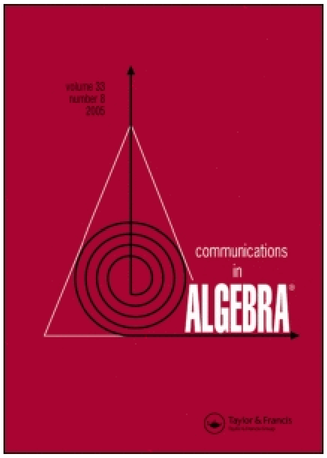

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Huang Zhao-Yong [a]; Cheng Fu-Chang [b]
[a] Department of mathematics, Beijing normal university. Beijing. P.R. China
[b] Department of mathematics, Guangxi normal university. Gulin. P.R. China

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ON HOMOLOGICAL DIMENSIONS OF SIMPLE MODULES OVER NON-COMMUTATIVE RINGS

HUANG ZHAO-YONG
DEPARTMENT OF MATHEMATICS, BEIJING NORMAL UNIVERSITY
BEIJING 100875, P. R. CHINA

CHENG FU-CHANG
DEPARTMENT OF MATHEMATICS, GUANGXI NORMAL UNIVERSITY
GUILIN 541004, P. R. CHINA

ABSTRACT. It is known that for a simple module S over a commutative ring R, $\mathrm{fd}_R(S) = \mathrm{id}_R(S)$. Let R, T be commutative rings and $R \to T$ a ring homomorphism, if T is a Noetherian ring and self-injective, then $\mathrm{fd}_R(T) = \mathrm{id}_R(T)$. In this paper we use the equalities of mixed functors to generalise these results over non-commutative rings.

1. Notations and Preliminaries.

In this paper all rings are associative rings with identity and all modules are unital. Let R be a ring and A a left (right) R-module. We use $l.\mathrm{fd}_R(A)$ $(r.\mathrm{fd}_R(A))$ and $l.\mathrm{id}_R(A)$ $(r.\mathrm{id}_R(A))$ to denote the left (right) flat dimension and the left (right) injective dimension of A, respectively. Recall that a left (right) R-module M is finitely presented if there is an exact sequence of left (right) R-modules $0 \to K \to F \to M \to 0$ with K, F finitely generated and F free. The left (right) FP-injective dimension of A, denoted by $l.\mathrm{FP}-\mathrm{id}_R(A)$ $(r.\mathrm{FP}-\mathrm{id}_R(A))$ is equal to the least integer $n \geq 0$ for which $\mathrm{Ext}_R^{n+1}(M, A) = 0$ for every finitely presented left (right) R-module M. If no such n exists set $l.\mathrm{FP}-\mathrm{id}_R(A)$ $(r.\mathrm{FP}-\mathrm{id}_R(A)) = \infty$. If $n = 0$, A is called a left (right) FP-injective module.

In the papers [9], [10], Xu, Yao prove respectively that for a simple module S over a commutative ring R, S is R-flat if and only if S is R-injective, $\mathrm{fd}_R(S) = \mathrm{id}_R(S)$. In the papers [5], [8], Jothilingam and Mangayarcarassy, Wang generalize the above results and prove respectively the following: let R, T be commutative rings and $R \to T$ a ring homomorphism. If T is a Noetherian ring and self-injective, then T is R-flat if and only if T is R-injective, $\mathrm{fd}_R(T) = \mathrm{id}_R(T)$. Recall that a ring is called a left (right) coherent ring if every finitely generated left (right) ideal is finitely presented. A left and right coherent ring is called a coherent ring. In the paper [3], Ding and Chen prove that for a simple module S over a commutative coherent ring R, $\mathrm{fd}_R(S) = \mathrm{id}_R(S) = \mathrm{FP} - \mathrm{id}_R(S)$. In this paper we give some equalities of homological dimensions of simple modules over non-commutative rings, generalizing some results in [3]–[5], [8]–[10].

2. Main Results.

We will prove the following three theorems.

Theorem I. *Let R be a ring, and I an ideal of R such that R/I is a semisimple artinian ring. Then $r.fd_R(R/I) = l.id_R(R/I)$ and $l.fd_R(R/I) = r.id_R(R/I)$.*

A ring R is called left (right) self FP-injective if it is left (right) FP-injective as an R-module. A left and right self FP-injective ring is called a self FP-injective ring.

Theorem II. *Let $R \to T$ be a ring homomorphism with R a left coherent ring and T a coherent self FP-injective ring. Then $r.fd_R(T) = l.FP - id_R(T)$.*

Theorem III. *Let $R \to T$ be a ring homomorphism, E an injective cogenerator in the category of left T-modules. Then $r.fd_R(T) = l.id_R(E)$.*

3. Proofs of Main Results.

The proof of Theorem I is analogous to that of [9, Theorem 1.1]. For the sake of completeness, we give here the proof.

Proof of Theorem I. Suppose $r.\mathrm{fd}_R(R/I) = n$ $(< \infty)$. Let $(R/I)^+ = \mathrm{Hom}_Z(R/I, Q/Z)$, where Q is the additive group of real numbers, and Z is the additive group of integers. If R/I is regarded as a left (right) R-module, then $(R/I)^+$ is a right (left) R-module. Suppose M is a left R-module. By [2, Chapter VI, Proposition 5.1], we have

$$\mathrm{Ext}_R^{n+1}(M, (R/I)^{++}) \cong \left[\mathrm{Tor}_{n+1}^R((R/I)^+, M)\right]^+ \qquad (1)$$

where R/I is a left R-module. Since $(R/I)^+ I = 0$, $(R/I)^+$ is a semisimple right R-module, and $(R/I)^+ \cong \oplus_{i \in T} S_i$, where every S_i is a simple right R-module and T is a set. Every S_i is a direct summand of R/I for the semisimplicity of R/I. Because $\mathrm{Tor}^R_{n+1}(R/I, M) = 0$, $\mathrm{Tor}^R_{n+1}(S_i, M) = 0$ for every S_i, and $\mathrm{Tor}^R_{n+1}((R/I)^+, M) = 0$. By (1), $\mathrm{Ext}^{n+1}_R(M, (R/I)^{++}) = 0$. Since Q/Z is an injective cogenerator in the category of Z-modules, the canonical valuation homomorphism $R/I \to (R/I)^{++}$ is monomorphic. It follows from $I(R/I)^{++} = 0$ that $(R/I)^{++}$ is also a semisimple module, and R/I is a direct summand of $(R/I)^{++}$. Hence $\mathrm{Ext}^{n+1}_R(M, R/I) = 0$ and therefore $l.\mathrm{id}_R(R/I) \le n$. This proves $l.\mathrm{id}_R(R/I) \le r.\mathrm{fd}_R(R/I)$.

Now we prove $r.\mathrm{fd}_R(R/I) \le l.\mathrm{id}_R(R/I)$. Suppose $l.\mathrm{id}_R(R/I) = n$ ($< \infty$). For any left R-module M, we have

$$\mathrm{Ext}^{n+1}_R(M, (R/I)^+) \cong \left[\mathrm{Tor}^R_{n+1}(R/I, M)\right]^+ \tag{2}$$

where R/I is a right R-module. Since $(R/I)^+$ is semisimple, $(R/I)^+ \cong \oplus_{i\in T'} S_i$, where every S_i is a simple left R-module and T' is a set. Since $l.\mathrm{id}_R(R/I) = n$, $\mathrm{Ext}^{n+1}_R(M, R/I) = 0$ and $\mathrm{Ext}^{n+1}_R(M, S_i) = 0$ for every S_i. So $\mathrm{Ext}^{n+1}_R(M, \Pi_{i\in T'} S_i) \cong \Pi_{i\in T'} \mathrm{Ext}^{n+1}_R(M, S_i) = 0$. Note that $\Pi_{i\in T'} S_i$ is also a semisimple module, so $(R/I)^+ \cong \oplus_{i\in T'} S_i$ is a direct summand of $\Pi_{i\in T'} S_i$. Thus $\mathrm{Ext}^{n+1}_R(M, (R/I)^+) = 0$ and hence $\left[\mathrm{Tor}^R_{n+1}(R/I, M)\right]^+ = 0$ by (2), $\mathrm{Tor}^R_{n+1}(R/I, M) = 0$. Therefore $r.\mathrm{fd}_R(R/I) \le n$. It follows that $r.\mathrm{fd}_R(R/I) \le l.\mathrm{id}_R(R/I)$. This completes the proof of the first equality.

The second equality is obtained by a similar proof. □

Corollary 1. *Let R be a semilocal ring with Jacobson radical J, i.e., R/J is semisimple. Then $r.fd_R(R/J) = l.id_R(R/J)$ and $l.fd_R(R/J) = r.id_R(R/J)$.*

Corollary 2. *(Hirano) Let R be a ring, and I an ideal of R such that R/I is a simple artinian ring. Then R/I is left (right) R-flat if and only if R/I is right (left) R-injective.*

Corollary 3. *(Yao) Let R be a commutative ring, S a simple R-module. Then $fd_R(S) = id_R(S)$. Particularly, S is R-flat if and only if S is R-injective.*

To prove Theorem II, we need some lemmas.

Lemma 1. *Let R be a left coherent, left self FP-injective ring and A a finitely presented right R-module. If $A^* = 0$, where $A^* = Hom_R(A, R)$, then $A = 0$.*

Proof. First, we show that there is no non-projective finitely presented left R-module with finite projective dimension. Let M be a finitely presented left R-

module with $l.\mathrm{pd}_R(M) = n$ ($< \infty$). We prove $n = 0$. Otherwise, if $n \neq 0$, then there is a left R-module N, such that $\mathrm{Ext}_R^n(M, N) \neq 0$. By the exactness of the sequence of left R-modules $0 \to K \to F \to N \to 0$, where F is free, we know that $\mathrm{Ext}_R^n(M, F) \to \mathrm{Ext}_R^n(M, N) \to \mathrm{Ext}_R^{n+1}(M, K) = 0$ is exact. So $\mathrm{Ext}_R^n(M, F) \neq 0$ and hence $l.\mathrm{FP}-\mathrm{id}_R(F) \geq n$. It follows that $l.\mathrm{FP}-\mathrm{id}_R(R) \geq n$ by [7, Theorem 3.2]. This contradiction shows that $n = 0$.

Secondly, suppose A is a finitely presented right R-module with $A^* = 0$. If $A \neq 0$, then there is an exact sequence of right R-modules $F_1 \xrightarrow{f} F_0 \to A \to 0$ with F_0, F_1 finitely generated free. Since $A^* = 0$, $0 \to F_0^* \xrightarrow{f^*} F_1^* \to \mathrm{Coker}\, f^* \to 0$ is an exact sequence of left R-modules. So $l.\mathrm{pd}_R(\mathrm{Coker}\, f^*) \leq 1$. Moreover, we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccc}
F_1 & \xrightarrow{f} & F_0 & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow{\scriptstyle \sigma_{F_1}} & & \downarrow{\scriptstyle \sigma_{F_0}} & & \downarrow{\scriptstyle \varphi} & & \\
F_1^{**} & \longrightarrow & F_0^{**} & \longrightarrow & \mathrm{Ext}_R^1(\mathrm{Coker}\, f^*, R) & \longrightarrow & 0
\end{array}$$

where σ_{F_0}, σ_{F_1} are the canonical valuation homomorphisms, φ is an induced homomorphism. It is known that σ_{F_0}, σ_{F_1} are isomorphisms. So φ is also an isomorphism, $\mathrm{Ext}_R^1(\mathrm{Coker}\, f^*, R) \cong A$, and $\mathrm{Ext}_R^1(\mathrm{Coker}\, f^*, R) \neq 0$. Thus $l.\mathrm{pd}_R(\mathrm{Coker}\, f^*) \geq 1$ and therefore $l.\mathrm{pd}_R(\mathrm{Coker}\, f^*) = 1$. In addition, $\mathrm{Coker}\, f^*$ is a finitely presented left R-module. It is a contradiction since there is no non-projective finitely presented left R-module with finite projective dimension. So $A = 0$. □

Lemma 2. *Let R be a left coherent ring and T a right coherent ring with the condition $({}_RA, {}_RB_T)$. If ${}_RA$ and B_T are finitely presented, then for $n \geq 0$, $\mathrm{Ext}_R^n(A, B)$ is a finitely presented right T-module.*

Proof. It is clear that for $n \geq 0$, $\mathrm{Ext}_R^n(A, B)$ is a right T-module. Suppose N is a finitely presented left R-module, there are integers $s, t \geq 1$, such that $R^s \to R^t \to N \to 0$ is exact. Then $0 \to \mathrm{Hom}_R(N, B) \to B^t \to B^s$ is an exact sequence of right T-modules. Since T is a right coherent ring and $\mathrm{Im}(B^t \to B^s)$ is a finitely generated right T-submodule of the finitely presented module B^s, $\mathrm{Im}(B^t \to B^s)$ is finitely presented. By [1, Proposition 1.6] $\mathrm{Hom}_R(N, B)$ is a finitely generated right T-submodule of the finitely presented module B^t and hence $\mathrm{Hom}_R(N, B)$ is also a finitely presented right T-module. Since ${}_RA$ is finitely presented, $0 \to K \to F \to A \to 0$ is exact, where K is a finitely presented left R-module and F is a finitely generated free left R-module. It follows that $\mathrm{Hom}_R(K,B) \to \mathrm{Ext}^1_R(A,B) \to 0$ is exact and for $n \geq 1$, $\mathrm{Ext}^{n+1}_R(A,B) \cong \mathrm{Ext}^n_R(K,B)$. Then we can obtain our conclusion by induction on n. □

The proof of the following lemma is similar to that of [6, Lemma 3.60 and Theorem 9.51], we omit it (Note: The proof is based on Lemma 2).

Lemma 3. *Let R be a left coherent ring and T a right coherent ring. In the condition $({}_RA, {}_RB_T, C_T)$, for $n \geq 0$, $\mathrm{Tor}^R_n(\mathrm{Hom}_T(B,C),A) \cong \mathrm{Hom}_T(\mathrm{Ext}^n_R(A,B),C)$, where ${}_RA$ and B_T are finitely presented, C_T is FP-injective.*

Now we prove Theorem II.

Proof of Theorem II. By Lemma 3, for any finitely presented left R-module A, we have that $\mathrm{Tor}^R_n(\mathrm{Hom}_T(T,T),A) \cong \mathrm{Hom}_T(\mathrm{Ext}^n_R(A,T),T)$. So $\mathrm{Tor}^R_n(T,A) \cong \mathrm{Hom}_T(\mathrm{Ext}^n_R(A,T),T)$. We know that $\mathrm{Ext}^n_R(A,T)$ is a finitely presented right T-module by Lemma 2. Then $\mathrm{Tor}^R_n(T,A) = 0$ if and only if $\mathrm{Ext}^n_R(A,T) = 0$ by Lemma 1. It is easy to see that $r.\mathrm{fd}_R(T) = l.\mathrm{FP}-\mathrm{id}_R(T)$. □

Corollary 4. *(Ding and Chen) Let R be a commutative coherent ring, S a simple R-module. Then $fd_R(S) = id_R(S) = FP - id_R(S)$.*

Proof. Since S is a simple R-module, there is a maximal ideal m of R such that $S \cong R/m$. So S is a simple ring. We get our conclusion by Corollary 3 and Theorem II. □

Finally, let's prove Theorem III.

Proof of Theorem III. For any left R-module A, we have that $\mathrm{Ext}^n_R(A,\mathrm{Hom}_T(T,E)) \cong \mathrm{Hom}_T(\mathrm{Tor}^R_n(T,A),E)$ by [2, Chapter VI, Proposition 5.1]. Then $\mathrm{Ext}^n_R(A,E) \cong \mathrm{Hom}_T(\mathrm{Tor}^R_n(T,A),E)$. Since E is an injective cogenerator in the category of left T-modules, $\mathrm{Ext}^n_R(A,E) = 0$ if and only if $\mathrm{Tor}^R_n(T,A) = 0$. It follows that $l.\mathrm{id}_R(E) = r.\mathrm{fd}_R(T)$. □

Corollary 5. *(Wang) Let $R \to T$ be a ring homomorphism with R a commutative ring, T a commutative noetherian ring and self-injective. Then $fd_R(T) = id_R(T)$. Particularly, T is R-flat if and only if T is R-injective.*

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