigoplus G_0 & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_1 \bigoplus G_1 & \longrightarrow & G_1 & \longrightarrow & 0
\end{array}$$

where F_i and G_i for i =0,1 are projective. Then we have the following exact commutative diagram;

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^* & \longrightarrow & B^* & \xrightarrow{f^*} & A^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_0^* & \longrightarrow & G_0^* \bigoplus F_0^* & \longrightarrow & F_0^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1^* & \longrightarrow & G_1^* \bigoplus F_1^* & \longrightarrow & F_1^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & TrC & & TrB & & TrA & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It follows from the snake lemma that $0 \to C^* \to B^* \xrightarrow{f^*} A^* \to TrC \to TrB \to TrA \to 0$ is exact. Since $Ext^i_\Lambda(Coker f^*, \Lambda) = 0$ for $0 \leq i \leq k-1$ and Tr C is a W^k-module, it follows that $Im(TrC \to TrB)$ is a W^k-module. Since Tr A is a W^k-module, we conclude that TrB is also a W^k-module and B is a k-torsionfree module. □

Remark. Let I_0' be injective envelope of Λ as a left Λ-module. We know that $l.fd_\Lambda(I_0') = 0$ if and only if $r.fd_\Lambda(I_0) = 0$. But in general we don't have the fact that $l.fd_\Lambda(I_0') \leq 1$ if and only if $r.fd_\Lambda(I_0) \leq 1$. In fact Hoshino M. had given the following example. Let Λ be a finite dimensional algebra which is given by quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3$$

modulo the ideal $\alpha\beta\alpha$. Then $l.fd_\Lambda(I_0') = 1$ and $r.fd_\Lambda(I_0) \geq 2$.

Corollary 2.4. *If $r.fd_\Lambda(I_i) \leq i+1$ for $i \leq k-1$, then each $\mathfrak{X}_i$ is extension closed for $i \leq k$.*

In next section, we will show that the converse to Corollary 2.4 holds.

Corollary 2.5. *If $l.id_\Lambda(\Lambda) = k$, then each $\mathfrak{X}_i$ is extension closed for $i \geq k$.*

Proof. By Theorem 2.3 and Iwanaga [8, Proposition 1]. $\square$

Lemma 2.6. *For any injective Λ^{op}-module I, $r.fd_\Lambda(I) \leq k$ if and only if $Hom_\Lambda(Ext_\Lambda^{k+1}(C, \Lambda), I) = 0$ for any C in mod Λ.*

Proof. It is easy from Cartan and Eilenberg [5, Chapter VI, Proposition 5.3]. $\square$

Corollary 2.7. *(1) If there is an injective Λ^{op}-module I satisfying $r.fd_\Lambda(I) = k$ and I a cogenerator for mod Λ^{op}, then $l.id_\Lambda(\Lambda) = k$.*

(2) If $r.fd_\Lambda(\bigoplus_{i=0}^k I_i) \leq k$, then $\bigoplus_{i=0}^k I_i$ is a cogenerator for mod Λ^{op} if and only if $l.id_\Lambda(\Lambda) \leq k$.

Proof. (1) Suppose I is an injective cogenerater for mod Λ^{op} and $r.fd_\Lambda(I) = k$. By Lemma 2.6 $Hom_\Lambda(Ext_\Lambda^{k+1}(M, \Lambda), I) = 0$ for any $M \in$ mod Λ, and there is a module $N \in$ mod Λ such that $Hom_\Lambda(Ext_\Lambda^k(N, \Lambda), I) \neq 0$. So $Ext_\Lambda^{k+1}(M, \Lambda) = 0$ and $Ext_\Lambda^k(N, \Lambda) \neq 0$. We conclude that $l.id_\Lambda(\Lambda) = k$.

(2) The necessity follows from (1), and the sufficiency from Iwanaga [7, Theorem 2] (Note: His argument remains valid in our assumption). $\square$

Theorem 2.8. *Let m be a non-negative integer. Then $r.fd_\Lambda(\bigoplus_{i=0}^{k-1} I_i) \leq k+m$ if and only if s.grade $Ext_\Lambda^{k+m+1}(M, \Lambda) \geq k$ for any M in mod Λ. Particularly, $r.fd_\Lambda(\bigoplus_{i=0}^{k-1} I_i) \leq k$ if and only if s.grade $Ext_\Lambda^{k+1}(M, \Lambda) \geq k$ for any M in mod Λ.*

Proof. "The sufficiency". We proceed by induction on i. Suppose s.grade $Ext_\Lambda^{k+m+1}(M, \Lambda) \geq k$ for any M in mod Λ. We first prove that $r.fd_\Lambda(I_0) \leq k+m$. Since s.grade $Ext_\Lambda^{k+m+1}(M, \Lambda) \geq k$, $Hom_\Lambda(Ext_\Lambda^{k+m+1}(M, \Lambda), \Lambda) = 0$. We claim that $Hom_\Lambda(Ext_\Lambda^{k+m+1}(M, \Lambda), I_0) = 0$. Otherwise, there is a non-zero homomorphism $f : Ext_\Lambda^{k+m+1}(M, \Lambda) \to I_0$ and $Im f \bigcap \Lambda \neq 0$ since Λ is essential in I_0. In this case there is a submodule X of $Ext_\Lambda^{k+m+1}(M,\Lambda)$ such that $Hom_\Lambda(X,\Lambda)\neq 0$, which contradicts that s.grade $Ext_\Lambda^{k+m+1}(M,\Lambda)\geq k$. So we conclude that $r.fd_\Lambda(I_0)\leq k+m$ by Lemma 2.6.

Now suppose $i\geq 1$. Consider the exact sequence $0\to K_{i-1}\to I_{i-1}\to K_i\to 0$, where $K_{i-1}=Ker(I_{i-1}\to I_i)$ and $K_i=Im(I_{i-1}\to I_i)$. Then for any submodule X of $Ext_\Lambda^{k+m+1}(M,\Lambda)$, we have an exact sequence $Hom_\Lambda(X,I_{i-1})\to Hom_\Lambda(X,K_i)\to Ext_\Lambda^1(X,K_{i-1})$. Since s.grade $Ext_\Lambda^{k+m+1}(M,\Lambda)\geq k$ and $1\leq i\leq k-1$, $Ext_\Lambda^1(X,K_{i-1})\cong Ext_\Lambda^i(X,\Lambda)=0$. By induction assumption and Lemma 2.6 we have that $Hom_\Lambda(Ext_\Lambda^{k+m+1}(M,\Lambda),I_{i-1})=0$. Since I_{i-1} is injective, $Hom_\Lambda(X,I_{i-1})=0$. It follows that $Hom_\Lambda(X,K_i)=0$. Noting that I_i is the injective envelope of K_i, then By a similar argument to the proof of the case $i=0$, we get that $Hom_\Lambda(Ext_\Lambda^{k+m+1}(M,\Lambda),I_i)=0$ and $r.fd_\Lambda(I_i)\leq k+m$. Hence we are done.

"The necessity". Suppose $r.fd_\Lambda(\bigoplus_{i=0}^{k-1}I_i)\leq k+m$. Then $Hom_\Lambda(Ext_\Lambda^{k+m+1}(M,\Lambda),\bigoplus_{i=0}^{k-1}I_i)=0$. Let X be a submodule of $Ext_\Lambda^{k+m+1}(M,\Lambda)$, we have that $Hom_\Lambda(X,\bigoplus_{i=0}^{k-1}I_i)=0$. Set $K_0=\Lambda$ and $K_i=Im(I_{i-1}\to I_i)$ for $1\leq i\leq k-1$. Then $Hom_\Lambda(X,K_i)=0$ for $0\leq i\leq k-1$. It is not difficult to prove that $Ext_\Lambda^{i+1}(X,K_0)\cong Ext_\Lambda^1(X,K_i)$ and $Ext_\Lambda^1(X,K_i)\cong Hom_\Lambda(X,K_{i+1})$ for $0\leq i\leq k-2$. So we have $Hom_\Lambda(X,\Lambda)=0=Ext_\Lambda^i(X,\Lambda)$ for $1\leq i\leq k-1$. □

Corollary 2.9. *Let m be a non-negative integer. Then $r.fd_\Lambda(I_i)\leq i+m+1$ for $0\leq i\leq k-1$ if and only if s.grade $Ext_\Lambda^{i+m+1}(M,\Lambda)\geq i$ for any M in mod Λ and $1\leq i\leq k$. Particularly, $r.fd_\Lambda(I_i)\leq i+1$ for $0\leq i\leq k-1$ if and only if s.grade $Ext_\Lambda^{i+1}(M,\Lambda)\geq i$ for any M in mod Λ and $1\leq i\leq k$.*

3. k-torsionfree Modules and k-th Syzygy Modules

In this section Λ is a Noetherian algebra. We will prove the following result.

Theorem 3.1. *$\mathfrak{X}_i$ is extension closed for any $1\leq i\leq k$ if and only if $\Omega^i(mod\Lambda)$ is extension closed for any $1\leq i\leq k$. In this case, $\mathfrak{X}_i=\Omega^i(mod\Lambda)$ for any $1\leq i\leq k$.*

To prove this theorem we need a lemma.

Lemma 3.2. *The following statements are equivalent.*

(1) $Ext_\Lambda^{i-1}(Ext_\Lambda^i(X,\Lambda),\Lambda)=0$ for any X in mod Λ^{op} and $1\leq i\leq k$.

(2) s.grade $Ext_\Lambda^{i+1}(M,\Lambda)\geq i$ for any M in mod Λ and $1\leq i\leq k$.

(3) $\mathfrak{X}_i$ is extension closed for $1\leq i\leq k$.

(4) s.grade $Ext_\Lambda^1(Y,\Lambda)\geq i$ for any i-torsionfree module Y in mod Λ and $1\leq i\leq k$.

Proof. (1) $\Rightarrow$ (2) Suppose $Ext_\Lambda^{i-1}(Ext_\Lambda^i(X,\Lambda),\Lambda) = 0$ for any X in mod Λ^{op} and $i \leq k$. We will prove that s.grade $Ext_\Lambda^{i+1}(M,\Lambda) \geq i$ for any $M \in$ mod Λ and $i \leq k$ by induction on k. If $k = 1$ and $Hom_\Lambda(Ext_\Lambda^1(X,\Lambda),\Lambda) = 0$, then by Theorem AR, s.grade $Ext_\Lambda^2(M,\Lambda) \geq 1$ for any $M \in$ mod Λ. Suppose $k \geq 2$ and s.grade $Ext_\Lambda^{i+1}(M,\Lambda) \geq i$ for any $M \in$ mod Λ and $i \leq k-1$. Then it follows from Auslander and Bridger [2, Proposition 2.26] that $\mathfrak{X}_i = \Omega^i(mod\Lambda)$ for $1 \leq i \leq k$. Suppose $M \in$ mod Λ and Y is a submodule of $Ext_\Lambda^{k+1}(M,\Lambda)$. By Corollary 2.9 and Lemma 2.6 we have that $Hom_\Lambda(Ext_\Lambda^{k+1}(M,\Lambda), \bigoplus_{i=0}^{k-2} I_i) = 0$. Since $\bigoplus_{i=0}^{k-2} I_i$ is injective, $Hom_\Lambda(Y, \bigoplus_{i=0}^{k-2} I_i) = 0$. It is easy to see that grade $Y \geq k-1$.

Consider an exact sequence of the form

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

where P_i' s are projective modules. By Hoshino [6, Lemma 1.5] $Ext_\Lambda^{k+1}(M,\Lambda)$ is isomorphic to a submodule of $Tr\Omega^k(M)$, so we have an exact sequence $0 \to Y \xrightarrow{g} Tr\Omega^k(M) \to Coker g \to 0$. Since $\Omega^k(M)$ is k-torsionfre by above argument, $Tr\Omega^k(M)$ is a W^k-module. So $Ext_\Lambda^{k-1}(Y,\Lambda) \cong Ext_\Lambda^k(Coker g,\Lambda)$. Then from the condition (1) we get that $Ext_\Lambda^{k-1}(Ext_\Lambda^{k-1}(Y,\Lambda),\Lambda) \cong Ext_\Lambda^{k-1}(Ext_\Lambda^k(Coker g,\Lambda),\Lambda) = 0$. Since s.grade $Ext_\Lambda^{i+1}(M,\Lambda) \geq i$ for any $M \in$ mod Λ and $i \leq k-1$ (induction assumption), it follows from Theorem AR that grade $Ext_\Lambda^{k-1}(Y,\Lambda) \geq k-1$. So grade $Ext_\Lambda^{k-1}(Y,\Lambda) \geq k$. Then by Hoshino [6, Lemma 6.2] we have grade $Y \geq k$ and s.grade $Ext_\Lambda^{k+1}(M,\Lambda) \geq k$.

(2) $\Rightarrow$ (3) By Auslander and Reiten [4, Theorem 1.7].

(3) $\Rightarrow$ (1) For case $k = 1$ the conclusion follows from Theorem AR. Now we assume $k \geq 2$ and $Ext_\Lambda^{i-1}(Ext_\Lambda^i(X,\Lambda),\Lambda) = 0$ for any $X \in$ mod Λ^{op} and $i \leq k-1$. Then by the proof of (1) implying (2) we know that $\mathfrak{X}_i = \Omega^i(mod\Lambda)$ for $1 \leq i \leq k$, and by the condition (3) $\Omega^i(mod\Lambda)$ is extension closed for $1 \leq i \leq k$. Then from Theorem AR it follows that grade $Ext_\Lambda^k(X,\Lambda) \geq k$ and $Ext_\Lambda^{k-1}(Ext_\Lambda^k(X,\Lambda),\Lambda) = 0$ for any $X \in$ mod Λ^{op}.

(3) $\Leftrightarrow$ (4) see [4, Corollary 1.5]. $\square$

Now Theorem 3.1 is an immediate consequence of [4, Theorem 1.7] and Lemma 3.2.

Putting the results from this section together with Theorem AR we have the following.

Theorem 3.3. *The following conditions are equivalent.*

(1) $\Omega^i(mod\Lambda)$ *is extension closed for* $1 \leq i \leq k$;

(2) add $\Omega^i(mod\Lambda)$ *is extension closed for* $1 \leq i \leq k$;

(3) $r.fd_\Lambda(I_i) \leq i+1$ *for* $0 \leq i \leq k-1$;

(4) s.grade $Ext_\Lambda^{i+1}(Y,\Lambda) \geq i$ *for all* Y *in mod* Λ *and* $1 \leq i \leq k$.

(5) grade $Ext^i_\Lambda(X,\Lambda) \geq i$ *for all* X *in mod* Λ^{op} *and* $1 \leq i \leq k$.

(6) $\mathfrak{X}_i$ *is extension closed for* $1 \leq i \leq k$.

(7) $Ext^{i-1}_\Lambda(Ext^i_\Lambda(X,\Lambda),\Lambda) = 0$ *for any* X *in mod* Λ^{op} *and* $1 \leq i \leq k$.

(8) s.grade $Ext^1_\Lambda(Y,\Lambda) \geq i$ *for any* i*-torsionfree module* Y *in mod* Λ *and* $1 \leq i \leq k$.

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