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SELF-ORTHOGONAL MODULES OF FINITE PROJECTIVE DIMENSION

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Let R be a ring and $_R\omega$ a self-orthogonal module. We introduce the notion of the right orthogonal dimension (relative to $_R\omega$) of modules. We give a criterion for computing this relative right orthogonal dimension of modules. For a left coherent and semilocal ring R and a finitely presented self-orthogonal module $_R\omega$, we show that the projective dimension of $_R\omega$ and the right orthogonal dimension (relative to $_R\omega$) of R/J are identical, where J is the Jacobson radical of R. As a consequence, we get that $_R\omega$ has finite projective dimension if and only if every left (finitely presented) R-module has finite right orthogonal dimension (relative to $_R\omega$). If ω is a tilting module, we then prove that a left R-module has finite right orthogonal dimension (relative to $_R\omega$) if and only if it has a special $\omega^{\perp}$-preenvelope.

Key Words: Projective dimension; Right orthogonal dimension; Self-orthogonal modules; Tilting modules; $\mathscr{X}$-coresolution dimension.

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Tilting modules and cotilting modules are very important research objects in representation theory of artin algebras, which are some special kinds of self-orthogonal modules. Let $_R\omega$ be a finitely generated self-orthogonal module over an Artinian algebra R. Huang (2000) introduced the notion of left orthogonal dimension (relative to $_R\omega$) of modules, and proved that the injective dimension of $_R\omega$ is finite if and only if every finitely generated left R-module has finite left orthogonal dimension (relative to $_R\omega$). Colpi and Trlifaj (1995) investigated the properties of tilting torsion theories. Furthermore, Angeleri Hügel and Coelho (2001) established the relationship between the tilting theory and relative homological theory.

Motivated by the articles mentioned above, in this article we introduce in Section 1 the notion of right orthogonal dimension (relative to a given self-orthogonal module) of modules. In Section 2, we first give a criterion for computing this relative right orthogonal dimension of modules and prove that for a left R-module M and a non-negative integer n, the right orthogonal dimension (relative to a self-orthogonal module $_R\omega$) of M is at most n if and only if the nth cosyzygy of M is right orthogonal with $_R\omega$. Let R be a left coherent and semi-local ring and $_R\omega$ a finitely presented self-orthogonal module. We show that the projective dimension of

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${}_R\omega$ and the right orthogonal dimension (relative to ${}_R\omega$) of R/J are identical, where J is the Jacobson radical of R. As a consequence of the obtained results, we get that ${}_R\omega$ has finite projective dimension if and only if every left (finitely presented) R-module has finite right orthogonal dimension (relative to ${}_R\omega$). In Section 3, we study the properties of right orthogonal dimension (relative to ${}_R\omega$) of modules when ω is a tilting module. We prove that a left R-module has finite right orthogonal dimension (relative to ${}_R\omega$) if and only if it has a special $\omega^{\perp}$-preenvelope if and only if it has a special $\mathscr{X}$-precover, where $\mathscr{X}$ denotes the subcategory of left R-modules consisting of all X with $\mathrm{Add}_R\omega$-coresol.$\dim_R(X)$ (see Section 3 for the defintion) at most n.

1. DEFINITIONS AND NOTATIONS

For a ring R, we use $\mathrm{Mod}\,R$ to denote the category of left R-modules. For a left R-module M, we use $l.\mathrm{pd}_R(M)$ to denote the left projective dimension of M.

Definition 1.1. Let $\omega \in \mathrm{Mod}\,R$. We call ω a *self-orthogonal module* if $\mathrm{Ext}^i_R(\omega, \omega) = 0$ for any $i \geq 1$.

Definition 1.2. Let $\omega \in \mathrm{Mod}\,R$ be a self-orthogonal module and $X \in \mathrm{Mod}\,R$. X is said to be *right orthogonal* with ω if $\mathrm{Ext}^i_R(\omega, X) = 0$ for any $i \geq 1$. We use $\omega^{\perp}$ to denote the subcategory of $\mathrm{Mod}\,R$ consisting of the modules which are right orthogonal with ω. An exact sequence $0 \to X \to X_0 \to \cdots \to X_n \to \cdots$ in $\mathrm{Mod}\,R$ is called a *$\omega^{\perp}$-coresolution* of X if all $X_i \in \omega^{\perp}$.

We now introduce the notion of the right orthogonal dimension (relative to a given module) of modules as follows.

Definition 1.3. Let ω and M be in Mod R. If M has a $\omega^{\perp}$-coresolution $0 \to M \to X_0 \to \cdots \to X_n \to \cdots$, then set $\omega^{\perp}\text{-}\dim_R(M) = \inf\{n \mid 0 \to M \to X_0 \to \cdots \to X_n \to 0$ is a right orthogonal coresolution of $M\}$. If no such an integer exists set $\omega^{\perp}\text{-}\dim_R(M) = \infty$. We call $\omega^{\perp}\text{-}\dim_R(M)$ the *right orthogonal dimension* of M.

For any $M \in \mathrm{Mod}\,R$, it is clear that $\omega^{\perp}\text{-}\dim_R(M)$ is at most the injective dimension of M, and $\omega^{\perp}\text{-}\dim_R(M) = 0$ if and only if $M \in \omega^{\perp}$. So the notion of this relative right orthogonal dimension can be regarded as a generalization of that of the injective dimension. On the other hand, there is a close relation between the right orthogonal dimension of modules (relative to a given self-orthogonal module ${}_R\omega$) and the projective dimension of ${}_R\omega$. In fact, in next section we will show that if ${}_R\omega$ is self-orthogonal, then for any $M \in \mathrm{Mod}\,R$, $\omega^{\perp}\text{-}\dim_R(M)$ is at most the projective dimension of ${}_R\omega$.

2. THE RIGHT ORTHOGONAL DIMENSION OF MODULES

In this section, R is a ring, and ${}_R\omega \in \mathrm{Mod}\,R$ is a self-orthogonal module. The right orthogonal dimension relative to ${}_R\omega$ of a module M is called the *right orthogonal dimension* of M for short.

Definition 2.1 (Rotman, 1979). Let $B \in \operatorname{Mod} R$ and $0 \to B \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} E_i \xrightarrow{d_i} \cdots$ be an injective coresolution; denote $\operatorname{Im} \varepsilon$ by $\operatorname{co}\Omega^0(B)$ and, for $n \geq 1$, denote $\operatorname{Im} d_{n-1}$ by $\operatorname{co}\Omega^n(B)$. For any $n \geq 0$, $\operatorname{co}\Omega^n(B)$ is called the nth *cosyzygy* of B.

The following theorem gives a criterion for computing the right orthogonal dimension of modules.

Theorem 2.2. *Let $M \in \operatorname{Mod} R$. Then $\omega^{\perp}\text{-}\dim_R(M) \leq n$ if and only if $\operatorname{co}\Omega^n(M) \in \omega^{\perp}$.*

Proof. Assume that

$$0 \to M \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{i-1}} E_i \xrightarrow{d_i} \cdots$$

is a minimal injective coresolution of ${}_RM$. The sufficiency is trivial. We next prove the necessity.

The case $n = 0$ is trivial. Now suppose that $n \geq 1$ and M has the following right orthogonal coresolution: $0 \to M \to Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots Y_{n-1} \xrightarrow{f_{n-1}} Y_n \longrightarrow 0$.

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & E_0 & \longrightarrow & \operatorname{co}\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_0 & \longrightarrow & E_0 \bigoplus \operatorname{Im} f_0 & \longrightarrow & \operatorname{co}\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Im} f_0 & = & \operatorname{Im} f_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the exactness of the middle row in the above diagram, we get that $\operatorname{Ext}_R^i(\omega, \operatorname{Im} f_0) \cong \operatorname{Ext}_R^i(\omega, \operatorname{co}\Omega^1(M))$ for any $i \geq 1$. On the other hand, it is easy to see that $\operatorname{Ext}_R^i(\omega, \operatorname{co}\Omega^n(M)) \cong \operatorname{Ext}_R^{i+n-1}(\omega, \operatorname{co}\Omega^1(M))$ for any $i \geq 1$ and $\operatorname{Ext}_R^{t+n-1}(\omega, \operatorname{Im} f_0) \cong \operatorname{Ext}_R^t(\omega, Y_n) = 0$ for any $t \geq 1$. So $\operatorname{Ext}_R^i(\omega, \operatorname{co}\Omega^n(M)) = 0$ for any $i \geq 1$ and $\operatorname{co}\Omega^n(M) \in \omega^{\perp}$. This completes the proof. $\square$

Lemma 2.3. *Let $M \in \operatorname{Mod} R$ and $\omega^{\perp}\text{-}\dim_R(M) < \infty$. Then $\omega^{\perp}\text{-}\dim_R(M) = \sup\{t \mid \operatorname{Ext}_R^t(\omega, M) \neq 0\}$.*

Proof. Suppose $\omega^{\perp}\text{-}\dim_R(M) = n < \infty$. By Theorem 2.2, $\operatorname{Ext}_R^k(\omega, M) \cong \operatorname{Ext}_R^{k-n}(\omega, \operatorname{co}\Omega^n(M)) = 0$ for any $k \geq n+1$. So $\sup\{t \mid \operatorname{Ext}_R^t(\omega, M) \neq 0\} \leq n$.

Suppose that

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

is a minimal injective coresolution of M. For any $n \geq 1$, from the exact sequence $0 \to \mathrm{co}\,\Omega^{n-1}(M) \to E_{n-1} \to \mathrm{co}\,\Omega^n(M) \to 0$, we get a long exact sequence:

$$\begin{aligned} \cdots &\to \mathrm{Ext}_R^{i-1}(\omega, \mathrm{co}\,\Omega^n(M)) \to \mathrm{Ext}_R^i(\omega, \mathrm{co}\,\Omega^{n-1}(M)) \to \mathrm{Ext}_R^i(\omega, E_{n-1}) \\ &\to \mathrm{Ext}_R^i(\omega, \mathrm{co}\,\Omega^n(M)) \to \mathrm{Ext}_R^{i+1}(\omega, \mathrm{co}\,\Omega^{n-1}(M)) \to \cdots . \end{aligned}$$

So $\mathrm{Ext}_R^i(\omega, \mathrm{co}\,\Omega^{n-1}(M)) = 0$ for any $i \geq 2$. We claim that $\mathrm{Ext}_R^1(\omega, \mathrm{co}\,\Omega^{n-1}(M)) \neq 0$. Otherwise, if $\mathrm{Ext}_R^1(\omega, \mathrm{co}\,\Omega^{n-1}(M)) = 0$, then $\mathrm{Ext}_R^i(\omega, \mathrm{co}\,\Omega^{n-1}(M)) = 0$ for any $i \geq 1$ and $\mathrm{co}\,\Omega^{n-1}(M) \in \omega^{\perp}$. It follows from Theorem 2.2 that $\omega^{\perp}\text{-dim}_R(M) \leq n-1$, which is a contradiction. In addition, $\mathrm{Ext}_R^n(\omega, M) \cong \mathrm{Ext}_R^1(\omega, \mathrm{co}\,\Omega^{n-1}(M))$, so $\mathrm{Ext}_R^n(\omega, M) \neq 0$, which implies $\sup\{t \mid \mathrm{Ext}_R^t(\omega, M) \neq 0\} \geq n$. This finishes the proof. □

Lemma 2.4. $\omega^{\perp}\text{-dim}_R(M) \leq l.\mathrm{pd}_R(\omega)$ *for any* $M \in \mathrm{Mod}\,R$.

Proof. Without loss of generalization, suppose $l.\mathrm{pd}_R(\omega) = n < \infty$. Then for any $M \in \mathrm{Mod}\,R$ and $i \geq n+1$, we have that $\mathrm{Ext}_R^i(\omega, \mathrm{co}\,\Omega^n(M)) \cong \mathrm{Ext}_R^{n+i}(\omega, M) = 0$ for any $i \geq 1$ and $\mathrm{co}\,\Omega^n(M) \in \omega^{\perp}$. It follows from Theorem 2.2 that $\omega^{\perp}\text{-dim}_R(M) \leq n$. We are done. □

Recall from Stenström (1975) that R is called a *semilocal ring* if R/J is an Artinian semisimple ring. It is well known that a semiperfect ring (more specially, an Artinian ring or a semiprimary ring) is semilocal (see Anderson and Fuller, 1992). Also recall from Stenström (1975) that a left R-module M is called *finitely presented* if there exists a finitely generated projective left R-module P and a finitely generated submodule N of P such that $P/N \cong M$. A left R-module M is said to *admit a finitely generated projective resolution* if there exists an exact sequence: $\cdots \longrightarrow P_i \xrightarrow{f_i} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \longrightarrow M \longrightarrow 0$, where P_i is a finitely generated projective left R-module for any $i \geq 0$ (see Glaz, 1989). It is clear that if M admits a finitely generated projective resolution as above, then M and $\mathrm{Im}\, f_i$ are finitely presented for any $i \geq 1$.

Theorem 2.5. *Let R be a semilocal ring. If ${}_R\omega$ admits a finitely generated projective resolution, then* $l.\mathrm{pd}_R(\omega) = \omega^{\perp}\text{-dim}_R(R/J)$.

Proof. By Lemma 2.4, we have that $\omega^{\perp}\text{-dim}_R(R/J) \leq l.\mathrm{pd}_R(\omega)$. Then we need to prove that $l.\mathrm{pd}_R(\omega) \leq \omega^{\perp}\text{-dim}_R(R/J)$. Without loss of generalization, suppose $\omega^{\perp}\text{-dim}_R(R/J) = n < \infty$. Then by Lemma 2.3, we have that $\mathrm{Ext}_R^i(\omega, R/J) = 0$ for any $i \geq n+1$.

Now suppose that

$$\cdots \to P_i \xrightarrow{f_i} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \to {}_R\omega \to 0$$

is a finitely generated projective resolution of ${}_R\omega$. Then $\mathrm{Ext}_R^i(\mathrm{Im}\, f_n, R/J) \cong \mathrm{Ext}_R^{n+i}(\omega, R/J) = 0$ for any $i \geq 1$. So by Xu and Cheng (1994, Lemma 3), we have that $\mathrm{Im}\, f_n$ is projective and hence $l.\mathrm{pd}_R(\omega) \leq n$. We are done. □

Recall from Stenström (1975) that R is called a *left coherent ring* if every finitely generated submodule of a finitely presented left R-module is also finitely presented. It is clear that R is left coherent if R is a left Noetherian ring. On the other hand, it is not difficult to see that if R is a left coherent ring, then every presented left R-module admits a finitely generated projective resolution. So, the following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Let R be a left coherent and semilocal ring. If ${}_R\omega$ is finitely presented, then* $l.\mathrm{pd}_R(\omega) = \omega^{\perp}\text{-}\dim_R(R/J)$.

The following result is a dual version of Huang (2000, Theorem 2).

Corollary 2.7. *Let R be a left coherent and semilocal ring and ${}_R\omega$ finitely presented. Then the following statements are equivalent*:

(1) $l.\mathrm{pd}_R(\omega) < \infty$;
(2) *Every (finitely presented) left R-module has finite right orthogonal dimension*;
(3) *Every cyclic left R-module has finite right orthogonal dimension*;
(4) $\omega^{\perp}\text{-}\dim_R(R/J) < \infty$.

Proof. By Lemma 2.4 and Corollary 2.6. □

3. TILTING CASE

Recall from Angeleri Hügel and Coelho (2001) that a module $M \in \mathrm{Mod}\, R$ is called a *tilting module* provided the following conditions are satisfied:

(1) $l.\mathrm{pd}_R(M) < \infty$;
(2) $\mathrm{Ext}^i_R(M, M^{(I)}) = 0$ for any $i \geq 1$ and index set I;
(3) There exists an exact sequence $0 \to {}_RR \to M_0 \to M_1 \to \cdots \to M_r \to 0$ in $\mathrm{Mod}\, R$ with $M_i \in \mathrm{Add}_R M$ for any $0 \leq i \leq r$, where $\mathrm{Add}_R M$ denotes the full subcategory of $\mathrm{Mod}\, R$ consisting of all modules isomorphic to direct summands of direct sums of copies of ${}_RM$.

In this section, $\omega \in \mathrm{Mod}\, R$ is a tilting module.

Let $\mathscr{A}$ be a subcategory of $\mathrm{Mod}\, R$. We use ${}^{\perp}\mathscr{A}$ to denote the subcategory of $\mathrm{Mod}\, R$ consisting of the modules X with $\mathrm{Ext}^i_R(X, A) = 0$ for any $A \in \mathscr{A}$ and $i \geq 1$.

Lemma 3.1 (Angeleri Hügel and Coelho, 2001, Lemma 2.4)**.**

(1) *For any $X \in \omega^{\perp}$, there exists an exact sequence $0 \to K \to M \to X \to 0$ in* $\mathrm{Mod}\, R$ *with $M \in \mathrm{Add}_R\omega$ and $K \in \omega^{\perp}$.*
(2) *Every homomorphism $A \to X$ with $A \in {}^{\perp}(\omega^{\perp})$ and $X \in \omega^{\perp}$ factors through* $\mathrm{Add}_R\omega$. *In particular,* $\mathrm{Add}_R\omega = \omega^{\perp} \bigcap {}^{\perp}(\omega^{\perp})$.

Definition 3.2 (Enochs and Jenda, 2000)**.** Assume that $\mathscr{C} \supset \mathscr{D}$ are subcategories of $\mathrm{Mod}\, R$ and $C \in \mathscr{C}$, $D \in \mathscr{D}$. The morphism $C \to D$ is said to be a *$\mathscr{D}$-preenvelope* of C if $\mathrm{Hom}_R(D, X) \to \mathrm{Hom}_R(C, X) \to 0$ is exact for all $X \in \mathscr{D}$. The subcategory

$\mathscr{D}$ is said to be *preenveloping* in $\mathscr{C}$ if every C in $\mathscr{C}$ has a $\mathscr{D}$-preenvelope. A *$\mathscr{D}$-preenvelope* f of C is called *special* if it is a monomorphism and $\mathrm{Ext}_R^1(\mathrm{Coker}\, f, X) = 0$ for any $X \in \mathscr{D}$. Dually, the morphism $D \to C$ is said to be a *$\mathscr{D}$-precover* of C if $\mathrm{Hom}_R(X, D) \to \mathrm{Hom}_R(X, C) \to 0$ is exact for all $X \in \mathscr{D}$. The subcategory $\mathscr{D}$ is said to be *precovering* in $\mathscr{C}$ if every C in $\mathscr{C}$ has a $\mathscr{D}$-precover. A *$\mathscr{D}$-precover* f of C is called *special* if it is an epimorphism and $\mathrm{Ext}_R^1(X, \mathrm{Ker}\, f) = 0$ for any $X \in \mathscr{D}$.

Lemma 3.3. *$\omega^{\perp}$-$\dim_R(M) \leq 1$ if and only if there exists an exact sequence $0 \to M \to V \to M' \to 0$ in $\mathrm{Mod}\, R$ with $V \in \omega^{\perp}$ and $M' \in \mathrm{Add}_R\omega$.*

Proof. Since ${}_R\omega$ is a tilting module, $\mathrm{Add}_R\omega \in \omega^{\perp}$. So the sufficiency is trivial. Hence it suffices to prove the necessity. Suppose $\omega^{\perp}$-$\dim_R(M) \leq 1$, then there exists an exact sequence $0 \to M \to V_0 \to V_1 \to 0$ in $\mathrm{Mod}\, R$ with $V_0, V_1 \in \omega^{\perp}$. By Lemma 3.1(1), there exists an exact sequence $0 \to K \to M' \to V_1 \to 0$ in $\mathrm{Mod}\, R$ with $M' \in \mathrm{Add}_R\omega$ and $K \in \omega^{\perp}$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & V & \to & M' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & V_0 & \to & V_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

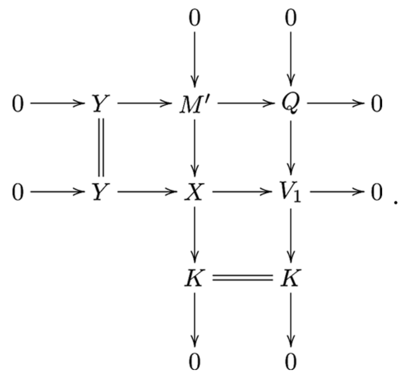
From the middle column in the above diagram, we know that $V \in \omega^{\perp}$, so the middle row is as desired. □

Let $\mathscr{X}$ be a full subcategory of Mod R and A a module in $\mathrm{Mod}\, R$. If there exists an exact sequence $0 \to A \to X_0 \to X_1 \to \cdots \to X_n \to \cdots$ in $\mathrm{Mod}\, R$ with $X_i \in \mathscr{X}$ for any $i \geq 0$, then we define the *$\mathscr{X}$-coresolution dimension* of A, denoted by $\mathscr{X}$-coresol.$\dim_R(A)$, as $\inf\{n \mid$ there exists an exact sequence $0 \to A \to X_0 \to X_1 \to \cdots \to X_n \to 0$ in $\mathrm{Mod}\, R$ with $X_i \in \mathscr{X}$ for any $0 \leq i \leq n\}$. We set $\mathscr{X}$-coresol.$\dim_R(A)$ infinity if no such an integer exists (see Auslander and Buchweitz, 1989).

Proposition 3.4. *Let n be a non-negative integer. For any $M \in \mathrm{Mod}\, R$, $\omega^{\perp}$-$\dim_R(M) \leq n$ if and only if there exists an exact sequence $0 \to M \to V \to M' \to 0$ in $\mathrm{Mod}\, R$ with $\mathrm{Add}_R\omega$-coresol.$\dim_R(M') \leq n - 1$ and $V \in \omega^{\perp}$.*

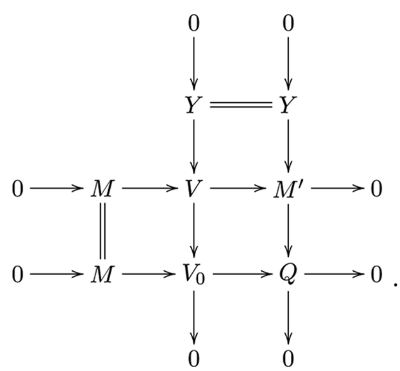
Proof. The sufficiency is trivial. In the following, we will prove the necessity by using induction on n. The case for $n \leq 1$ follows from Lemma 3.3. Now suppose $n \geq 2$. We have an exact sequence $0 \to M \to V_0 \to Q \to 0$ in $\mathrm{Mod}\, R$ with $V_0 \in \omega^{\perp}$ and $\omega^{\perp}$-$\dim_R(Q) \leq n - 1$. By the induction hypothesis, we have an exact sequence $0 \to Q \to V_1 \to K \to 0$ in $\mathrm{Mod}\, R$ with $\mathrm{Add}_R\omega$-coresol.$\dim_R(K) \leq n - 2$ and $V_1 \in \omega^{\perp}$.

Since $V_1 \in \omega^{\perp}$, by Lemma 3.1(1) there exists an exact sequence $0 \to Y \to X \to V_1 \to 0$ in $\operatorname{Mod} R$ with $X \in \operatorname{Add}_R \omega$ and $Y \in \omega^{\perp}$. First, consider the following pullback diagram:



From the middle column in the above diagram, we have that $\operatorname{Add}_R\omega$-coresol. $\dim_R(M') \leq n-1$.

Next, consider the following pullback diagram:



From the middle column in the above diagram, we know that $E \in \omega^{\perp}$, so the middle row is as desired. □

Proposition 3.5. *The following statements are equivalent for a non-negative integer n:*

(1) *For any $M \in \operatorname{Mod} R$, there exists an exact sequence $0 \to M \to V \to K \to 0$ in $\operatorname{Mod} R$ with $\operatorname{Add}_R\omega$-coresol.$\dim_R(K) \leq n-1$ and $V \in \omega^{\perp}$;*

(2) *For any $M \in \operatorname{Mod} R$, there exists an exact sequence $0 \to V' \to K' \to M \to 0$ in $\operatorname{Mod} R$ with $\operatorname{Add}_R\omega$-coresol.$\dim_R(K') \leq n$ and $V' \in \omega^{\perp}$.*

Proof. (1) ⇒ (2) Let $M \in \operatorname{Mod} R$. Then by (1), there exists an exact sequence $0 \to M \to V \to K \to 0$ in $\operatorname{Mod} R$ with $\operatorname{Add}_R\omega$-coresol.$\dim_R(K) \leq n-1$ and $V \in$

$\omega^{\perp}$. By Lemma 3.1(1), there exists an exact sequence $0 \to V' \to X \to V \to 0$ in $\operatorname{Mod} R$ with $X \in \operatorname{Add}_R \omega$ and $V' \in \omega^{\perp}$.

Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & K' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & X & \longrightarrow & V & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle column in the above diagram, we have $\operatorname{Add}_R\omega$-coresol.$\dim_R(K') \leq n$. So the first row is as desired.

(2) $\Rightarrow$ (1) Let $M \in \operatorname{Mod} R$. Then by (2), there exists an exact sequence $0 \to V' \to K' \to M \to 0$ in $\operatorname{Mod} R$ with $\operatorname{Add}_R\omega$-coresol.$\dim_R(K') \leq n$ and $V' \in \omega^{\perp}$. So there exists an exact sequence $0 \to K' \to X \to K \to 0$ in $\operatorname{Mod} R$ with $X \in \operatorname{Add}_R \omega$ and $\operatorname{Add}_R\omega$-coresol.$\dim_R(K) \leq n - 1$.

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & V' & = & V' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K' & \longrightarrow & X & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the middle column in the above diagram, we have $V \in \omega^{\perp}$, so the last row is as desired. □

It is easy to verify that the exact sequence in Proposition 3.5(1) is a special $\omega^{\perp}$-preenvelope of M, and that of Proposition 3.5(2) is a special $\mathscr{X}$-precover of M, where $\mathscr{X}$ denotes the subcategory of $\operatorname{Mod} R$ consisting of all X with $\operatorname{Add}_R\omega$-coresol.$\dim_R(X) \leq n$. Thus we get the main result in this section as follows.

Theorem 3.6. *For any $M \in \operatorname{Mod} R$, the following are equivalent*:

(1) $\omega^{\perp}\text{-}\dim_R(M) \leq n$;
(2) $\operatorname{co}\Omega^n(M) \in \omega^{\perp}$;
(3) *M has a special $\omega^{\perp}$-preenvelope $f: M \to V$ with $\operatorname{Add}_R\omega\text{-coresol.dim}_R(\operatorname{Coker} f) \leq n-1$*;
(4) *M has a special $\mathscr{X}$-precover $g: K' \to M$ with $Ker g \in \omega^{\perp}$, where $\mathscr{X}$ denotes the subcategory of $\operatorname{Mod} R$ consisting of all X with $\operatorname{Add}_R\omega\text{-coresol.dim}_R(X) \leq n$.*

Proof. By Theorem 2.2, we have (1) $\Leftrightarrow$ (2). By Propositions 3.4 and 3.5, we have (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). □

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