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Homological Characterizations of Rings with Property (P)

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ABSTRACT

A commutative ring R is said to satisfy property (P) if every finitely generated proper ideal of R admits a non-zero annihilator. In this paper we give some necessary and sufficient conditions that a ring satisfies property (P). In particular, we characterize coherent rings, noetherian rings and Π-coherent rings with property (P).

Key Words: Property (P); Coherent rings; Noetherian rings; Π-Coherent rings.

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1. INTRODUCTION

It is well known that the notion of annihilators plays an important role in the study of rings and modules. Following Morita (1966), a ring R is called a left S-ring if each proper right ideal of R admits a non-zero annihilator. The definition of right S-rings may be given dually. An S-ring means a left and right S-ring. Kato (1968) showed that a ring R is a left S-ring if and only if the envelope of R as a right R-module is an injective cogenerator in the category of right R-modules. Glaz (1989) introduced the notion of rings with property (P), that is; a commutative ring R is said to satisfy property (P) if every finitely generated proper ideal of R admits a non-zero annihilator. She then studied the homological properties of local rings with property (P) and proved that a commutative local ring R satisfies property (P) if and only if M/N is free for any finitely generated free module M and its finitely generated free submodule N (see Glaz, 1989, Theorem 3.3.16).

In this paper we first generalize Glaz's result above and show that a commutative ring R satisfies property (P) if and only if M/N is projective for any finitely generated projective module M and its finitely generated projective submodule N if and only if $M^* \neq 0$ for any non-zero finitely presented R-module M if and only if $\text{fp.dim}\, R = 0$ (Theorem 1). As applications to the result obtained, we then characterize coherent rings, noetherian rings and Π-coherent rings with property (P) respectively.

Throughout this paper, R is a commutative ring with unit and all modules are unitary.

2. MAIN RESULTS

Definition 1 (Glaz, 1989). *R is said to satisfy property* (P) *if every finitely generated proper ideal of R admits a non-zero annihilator.*

Remark. 1. Clearly, an S-ring satisfies property (P).

2. Assume that R is a local ring with maximal ideal m. Then R satisfies property (P) if m belongs to either of the following:

(1) Associated primes of R (see Glaz, 1989, Corollary 3.3.3).
(2) Associated primes of a flat R-module (see Glaz, 1989, Lemma 3.3.6).

Let M be an R-module. We denote $\text{Hom}_R(M, R)$ and the projective dimension of M by M^* and $\text{pd}_R(M)$ respectively. M is called finitely

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presented if there is a finitely generated projective R-module P and a finitely generated submodule N of P such that $P/N \cong M$. Set fp.dim $R = \sup\{\mathrm{pd}_R(A) \mid A$ admits a resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$ with each P_i finitely generated projective for any $0 \leq i \leq n\}$ (Glaz, 1989).

The main result in this paper is the following

Theorem 1. *Let R be any ring. The following statements are equivalent.*

(1) *R satisfies property* (P).
(2) *$M^* \neq 0$ for any non-zero finitely presented R-module M.*
(3) *If $N \subseteq M$ are finitely generated projective modules, then M/N is projective.*
(4) fp.dim $R = 0$.

Proof. (1) ⇒ (2) Let M be a non-zero finitely presented R-module. Then there is an exact sequence of R-modules:

$$0 \to K \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$$

with K finitely generated. It is easy to see that we may assume that R is not a direct summand of K.

Let $\pi : R^n \to R$ be the natural projection. Then $\pi\alpha(K) \neq R$. Otherwise, R will be a direct summand of K, which is a contradiction. Note that K is finitely generated. So $\pi\alpha(K)$ is a finitely generated proper ideal of R. By (1) there is a non-zero element r in R such that $r\pi\alpha(K) = 0$. Now define $\pi' : R^n \to R$ via $\pi'(x) = r\pi(x)$ for any $x \in R^n$. Then π' is a non-zero homomorphism of R-modules and K is isomorphic to a submodule of $\mathrm{Ker}\,\pi'$. By Theorem 3.6 (Anderson and Fuller, 1992) there is a non-zero homomorphism $\gamma : M \to R$ such that $\gamma\beta = \pi'$. So we have $M^* \neq 0$.

(2) ⇒ (3) Let

$$0 \to N \xrightarrow{f} M \to M/N \to 0 \qquad (2.1)$$

be an exact sequence of R-modules with N and M finitely generated projective. Then we get an exact sequence $M^* \xrightarrow{f^*} N^* \to \mathrm{Ext}^1_R(M/N, R) \to 0$ with M^* and N^* finitely generated projective. It follows that $\mathrm{Ext}^1_R(M/N, R)$ is finitely presented.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & 0 & \to & N & \xrightarrow{f} & M & \to & M/N & \to & 0 \\ & & & & \downarrow \sigma_N & & \downarrow \sigma_M & & & & \\ 0 & \to & [\mathrm{Ext}_R^1(M/N, R)]^* & \longrightarrow & N^{**} & \xrightarrow{f^{**}} & M^{**} & & & & \end{array}$$

where σ_N and σ_M are isomorphisms. So $[\mathrm{Ext}_R^1(M/N,\ R)]^* = \mathrm{Ker} f^{**} \cong \mathrm{Ker} f = 0$. By (2) we then have that $\mathrm{Ext}_R^1(M/N,\ R) = 0$ and the exact sequence (2.1) splits, which implies that M/N is projective.

(3) $\Rightarrow$ (4) Assume that M is an R-module and there is an exact sequence

$$0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_0} P_0 \to M \to 0$$

with each P_i finitely generated projective for any $0 \le i \le n$. Then we have exact sequences

$$0 \to P_n \to P_{n-1} \to \mathrm{Im}\, d_{n-1} \to 0,$$
$$0 \to \mathrm{Im}\, d_{n-1} \to P_{n-2} \to \mathrm{Im}\, d_{n-2} \to 0,$$
$$\cdots\cdots\cdots\cdots$$
$$0 \to \mathrm{Im}\, d_2 \to P_1 \to \mathrm{Im}\, d_1 \to 0,$$
$$0 \to \mathrm{Im}\, d_1 \to P_0 \to M \to 0.$$

By (3) it is easy to see that $\mathrm{Im}\, d_{n-1}, \mathrm{Im}\, d_{n-2}, \ldots, \mathrm{Im}\, d_1$ and M are projective. So we conclude that fp.dim $R = 0$.

(4) $\Rightarrow$ (1) Assume that I is a finitely generated ideal and $\{a_1, \ldots, a_n\}$ is a set of generators of I. Then I is contained in some maximal ideal m of R. Let $f: R \to R^{(n)}$ be a homomorphism via $f(r) = (a_1 r, \ldots, a_n r)$ for any $r \in R$. Clearly $\mathrm{Ker} f = 0 :_R I$. We claim that $\mathrm{Ker} f = 0 :_R I \neq 0$. Otherwise, if $\mathrm{Ker} f = 0$ then we have an exact sequence $0 \to R \xrightarrow{f} R^{(n)} \to R^{(n)}/R \to 0$. By (4), $R^{(n)}/R$ is projective. So we get an exact sequence

$$0 \to R/m \bigotimes_R R \xrightarrow{1 \otimes f} R/m \bigotimes_R R^{(n)} \to R/m \bigotimes_R R^{(n)}/R \to 0$$

and hence we get a monomorphism $f': R/mR \to R^{(n)}/mR^{(n)}$ via $f'(r + mR) = f(r) + mR^{(n)}$ for any $r \in R$. But $I \subset m$, so $a_i \in m$ for any $1 \le i \le n$. Thus $f(r) \in mR^{(n)}$ and $R/mR = 0$, which is a contradiction. Consequently we conclude that R satisfies property (P). □

Definition 2 (Cheng and Zhao, 1991). *R is called an FP-ring if every finitely generated projective module is free.*

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Remark. $\mathbb{Z}[x]$ (where $\mathbb{Z}$ is the integer ring), Bezout domains, indecomposable semi-local rings and local rings are FP-rings (see Cheng and Zhao, 1991).

The following results generalize Theorem 3.3.16 (Glaz, 1989) and Corollary 3.3.18 (Glaz, 1989), respectively.

Corollary 1. *Let R be an FP-ring. Then R satisfies property* (P) *if and only if M/N is free provided $N \subseteq M$ are finitely generated free modules.*

Recall that R is called a semi-local ring if R has finitely many maximal ideals.

Corollary 2. *Let R be a semi-local ring. Then R satisfies property* (P) *if and only if M/N is flat provided $N \subseteq M$ are finitely generated flat modules.*

Proof. Suppose M is a finitely generated flat module and $0 \to K \to F \to M \to 0$ is an exact sequence with F finitely generated free. For any maximal ideal m of R, we know that R_m is a local ring and M_m is a finitely generated flat R_m-module. So M_m is a projective R_m-module by Theorem 1.2.2 (Glaz, 1989). So the exact sequence $0 \to K_m \to F_m \to M_m \to 0$ of R_m-modules splits and hence K_m is a direct summand of the finitely generated free R_m-module F_m, which implies that K_m is finitely generated projective as an R_m-module. Then, by Lemma 4.7 (Ishikawa, 1964), K is finitely generated as an R-module and M is finitely presented. Thus M is projective. Now our conclusion follows from Theorem 1. □

Recall that R is called a coherent ring if each finitely generated ideal of R is finitely presented (Glaz, 1989); also recall that $\text{f.fp.dim}\, R = \sup\{\text{pd}_R(A) \mid A$ is a finitely presented R-module with finite projective dimension$\}$ (Ding, 1991). We have the following

Theorem 2. *Let R be a coherent ring. The following statements are equivalent.*

(1) *R satisfies property* (P).
(2) *$M^* \neq 0$ for any non-zero finitely presented R-module M.*
(3) *M/N is projective if $N \subseteq M$ are finitely generated projective modules.*
(4) $\text{f.fp.dim}\, R = 0$.

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Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By Theorem 1.

(1) $\Leftrightarrow$ (4) We know that a finitely presented R-module M with finite projective dimension admits a resolution:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i finitely generated projective for any $0 \leq i \leq n$ because R is a coherent ring (Glaz, 1989). So f.fp.dim R = fp.dim R. Then by Theorem 1 we are done. □

Let A be an R-module and $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. A is called torsionless if σ_A is a monomorphism.

Corollary 3. *Let R be a coherent ring with property* (P). *Then any finitely generated projective submodule of a finitely presented torsionless R-module is a direct summand.*

Proof. Assume that M is a finitely presented torsionless R-module and P is a finitely generated projective submodule of M. Then we have an exact sequence $0 \to P \xrightarrow{f} M$ and the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & 0 & \to & P & \xrightarrow{f} & M \\ & & & & \downarrow \sigma_P & & \downarrow \sigma_M \\ 0 & \to & (\mathrm{Coker} f^*)^* & \longrightarrow & P^{**} & \xrightarrow{f^{**}} & M^{**} \end{array}$$

where σ_P is an isomorphism and σ_M is a monomorphism. So $f^{**} = \sigma_M f \sigma_P^{-1}$ is a monomorphism and $(\mathrm{Coker} f^*)^* = 0$. On the other hand, because R is a coherent ring and M is a finitely presented R-module, M^* is also a finitely presented R-module by Lemma 2 (Huang and Cheng 1996). It follows that $\mathrm{Coker} f^*$ is a finitely presented R-module and hence $\mathrm{Coker} f^* = 0$ by Theorem 2. Then we get an exact sequence $M^* \xrightarrow{f^*} P^* \to 0$ with P^* projective. It follows that $M^* \cong P^* \oplus Q$ for some R-module Q and $M^{**} \cong P^{**} \oplus Q^* \cong P \oplus Q^*$. Since M is torsionless, $M \subseteq M^{**}$ and $M = M \cap M^{**} = M \cap (P \oplus Q^*) \cong P \oplus (M \cap Q^*)$. We are done. □

The following result characterizes noetherian rings with property (P), which develops Theorem 1 (Morita, 1966) and Proposition 2 (Kato, 1968).

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Theorem 3. *Let R be a noetherian ring. Then the following statements are equivalent.*

(1) *R satisfies property* (P).
(2) $M^* \neq 0$ *for any non-zero finitely generated R-module M.*
(3) $T^* \neq 0$ *for any non-zero simple R-module T.*
(4) *Every maximal ideal of R admits a non-zero annihilator.*
(5) *The injective envelope of R is an injective cogenerator in the category of R-modules.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (4) Assume that m is a maximal ideal of R and there is an exact sequence $0 \to m \to R \to R/m \to 0$. Then $0:_R m = (R/m)^*$ by Proposition 23.12 (Faith, 1976). Since R/m is a simple R-module, $(R/m)^* \neq 0$ by (3). So $0:_R m \neq 0$.

(4) $\Rightarrow$ (1) Assume that I is a proper ideal of R. Then I is contained in some maximal ideal m of R. So $0:_R m \subseteq 0:_R I$. By (4), $0:_R m \neq 0$ and $0:_R I \neq 0$.

(1) $\Leftrightarrow$ (5) By Proposition 2 (Kato, 1968). □

We use $\prod R$ to denote any direct product of the ring R.

Defintion 3 (Camillo, 1990)**.** *R is called a* Π*-coherent ring if every finitely generated submodule of* $\prod R$ *is finitely presented.*

Remark. We know that noetherian rings $\Rightarrow$ Π-coherent rings $\Rightarrow$ coherent rings. But, in general, the converses do not hold (see Camillo, 1990; Wang, 1993).

We define $\text{f.FGT-Pdim}\, R = \sup\{\text{pd}_R(M) \mid M$ is finitely generated torsionless R-module with finite projective dimension$\}$. Recall that R is called FP-selfinjective if $\text{Ext}^1_R(X, R) = 0$ for any finitely presented R-module X. We have the following

Theorem 4. *Let R be a* Π*-coherent ring. Consider the following conditions.*

(1) *R satisfies property* (P).
(2) *For any finitely generated ideal I,* I^* *is projective implies that I is projective.*

(3) *For any finitely generated torsionless R-module M, M^* is projective implies that M is projective.*

(4) f.FGT-Pdim $R = 0$.

We have that $(1) \Rightarrow (2) \Leftrightarrow (3)$ *and* $(1) \Rightarrow (4)$. *If R is* FP-*selfinjective, then the conditions above are equivalent.*

Proof. $(1) \Rightarrow (2)$ Assume that I is a finitely generated ideal of R with I^* projective. Then I is finitely presented since a Π-coherent ring is coherent. By Theorem 1 (Camillo, 1990) I^* is finitely generated. So I^{**} is finitely generated projective.

On the other hand, we have an exact sequence $F_1 \xrightarrow{f} F_0 \to I \to 0$ with F_0 and F_1 finitely generated free, which induces an exact sequence $0 \to I^* \to F_0^* \xrightarrow{f^*} F_1^* \to N \to 0$ where $N = \mathrm{Coker} f^*$. Since I^* is projective, N is also projective by Theorem 2. Moreover, by Lemma 2.1 (Huang and Tang, 2001) we have an exact sequence:

$$0 \to \mathrm{Ext}_R^1(N, R) \to I \xrightarrow{\sigma_I} I^{**} \to \mathrm{Ext}_R^2(N, R) \to 0.$$

Since N is projective, $\mathrm{Ext}_R^1(N, R) = 0 = \mathrm{Ext}_R^2(N, R)$ and $I \cong I^{**}$ is projective.

$(2) \Rightarrow (3)$ Let M be a finitely generated torsionless R-module.

The First Case. Assume that M^* is finitely generated free with rank $M^* = n$. We proceed by induction on n. When $n = 1$, then M is isomorphic to an ideal of R. Our conclusion follows from (2). Now suppose $n > 1$. Then $M^* \cong R \oplus F$ with F a finitely generated free module and rank$F < n$. Note that $0 :_R (0 :_M R) = (M/(0 :_M R))^*$ by Proposition 23.12 (Faith, 1976). On the other hand, $M^*/R \cong F$ is finitely generated torsionless, so R is a close submodule of M^* and $R = 0 :_R (0 :_M R)$. Hence $0 :_M R$ is also a close submodule of M and $M/(0 :_M R)$ is finitely generated torsionless, which implies that $\sigma_{M/(0 :_M R)} : M/(0 :_M R) \to (M/(0 :_M R))^{**} = (0 :_R (0 :_M R))^{**} = R^{**} \cong R$ is a monomorphism. Thus $M/(0 :_M R)$ is isomorphic to some finitely generated ideal of R and therefore $M/(0 :_M R)$ is finitely generated projective by (2). Then we have that $M \cong (0 :_M R) \oplus M/(0 :_M R)$ and $M^* \cong (0 :_M R)^* \oplus (M/(0 :_M R))^* \cong (0 :_M R)^* \oplus (0 :_R (0 :_M R)) \cong (0 :_M R)^* \oplus R$ and so $(0 :_M R)^* \cong F$. Clearly $0 :_M R$ is torsionless since it is a submodule of a finitely generated torsionless module M. Moreover, since R is Π-coherent and $M/(0 :_M R)$ is finitely generated torsionless, $M/(0 :_M R)$ is finitely presented by Theorem 1 (Wang, 1991). It follows that $0 :_M R$ is finitely generated. By induction assumption, $0 :_M R$ is projective. Thus M is also projective.

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The Second Case. Assume that M^* is finitely generated projective. Then there is a finitely generated projective module N such that $M^* \oplus N$ is finitely generated free. On the other hand, $(M \oplus N^*)^* \cong M^* \oplus N^{**} \cong M^* \oplus N$. By the argument in the first case, $M \oplus N^*$ is projective and M is also projective.

$(1) \Rightarrow (4)$ Since any finitely generated torsionless module is finitely presented over Π-coherent rings (see Wang, 1991, Theorem 1). It is easy to see that f.FGT-Pdim $R = 0$ by Theorem 2.

Now assume that R is FP-selfinjective.

$(3) \Rightarrow (1)$ Let M be a finitely presented R-module and assume there is an exact sequence:

$$0 \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

with each P_i finitely generated projective for any $0 \le i \le n$. Then we have exact sequences:

$$\begin{aligned} &0 \to P_n \to P_{n-1} \to \operatorname{Im} d_{n-1} \to 0,\\ &0 \to \operatorname{Im} d_{n-1} \to P_{n-2} \to \operatorname{Im} d_{n-2} \to 0,\\ &\qquad\qquad \cdots\cdots\cdots\cdots\\ &0 \to \operatorname{Im} d_2 \to P_1 \to \operatorname{Im} d_1 \to 0,\\ &0 \to \operatorname{Im} d_1 \to P_0 \to M \to 0. \end{aligned}$$

By Lemma 2.6 (Huang, 1999), $P_n \cong [P^*_{n-1}/(\operatorname{Im} d_{n-1})^*]^*$ and $P^*_{n-1}/(\operatorname{Im} d_{n-1})^*$ is finitely generated torsionless. By (3) $P^*_{n-1}/(\operatorname{Im} d_{n-1})^*$ is projective. It follows that $(\operatorname{Im} d_{n-1})^*$ is also projective. However, $\operatorname{Im} d_{n-1}$ is finitely generated torsionless, $\operatorname{Im} d_{n-1}$ is projective by (3). Similarly, we know that $\operatorname{Im} d_{n-2}, \ldots, \operatorname{Im} d_1$ are finitely generated projective. Moreover, we have an exact sequence:

$$0 \to M^* \to P_0^* \to (\operatorname{Im} d_1)^* \to \operatorname{Ext}^1_R(M, R) \to 0.$$

Since R is FP-selfinjective, $\operatorname{Ext}^1_R(M, R) = 0$ and $0 \to M^* \to P_0^* \to (\operatorname{Im} d_1)^* \to 0$ is exact. But P_0^* and $(\operatorname{Im} d_1)^*$ are projective, so M^* is projective and hence M is also projective by (3). It follows that f.fp.dim $R = 0$. Then R satisfies property (P) by Theorem 2.

$(4) \Rightarrow (1)$ Let M be a finitely presented R-module with $\operatorname{pd}_R(M) < \infty$. Then there is an exact sequence:

$$0 \to K \to R^{(n)} \to M \to 0$$

with K finitely generated torsionless and $\operatorname{pd}_R(K) < \infty$. By (4) K is projective and $\operatorname{pd}_R(M) \le 1$. If $\operatorname{pd}_R(M) = 1$, then $\operatorname{Ext}^1_R(M, R) \neq 0$ by

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Corollary 2.5 (Ding, 1991), which is a contradiction because R is FP-selfinjective. So M is projective and f.fp.dim $R = 0$ and hence R satisfies property (P) by Theorem 2. □

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REFERENCES

Anderson, F. W., Fuller, K. R. (1992). *Rings and Categories of Modules*. 2nd ed. Graduate Texts in Mathematics 13, Berlin-Heidelberg-New York: Springer-Verlag.

Camillo, V. (1990). Coherence for polynomial rings. *J. Algebra* 132:72–76.

Cheng, F. C., Zhao, Y. C. (1991). The structure of coherent FP-rings. *Chinese Annals of Math.* 12A:254–257, (in Chinese).

Ding, N. Q. (1991). f.f.p. dimension. *Acta Math. Sinica* 34:40–47, (in Chinese).

Faith, C. (1976). *Algebra II, Ring Theory*. Grundlehren der Mathematischen Wissenchaften 191 (A Series of Comprehensive Studies in Mathematics), Berlin-Heidelberg-New York: Springer-Verlag.

Glaz, S. (1989). *Commutative Coherent Rings*. Lecture Notes in Mathematics 1371, Berlin-Heidelberg-New York: Springer-Verlag.

Huang, Z. Y. (1999). On a generalization of the Auslander-Bridger transpose. *Comm. Algebra* 27:5791–5812.

Huang, Z. Y., Cheng, F. C. (1996). On homological dimensions of simple modules over non-commutative rings. *Comm. Algebra* 24:3259–3264.

Huang, Z. Y., Tang, G. H. (2001). Self-orthogonal modules over coherent rings. *J. Pure and Appl. Algebra* 161:167–176.

Ishikawa, T. (1964). Faithfully exact functors and their applications to projective modules and injective modules. *Nagoya Math. J.* 24:29–42.

Kato, T. (1968). Torsionless modules. *Tôhoku Math. J.* 20:234–243.

Morita, K. (1966). On S-rings in the sense of F. Kasch. *Nagoya Math. J.* 27:687–695.
Wang, M. Y. (1991). The dual modules over GN-rings. *J. Guangxi Normal Univ. (Natural Science Edition)* 9:11–15, (in Chinese).
Wang, M. Y. (1993). Some studies on Π-coherent rings. *Proc. Amer. Math. Soc.* 119:71–76.

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