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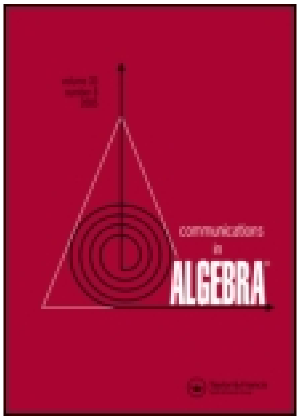

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n-Strongly Gorenstein Projective, Injective and Flat Modules

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n-STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

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In this article, we study the relation between m-strongly Gorenstein projective (resp., injective) modules and n-strongly Gorenstein projective (resp., injective) modules whenever $m \neq n$, and the homological behavior of n-strongly Gorenstein projective (resp., injective) modules. We introduce the notion of n-strongly Gorenstein flat modules. Then we study the homological behavior of n-strongly Gorenstein flat modules, and the relation between these modules and n-strongly Gorenstein projective (resp., injective) modules.

Key Words: *n*-Strongly Gorenstein flat modules; *n*-Strongly Gorenstein injective modules; *n*-Strongly Gorenstein projective modules.

2000 Mathematics Subject Classification: 16E05; 16E10; 16E30.

1. INTRODUCTION

As a nice generalization of the notion of finitely generated projective modules, Auslander and Bridger introduced in [1] the notion of finitely generated modules having Gorenstein dimension zero over left and right Noetherian rings. For any module over a general ring, Enochs and Jenda introduced in [8] the notion of Gorenstein projective modules, which coincides with that of modules having Gorenstein dimension zero for finitely generated modules over left and right Noetherian rings. In [8] Enochs and Jenda also introduced the dual notion of Gorenstein projective modules, which is called Gorenstein injective modules. As a generalization of the notion of flat modules, Enochs, Jenda, and Torrecillas introduced in [10] the notion of Gorenstein flat modules. These modules have been studied extensively by many authors (see [1, 2, 6–10, 15, 17], and so on). In particular, it was proved that these modules share many nice properties of the classical modules: projective, injective, and flat modules, respectively.

In 2007, Bennis and Mahdou introduced in [3] the notion of strongly Gorenstein projective, injective, flat modules, which situate between projective, injective, flat modules and Gorenstein projective, injective, flat modules, respectively. Then they proved that a module is Gorenstein projective (resp., injective) if and only

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if it is a direct summand of a strongly Gorenstein projective (resp., injective) module, and that every Gorenstein flat module is a direct summand of a strongly Gorenstein flat module. Yang and Liu proved in [18] that a module M is strongly Gorenstein projective (resp., injective, flat) if and only if so is $M \oplus H$ for any projective (resp., injective, flat) module H. Gao and Zhang gave in [13] a concrete construction of strongly Gorenstein projective modules, via the existed construction of upper triangular matrix Artinian algebras of degree two.

In a recent article [4], for any $n \geq 1$, Bennis and Mahdou introduced the notion of n-strongly Gorenstein projective and injective modules, in which 1-strongly Gorenstein projective (resp., injective) modules are just strongly Gorenstein projective (resp., injective) modules. Then they proved that an n-strongly Gorenstein projective module is projective if and only if it has finite flat dimension. They also gave some equivalent characterizations of n-strongly Gorenstein projective modules in terms of the vanishing of some homological groups.

In this article, based on the results mentioned above, we mainly study the homological behavior of n-strongly Gorenstein projective, injective, and flat modules, and investigate the relation among them. This article is organized as follows.

In Section 2, we give the definitions of (strongly) Gorenstein projective, injective, and flat modules.

In Section 3, we study the relation between m-strongly Gorenstein projective modules and n-strongly Gorenstein projective modules whenever $m \neq n$, and the closure of some special direct summand of an n-strongly Gorenstein projective module. For any $n \geq 1$, we give an example of an n-strongly Gorenstein projective module, which is not m-strongly Gorenstein projective whenever $n \nmid m$. For any $m, n \geq 1$, we prove that the intersection of the subcategory of m-strongly Gorenstein projective modules and that of n-strongly Gorenstein projective modules is the subcategory of (m, n)-strongly Gorenstein projective modules, where (m, n) is the greatest common divisor of m and n. We give a method how to construct a 1-strongly Gorenstein projective module from n-strongly Gorenstein projective modules. In addition, we prove that a module M is n-strongly Gorenstein projective if and only if so is $M \oplus H$ for any projective module H, which is a generalization of [18, Theorem 2.1]. We remark that all the dual results hold for n-strongly Gorenstein injective modules.

In Section 4, for any $n \geq 1$, we introduce the notion of n-strongly Gorenstein flat modules, and then give an example of an n-strongly Gorenstein flat module, which is not m-strongly Gorenstein flat whenever $n \nmid m$. We prove that a module M is n-strongly Gorenstein flat if and only if so is $M \oplus H$ for any flat module H. We also investigate the relation between n-strongly Gorenstein flat modules and n-strongly Gorenstein projective (resp., injective) modules. We prove that a finitely generated n-strongly Gorenstein projective module is finitely presented n-strongly Gorenstein flat. In addition, we prove that the character module of an n-strongly Gorenstein flat module is n-strongly Gorenstein injective; and that the character module of an n-strongly Gorenstein injective module is n-strongly Gorenstein flat over an Artinian algebra. These results generalize some results in [18].

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2. PRELIMINARIES

Throughout this article, R is an associative ring with identity and $\mathrm{Mod}\, R$ is the category of left R-modules.

Definition 2.1 ([8]). A module $G \in \mathrm{Mod}\, R$ is called *Gorenstein projective* (*G-projective* for short) if there exists an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots,$$

in $\mathrm{Mod}\, R$, such that: (1) All P_i and P^i are projective; (2) $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \mathrm{Mod}\, R$ is projective; and (3) $G \cong \mathrm{Im}(P_0 \to P^0)$. Dually, the notion of *Gorenstein injective modules* (*G-injective modules* for short) is defined.

Definition 2.2 ([10]). A module $F \in \mathrm{Mod}\, R$ is called *Gorenstein flat* (*G-flat* for short) if there exists an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots,$$

in $\mathrm{Mod}\, R$, such that: (1) All F_i and F^i are flat; (2) $I \otimes_R -$ leaves the sequence exact whenever $I \in \mathrm{Mod}\, R^{op}$ is injective; and (3) $F \cong \mathrm{Im}(F_0 \to F^0)$.

For a module $M \in \mathrm{Mod}\, R$, we denote $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, which is called the *character module* of M, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers (see [12]). The following result is an analog of [15, Proposition 2.27], which is maybe known.

Lemma 2.3. *A G-flat module with finite flat dimension is flat.*

Proof. Let $M \in \mathrm{Mod}\, R$ be a G-flat module with finite flat dimension. Then by [15, Theorem 3.6] and [12, Theorem 2.1], we have that $M^+ \in \mathrm{Mod}\, R^{op}$ is G-injective with finite injective dimension. So M^+ is injective by the dual version of [15, Proposition 2.27], and hence M is flat by [12, Theorem 2.1]. □

Definition 2.4 ([3]). (1) A module $M \in \mathrm{Mod}\, R$ is called *strongly Gorenstein projective* (*SG-projective* for short), if there exists an exact sequence

$$0 \to M \to P_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with P_0 projective, such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \mathrm{Mod}\, R$ is projective. Dually, the notion of *strongly Gorenstein injective modules* (*SG-injective modules* for short) is defined.

(2) A module $M \in \mathrm{Mod}\, R$ is called *strongly Gorenstein flat* (*SG-flat* for short), if there exists an exact sequence

$$0 \to M \to F_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with F_0 flat, such that $I \otimes_R$–leaves the sequence exact whenever $I \in \mathrm{Mod}\, R^{op}$ is injective.

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It is trivial that {projective modules} $\subseteq$ {SG-projective modules} $\subseteq$ {G-projective modules}, {injective modules} $\subseteq$ {SG-injective modules} $\subseteq$ {G-injective modules} and {flat modules} $\subseteq$ {SG-flat modules} $\subseteq$ {G-flat modules}. By [3], all of the inclusions are strict in general.

3. *n*-STRONGLY GORENSTEIN PROJECTIVE AND INJECTIVE MODULES

In this section we study the properties of n-strongly Gorenstein projective modules. All the dual results hold for the n-strongly Gorenstein injective modules, and we omit this dual part.

Definition 3.1 ([4]). Let n be a positive integer. A module $M \in \operatorname{Mod} R$ is called *n-strongly Gorenstein projective* (*n-SG-projective* for short), if there exists an exact sequence

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

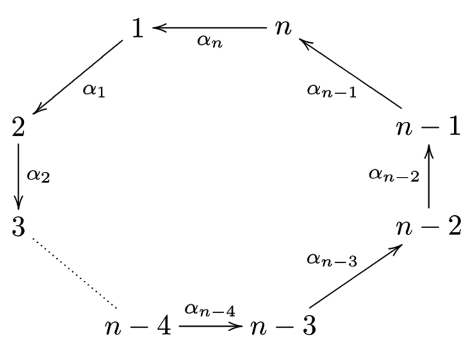
in $\operatorname{Mod} R$ with P_i projective for any $0 \leq i \leq n-1$, such that $\operatorname{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \operatorname{Mod} R$ is projective. Dually, the notion of *n-strongly Gorenstein injective modules* (*n-SG-injective modules* for short) is defined.

It is clear that the global dimension of R is infinite if there exists a nonprojective n-SG-projective R-module for some $n \geq 1$.

In the following, we first study the relation between m-SG-projective modules and n-SG-projective modules whenever $m \neq n$.

Note that 1-SG-projective modules are just SG-projective modules. In addition, for any $1 \leq i \leq n$, $\operatorname{Im} f_i$ in the above exact sequence is also n-SG-projective. It is trivial that a 1-SG-projective module is n-SG-projective for any $n \geq 1$. However, for any $n \geq 2$, an n-SG-projective module is not necessarily m-SG-projective whenever $n \nmid m$, as showed in the following example.

Example 3.2. Let R be a finite-dimensional algebra over a field given by the quiver



modulo the ideal generated by $\{\alpha_{i+1}\alpha_i, \alpha_1\alpha_n \mid 1 \leq i \leq n-1\}$. For any $1 \leq i \leq n$, we use S_i, P_i, and I^i to denote the simple R-module, the indecomposable projective R-module and the indecomposable injective R-module corresponding to the vertex i, respectively. Then R is a self-injective algebra with infinite global dimension,

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and $P_n = I^1$, $P_i = I^{i+1}$ for any $1 \leq i \leq n-1$. In addition, for any $1 \leq i \leq n$, we have:

(1) The following exact sequence

$$0 \to S_i \to P_{i-1} \to \cdots \to P_1 \to P_n \to P_{n-1} \to \cdots \to P_i \to S_i \to 0$$

is a minimal projective resolution of S_i.

(2) For any $m \geq 1$, if $n \nmid m$, then $\mathrm{Ext}_R^m(S_i, S_i) = 0$; if $n \mid m$, then $\mathrm{Ext}_R^m(S_i, S_i) \neq 0$.

(3) S_i is n-SG-projective.

(4) S_i is not m-SG-projective whenever $n \nmid m$.

For any $n \geq 1$, we use n-SG-Proj(R) to denote the subcategory of $\mathrm{Mod}\, R$ consisting of n-SG-projective modules. In the following, assume that m and n are positive integers with $n \leq m$.

Lemma 3.3. *If $n \mid m$, then n-SG-Proj$(R) \subseteq m$-SG-Proj(R).*

We state a crucial result as follows.

Proposition 3.4.

(1) *If $n \mid m$, then m-SG-Proj$(R) \cap n$-SG-Proj$(R) = n$-SG-Proj(R).*

(2) *If $n \nmid m$ and $m = kn + j$, where k is a positive integer and $0 < j < n$, then m-SG-Proj$(R) \cap n$-SG-Proj$(R) \subseteq j$-SG-Proj(R).*

Proof. (1) It is trivial by Lemma 3.3.

(2) By Lemma 3.3, we have that m-SG-Proj$(R) \cap n$-SG-Proj$(R) \subseteq m$-SG-Proj$(R) \cap kn$-SG-Proj(R). Assume that $M \in m$-SG-Proj$(R) \cap kn$-SG-Proj(R). Then there exists an exact sequence

$$0 \to M \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \qquad (1)$$

in $\mathrm{Mod}\, R$ with P_i projective for any $0 \leq i \leq m-1$. Put $L_i = \mathrm{Ker}(P_{i-1} \to P_{i-2})$ for any $2 \leq i \leq m$. Because $M \in kn$-SG-Proj(R), it is easy to see that M and L_{kn} are projectively equivalent, that is, there exist projective modules P and Q in $\mathrm{Mod}\, R$, such that $M \oplus P \cong Q \oplus L_{kn}$.

First, consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & \\
0 \arrow[r] & L_{kn+1} \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M \oplus P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_{kn+1} \arrow[r] & P_{kn} \arrow[r] \arrow[d] & L_{kn} \arrow[r] \arrow[d] & 0. \\
 & & 0 & 0 &
\end{tikzcd}$$

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Then X is projective. Next, consider the following pullback diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & L_{kn+1} \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L_{kn+1} \arrow[r] & X \arrow[r] \arrow[d] & M \oplus P \arrow[r] \arrow[d] & 0. \\ & & P \arrow[r, equal] \arrow[d] & P \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

Thus Y is also projective. Combining the exact sequence (1) and the first row in the above diagram, we get the following exact sequence:

$$0 \to M \to P_{m-1} \to \cdots \to P_{kn+1} \to Y \to M \to 0,$$

which is still exact after applying the functor $\mathrm{Hom}_R(-, P)$ for any projective R-module P. So M is j-SG-projective, and hence m-SG-Proj$(R) \cap n$-SG-Proj$(R) \subseteq j$-SG-Proj(R). □

We use (m, n) to denote the greatest common divisor of m and n.

Theorem 3.5. *m-SG-Proj$(R) \cap n$-SG-Proj$(R) = (m, n)$-SG-Proj(R).*

Proof. If $n \mid m$, then the assertion follows from Proposition 3.4(1).

Now assume that $n \nmid m$ and $m = k_0 n + j_0$, where k_0 is a positive integer and $0 < j_0 < n$. By Proposition 3.4(2), we have m-SG-Proj$(R) \cap n$-SG-Proj$(R) \subseteq j_0$-SG-Proj(R). If $j_0 \nmid n$ and $n = k_1 j_0 + j_1$ with $0 < j_1 < j_0$, then by Proposition 3.4(2) again, we have that m-SG-Proj$(R) \cap n$-SG-Proj$(R) \subseteq n$-SG-Proj$(R) \cap j_0$-SG-Proj$(R) \subseteq j_1$-SG-Proj(R). Continuing the above procedure, after finite steps, there exists a positive integer t such that $j_t = k_{t+2} j_{t+1}$ and $j_{t+1} = (m, n)$. Thus m-SG-Proj$(R) \cap n$-SG-Proj$(R) \subseteq j_t$-SG-Proj$(R) \cap j_{t+1}$-SG-Proj$(R) = j_{t+1}$-SG-Proj$(R) = (m, n)$-SG-Proj(R). On the other hand, we always have (m, n)-SG-Proj$(R) \subseteq m$-SG-Proj$(R) \cap n$-SG-Proj(R), so they are identical. □

As an immediate consequence of Theorem 3.5, we have the following corollary.

Corollary 3.6. *n-SG-Proj$(R) \cap (n+1)$-SG-Proj$(R) = 1$-SG-Proj(R). In particular, $\bigcap_{n \geq 2} n$-SG-Proj$(R) = 1$-SG-Proj(R).*

The following result shows that the difference between the projectivity and n-SG-projectivity of modules is the self-orthogonality of modules.

Proposition 3.7. *Let $M \in \mathrm{Mod}\, R$ be n-SG-projective and $n \geq 1$. Then the following statements are equivalent:*

(1) *M is projective.*

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(2) $\mathrm{Ext}^i_R(M, M) = 0$ *for any* $i \geq 1$.
(3) $\mathrm{Ext}^i_R(M, M) = 0$ *for any* $1 \leq i \leq n$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. By the dimension shifting, it is easy to get (3) $\Rightarrow$ (1). □

In the rest of this section, we will study the homological behavior of n-SG-projective modules.

Proposition 3.8. *For any* $n \geq 1$, *n-SG-Proj(R) is closed under direct sums.*

Proof. Let $\{M_j\}_{j\in J}$ be a family of n-SG-projective modules in $\mathrm{Mod}\, R$. Then for any $j \in J$, there exists an exact sequence

$$0 \to M_j \to P^{(j)}_{n-1} \to \cdots \to P^{(j)}_0 \to M_j \to 0$$

in $\mathrm{Mod}\, R$ with $P^{(j)}_i$ projective for any $0 \leq i \leq n-1$, such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \mathrm{Mod}\, R$ is projective. So we get an exact sequence

$$0 \to \bigoplus_{j\in J} M_j \to \bigoplus_{j\in J} P^{(j)}_{n-1} \to \cdots \to \bigoplus_{j\in J} P^{(j)}_0 \to \bigoplus_{j\in J} M_j \to 0$$

in $\mathrm{Mod}\, R$. Because $\bigoplus_{j\in J} P^{(j)}_{n-1}, \ldots, \bigoplus_{j\in J} P^{(j)}_0$ are projective and the obtained exact sequence is still exact after applying the functor $\mathrm{Hom}_R(-, P)$ whenever $P \in \mathrm{Mod}\, R$ is projective, $\bigoplus_{j\in J} M_j$ is n-SG-projective and the assertion follows. □

The following result gives some characterizations of n-SG-projective modules, which also gives a method how to construct a 1-SG-projective module from n-SG-projective modules.

Theorem 3.9. *For any* $M \in \mathrm{Mod}\, R$ *and* $n \geq 1$, *the following statements are equivalent*:

(1) *M is n-SG-projective.*
(2) *There exists an exact sequence*

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

in $\mathrm{Mod}\, R$ *with* P_i *projective for any* $0 \leq i \leq n-1$, *such that* $\bigoplus_{i=1}^n \mathrm{Im}\, f_i$ *is* 1*-SG-projective.*
(3) *There exists an exact sequence*:

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

in $\mathrm{Mod}\, R$ *with* P_i *projective for any* $0 \leq i \leq n-1$, *such that* $\bigoplus_{i=1}^n \mathrm{Im}\, f_i$ *is G-projective.*

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(4) *There exists an exact sequence*

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

in $\operatorname{Mod} R$, *where* P_i *has finite projective dimension for any* $0 \le i \le n-1$, *such that* $\bigoplus_{i=1}^n \operatorname{Im} f_i$ *is* 1*-SG-projective.*

(5) *There exists an exact sequence*:

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

in $\operatorname{Mod} R$, *where* P_i *has finite projective dimension for any* $0 \le i \le n-1$, *such that* $\bigoplus_{i=1}^n \operatorname{Im} f_i$ *is G-projective.*

Proof. (1) $\Rightarrow$ (2) Let $M \in \operatorname{Mod} R$ be n-SG-projective. Then we have an exact sequence

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

in $\operatorname{Mod} R$ with P_i projective for any $0 \le i \le n-1$, such that $\operatorname{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \operatorname{Mod} R$ is projective. Thus, for any $1 \le i \le n$, we have an exact sequence

$$0 \to \operatorname{Im} f_i \xrightarrow{\alpha_i} P_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} \operatorname{Im} f_i \to 0$$

in $\operatorname{Mod} R$. By adding these exact sequences, we get the following exact sequence

$$0 \to \bigoplus_{i=1}^{n} \operatorname{Im} f_i \xrightarrow{\alpha} \bigoplus_{i=0}^{n-1} P_i \xrightarrow{f} P_{n-1} \oplus P_0 \oplus \cdots \oplus P_{n-2} \to \ldots,$$

where $\alpha = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $f = \operatorname{diag}\{f_n f_0, f_1, \ldots, f_{n-1}\}$. It is easy to see that $\operatorname{Im} f \cong \bigoplus_{i=1}^n \operatorname{Im} f_i$ and $\operatorname{Ext}_R^1(\bigoplus_{i=1}^n \operatorname{Im} f_i, P) = 0$ for any projective module $P \in \operatorname{Mod} R$, which implies $\bigoplus_{i=1}^n \operatorname{Im} f_i$ is 1-SG-projective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1) Assume that

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is an exact sequence in $\operatorname{Mod} R$, where P_i has finite projective dimension for any $0 \le i \le n-1$, such that $\bigoplus_{i=1}^n \operatorname{Im} f_i$ is G-projective. Then, for any $0 \le i \le n-1$, we have the exact sequence

$$0 \to \operatorname{Im} f_{i+1} \to P_i \to \operatorname{Im} f_i \to 0.$$

Because $\bigoplus_{i=1}^n \operatorname{Im} f_i$ is G-projective, so is each P_i by [15, Theorem 2.5]. Thus each P_i is projective by [15, Proposition 2.27]. In particular, M is also G-projective by [15, Theorem 2.5], so $\operatorname{Ext}_R^i(M, P) = 0$ for any projective module $P \in \operatorname{Mod} R$ and $i \ge 1$. It follows that M is n-SG-projective. $\square$

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From [18] we know that 1-SG-Proj(R) is not closed under direct summands. The following example illustrates that for any $n \geq 1$, n-SG-Proj(R) is not closed under direct summands.

Example 3.10. Under the assumption of Example 3.2, we have that $\bigoplus_{i=1}^{n} S_i$ is 1-SG-projective by Theorem 3.9, and hence $(n-1)$-SG-projective. However, for any $1 \leq i \leq n$, S_i is not $(n-1)$-SG-projective.

The following result is a generalization of [18, Theorem 2.1], which shows that some special direct summand of an n-SG-projective module is again n-SG-projective. For a module $M \in \operatorname{Mod} R$, we use $\underline{M}$ to denote the maximal direct summand of M without projective summands.

Theorem 3.11. *For any $n \geq 1$, a module $M \in \operatorname{Mod} R$ is n-SG-projective if and only if so is $\underline{M}$.*

Proof. Let $M = \underline{M} \oplus P$ with P a projective module in $\operatorname{Mod} R$. If $\underline{M}$ is n-SG-projective, then M is also n-SG-projective by Proposition 3.8.

Conversely, assume that $M \in \operatorname{Mod} R$ is n-SG-projective. Then there exists an exact sequence

$$0 \to (M=)\underline{M} \oplus P \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} \underline{M} \oplus P(=M) \to 0$$

in $\operatorname{Mod} R$ with P_i projective for any $0 \leq i \leq n-1$, such that $\operatorname{Hom}_R(-, Q)$ leaves the sequence exact whenever $Q \in \operatorname{Mod} R$ is projective.

Put $\operatorname{Im} f_i = K_i$ for any $0 \leq i \leq n$. First, consider the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "f_n"] & \underline{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r] & P_{n-1} \arrow[r] \arrow[d] & Q_{n-1} \arrow[r] \arrow[d] & 0. \\
 & & K_{n-1} \arrow[r, equal] \arrow[d] & K_{n-1} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Because M is G-projective, both $\underline{M}$ and Q_{n-1} are also G-projective by [15, Theorem 2.5]. So $\operatorname{Ext}_R^1(Q_{n-1}, P) = 0$ and the middle row $0 \to P \to P_{n-1} \to Q_{n-1} \to 0$ in the above diagram splits, which implies that Q_{n-1} is projective. Because K_{n-1} is also n-SG-projective, the third column

$$0 \to \underline{M} \to Q_{n-1} \to K_{n-1} \to 0$$

in the above diagram is still exact after applying the functor $\operatorname{Hom}_R(-, Q)$ whenever $Q \in \operatorname{Mod} R$ is projective.

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Next, consider the following pullback diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d] & & \\
0 \arrow[r] & Q_0 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "f_0"] & P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{M} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & P \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}
$$

Then $0 \to K_1 \to Q_0 \to \underline{M} \to 0$ is exact and Q_0 is projective. Thus we obtain the following exact sequence

$$0 \to \underline{M} \to Q_{n-1} \to P_{n-2} \to \cdots \to P_1 \to Q_0 \to \underline{M} \to 0.$$

Note that both K_1 and $\underline{M}$ are G-projective. Thus the above exact sequence is still exact after applying the functor $\mathrm{Hom}_R(-, Q)$ whenever $Q \in \mathrm{Mod}\, R$ is projective, which implies $\underline{M}$ is n-SG-projective. □

By Theorem 3.11, we immediately have the following corollary.

Corollary 3.12. *Assume that $M, N \in \mathrm{Mod}\, R$ are projectively equivalent. Then, for any $n \geq 1$, M is n-SG-projective if and only if so is N.*

We denote $\mathrm{mod}\, R$ the category of finitely generated left R-modules, and n-SG-proj$(R) = \{M \in \mathrm{mod}\, R \mid$ there exists an exact sequence $0 \to M \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$ in $\mathrm{mod}\, R$ with P_i projective for any $0 \leq i \leq n-1$, such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \mathrm{mod}\, R$ is projective$\}$.

The following fact is useful, which is a generalization of [13, Proposition 1.1].

Lemma 3.13. *For any $n \geq 1$, n-SG-Proj$(R) \cap \mathrm{mod}\, R =$ n-SG-proj(R).*

Proof. Let $M \in n$-SG-proj(R). By using an argument similar to that of [13, Proposition 1.1], we have that $M \in n$-SG-Proj$(R) \cap \mathrm{mod}\, R$.

Conversely, let $M \in$ n-SG-Proj$(R) \cap \mathrm{mod}\, R$. Then there exists an exact sequence

$$0 \to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \tag{2}$$

in $\mathrm{Mod}\, R$ with P_i projective for any $0 \leq i \leq n-1$, such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in \mathrm{Mod}\, R$ is projective. Put $\mathrm{Im}\, f_i = K_i$ for any $0 \leq i \leq n$. There exists a projective module $P'_{n-1} \in \mathrm{Mod}\, R$ such that $P_{n-1} \oplus P'_{n-1} = Q$ is free, so we have an exact sequence

$$0 \to M \xrightarrow{f'_n} P_{n-1} \oplus P'_{n-1} \xrightarrow{f'_{n-1}} P_{n-2} \oplus P'_{n-1} \xrightarrow{f'_{n-2}} P_{n-3} \xrightarrow{f_{n-3}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0.$$

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Then $\operatorname{Im} f'_{n-1} \cong K_{n-1} \oplus P'_{n-1}$ and $\operatorname{Im} f'_{n-2} \cong K_{n-2}$. Since M is finitely generated, one can write $Q = Q_{n-1} \oplus Q'_{n-1}$ with $Q_{n-1} \in \operatorname{mod} R$ and $\operatorname{Im} f'_n \subseteq Q_{n-1}$. So we get an exact sequence

$$0 \longrightarrow M \xrightarrow{f'_n} Q_{n-1} \longrightarrow K'_{n-1} \longrightarrow 0 \tag{3}$$

with $K'_{n-1} \oplus Q'_{n-1} \cong \operatorname{Im} f'_{n-1}$, and hence $K'_{n-1} \in n\text{-SG-Proj}(R) \cap \operatorname{mod} R$ by Corollary 3.12.

Consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Q'_{n-1} \arrow[r, equal] \arrow[d] & Q'_{n-1} \arrow[d] & & \\
0 \arrow[r] & \operatorname{Im} f'_{n-1} \arrow[r] \arrow[d] & P_{n-2} \oplus P'_{n-1} \arrow[r] \arrow[d] & K_{n-2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K'_{n-1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & K_{n-2} \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

Then X is G-projective by [15, Theorem 2.5], and so the middle column $0 \to Q'_{n-1} \to P_{n-2} \oplus P'_{n-1} \to X \to 0$ in the above diagram splits, which implies that X is projective. Combining the exact sequences (2), (3) with the third row in the above diagram, we get an exact sequence

$$0 \to M \xrightarrow{f'_n} Q_{n-1} \to X \to P_{n-3} \to \cdots \to P_0 \to M \to 0$$

with $Q_{n-1} \in \operatorname{mod} R$. Repeating the above procedure with $K'_{n-1}(\cong \operatorname{Coker} f'_n)$ replacing M, we finally obtain the following exact sequence

$$0 \to M \to Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to M \to 0$$

in $\operatorname{mod} R$ with Q_i projective for any $0 \le i \le n-1$, which implies $M \in n\text{-SG-proj}(R)$. $\square$

The following result gives some equivalent characterizations of finitely generated n-SG-projective modules.

Theorem 3.14. *For any $M \in \operatorname{mod} R$ and $n \ge 1$, the following statements are equivalent*:

(1) *M is n-SG-projective.*
(2) *There exists an exact sequence*

$$0 \to M \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

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in $\operatorname{mod} R$ *with* P_i *projective for any* $0 \le i \le n-1$, *and* $\operatorname{Ext}^i_R(M, R) = 0$ *for any* $i \ge 1$.

(3) *There exists an exact sequence*

$$0 \to M \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

in $\operatorname{mod} R$ *with* P_i *projective for any* $0 \le i \le n-1$, *and* $\operatorname{Ext}^i_R(M, F) = 0$ *for any flat module* $F \in \operatorname{Mod} R$ *and* $i \ge 1$.

(4) *There exists an exact sequence*

$$0 \to M \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

in $\operatorname{mod} R$ *with* P_i *projective for any* $0 \le i \le n-1$, *and* $\operatorname{Ext}^i_R(M, F) = 0$ *for any* $F \in \operatorname{Mod} R$ *with finite flat dimension and* $i \ge 1$.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 3.13. The proofs of other implications are similar to that of [3, Proposition 2.12], so we omit them. □

We have obtained some properties of the intersection between m-SG-projective modules and n-SG-projective modules (see 3.3–3.6). We end this section with some properties of the union of these modules.

It has been known that $\bigcup_{n\ge 1} n$-SG-Proj$(R) \subseteq \{$G-projective R-modules$\}$. We will show that this inclusion is strict in general, and also investigate when the equality holds true.

In the rest of this section, R is a finite-dimensional k-algebra over an algebraically closed field k. Let $M \in \operatorname{mod} R$ and

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution of M in $\operatorname{mod} R$. Recall from [11] that the *complexity* of M is defined as $\operatorname{cx}(M) = \inf\{b \ge 0 \mid$ there exists a $c > 0$ such that $\dim_k P_n \le cn^{b-1}$ for all $n\}$ if it exists, otherwise $\operatorname{cx}(M) = \infty$. It is easy to see that $\operatorname{cx}(M) = 0$ implies M is of finite projective dimension, and $\operatorname{cx}(M) \le 1$ if and only if the dimensions of P_n are bounded.

Proposition 3.15. *Let R be a self-injective algebra.*

(1) *If* R *is of infinite representation type with vanishing radical cube, then* $\bigcup_{n\ge 1} n$*-SG-Proj*$(R) \subsetneqq \{$*G-projective R-modules*$\}$.

(2) *If* R *is of finite representation type, then* $\bigcup_{n\ge 1} n$*-SG-proj*$(R) = \{$*finitely generated G-projective R-modules*$\}$.

Proof. Let R be a self-injective algebra. Then $\operatorname{mod} R = \{$finitely generated G-projective R-modules$\}$.

(1) Assume that R is of infinite representation type with vanishing radical cube. Then by [14, Theorem 7.1], there exists a module $M \in \operatorname{mod} R$ such that $\operatorname{cx}(M) \ge 2$. It is easy to see that M is not n-SG-projective for any $n \ge 1$. Thus

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$\bigcup_{n\geq 1}$ n-SG-proj$(R) \subsetneqq$ {finitely generated G-projective R-modules}, and therefore, $\bigcup_{n\geq 1}$ n-SG-Proj$(R) \subsetneqq$ {G-projective R-modules} by Lemma 3.13.

(2) Assume that R is of finite representation type. We claim that any indecomposable module $M \in \operatorname{mod} R$ is n-SG-projective for some $n \geq 1$. Otherwise, if $M \in \operatorname{mod} R$ is not n-SG-projective for any $n \geq 1$. Then there exists a minimal projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M in $\operatorname{mod} R$, which is of infinite length. Because R is self-injective, all P_i are also injective. Then by [16, Lemma 2.6], all syzysy modules in the above exact sequence are indecomposable. It is not difficult to see that any two of these syzysy modules are not isomorphic, which implies that R is of infinite representation type. This is a contradiction. The claim is proved. So it follows from Proposition 3.8 that any module $M \in \operatorname{mod} R$ is n-SG-projective for some $n \geq 1$. Thus we get that $\bigcup_{n\geq 1}$ n-SG-proj$(R) =$ {finitely generated G-projective R-modules}. □

4. *n*-STRONGLY GORENSTEIN FLAT MODULES

In this section, we introduce the notion of n-strongly Gorenstein flat modules. Then we study the homological behavior of n-strongly Gorenstein flat modules, and the relation between these modules and n-strongly Gorenstein projective (resp., injective) modules.

Definition 4.1. Let n be a positive integer. A module $M \in \operatorname{Mod} R$ is called *n-strongly Gorenstein flat* (*n-SG-flat* for short), if there exists an exact sequence

$$0 \to M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} F_{n-2} \xrightarrow{h_{n-2}} \cdots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \to 0$$

in $\operatorname{Mod} R$ with F_i flat for any $0 \leq i \leq n-1$, such that $I \otimes_R$–leaves the sequence exact whenever $I \in \operatorname{Mod} R^{op}$ is injective.

Note that 1-SG-flat modules are just SG-flat modules. For any $1 \leq i \leq n$, $\operatorname{Im} h_i$ in the above exact sequence is also n-SG-flat.

Let n be a positive integer. It is trivial that a 1-SG-flat (especially, flat) module is n-SG-flat, and an n-SG-flat module is G-flat. It is clear that the weak global dimension of R is infinite if there exists a non-flat n-SG-flat R-module for some $n \geq 1$. On the other hand, for a quasi-Frobenius ring R, it is easy to see that a module in $\operatorname{Mod} R$ is n-SG-flat if and only if it is n-SG-projective, if and only if it is n-SG-injective. So we have the following example which illustrates that there exists an n-SG-flat module, but it is not m-SG-flat whenever $n \nmid m$.

Example 4.2. Under the assumption of Example 3.2, because R is quasi-Frobenius, for any $1 \leq i \leq n$, we have the following facts: (1) S_i is n-SG-flat; and (2) S_i is not m-SG-flat whenever $n \nmid m$.

Proposition 4.3. *For any $n \geq 1$, the subcategory n-SG-Flat(R) of* $\operatorname{Mod} R$ *consisting of n-SG-flat modules is closed under direct sums.*

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Proof. The proof is similar to that of Proposition 3.8, so we omit it. □

The following result is an analog of Theorem 3.9, which gives some characterizations of n-SG-flat modules, and also gives a method how to construct a 1-SG-flat module from n-SG-flat modules.

Theorem 4.4. *For any $M \in \operatorname{Mod} R$ and $n \geq 1$, consider the following conditions:*

(1) *M is n-SG-flat;*
(2) *There exists an exact sequence*

$$0 \to M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \to 0$$

in $\operatorname{Mod} R$ with F_i flat for any $0 \leq i \leq n-1$, such that $\bigoplus_{i=1}^{n} \operatorname{Im} h_i$ is 1-SG-flat;
(3) *There exists an exact sequence*

$$0 \to M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \to 0$$

in $\operatorname{Mod} R$ with F_i flat for any $0 \leq i \leq n-1$, such that $\bigoplus_{i=1}^{n} \operatorname{Im} h_i$ is G-flat;
(4) *There exists an exact sequence*

$$0 \to M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \to 0$$

in $\operatorname{Mod} R$, where F_i has finite flat dimension for any $0 \leq i \leq n-1$, such that $\bigoplus_{i=1}^{n} \operatorname{Im} h_i$ is 1-SG-flat;
(5) *There exists an exact sequence*

$$0 \to M \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_1} F_0 \xrightarrow{h_0} M \to 0$$

in $\operatorname{Mod} R$, where F_i has finite flat dimension for any $0 \leq i \leq n-1$, such that $\bigoplus_{i=1}^{n} \operatorname{Im} h_i$ is G-flat.

In general, we have (1) ⇔ (2) ⇔ (3) ⇒ (4) ⇒ (5). If R is a right coherent ring, then all of these conditions are equivalent.

Proof. (1) ⇒ (2) By using an argument similar to that in the proof of (1) ⇒ (2) in Theorem 3.9, we get the assertion.

(2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are trivial, and it is easy to get (3) ⇒ (1).

Assume that R is a right coherent ring. Notice that the subcategory of $\operatorname{Mod} R$ consisting of G-flat modules is closed under extensions and direct summands by [15, Theorem 3.7], and also notice that a G-flat module in $\operatorname{Mod} R$ with finite flat dimension is flat by Lemma 2.3, so we get (5) ⇒ (1) by using an argument similar to that in the proof of (5) ⇒ (1) in Theorem 3.9. □

From the above argument, we see that n-SG-Flat(R) is not closed under direct summands in general. However, the following result, which is a generalization of

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[18, Lemma 2.3], shows that some special direct summand of an n-SG-flat module is again n-SG-flat. For a module $M \in \operatorname{Mod} R$, we use $\widetilde{M}$ to denote the maximal direct summand of M without flat direct summands.

Theorem 4.5. *For any $n \geq 1$, a module $M \in \operatorname{Mod} R$ is n-SG-flat if and only if so is $\widetilde{M}$.*

Proof. The sufficiency follows from Proposition 4.3. In the following, we will prove the necessity.

Assume that $M \in \operatorname{Mod} R$ is n-SG-flat and $I \in \operatorname{Mod} R^{op}$ is any injective module. Then there exists an exact sequence

$$0 \to (M=)\widetilde{M} \oplus F \xrightarrow{h_n} F_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_1} F_0 \xrightarrow{h_0} \widetilde{M} \oplus F(=M) \to 0$$

in $\operatorname{Mod} R$ with F and F_i flat for any $0 \leq i \leq n-1$, such that $I \otimes_R -$leaves the sequence exact. Put $\operatorname{Im} h_i = L_i$ for any $0 \leq i \leq n$. We first consider the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "h_n"] & \widetilde{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] & F_{n-1} \arrow[r] \arrow[d] & H_{n-1} \arrow[r] \arrow[d] & 0. \\
 & & L_{n-1} \arrow[r, equal] \arrow[d] & L_{n-1} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Then we have the following diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L_{n-1}^{+} \arrow[r, equal] \arrow[d] & L_{n-1}^{+} \arrow[d] & & \\
0 \arrow[r] & H_{n-1}^{+} \arrow[r] \arrow[d] & F_{n-1}^{+} \arrow[r] \arrow[d] & F^{+} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\widetilde{M})^{+} \arrow[r] \arrow[d] & M^{+} \arrow[r] \arrow[d] & F^{+} \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

where both F^+ and F_{n-1}^+ are injective by [12, Theorem 2.1]. Because both M^+ and L_{n-1}^+ are G-injective by [15, Theorem 3.6], both $(\widetilde{M})^+$ and H_{n-1}^+ are also G-injective

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by [15, Theorem 2.6]. Thus $\mathrm{Ext}_R^1(F^+, H_{n-1}^+) = 0$ and the middle row

$$0 \to H_{n-1}^+ \to F_{n-1}^+ \to F^+ \to 0$$

in the above diagram splits. So H_{n-1}^+ is injective and hence H_{n-1} is flat again by [12, Theorem 2.1]. Because L_{n-1} is n-SG-flat, the third column

$$0 \to \widetilde{M} \to H_{n-1} \to L_{n-1} \to 0$$

in the former diagram is still exact after applying the functor $I \otimes_R -$.

Next, we consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L_1 \arrow[r, equal] \arrow[d] & L_1 \arrow[d] & & \\
0 \arrow[r] & H_0 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{M} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & F \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

From the middle row in the above diagram we know that H_0 is flat. Then from the third row in the above diagram, we get an exact sequence

$$0 = \mathrm{Tor}_{i+1}^R(I, F) \to \mathrm{Tor}_i^R(I, \widetilde{M}) \to \mathrm{Tor}_i^R(I, M) = 0$$

for any $i \geq 1$. Thus $\mathrm{Tor}_i^R(I, \widetilde{M}) = 0$ for any $i \geq 1$, and therefore, $0 \to I \otimes_R L_1 \to I \otimes_R H_0 \to I \otimes_R \widetilde{M} \to 0$ is exact. So we obtain the following exact sequence

$$0 \to \widetilde{M} \to H_{n-1} \to F_{n-2} \to \cdots \to F_1 \to H_0 \to \widetilde{M} \to 0,$$

which is still exact after applying the functor $I \otimes_R -$, which implies that $\widetilde{M}$ is n-SG-flat. □

We call two modules $M, N \in \mathrm{Mod}\, R$ *flatly equivalent* if there exist flat modules F_1, F_2 in $\mathrm{Mod}\, R$, such that $M \oplus F_1 \cong N \oplus F_2$. By Theorems 4.5, we immediately have the following corollary.

Corollary 4.6. *Assume that $M, N \in \mathrm{Mod}\, R$ are flatly equivalent. Then, for any $n \geq 1$, M is n-SG-flat if and only if so is N.*

In the rest of this section, we will investigate the relation between n-SG-flat modules and n-SG-projective (resp., injective) modules. We first have the following result, which is a generalization of [3, Proposition 3.9].

Proposition 4.7. *For any $n \geq 1$, a finitely generated n-SG-projective R-module is finitely presented n-SG-flat.*

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Proof. Assume that M is a finitely generated n-SG-projective R-module. By Theorem 3.14, there exists an exact sequence

$$0 \to M \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

in $\operatorname{mod} R$ with P_i projective for any $0 \leq i \leq n-1$, and $\operatorname{Ext}_R^i(M, R) = 0$ for any $i \geq 1$. Let $I \in \operatorname{Mod} R^{op}$ be injective. By [5, Chapter VI, Proposition 5.3 "Remark"], we have an isomorphism

$$\operatorname{Tor}_i^R(I, M) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^i(M, R), I)$$

for any $i \geq 1$. Thus $\operatorname{Tor}_i^R(I, M) = 0$ for any $i \geq 1$, and therefore, M is finitely presented n-SG-flat. □

As an application of Proposition 4.7, we give another example of 2-SG-flat modules, but not 1-SG-flat.

Example 4.8. Consider a Noetherian local ring $R = k[[X, Y]]/(XY)$, where k is a field. Then the ideals $(X + (XY))$ and $(Y + (XY))$ of R are finitely generated 2-SG-projective R-modules by [6, Example 4.15], where $(X + (XY))$ and $(Y + (XY))$ are the residue classes in R of X and Y respectively. By Proposition 4.7, both $(X + (XY))$ and $(Y + (XY))$ are 2-SG-flat, but neither of them are 1-SG-flat by [3, Example 3.11].

The following result generalizes [18, Theorems 2.4 and 2.12].

Proposition 4.9.

(1) *If $M \in \operatorname{Mod} R$ is n-SG-flat, then $M^+ \in \operatorname{Mod} R^{op}$ is n-SG-injective.*
(2) *For an Artinian algebra R, if $M \in \operatorname{Mod} R$ is n-SG-injective, then $M^+ \in \operatorname{Mod} R^{op}$ is n-SG-flat.*

Proof. (1) Assume that $M \in \operatorname{Mod} R$ is n-SG-flat. Then there exists an exact sequence

$$0 \to M \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$$

in $\operatorname{Mod} R$ with F_i flat for any $0 \leq i \leq n-1$, and $\operatorname{Tor}_i^R(E, M) = 0$ for any injective right R-module E and $i \geq 1$. So we get the following exact sequence

$$0 \to M^+ \to F_0^+ \to \cdots \to F_{n-2}^+ \to F_{n-1}^+ \to M^+ \to 0$$

in $\operatorname{Mod} R^{op}$ with F_i^+ injective by [12, Theorem 2.1]. Because $\operatorname{Ext}_R^i(E, M^+) \cong \operatorname{Tor}_i^R(E, M)^+ = 0$ for any $i \geq 1$ by [5, Chapter VI, Proposition 5.1], M^+ is n-SG-injective.

(2) Assume that $M \in \operatorname{Mod} R$ is n-SG-injective. Then there exists an exact sequence

$$0 \to M \to I^{n-1} \to I^{n-2} \to \cdots \to I^0 \to M \to 0$$

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in $\operatorname{Mod} R$ with I^i injective for any $0 \leq i \leq n-1$. So we get the following exact sequence:

$$0 \to M^+ \to (I^0)^+ \to \cdots \to (I^{n-2})^+ \to (I^{n-1})^+ \to M^+ \to 0$$

in $\operatorname{Mod} R^{op}$ with $(I^i)^+$ flat for any $0 \leq i \leq n-1$.

Let $E \in \operatorname{Mod} R$ be any injective module. Then $E = \bigoplus_{\gamma \in \Gamma} E_\gamma$ with each $E_\gamma \in \operatorname{mod} R$ injective. By [5, Chapter VI, Proposition 5.3 "Remark"], we have the following isomorphism

$$\operatorname{Tor}_i^R(M^+, E) \cong \operatorname{Tor}_i^R\left(M^+, \bigoplus_{\gamma \in \Gamma} E_\gamma\right) \cong \bigoplus_{\gamma \in \Gamma} \operatorname{Tor}_i^R(M^+, E_\gamma) \cong \bigoplus_{\gamma \in \Gamma} (\operatorname{Ext}_R^i(E_\gamma, M))^+ = 0$$

for any $i \geq 1$, which implies that M^+ is n-SG-flat. □

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