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Research Article

Zhaoyong Huang*

Homological dimensions relative to preresolving subcategories II

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Abstract: Let $\mathscr{A}$ be an abelian category having enough projective and injective objects, and let $\mathscr{T}$ be an additive subcategory of $\mathscr{A}$ closed under direct summands. A known assertion is that in a short exact sequence in $\mathscr{A}$, the $\mathscr{T}$-projective (resp. $\mathscr{T}$-injective) dimensions of any two terms can sometimes induce an upper bound of that of the third term by using the same comparison expressions. We show that if $\mathscr{T}$ contains all projective (resp. injective) objects of $\mathscr{A}$, then the above assertion holds true if and only if $\mathscr{T}$ is resolving (resp. coresolving). As applications, we get that a left and right Noetherian ring R is n-Gorenstein if and only if the Gorenstein projective (resp. injective, flat) dimension of any left R-module is at most n. In addition, in several cases, for a subcategory $\mathscr{C}$ of $\mathscr{T}$, we show that the finitistic $\mathscr{C}$-projective and $\mathscr{T}$-projective dimensions of $\mathscr{A}$ are identical.

Keywords: Relative projective dimension, relative injective dimension, finitistic dimension, Gorenstein rings, Gorenstein projective dimension, Gorenstein injective dimension, Gorenstein flat dimension

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1 Introduction

Homological dimensions are fundamental invariants in homological theory, which play a crucial role in studying the structures of modules and rings. Let R be an arbitrary ring, let $\operatorname{Mod} R$ be the category of left R-modules, and let $\mathscr{T}$ be a subcategory of $\operatorname{Mod} R$. For a module $A \in \operatorname{Mod} R$, we use $\mathscr{T}\text{-pd}\, A$ to denote the $\mathscr{T}$-projective dimension of A. Let

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

be an exact sequence in $\operatorname{Mod} R$. Consider the following assertions:
(1) $\mathscr{T}\text{-pd}\, A_2 \leq \max\{\mathscr{T}\text{-pd}\, A_1, \mathscr{T}\text{-pd}\, A_3\}$ with equality if $\mathscr{T}\text{-pd}\, A_1 + 1 \neq \mathscr{T}\text{-pd}\, A_3$.
(2) $\mathscr{T}\text{-pd}\, A_1 \leq \max\{\mathscr{T}\text{-pd}\, A_2, \mathscr{T}\text{-pd}\, A_3 - 1\}$ with equality if $\mathscr{T}\text{-pd}\, A_2 \neq \mathscr{T}\text{-pd}\, A_3$.
(3) $\mathscr{T}\text{-pd}\, A_3 \leq \max\{\mathscr{T}\text{-pd}\, A_1 + 1, \mathscr{T}\text{-pd}\, A_2\}$ with equality if $\mathscr{T}\text{-pd}\, A_1 \neq \mathscr{T}\text{-pd}\, A_2$.
It has been known that these assertions hold true if $\mathscr{T}$ is the subcategory of $\operatorname{Mod} R$ consisting of one kind of the following modules (among others):
- Projective modules.
- Flat modules.
- Gorenstein projective modules [9, Lemma 2.4].
- C-Gorenstein projective modules with C a semidualizing bimodule [27, Lemma 3.2].
- Gorenstein flat modules (see [7, Theorem 2.11] and [31, Theorem 4.11]).
- Auslander classes [25, Corollary 4.5].

***Corresponding author: Zhaoyong Huang,** Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P. R. China, e-mail: huangzy@nju.edu.cn

It is natural to ask the following question: what properties should a subcategory of Mod R have in order for properties (1)–(3) to hold? One of the aims in this paper is to study this question. In fact, we will show that if $\mathscr{T}$ is an additive subcategory of Mod R which is closed under direct summands and contains all projective left R-modules, then the above assertions hold true if and only if $\mathscr{T}$ is resolving.

On the other hand, Auslander and Bridger proved that a commutative Noetherian local ring R is Gorenstein if and only if any finitely generated R-module has finite Gorenstein dimension (or Gorenstein projective dimension, in more popular terminology) [3, Theorem 4.20]. Then Hoshino developed Auslander and Bridger's arguments to prove that an artin algebra R is Gorenstein if and only if any finitely generated left R-module has finite Gorenstein dimension [21, Theorem]. Furthermore, Huang and Huang generalized it to left and right Noetherian rings [22, Theorem 1.4]. By applying the results obtained by studying the question mentioned above, our other aim is to generalize this result to arbitrary modules over left and right Noetherian rings. Note that for a left and right Noetherian ring R, if R is n-Gorenstein (that is, the left and right self-injective dimensions of R are at most n), then the Gorenstein projective dimension of any left R-module is at most n [13, Theorem 11.5.1]. However, the converse seems to be far from clear.

The paper is organized as follows. In Section 2, we give some notions and notations that will be used in the sequel. Let $\mathscr{A}$ be an abelian category having enough projective objects. In Section 3, we first prove the following result.

Theorem 1.1 (Theorem 3.2). *Let $\mathscr{T}$ be an additive subcategory of $\mathscr{A}$ which is closed under direct summands and contains all projective objects of $\mathscr{A}$. Then the following statements are equivalent:*
(1) *$\mathscr{T}$ is resolving.*
(2) *For any exact sequence*
$$0 \to A_1 \to A_2 \to A_3 \to 0$$
in $\mathscr{A}$, we have
(2.1) $\mathscr{T}\text{-pd}\,A_2 \leq \max\{\mathscr{T}\text{-pd}\,A_1, \mathscr{T}\text{-pd}\,A_3\}$ *with equality if* $\mathscr{T}\text{-pd}\,A_1 + 1 \neq \mathscr{T}\text{-pd}\,A_3$*;*
(2.2) $\mathscr{T}\text{-pd}\,A_1 \leq \max\{\mathscr{T}\text{-pd}\,A_2, \mathscr{T}\text{-pd}\,A_3 - 1\}$ *with equality if* $\mathscr{T}\text{-pd}\,A_2 \neq \mathscr{T}\text{-pd}\,A_3$*;*
(2.3) $\mathscr{T}\text{-pd}\,A_3 \leq \max\{\mathscr{T}\text{-pd}\,A_1 + 1, \mathscr{T}\text{-pd}\,A_2\}$ *with equality if* $\mathscr{T}\text{-pd}\,A_1 \neq \mathscr{T}\text{-pd}\,A_2$.

Then we apply it to prove that if $\mathscr{T}$ is a resolving subcategory of $\mathscr{A}$ which is closed under direct summands and admits an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$ with $\mathscr{E}$ a subcategory of $\mathscr{A}$, then the finitistic $\mathscr{T}$-projective dimension of $\mathscr{A}$ is at most its finitistic $\mathscr{C}$-projective dimension, and with equality when $\operatorname{Ext}^{\geq 1}_{\mathscr{A}}(T, C) = 0$ for any $T \in \mathscr{T}$ and $C \in \mathscr{C}$ (Corollary 3.5). We also list the duals of these results without proofs (Theorem 3.9 and Corollary 3.12).

In Section 4, we first present a partial list of examples of how the results obtained in Section 3 can be applied (Remark 4.4). Then it is shown that Corollaries 3.5 and 3.12 can be applied in many cases for module categories (Corollaries 4.5–4.7). Some known results are obtained as corollaries. The main result in this section is the following theorem.

Theorem 1.2 (Theorems 4.9, 4.11 and 4.13). *Let R be a left and right Noetherian ring and let $n \geq 0$. Then the following statements are equivalent:*
(1) *R is n-Gorenstein.*
(2) *The Gorenstein projective dimension of any left R-module is at most n.*
(3) *The Gorenstein injective dimension of any left R-module is at most n.*
(4) *The Gorenstein flat dimension of any left R-module is at most n.*
(5) *The strongly Gorenstein flat dimension of any left R-module is at most n.*
(6) *The projectively coresolved Gorenstein flat dimension of any left R-module is at most n.*
(i)$^{\mathrm{op}}$ *Opposite side version of* (i)*, with $2 \leq i \leq 6$.*

The Gorenstein symmetric conjecture states that for any artin algebra R, the left self-injective dimension of R is finite implies that so is its right self-injective dimension (see [4, p. 410]). By Theorem 1.2, we have that the Gorenstein symmetric conjecture holds true is equivalent to that for any artin algebra R, the left self-injective dimension of R being at most n implies that any of (2)–(6) (resp. (2)$^{\mathrm{op}}$–(6)$^{\mathrm{op}}$) is satisfied.

Let R, S be arbitrary rings, let ${}_RC_S$ be a semidualizing bimodule and let $M \in \operatorname{Mod} R$. We show that M is C-flat if and only if its character module is C-injective, and that M is C-Gorenstein flat implies that its character module is C-Gorenstein injective (Theorem 4.17), which are the C-versions of [8, Theorem 2.2] and [18, Theorem 3.6], respectively. As a consequence, we get that the C-Gorenstein flat dimension of M is at most its C-flat dimension with equality if the C-flat dimension of M is finite; moreover, the finitistic flat and Gorenstein flat dimensions of R are identical (Theorem 4.19). It extends [16, Theorem 2.1] and [18, Theorem 3.24].

2 Preliminaries

Throughout this paper, $\mathscr{A}$ is an abelian category and all subcategories of $\mathscr{A}$ involved are full, additive and closed under isomorphisms and direct summands. We use $\mathcal{P}(\mathscr{A})$ (resp. $\mathcal{I}(\mathscr{A})$) to denote the subcategory of $\mathscr{A}$ consisting of projective (resp. injective) objects.

Let $\mathscr{X}$ be a subcategory of $\mathscr{A}$. We write

$$
\begin{aligned}
{}^{\perp}\mathscr{X} &:= \{A \in \mathscr{A} \mid \mathrm{Ext}^{\geq 1}_{\mathscr{A}}(A, X) = 0 \text{ for any } X \in \mathscr{X}\},\\
\mathscr{X}^{\perp} &:= \{A \in \mathscr{A} \mid \mathrm{Ext}^{\geq 1}_{\mathscr{A}}(X, A) = 0 \text{ for any } X \in \mathscr{X}\}.
\end{aligned}
$$

Let $M \in \mathscr{A}$. The *$\mathscr{X}$-projective dimension* $\mathscr{X}\text{-pd}\, M$ of M is defined by

$$\inf\{n \mid \text{there exists an exact sequence } 0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0 \text{ in } \mathscr{A} \text{ with all } X_i \in \mathscr{X}\},$$

and set $\mathscr{X}\text{-pd}\, M = \infty$ if no such integer exists. Dually, the *$\mathscr{X}$-injective dimension* $\mathscr{X}\text{-id}\, M$ of M is defined by

$$\inf\{n \mid \text{there exists an exact sequence } 0 \to M \to X^0 \to X^1 \to \cdots \to X^n \to 0 \text{ in } \mathscr{A} \text{ with all } X^i \in \mathscr{X}\},$$

and set $\mathscr{X}\text{-id}\, M = \infty$ if no such integer exists. We use $\mathscr{X}\text{-pd}^{<\infty}$ (resp. $\mathscr{X}\text{-id}^{<\infty}$) to denote the subcategory of $\mathscr{A}$ consisting of objects with finite $\mathscr{X}$-projective (resp. $\mathscr{X}$-injective) dimension. We write

$$
\begin{aligned}
\mathscr{X}\text{-FPD} &:= \sup\{\mathscr{X}\text{-pd}\, M \mid M \in \mathscr{X}\text{-pd}^{<\infty}\},\\
\mathscr{X}\text{-FID} &:= \sup\{\mathscr{X}\text{-id}\, M \mid M \in \mathscr{X}\text{-id}^{<\infty}\}.
\end{aligned}
$$

Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. Recall from [13] that a sequence

$$\mathbb{S} : \cdots \to S_1 \to S_2 \to S_3 \to \cdots$$

in $\mathscr{A}$ is called $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)$*-exact* (resp. $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{E})$*-exact*) if $\mathrm{Hom}_{\mathscr{A}}(E, \mathbb{S})$ (resp. $\mathrm{Hom}_{\mathscr{A}}(\mathbb{S}, E)$) is exact for any $E \in \mathscr{E}$. Let $\mathscr{C} \subseteq \mathscr{T}$ be subcategories of $\mathscr{A}$. Recall from [24] that $\mathscr{C}$ is called an *$\mathscr{E}$-proper generator* (resp. *$\mathscr{E}$-coproper cogenerator*) for $\mathscr{T}$ if for any $T \in \mathscr{T}$ there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)$ (resp. $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{E})$)-exact exact sequence

$$0 \to T' \to C \to T \to 0 \quad (\text{resp. } 0 \to T \to C \to T' \to 0)$$

in $\mathscr{A}$ with $C \in \mathscr{C}$ and $T' \in \mathscr{T}$. When $\mathscr{E} = \mathcal{P}(\mathscr{A})$ (resp. $\mathcal{I}(\mathscr{A})$), an $\mathscr{E}$-proper generator (resp. $\mathscr{E}$-coproper cogenerator) is exactly a usual generator (resp. cogenerator).

We define

$$
\begin{aligned}
\widetilde{\mathrm{res}_{\mathscr{E}}\mathscr{C}} := \{M \in \mathscr{A} \mid {} & \text{there exists a } \mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)\text{-exact exact sequence}\\
& \cdots \to C_i \to \cdots \to C_1 \to C_0 \to M \to 0 \text{ in } \mathscr{A} \text{ with all } C_i \in \mathscr{C}\}.
\end{aligned}
$$

Dually, we define

$$
\begin{aligned}
\widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}} := \{M \in \mathscr{A} \mid {} & \text{there exists a } \mathrm{Hom}_{\mathscr{A}}(-, \mathscr{E})\text{-exact exact sequence}\\
& 0 \to M \to C^0 \to C^1 \to \cdots \to C^i \to \cdots \text{ in } \mathscr{A} \text{ with all } C^i \text{ in } \mathscr{C}\}.
\end{aligned}
$$

Definition 2.1 ([24]). Let $\mathscr{E}$ and $\mathscr{T}$ be subcategories of $\mathscr{A}$.

(1) The subcategory $\mathscr{T}$ is called *$\mathscr{E}$-preresolving* in $\mathscr{A}$ if the following conditions are satisfied:

(1.1) $\mathscr{T}$ admits an $\mathscr{E}$-proper generator.

(1.2) $\mathscr{T}$ is *closed under $\mathscr{E}$-proper extensions*, that is, for any $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, -)$-exact exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\mathscr{A}$, if both A_1 and A_3 are in $\mathscr{T}$, then A_2 is also in $\mathscr{T}$.

(2) The subcategory $\mathscr{T}$ is called *$\mathscr{E}$-precoresolving* in $\mathscr{A}$ if the following conditions are satisfied:
(2.1) $\mathscr{T}$ admits an $\mathscr{E}$-coproper cogenerator.
(2.2) $\mathscr{T}$ is *closed under $\mathscr{E}$-coproper extensions*, that is, for any $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\mathscr{A}$, if both A_1 and A_3 are in $\mathscr{T}$, then A_2 is also in $\mathscr{T}$.

The following definition is cited from [14].

Definition 2.2. Let $\mathscr{U}$, $\mathscr{V}$ be subcategories of $\mathscr{A}$.
(1) The pair $(\mathscr{U},\mathscr{V})$ is called a *cotorsion pair* in $\mathscr{A}$ if

$$\mathscr{U} = \{A \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(A, V) = 0 \text{ for any } V \in \mathscr{V}\}$$

and

$$\mathscr{V} = \{A \in \mathscr{A} \mid \mathrm{Ext}^1_{\mathscr{A}}(U, A) = 0 \text{ for any } U \in \mathscr{U}\}.$$

(2) A cotorsion pair $(\mathscr{U},\mathscr{V})$ is called *hereditary* if one of the following equivalent conditions is satisfied:
(2.1) $\mathrm{Ext}^{\geq 1}_{\mathscr{A}}(U, V) = 0$ for any $U \in \mathscr{U}$ and $V \in \mathscr{V}$.
(2.2) $\mathscr{U}$ is resolving in the sense that $\mathcal{P}(\mathscr{A}) \subseteq \mathscr{U}$ and $\mathscr{U}$ is closed under extensions and kernels of epimorphisms.
(2.3) $\mathscr{V}$ is coresolving in the sense that $\mathcal{I}(\mathscr{A}) \subseteq \mathscr{V}$ and $\mathscr{V}$ is closed under extensions and cokernels of monomorphisms.

3 General results

3.1 Projective dimension relative to resolving subcategories

We begin with the following observation.

Lemma 3.1. *Let $M \in \mathscr{A}$ and $n \geq 0$.*
(1) *Assume that $\mathscr{A}$ has enough projective objects. If $\mathscr{T}$ is a resolving subcategory of $\mathscr{A}$, then the following statements are equivalent:*
(1.1) $\mathscr{T}$-pd $M \leq n$.
(1.2) *There exists an exact sequence*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\mathscr{A}$ with all P_i projective and $K_n \in \mathscr{T}$.
(1.3) *For any exact sequence*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\mathscr{A}$, if all P_i are projective, then $K_n \in \mathscr{T}$.
(1.4) *For any exact sequence*

$$0 \to K_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to M \to 0$$

in $\mathscr{A}$, if all T_i are in $\mathscr{T}$, then $K_n \in \mathscr{T}$.
(2) *Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. If $\mathscr{T}$ is an $\mathscr{E}$-precoresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$, then the following statements are equivalent:*
(2.1) $\mathscr{T}$-pd $M \leq n$.
(2.2) *There exists an exact sequence*

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to T_0 \to M \to 0$$

in $\mathscr{A}$ with all C_i in $\mathscr{C}$ and $T_0 \in \mathscr{T}$; that is, there exists an exact sequence

$$0 \to K \to T \to M \to 0$$

in $\mathscr{A}$ with $T \in \mathscr{T}$ and $\mathscr{C}$-pd $K \leq n-1$.

Proof. (1) The implications (1.4) $\Longrightarrow$ (1.3) $\Longrightarrow$ (1.2) are trivial. By [24, Theorem 3.6], we have that (1.1) is equivalent to (1.2). By [3, Lemma 3.12], we have that (1.1) implies (1.4).

(2) It follows from [24, Theorem 4.7]. □

The main result in this subsection is as follows.

Theorem 3.2. *Assume that $\mathscr{A}$ has enough projective objects and $\mathscr{T}$ is a subcategory of $\mathscr{A}$ containing $\mathcal{P}(\mathscr{A})$. Then the following statements are equivalent:*
(1) *$\mathscr{T}$ is resolving.*
(2) *For any exact sequence*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\mathscr{A}$, we have
(2.1) (a) $\mathscr{T}\text{-pd}\, A_2 \leq \max\{\mathscr{T}\text{-pd}\, A_1, \mathscr{T}\text{-pd}\, A_3\}$, (b) *the equality holds if* $\mathscr{T}\text{-pd}\, A_1 + 1 \neq \mathscr{T}\text{-pd}\, A_3$;
(2.2) (a) $\mathscr{T}\text{-pd}\, A_1 \leq \max\{\mathscr{T}\text{-pd}\, A_2, \mathscr{T}\text{-pd}\, A_3 - 1\}$, (b) *the equality holds if* $\mathscr{T}\text{-pd}\, A_2 \neq \mathscr{T}\text{-pd}\, A_3$;
(2.3) (a) $\mathscr{T}\text{-pd}\, A_3 \leq \max\{\mathscr{T}\text{-pd}\, A_1 + 1, \mathscr{T}\text{-pd}\, A_2\}$, (b) *the equality holds if* $\mathscr{T}\text{-pd}\, A_1 \neq \mathscr{T}\text{-pd}\, A_2$.

Proof. "(2) $\Longrightarrow$ (1)": By (2.1) (a) and (2.2) (a), we have that $\mathscr{T}$ is closed under extensions and kernels of epimorphisms, respectively, and so $\mathscr{T}$ is resolving.

"(1) $\Longrightarrow$ (2)": (2.1) (a) If $\max\{\mathscr{T}\text{-pd}\, A_1, \mathscr{T}\text{-pd}\, A_3\} = 0$, that is, both A_1 and A_3 are in $\mathscr{T}$, then A_2 is also in $\mathscr{T}$ by (1), and the assertion follows. Now suppose $\max\{\mathscr{T}\text{-pd}\, A_1, \mathscr{T}\text{-pd}\, A_3\} = n \geq 1$. By Lemma 3.1 (1), we have the following two exact sequences:

$$0 \to K_n' \to P_{n-1}' \to \cdots \to P_1' \to P_0' \to A_1 \to 0,$$
$$0 \to K_n'' \to P_{n-1}'' \to \cdots \to P_1'' \to P_0'' \to A_3 \to 0$$

in $\mathscr{A}$ with all P_i', P_i'' projective and $K_n', K_n'' \in \mathscr{T}$. Then, by the horseshoe lemma, we get the following two exact sequences:

$$0 \to K_n \to P_{n-1}' \oplus P_{n-1}'' \to \cdots \to P_1' \oplus P_1'' \to P_0' \oplus P_0'' \to A_2 \to 0, \tag{3.1}$$
$$0 \to K_n' \to K_n \to K_n'' \to 0. \tag{3.2}$$

By the exact sequence (3.2) and by (1), we have $K_n \in \mathscr{T}$. Then the exact sequence (3.1) implies $\mathscr{T}\text{-pd}\, A_2 \leq n$.

(2.2) (a) Let $\mathscr{T}\text{-pd}\, A_2 = n_2$ and $\mathscr{T}\text{-pd}\, A_3 = n_3$ with $n_2, n_3 < \infty$.

We first suppose $n_3 = 0$ (that is, $A_3 \in \mathscr{T}$). If $n_2 = 0$ (that is, $A_2 \in \mathscr{T}$), then $A_1 \in \mathscr{T}$ by (1). If $n_2 \geq 1$, then by Lemma 3.1 (1), there exists an exact sequence

$$0 \to A_2' \to P \to A_2 \to 0$$

in $\mathscr{A}$ with P projective and $\mathscr{T}\text{-pd}\, A_2' \leq n_2 - 1$. Consider the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A_2' \arrow[r, equal] \arrow[d, dashed] & A_2' \arrow[d] & & \\
0 \arrow[r, dashed] & T \arrow[r, dashed] \arrow[d, dashed] & P \arrow[r, dashed] \arrow[d] & A_3 \arrow[r, dashed] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, dashed] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

By (1) and the middle row in the above diagram, we have $T \in \mathscr{T}$. Then the leftmost column in that diagram implies $\mathscr{T}\text{-pd}\,A_1 \leq n_2$.

Now suppose $n_3 \geq 1$. Then, by Lemma 3.1 (1), there exists an exact sequence

$$0 \to A_3' \to Q \to A_3 \to 0$$

in $\mathscr{A}$ with Q projective and $\mathscr{T}\text{-pd}\,A_3' \leq n_3 - 1$. Consider the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A_3' \arrow[r, equal] \arrow[d] & A_3' \arrow[d] & \\
0 \arrow[r, dashed] & A_1 \arrow[r, dashed] \arrow[d, equal] & A_1 \oplus Q \arrow[r, dashed] \arrow[d] & Q \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

By (2.1) (a) and the middle column in the above diagram, we have $\mathscr{T}\text{-pd}(A_1 \oplus Q) \leq \max\{n_2, n_3 - 1\}$. It follows from [24, Corollary 3.9] that $\mathscr{T}\text{-pd}\,A_1 \leq \max\{n_2, n_3 - 1\}$.

(2.3) (a) Let $\mathscr{T}\text{-pd}\,A_1 = n_1$ and $\mathscr{T}\text{-pd}\,A_2 = n_2$ with $n_1, n_2 < \infty$. If $n_2 = 0$, that is, $A_2 \in \mathscr{T}$, then

$$\mathscr{T}\text{-pd}\,A_3 = n_1 + 1.$$

Now suppose $n_2 \geq 1$. Then, by Lemma 3.1 (1), there exists an exact sequence

$$0 \to A_2' \to P \to A_2 \to 0$$

in $\mathscr{A}$ with P projective and $\mathscr{T}\text{-pd}\,A_2' \leq n_2 - 1$. Consider the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A_2' \arrow[r, equal] \arrow[d] & A_2' \arrow[d] & & \\
0 \arrow[r, dashed] & K \arrow[r, dashed] \arrow[d] & P \arrow[r, dashed] \arrow[d] & A_3 \arrow[r, dashed] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

By (1) and the leftmost column in the above diagram, we have

$$\mathscr{T}\text{-pd}\,K \leq \max\{\mathscr{T}\text{-pd}\,A_1, \mathscr{T}\text{-pd}\,A_2'\} \leq \max\{n_1, n_2 - 1\}.$$

Then the middle row in that diagram implies

$$\mathscr{T}\text{-pd}\,A_3 \leq \mathscr{T}\text{-pd}\,K + 1 \leq \max\{n_1, n_2 - 1\} + 1 = \max\{n_1 + 1, n_2\}.$$

(2.1) (b) If $\mathscr{T}\text{-pd}\,A_1 + 1 < \mathscr{T}\text{-pd}\,A_3$, then we have $\mathscr{T}\text{-pd}\,A_2 \leq \mathscr{T}\text{-pd}\,A_3$ and $\mathscr{T}\text{-pd}\,A_3 \leq \mathscr{T}\text{-pd}\,A_2$ by (2.1) (a) and (2.3) (a), respectively. Thus, $\mathscr{T}\text{-pd}\,A_2 = \mathscr{T}\text{-pd}\,A_3$.

If $\mathscr{T}\text{-pd}\,A_3 < \mathscr{T}\text{-pd}\,A_1 + 1$, then we have $\mathscr{T}\text{-pd}\,A_2 \leq \mathscr{T}\text{-pd}\,A_1$ and $\mathscr{T}\text{-pd}\,A_1 \leq \mathscr{T}\text{-pd}\,A_2$ by (2.1) (a) and (2.2) (a), respectively. Thus, $\mathscr{T}\text{-pd}\,A_2 = \mathscr{T}\text{-pd}\,A_1$.

(2.2) (b) If $\mathscr{T}\text{-pd}\,A_2 < \mathscr{T}\text{-pd}\,A_3$, then we have $\mathscr{T}\text{-pd}\,A_1 \leq \mathscr{T}\text{-pd}\,A_3 - 1$ and $\mathscr{T}\text{-pd}\,A_3 \leq \mathscr{T}\text{-pd}\,A_1 + 1$ by (2.2) (a) and (2.3) (a), respectively. Thus, $\mathscr{T}\text{-pd}\,A_1 = \mathscr{T}\text{-pd}\,A_3 - 1$.

If $\mathscr{T}\text{-pd}\,A_3 < \mathscr{T}\text{-pd}\,A_2$, then we have $\mathscr{T}\text{-pd}\,A_1 \leq \mathscr{T}\text{-pd}\,A_2$ and $\mathscr{T}\text{-pd}\,A_2 \leq \mathscr{T}\text{-pd}\,A_1$ by (2.2) (a) and (2.1) (a), respectively. Thus, $\mathscr{T}\text{-pd}\,A_1 = \mathscr{T}\text{-pd}\,A_2$.

(2.3) (b) If $\mathscr{T}\text{-pd}\,A_1 < \mathscr{T}\text{-pd}\,A_2$, then we have $\mathscr{T}\text{-pd}\,A_3 \leq \mathscr{T}\text{-pd}\,A_2$ and $\mathscr{T}\text{-pd}\,A_2 \leq \mathscr{T}\text{-pd}\,A_3$ by (2.3) (a) and (2.1) (a), respectively. Thus, $\mathscr{T}\text{-pd}\,A_3 = \mathscr{T}\text{-pd}\,A_2$.

If $\mathscr{T}\text{-pd}\,A_2 < \mathscr{T}\text{-pd}\,A_1$, then we have $\mathscr{T}\text{-pd}\,A_3 \leq \mathscr{T}\text{-pd}\,A_1 + 1$ and $\mathscr{T}\text{-pd}\,A_1 \leq \mathscr{T}\text{-pd}\,A_3 - 1$ by (2.3) (a) and (2.2) (a), respectively. Thus, $\mathscr{T}\text{-pd}\,A_3 = \mathscr{T}\text{-pd}\,A_1 + 1$. □

As an immediate consequence, we get the following result.

Corollary 3.3. *Assume that $\mathscr{A}$ has enough projective objects and $\mathscr{T}$ is a resolving subcategory of $\mathscr{A}$. Then $\mathscr{T}\text{-pd}^{<\infty}$ satisfies the two-out-of-three property; that is, in a short exact sequence in $\mathscr{A}$, if any two terms are in $\mathscr{T}\text{-pd}^{<\infty}$, then so is the third term.*

The following result shows that if the resolving subcategory $\mathscr{T}$ of $\mathscr{A}$ admits an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$, then any object in $\mathscr{A}$ with finite $\mathscr{T}$-projective dimension is isomorphic to a kernel (resp. a cokernel) of a morphism from an object in $\mathscr{A}$ with finite $\mathscr{C}$-projective dimension to an object in $\mathscr{T}$.

Corollary 3.4. *Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. If $\mathscr{T}$ is an $\mathscr{E}$-precoresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$, then, for any $M \in \mathscr{A}$ with $\mathscr{T}\text{-pd}\,M = n < \infty$, the following assertions hold:*

(1) *There exists an exact sequence*
$$0 \to K \to T \to K' \to T' \to 0$$
in $\mathscr{A}$ with $\mathscr{C}\text{-pd}\,K \leq n-1$, $\mathscr{C}\text{-pd}\,K' \leq n$ and $T, T' \in \mathscr{T}$ such that $M \cong \operatorname{Im}(T \to K')$.

(2) *If $\mathscr{A}$ has enough projective objects and $\mathscr{T}$ is resolving in $\mathscr{A}$, then the two "$\leq$" in (1) are "$=$".*

Proof. (1) Let $M \in \mathscr{A}$ with $\mathscr{T}\text{-pd}\,M = n < \infty$. The case for $n = 0$ is trivial. Now suppose $n \geq 1$. By Lemma 3.1 (2), there exists an exact sequence
$$0 \to K \to T \to M \to 0 \tag{3.3}$$
in $\mathscr{A}$ with $\mathscr{C}\text{-pd}\,K \leq n-1$ and $T \in \mathscr{T}$. Thus, there exists an exact sequence
$$0 \to T \to C \to T' \to 0$$
in $\mathscr{A}$ with $C \in \mathscr{C}$ and $T' \in \mathscr{T}$. Consider the following push-out diagram:
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r, dashed] & K \arrow[r, dashed] & C \arrow[r, dashed] \arrow[d] & K' \arrow[r, dashed] \arrow[d, dashed] & 0 \\
 & & T' \arrow[r, equal] \arrow[d] & T' \arrow[d, dashed] & \\
 & & 0 & 0. &
\end{tikzcd}$$
By the middle row in the above diagram, we have $\mathscr{C}\text{-pd}\,K' \leq n$. Now splicing (3.3) and the rightmost column
$$0 \to M \to K' \to T' \to 0, \tag{3.4}$$
we get the desired exact sequence.

(2) Assume that $\mathscr{A}$ has enough projective objects and $\mathscr{T}$ is resolving in $\mathscr{A}$. Then, by (3.3) and Theorem 3.2 (2.2), we have $\mathscr{T}\text{-pd}\,K = n-1$. Since $\mathscr{C}\text{-pd}\,K \geq \mathscr{T}\text{-pd}\,K$, we have $\mathscr{C}\text{-pd}\,K = n-1$. By (3.4) and Theorem 3.2 (2.1), we have $\mathscr{T}\text{-pd}\,K' = n$, and so $\mathscr{C}\text{-pd}\,K' = n$. □

Furthermore, we get the following result.

Corollary 3.5. *Assume that $\mathscr{A}$ has enough projective objects and $\mathscr{T}$ is a resolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$. Then the following assertions hold:*
(1) $\mathscr{T}$-FPD $\leq \mathscr{C}$-FPD.
(2) *If $\mathscr{T} \subseteq {}^{\perp}\mathscr{C}$, then $\mathscr{T}$-pd $M = \mathscr{C}$-pd M for any $M \in \mathscr{A}$ with $\mathscr{C}$-pd $M < \infty$.*
(3) *If $\mathscr{T} \subseteq {}^{\perp}\mathscr{C}$, then $\mathscr{T}$-FPD $= \mathscr{C}$-FPD.*

Proof. (1) Let $M \in \mathscr{A}$ with $\mathscr{T}$-pd $M = n < \infty$. By Corollary 3.4, there exists $K' \in \mathscr{A}$ such that $\mathscr{C}$-pd $K' = n$. It follows that $\mathscr{T}$-FPD $\leq \mathscr{C}$-FPD.

(2) Let $M \in \mathscr{A}$ with $\mathscr{C}$-pd $M = n < \infty$. Then $\mathscr{T}$-pd $M = m \leq n$. By Corollary 3.4, there exists an exact sequence

$$0 \to M \to K' \to T' \to 0$$

in $\mathscr{A}$ with $\mathscr{C}$-pd $K' = m$ and $T' \in \mathscr{T}$. Since $\mathscr{T} \subseteq {}^{\perp}\mathscr{C}$ by assumption, we have $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(T', M) = 0$ by dimension shifting. So, the above exact sequence splits and M is a direct summand of K'. So, $n = \mathscr{C}$-pd $M \leq m$ by [24, Corollary 3.9], and hence $m = n$ and $\mathscr{T}$-pd $M = n$.

(3) By (2), we have $\mathscr{C}$-FPD $\leq \mathscr{T}$-FPD. So, the assertion follows from (1). □

In the next section, we need the following two propositions.

Proposition 3.6. *Let $\mathscr{E}$ and $\mathscr{C}$ be subcategories of $\mathscr{A}$. If ${}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}}$ is closed under ($\mathscr{E}$-coproper) extensions, then it is closed under kernels of epimorphisms. In particular, if $\mathrm{cores}\,\mathscr{C} := \widetilde{\mathrm{cores}_{\mathcal{I}(\mathscr{A})}\mathscr{C}}$ is closed under extensions, then it is closed under kernels of epimorphisms.*

Proof. Let

$$0 \to A \to T_1 \to T_2 \to 0$$

be an exact sequence in $\mathscr{A}$ with $T_1, T_2 \in {}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}}$. Then there exists a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{E})$-exact exact sequence

$$0 \to T_1 \to C \to T_1' \to 0$$

in $\mathscr{A}$ with $C \in \mathscr{C}$ and $T_1' \in {}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}}$. Consider the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & T_1 \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r, dashed] & A \arrow[r, dashed] & C \arrow[r, dashed] \arrow[d] & T \arrow[r, dashed] \arrow[d, dashed] & 0 \\
 & & T_1' \arrow[r, equal] \arrow[d] & T_1' \arrow[d, dashed] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By [23, Lemma 2.4 (2)], all columns and rows in this diagram are $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{E})$-exact exact sequences. If ${}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}}$ is closed under $\mathscr{E}$-coproper extensions, then the rightmost column implies $T \in {}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}}$, and thus the middle row yields $A \in {}^{\perp}\mathscr{E} \cap \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}}$.

The latter assertion follows from the former one by putting $\mathscr{E} = \mathcal{I}(\mathscr{A})$. □

Proposition 3.7. *Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. If $\mathscr{T}$ is an $\mathscr{E}$-precoresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$, then $\widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}} = \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{T}}$.*

Proof. It is trivial that $\widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{C}} \subseteq \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{T}}$. Now let $M \in \widetilde{\mathrm{cores}_{\mathscr{E}}\mathscr{T}}$ and let

$$0 \to M \to T \to M' \to 0$$

be a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequence in $\mathscr{A}$ with $T \in \mathscr{T}$ and $M^{'} \in \widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{T}$. Since $\mathscr{T}$ admits an $\mathscr{E}$-coproper cogenerator $\mathscr{C}$ by assumption, there exists a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequence

$$0 \to T \to C^0 \to T^{'} \to 0$$

in $\mathscr{A}$ with $C^0 \in \mathscr{C}$ and in $T^{'} \in \mathscr{T}$. Then we have the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & M^{'} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r, dashed] & M \arrow[r, dashed] & C^0 \arrow[r, dashed] \arrow[d] & M^1 \arrow[r, dashed] \arrow[d, dashed] & 0 \\
 & & T^{'} \arrow[r, equal] \arrow[d] & T^{'} \arrow[d, dashed] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Since there also exists a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequence

$$0 \to M^{'} \to T^{''} \to M^{''} \to 0$$

in $\mathscr{A}$ with $T^{''} \in \mathscr{T}$ and $M^{''} \in \widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{T}$, we have the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M^{'} \arrow[r] \arrow[d] & T^{''} \arrow[r] \arrow[d, dashed] & M^{''} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dashed] & M^1 \arrow[r, dashed] \arrow[d] & T^1 \arrow[r, dashed] \arrow[d, dashed] & M^{''} \arrow[r, dashed] & 0 \\
 & T^{'} \arrow[r, equal] \arrow[d] & T^{'} \arrow[d, dashed] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Due to [23, Lemma 2.4 (2)], all columns and rows in the above two diagrams are $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequences. Since $\mathscr{T}$ is closed under $\mathscr{E}$-coproper extensions by assumption, the middle column in the second diagram implies $T^1 \in \mathscr{T}$, and hence the middle row in that diagram implies $M^1 \in \widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{T}$. Similarly, we get a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequence

$$0 \to M^1 \to C^1 \to M^2 \to 0$$

in $\mathscr{A}$ with $C^1 \in \mathscr{C}$ and $M^2 \in \widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{T}$. Continuing this process, we get a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{E})$-exact exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^i \to \cdots$$

in $\mathscr{A}$ with all C^i in $\mathscr{C}$. It follows that $M \in \widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{C}$ and $\widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{T} \subseteq \widetilde{\mathrm{cores}_{\mathscr{E}}}\,\mathscr{C}$. □

3.2 Injective dimension relative to coresolving subcategories

All results and their proofs in this subsection are completely dual to those in Subsection 3.1, so we only list the results without proofs.

Lemma 3.8. *Let $M \in \mathscr{A}$ and $n \geq 0$.*

(1) *Assume that $\mathscr{A}$ has enough injective objects. If $\mathscr{T}$ is a coresolving subcategory of $\mathscr{A}$, then the following statements are equivalent:*
   (1.1) $\mathscr{T}$-id $M \leq n$.
   (1.2) *There exists an exact sequence*
$$0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to K^n \to 0$$
   in $\mathscr{A}$ with all I^i injective and $K^n \in \mathscr{T}$.
   (1.3) *For any exact sequence*
$$0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to K^n \to 0$$
   in $\mathscr{A}$, if all I^i are injective, then $K^n \in \mathscr{T}$.
   (1.4) *For any exact sequence*
$$0 \to M \to T^0 \to T^1 \to \cdots \to T^{n-1} \to K^n \to 0$$
   in $\mathscr{A}$, if all T^i are in $\mathscr{T}$, then $K^n \in \mathscr{T}$.

(2) *Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. If $\mathscr{T}$ is an $\mathscr{E}$-preresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-proper generator $\mathscr{C}$, then the following statements are equivalent:*
   (2.1) $\mathscr{T}$-id $M \leq n$.
   (2.2) *There exists an exact sequence*
$$0 \to M \to T^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$$
   in $\mathscr{A}$ with $T^0 \in \mathscr{T}$ and all C^i in $\mathscr{C}$; that is, there exists an exact sequence
$$0 \to M \to T \to K \to 0$$
   in $\mathscr{A}$ with $T \in \mathscr{T}$ and $\mathscr{C}$-id $K \leq n-1$.

The main result in this subsection is as follows.

Theorem 3.9. *Assume that $\mathscr{A}$ has enough injective objects and $\mathscr{T}$ is a subcategory of $\mathscr{A}$ containing $\mathfrak{I}(\mathscr{A})$. Then the following statements are equivalent:*

(1) *$\mathscr{T}$ is coresolving.*

(2) *For any exact sequence*
$$0 \to A_1 \to A_2 \to A_3 \to 0$$
   in $\mathscr{A}$, we have
   (2.1) (a) $\mathscr{T}$-id $A_2 \leq \max\{\mathscr{T}\text{-id}\, A_1, \mathscr{T}\text{-id}\, A_3\}$, (b) *the equality holds if* $\mathscr{T}$-id $A_1 \neq \mathscr{T}$-id $A_3 + 1$;
   (2.2) (a) $\mathscr{T}$-id $A_3 \leq \max\{\mathscr{T}\text{-id}\, A_1 - 1, \mathscr{T}\text{-id}\, A_2\}$, (b) *the equality holds if* $\mathscr{T}$-id $A_1 \neq \mathscr{T}$-id A_2;
   (2.3) (a) $\mathscr{T}$-id $A_1 \leq \max\{\mathscr{T}\text{-id}\, A_2, \mathscr{T}\text{-id}\, A_3 + 1\}$, (b) *the equality holds if* $\mathscr{T}$-id $A_2 \neq \mathscr{T}$-id A_3.

As an immediate consequence, we get the following result.

Corollary 3.10. *Assume that $\mathscr{A}$ has enough injective objects and $\mathscr{T}$ is a coresolving subcategory of $\mathscr{A}$. Then $\mathscr{T}$-id$^{<\infty}$ satisfies the two-out-of-three property; that is, in a short exact sequence in $\mathscr{A}$, if any two terms are in $\mathscr{T}$-id$^{<\infty}$, then so is the third term.*

The following result shows that if the coresolving subcategory $\mathscr{T}$ of $\mathscr{A}$ admits an $\mathscr{E}$-proper generator $\mathscr{C}$, then any object in $\mathscr{A}$ with finite $\mathscr{T}$-injective dimension is isomorphic to a kernel (resp. a cokernel) of a morphism from an object in $\mathscr{T}$ to an object in $\mathscr{A}$ with finite $\mathscr{C}$-injective dimension.

Corollary 3.11. *Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. If $\mathscr{T}$ is an $\mathscr{E}$-preresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-proper generator $\mathscr{C}$, then, for any $M \in \mathscr{A}$ with $\mathscr{T}$-id $M = n < \infty$, the following assertions hold:*

(i) *There exists an exact sequence*
$$0 \to T' \to K' \to T \to K \to 0$$
   in $\mathscr{A}$ with $\mathscr{C}$-id $K' \leq n$, $\mathscr{C}$-id $K \leq n-1$ and $T', T \in \mathscr{T}$ such that $M \cong \mathrm{Im}(K' \to T)$.

(ii) *If $\mathscr{A}$ has enough injective objects and $\mathscr{T}$ is coresolving in $\mathscr{A}$, then the two "$\leq$" in (1) are "$=$".*

Furthermore, we get the following result.

Corollary 3.12. *Assume that $\mathscr{A}$ has enough injective objects and $\mathscr{T}$ is a coresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-proper generator $\mathscr{C}$. Then the following assertions hold:*
(1) $\mathscr{T}$-FID $\leq \mathscr{C}$-FID.
(2) *If $\mathscr{T} \subseteq \mathscr{C}^{\perp}$, then $\mathscr{T}$-id $M = \mathscr{C}$-id M for any $M \in \mathscr{A}$ with $\mathscr{C}$-id $M < \infty$.*
(3) *If $\mathscr{T} \subseteq \mathscr{C}^{\perp}$, then $\mathscr{T}$-FID $= \mathscr{C}$-FID.*

Proposition 3.13. *Let $\mathscr{E}$ and $\mathscr{C}$ be subcategories of $\mathscr{A}$. If $\mathscr{E}^{\perp} \cap \widetilde{\operatorname{res}_{\mathscr{E}} \mathscr{C}}$ is closed under ($\mathscr{E}$-proper) extensions, then it is closed under cokernels of monomorphisms. In particular, if $\widetilde{\operatorname{res}\mathscr{C}} := \widetilde{\operatorname{res}_{\mathcal{P}(\mathscr{A})}\mathscr{C}}$ is closed under extensions, then it is closed under cokernels of monomorphisms.*

Proposition 3.14. *Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. If $\mathscr{T}$ is an $\mathscr{E}$-preresolving subcategory of $\mathscr{A}$ admitting an $\mathscr{E}$-proper generator $\mathscr{C}$, then $\widetilde{\operatorname{res}_{\mathscr{E}} \mathscr{C}} = \widetilde{\operatorname{res}_{\mathscr{E}} \mathscr{T}}$.*

4 Applications to module categories

In this section, all rings are associative rings with unit and all modules are unital. For a ring R, we use $\operatorname{Mod} R$ to denote the category of left R-modules, and we use $\operatorname{mod} R$ to denote the category of finitely generated left R-modules.

Definition 4.1 ([2, 20]). Let R and S be arbitrary rings. An (R-S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied:
(1) ${}_RC$ admits a degreewise finite R-projective resolution.
(2) C_S admits a degreewise finite S^{op}-projective resolution.
(3) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \operatorname{Hom}_{S^{\mathrm{op}}}(C, C)$ is an isomorphism.
(4) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_R(C, C)$ is an isomorphism.
(5) $\operatorname{Ext}_R^{\geq 1}(C, C) = 0$.
(6) $\operatorname{Ext}_{S^{\mathrm{op}}}^{\geq 1}(C, C) = 0$.

Wakamatsu [37] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*; see [6, 29]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if it is Wakamatsu tilting [39, Corollary 3.2]. Typical examples of semidualizing bimodules include the free module of rank one and the dualizing module over a Cohen–Macaulay local ring. For more examples of semidualizing bimodules, the reader is referred to [20, 35, 38].

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. We write

$$(-)_* := \operatorname{Hom}(C, -),$$

and write

$$\begin{aligned}
\mathcal{P}_C(R) &:= \{C \otimes_S P \mid P \text{ is projective in } \operatorname{Mod} S\},\\
\mathcal{F}_C(R) &:= \{C \otimes_S F \mid F \text{ is flat in } \operatorname{Mod} S\},\\
\mathcal{I}_C(R^{\mathrm{op}}) &:= \{I_* \mid I \text{ is injective in } \operatorname{Mod} S^{\mathrm{op}}\}.
\end{aligned}$$

The modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R^{\mathrm{op}})$ are called *C-projective*, *C-flat* and *C-injective*, respectively. When ${}_RC_S = {}_RR_R$, C-projective, C-flat and C-injective modules are exactly projective, flat and injective modules, respectively.

Let $\mathscr{B}$ be a subcategory of $\operatorname{Mod} R^{\mathrm{op}}$. Recall that a sequence in $\operatorname{Mod} R$ is called ($\mathscr{B} \otimes_R -$)*-exact* if it is exact after applying the functor $B \otimes_R -$ for any $B \in \mathscr{B}$. We write

$$\mathscr{B}^{\top} := \{M \in \operatorname{Mod} R \mid \operatorname{Tor}_{\geq 1}^R(B, M) = 0 \text{ for any } B \in \mathscr{B}\}.$$

The following notions were introduced by Holm and Jørgensen [19] for commutative rings. The following are their non-commutative versions.

Definition 4.2. (1) A module $M \in \operatorname{Mod} R$ is called *C-Gorenstein projective* if $M \in {}^{\perp}\mathcal{P}_C(R)$ and there exists a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^i \to \cdots$$

in $\operatorname{Mod} R$ with all G^i in $\mathcal{P}_C(R)$.

(2) A module $M \in \operatorname{Mod} R$ is called *C-Gorenstein flat* if $M \in \mathcal{I}_C(R^{\mathrm{op}})^{\top}$ and there exists an $(\mathcal{I}_C(R^{\mathrm{op}}) \otimes_R -)$-exact exact sequence

$$0 \to M \to Q^0 \to Q^1 \to \cdots \to Q^i \to \cdots$$

in $\operatorname{Mod} R$ with all Q^i in $\mathcal{F}_C(R)$.

(3) A module $N \in \operatorname{Mod} R^{\mathrm{op}}$ is called *C-Gorenstein injective* if $N \in \mathcal{I}_C(R^{\mathrm{op}})^{\perp}$ and there exists a $\operatorname{Hom}_{R^{\mathrm{op}}}(\mathcal{I}_C(R^{\mathrm{op}}), -)$-exact exact sequence

$$\cdots \to E_i \to \cdots \to E_1 \to E_0 \to N \to 0$$

in $\operatorname{Mod} R^{\mathrm{op}}$ with all E_i in $\mathcal{I}_C(R^{\mathrm{op}})$.

We use $\mathcal{GP}_C(R)$ (resp. $\mathcal{GF}_C(R)$) to denote the subcategory of $\operatorname{Mod} R$ consisting of C-Gorenstein projective (resp. flat) modules, and we use $\mathcal{GI}_C(R^{\mathrm{op}})$ to denote the subcategory of $\operatorname{Mod} R^{\mathrm{op}}$ consisting of C-Gorenstein injective modules. When ${}_RC_S = {}_RR_R$, C-Gorenstein projective, flat and injective modules are exactly Gorenstein projective, flat and injective modules, respectively [13, 18]; in this case, we write

$$\mathcal{P}(R) := \mathcal{P}_C(R), \qquad \mathcal{I}(R^{\mathrm{op}}) := \mathcal{I}_C(R^{\mathrm{op}}), \qquad \mathcal{F}(R) := \mathcal{F}_C(R),$$
$$\mathcal{GP}(R) := \mathcal{GP}_C(R), \quad \mathcal{GI}(R^{\mathrm{op}}) := \mathcal{GI}_C(R^{\mathrm{op}}), \quad \mathcal{GF}(R) := \mathcal{GF}_C(R).$$

Definition 4.3 ([20]). (1) The *Auslander class* $\mathcal{A}_C(R^{\mathrm{op}})$ with respect to C consists of all modules N in $\operatorname{Mod} R^{\mathrm{op}}$ satisfying the following conditions:

(1.1) $\operatorname{Tor}^R_{\geq 1}(N, C) = 0$.

(1.2) $\operatorname{Ext}^{\geq 1}_{S^{\mathrm{op}}}(C, N \otimes_R C) = 0$.

(1.3) The canonical evaluation homomorphism

$$\mu_N : N \to (N \otimes_R C)_*,$$

defined by $\mu_N(x)(c) = x \otimes c$ for any $x \in N$ and $c \in C$, is an isomorphism in $\operatorname{Mod} R^{\mathrm{op}}$.

(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all modules M in $\operatorname{Mod} R$ satisfying the following conditions:

(2.1) $\operatorname{Ext}^{\geq 1}_R(C, M) = 0$.

(2.2) $\operatorname{Tor}^S_{\geq 1}(C, M_*) = 0$.

(2.3) The canonical evaluation homomorphism

$$\theta_M : C \otimes_S M_* \to M,$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$, is an isomorphism in $\operatorname{Mod} R$.

For a subcategory $\mathscr{X}$ of $\operatorname{Mod} R$ (or $\operatorname{Mod} R^{\mathrm{op}}$), we write

$$\mathscr{X}^+ := \{X^+ \mid X \in \mathscr{X}\},$$

where $(-)^+ = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ with $\mathbb{Z}$ being the additive group of integers and $\mathbb{Q}$ being the additive group of rational numbers. For simplicity, we write

$$\widetilde{\operatorname{res}\mathscr{C}} := \widetilde{\operatorname{res}_{\mathscr{C}}\mathscr{C}} \quad \text{and} \quad \widetilde{\operatorname{cores}\mathscr{C}} := \widetilde{\operatorname{cores}_{\mathscr{C}}\mathscr{C}}.$$

In the following, we present a partial list of examples of how the results obtained in Section 3 can be applied.

Remark 4.4. (1) It is well known that $\mathcal{P}(R)$ and $\mathcal{F}(R)$ are resolving and $\mathcal{I}(R)$ is coresolving in $\operatorname{Mod} R$.

(2) Let $(\mathscr{U}, \mathscr{V})$ be a hereditary cotorsion pair in Mod R, and let $\mathscr{C} := \mathscr{U} \cap \mathscr{V}$ be its *kernel*. Then the following assertions hold:
  (a) $\mathscr{U}$ is resolving in Mod R admitting a $\mathscr{C}$-coproper cogenerator $\mathscr{C}$ (see [33, Lemma 4.4]).
  (b) Dually, $\mathscr{V}$ is coresolving in Mod R admitting a $\mathscr{C}$-proper generator $\mathscr{C}$.

(3) (a) It holds that

$$\mathcal{GP}_C(R) = {}^{\perp}\mathcal{P}_C(R) \cap \widetilde{\operatorname{cores}\mathcal{P}_C}(R)$$

is resolving in Mod R admitting a $\mathcal{P}_C(R)$-coproper cogenerator $\mathcal{P}_C(R)$ (see [33, Example 3.2 (2) and Proposition 3.3]). In particular,

$$\mathcal{GP}(R) = {}^{\perp}\mathcal{P}(R) \cap \widetilde{\operatorname{cores}\mathcal{P}}(R)$$

is resolving in Mod R admitting a $\mathcal{P}(R)$-coproper cogenerator $\mathcal{P}(R)$.
  (b) Dually,

$$\mathcal{GI}_C(R^{\mathrm{op}}) = \mathcal{I}_C(R^{\mathrm{op}})^{\perp} \cap \operatorname{res}\widetilde{\mathcal{I}_C(R^{\mathrm{op}}})$$

is coresolving in Mod R^{op} admitting an $\mathcal{I}_C(R^{\mathrm{op}})$-proper generator $\mathcal{I}_C(R^{\mathrm{op}})$ (see [33, Example 3.2 (2) and the dual of Proposition 3.3]). In particular,

$$\mathcal{GI}(R^{\mathrm{op}}) = \mathcal{I}(R^{\mathrm{op}})^{\perp} \cap \operatorname{res}\widetilde{\mathcal{I}(R^{\mathrm{op}}})$$

is coresolving in Mod R^{op} admitting an $\mathcal{I}(R^{\mathrm{op}})$-proper generator $\mathcal{I}(R^{\mathrm{op}})$.
  (c) Let R be a left and right Noetherian ring, and let $p(R)$ be the subcategory of mod R consisting of projective modules. Recall that a module $M \in \operatorname{mod} R$ is said to *have Gorenstein dimension zero* [3] or be *totally reflexive* [5] if $M \in \mathcal{G}p(R)$, where

$$\mathcal{G}p(R) = {}^{\perp_R}R \cap \widetilde{\operatorname{cores} p}(R),$$

which is resolving in mod R admitting a $p(R)$-coproper cogenerator $p(R)$.

(4) (a) Recall from [12] that a module $M \in \operatorname{Mod} R$ is called *strongly Gorenstein flat* if $M \in \mathcal{SGF}(R)$, where

$$\mathcal{SGF}(R) = {}^{\perp}\mathcal{F}(R) \cap \widetilde{\operatorname{cores}_{\mathcal{F}(R)}}\mathcal{P}(R).$$

It is trivial that ${}^{\perp}\mathcal{F}(R)$ is closed under extensions. By the dual version of [13, Lemma 8.2.1] (cf. [18, Horseshoe Lemma 1.7]), it is easy to see that $\mathcal{SGF}(R)$ is closed under extensions. It follows from Proposition 3.6 that $\mathcal{SGF}(R)$ is resolving in Mod R admitting an $\mathcal{F}(R)$-coproper cogenerator $\mathcal{P}(R)$, which generalizes [12, Proposition 2.10 (1) and (2)].
  (b) Recall from [28, 34] that a module $M \in \operatorname{Mod} R$ is called *FP-injective* (or *absolutely pure*) if $M \in \mathcal{FI}(R)$, where

$$\mathcal{FI}(R) = \{M \in \operatorname{Mod} R \mid \operatorname{Ext}^1_R(X, M) = 0 \text{ for all finitely presented left } R\text{-modules } X\}.$$

Recall from [30] that a module $M \in \operatorname{Mod} R$ is called *Gorenstein FP-injective* if $M \in \mathcal{GFI}(R)$, where

$$\mathcal{GFI}(R) = \mathcal{FI}(R)^{\perp} \cap \widetilde{\operatorname{res}_{\mathcal{FI}(R)}}\mathcal{I}(R).$$

It is trivial that $\mathcal{FI}(R)^{\perp}$ is closed under extensions. By [13, Lemma 8.2.1], it is easy to see that $\mathcal{GFI}(R)$ is closed under extensions. It follows from Proposition 3.13 that $\mathcal{GFI}(R)$ is coresolving in Mod R admitting an $\mathcal{FI}(R)$-proper generator $\mathcal{I}(R)$, which generalizes [30, Proposition 2.6 (1) and (2)].

(5) (a) Recall from [10] that a module $M \in \operatorname{Mod} R$ is called *level* if $M \in \mathcal{L}(R)$, where

$$\mathcal{L}(R) = \{M \in \operatorname{Mod} R \mid \operatorname{Tor}^R_1(X, M) = 0 \text{ for all right } R\text{-modules } X \text{ admitting a degreewise finite } R^{\mathrm{op}}\text{-projective resolution}\}.$$

Also, recall that a module $M \in \operatorname{Mod} R$ is called *Gorenstein AC-projective* if $M \in \mathcal{GP}_{ac}(R)$, where

$$\mathcal{GP}_{ac}(R) = {}^{\perp}\mathcal{L}(R) \cap \widetilde{\operatorname{cores}_{\mathcal{L}(R)}}\mathcal{P}(R).$$

By [10, Lemma 8.6], we have that $\mathcal{GP}_{ac}(R)$ is resolving in Mod R admitting a $\mathcal{L}(R)$-coproper cogenerator $\mathcal{P}(R)$.

(b) Recall from [10] that a module $M \in \operatorname{Mod} R$ is called *absolutely clean* if $M \in \mathcal{AC}(R)$, where

$$\mathcal{AC}(R) = \{M \in \operatorname{Mod} R \mid \operatorname{Ext}^1_R(X, M) = 0 \text{ for all left } R\text{-modules } X \text{ admitting a degreewise finite } R\text{-projective resolution}\}.$$

Also, recall that a module $M \in \operatorname{Mod} R$ is called *Gorenstein AC-injective* if $M \in \mathcal{GI}_{ac}(R)$, where

$$\mathcal{GI}_{ac}(R) = \mathcal{AC}(R)^{\perp} \cap \widetilde{\operatorname{res}_{\mathcal{AC}(R)} \mathcal{I}}(R).$$

By [10, Lemma 5.6], we have that $\mathcal{GI}_{ac}(R)$ is coresolving in $\operatorname{Mod} R$ admitting an $\mathcal{AC}(R)$-proper generator $\mathcal{I}(R)$.

(6) (a) It holds that

$$\mathcal{A}_C(R^{\mathrm{op}}) = {}^{\perp}\mathcal{I}_C(R^{\mathrm{op}}) \cap \widetilde{\operatorname{cores} \mathcal{I}_C(R^{\mathrm{op}})},$$

which is resolving in $\operatorname{Mod} R^{\mathrm{op}}$ admitting an $\mathcal{I}_C(R^{\mathrm{op}})$-coproper cogenerator $\mathcal{I}_C(R^{\mathrm{op}})$ (see [33, Example 3.2 (2) and Proposition 3.3]; also, cf. [20, Theorem 2]).

(b) Dually,

$$\mathcal{B}_C(R) = \mathcal{P}_C(R)^{\perp} \cap \widetilde{\operatorname{res} \mathcal{P}_C(R)},$$

which is coresolving in $\operatorname{Mod} R$ admitting a $\mathcal{P}_C(R)$-proper generator $\mathcal{P}_C(R)$ (see [33, Example 3.2 (2) and the dual of Proposition 3.3]; also cf. [20, Theorem 6.1]).

(7) Let $\mathscr{B}$ be a subcategory of $\operatorname{Mod} R^{\mathrm{op}}$. Recall from [15] that a module $M \in \operatorname{Mod} R$ is called *Gorenstein $\mathscr{B}$-flat* (resp. *projectively coresolved Gorenstein $\mathscr{B}$-flat*) if $M \in \mathscr{B}^{\top}$ and there exists a $(\mathscr{B} \otimes_R -)$-exact exact sequence

$$0 \to M \to Q^0 \to Q^1 \to \cdots \to Q^i \to \cdots$$

in $\operatorname{Mod} R$ with all Q^i in $\mathcal{F}(R)$ (resp. $\mathcal{P}(R)$). We use $\mathcal{GF}_{\mathscr{B}}(R)$ (resp. $\mathcal{PGF}_{\mathscr{B}}(R)$) to denote the subcategory of $\operatorname{Mod} R$ consisting of Gorenstein $\mathscr{B}$-flat modules (resp. projectively coresolved Gorenstein $\mathscr{B}$-flat modules). Also recall from [15] that $\mathscr{B}$ is *semi-definable* if $\mathscr{B}$ is closed under direct products and its definable closure $\langle \mathscr{B} \rangle$ (the smallest subcategory of $\operatorname{Mod} R^{\mathrm{op}}$ containing $\mathscr{B}$ which is closed under direct products, direct limits and pure submodules) contains a pure injective module D such that any module in $\langle \mathscr{B} \rangle$ is a pure submodule of some direct product of copies of D.

Let $B \in \operatorname{Mod} R^{\mathrm{op}}$, $M \in \operatorname{Mod} R$ and $n \geq 1$. By [17, Lemma 2.16 (a) und (b)], we have

$$(B \otimes_R -)^{+} \cong \operatorname{Hom}_R(-, B^{+}), \tag{4.1}$$

$$[\operatorname{Tor}^R_n(B, M)]^{+} \cong \operatorname{Ext}^n_R(M, B^{+}). \tag{4.2}$$

This yields that

$$\mathcal{GF}_{\mathscr{B}}(R) = {}^{\perp}(\mathscr{B}^{+}) \cap \widetilde{\operatorname{cores}_{\mathscr{B}^{+}} \mathcal{F}}(R),$$

$$\mathcal{PGF}_{\mathscr{B}}(R) = {}^{\perp}(\mathscr{B}^{+}) \cap \widetilde{\operatorname{cores}_{\mathscr{B}^{+}} \mathcal{P}}(R).$$

By [15, Theorem 2.8], we have that $\mathcal{PGF}_{\mathscr{B}}(R)$ is resolving in $\operatorname{Mod} R$ admitting an $\mathcal{I}_C(R^{\mathrm{op}})^{+}$-coproper cogenerator $\mathcal{P}(R)$. When $\mathscr{B} = \mathcal{I}(R^{\mathrm{op}})$, projectively coresolved Gorenstein $\mathscr{B}$-flat modules are called *projectively coresolved Gorenstein flat* [31]; in this case, we write

$$\mathcal{PGF}(R) := \mathcal{PGF}_{\mathscr{B}}(R).$$

We have (see [26, Lemma 3])

$$\mathcal{P}(R) \subseteq \mathcal{PGF}(R) = \mathcal{SGF}(R)(R) \cap \mathcal{GF}(R).$$

On the other hand, it follows from [15, Theorem 2.12 and Corollary 2.14] that if $\mathscr{B}$ is semi-definable, then $\mathcal{GF}_{\mathscr{B}}(R)$ is resolving in $\operatorname{Mod} R$ admitting a $\mathscr{B}^{+}$-coproper cogenerator $\mathcal{F}(R)$. In particular, $\mathcal{GF}(R)$ is resolving in $\operatorname{Mod} R$ admitting an $\mathcal{I}_C(R^{\mathrm{op}})^{+}$-coproper cogenerator $\mathcal{F}(R)$ (also, cf. [31, Theorem 4.11]).

(8) By (4.1) and (4.2), we have that

$$\mathcal{GF}_C(R) = {}^{\perp}(\mathcal{I}_C(R^{\mathrm{op}})^+) \cap \mathrm{cores}\,\widetilde{\mathcal{I}_C(R^{\mathrm{op}})^+}\,\mathcal{F}_C(R),$$

which admits an $\mathcal{I}_C(R^{\mathrm{op}})^+$-coproper cogenerator $\mathcal{F}_C(R)$. It is trivial that

$$\mathcal{P}(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{GF}_C(R).$$

By Proposition 3.6, we have that if $\mathcal{GF}_C(R)$ is closed under extensions, then it is resolving in $\mathrm{Mod}\,R$.

4.1 Finitistic dimensions

In this subsection, R is an arbitrary associative ring.

By Corollaries 3.5 and 3.12 and Remark 4.4 (2), we immediately get the following result.

Corollary 4.5. *Let $(\mathscr{U}, \mathscr{V})$ be a hereditary cotorsion pair in* $\mathrm{Mod}\,R$ *with the kernel $\mathscr{C}$. Then the following assertions hold:*

(1) *For any $M \in \mathrm{Mod}\,R$ with $\mathscr{C}$-$\mathrm{pd}\,M < \infty$, we have*

$$\mathscr{U}\text{-}\mathrm{pd}\,M = \mathscr{C}\text{-}\mathrm{pd}\,M.$$

Moreover, we have

$$\mathscr{U}\text{-FPD} = \mathscr{C}\text{-FPD}\,.$$

(2) *For any $M \in \mathrm{Mod}\,R$ with $\mathscr{C}$-$\mathrm{id}\,M < \infty$, we have*

$$\mathscr{V}\text{-}\mathrm{id}\,M = \mathscr{C}\text{-}\mathrm{id}\,M.$$

Moreover, we have

$$\mathscr{V}\text{-FID} = \mathscr{C}\text{-FID}\,.$$

Following the usual customary notation, we write

$$\begin{array}{lll} \mathrm{pd}_R\,M := \mathcal{P}(R)\text{-}\mathrm{pd}\,M, & \mathrm{id}_R\,M := \mathcal{I}(R)\text{-}\mathrm{id}\,M, & \mathrm{fd}_R\,M := \mathcal{F}(R)\text{-}\mathrm{pd}\,M, \\ \text{G-}\mathrm{pd}_R\,M := \mathcal{GP}(R)\text{-}\mathrm{pd}\,M, & \text{G-}\mathrm{id}_R\,M := \mathcal{GI}(R)\text{-}\mathrm{id}\,M, & \text{G-}\mathrm{fd}_R\,M := \mathcal{GF}(R)\text{-}\mathrm{pd}\,M, \\ \mathrm{G}_C\text{-}\mathrm{pd}_R M := \mathcal{GP}_C(R)\text{-}\mathrm{pd}\,M, & \mathrm{G}_C\text{-}\mathrm{id}_R M := \mathcal{GI}_C(R)\text{-}\mathrm{id}\,M, & \mathrm{G}_C\text{-}\mathrm{fd}_R M := \mathcal{GF}_C(R)\text{-}\mathrm{pd}\,M. \end{array}$$

By Corollary 3.5 and Remark 4.4 (3)–(7), we immediately get the following result, in which assertion (2) extends [18, Proposition 2.27 and Theorem 2.28], and assertion (3) generalizes [40, Lemma 4.6].

Corollary 4.6. (1) *For any $M \in \mathrm{Mod}\,R$ with $\mathcal{P}_C(R)$-$\mathrm{pd}\,M < \infty$, we have*

$$\mathrm{G}_C\text{-}\mathrm{pd}_R M = \mathcal{P}_C(R)\text{-}\mathrm{pd}\,M.$$

Moreover, we have

$$\mathcal{GP}_C(R)\text{-FPD} = \mathcal{P}_C(R)\text{-FPD}\,.$$

(2) *For any $M \in \mathrm{Mod}\,R$ with $\mathrm{pd}_R\,M < \infty$, we have*

$$\text{G-}\mathrm{pd}_R\,M = \mathcal{GP}_{\mathrm{ac}}(R)\text{-}\mathrm{pd}\,M = \mathcal{SGF}(R)\text{-}\mathrm{pd}\,M = \mathcal{PGF}(R)\text{-}\mathrm{pd}\,M = \mathrm{pd}_R\,M.$$

Moreover, we have

$$\mathcal{GP}(R)\text{-FPD} = \mathcal{GP}_{\mathrm{ac}}(R)\text{-FPD} = \mathcal{SGF}(R)\text{-FPD} = \mathcal{PGF}(R)\text{-FPD} = \mathcal{P}(R)\text{-FPD}\,.$$

(3) *Let R be a left and right Noetherian ring. Then, for any $M \in \mathrm{mod}\,R$ with $\mathrm{pd}_R\,M < \infty$, we have*

$$\mathcal{G}p(R)\text{-}\mathrm{pd}\,M = \mathrm{pd}_R\,M.$$

Moreover, we have

$$\mathcal{G}p(R)\text{-FPD} = p(R)\text{-FPD}\,.$$

(4) *For any* $N \in \operatorname{Mod} R^{\mathrm{op}}$ *with* $\mathcal{I}_C(R^{\mathrm{op}})$-pd $N < \infty$, *we have*

$$\mathcal{A}_C(R^{\mathrm{op}})\text{-pd}\, N = \mathcal{I}_C(R^{\mathrm{op}})\text{-pd}\, N.$$

Moreover, we have

$$\mathcal{A}_C(R^{\mathrm{op}})\text{-FPD} = \mathcal{I}_C(R^{\mathrm{op}})\text{-FPD}\,.$$

By Corollary 3.12 and Remark 4.4 (3)–(6), we immediately get the following result, in which assertion (2) extends [18, Theorem 2.29].

Corollary 4.7. (1) *For any* $M \in \operatorname{Mod} R$ *with* $\mathcal{I}_C(R)$-id $M < \infty$, *we have*

$$\mathrm{G}_C\text{-id}_R M = \mathcal{I}_C(R)\text{-id}\, M.$$

Moreover, we have

$$\mathcal{G}\mathcal{I}_C(R)\text{-FID} = \mathcal{I}_C(R)\text{-FID}\,.$$

(2) *For any* $M \in \operatorname{Mod} R$ *with* $\mathrm{id}_R\, M < \infty$, *we have*

$$\text{G-id}_R\, M = \mathcal{G}\mathcal{I}_{\mathrm{ac}}(R)\text{-id}\, M = \mathcal{G}\mathcal{F}\mathcal{I}(R)\text{-id}\, M = \mathrm{id}_R\, M.$$

Moreover, we have

$$\mathcal{G}\mathcal{I}(R)\text{-FID} = \mathcal{G}\mathcal{I}_{\mathrm{ac}}(R)\text{-FID} = \mathcal{G}\mathcal{F}\mathcal{I}(R)\text{-FID} = \mathcal{I}(R)\text{-FID}\,.$$

(3) *For any* $M \in \operatorname{Mod} R$ *with* $\mathcal{P}_C(R)$-id $M < \infty$, *we have*

$$\mathcal{B}_C(R)\text{-id}\, M = \mathcal{P}_C(R)\text{-id}\, M.$$

Moreover, we have

$$\mathcal{B}_C(R)\text{-FID} = \mathcal{P}_C(R)\text{-FID}\,.$$

4.2 Equivalent characterizations of Gorenstein rings

In this subsection, R is a left and right Noetherian ring and $n \geq 0$. Recall that R is called *n-Gorenstein* if $\mathrm{id}_R\, R = \mathrm{id}_{R^{\mathrm{op}}}\, R \leq n$.

The following lemma plays a crucial role in the sequel.

Lemma 4.8. *Let* $\mathscr{T}$ *be an* $\mathscr{E}$*-precoresolving subcategory of* $\operatorname{Mod} R$ *admitting an* $\mathscr{E}$*-coproper cogenerator* $\mathscr{C}$, *where* $\mathscr{E}$ *is a subcategory of* $\operatorname{Mod} R$ *and* $\mathscr{C} \subseteq \mathcal{F}(R)$. *If* $\mathscr{T}$-pd $M \leq n$ *for any* $M \in \operatorname{mod} R$, *then* $\mathrm{id}_{R^{\mathrm{op}}}\, R \leq n$.

Proof. Let $M \in \operatorname{mod} R$. If $\mathscr{T}$-pd $M \leq n$, then, by assumption and by Corollary 3.4 (1), there exists an exact sequence

$$0 \to M \to K' \to T' \to 0$$

in $\operatorname{Mod} R$ with $\mathscr{C}$-pd $K' \leq n$ and $T' \in \mathscr{T}$. Since $\mathscr{C} \subseteq \mathcal{F}(R)$, we have $\mathrm{fd}_R\, K' \leq n$. Therefore, $\mathrm{id}_{R^{\mathrm{op}}}\, R \leq n$ by [22, Lemma 3.8]. □

Recall from Remark 4.4 (3) and (4) that

$$^{\perp}\mathcal{P}(R) \cap \mathrm{cores}\,\widetilde{\mathcal{P}}(R) = \mathcal{G}\mathcal{P}(R) \supseteq \mathcal{S}\mathcal{G}\mathcal{F}(R) = {}^{\perp}\mathcal{F}(R) \cap \widetilde{\mathrm{cores}_{\mathcal{F}(R)}}\,\mathcal{P}(R).$$

In terms of the projective dimensions relative to all six subcategories of $\operatorname{Mod} R$ that appear in this relation, we give some equivalent characterizations of n-Gorenstein rings as follows.

Theorem 4.9. *The following statements are equivalent:*

(1) *R is n-Gorenstein.*

(2) $\text{G-pd}_R\, M \leq n$ *for any* $M \in \operatorname{Mod} R$.

$(2)^{\mathrm{op}}$ $\text{G-pd}_{R^{\mathrm{op}}}\, N \leq n$ *for any* $N \in \operatorname{Mod} R^{\mathrm{op}}$.

(3) $^{\perp}\mathcal{P}(R)$-pd $M \leq n$ *and* $^{\perp}\mathcal{P}(R^{\mathrm{op}})$-pd $N \leq n$ *for any* $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} R^{\mathrm{op}}$.

(4) $\operatorname{cores}\widetilde{\mathcal{P}(R)}$-pd $M \le n$ *and* $\operatorname{cores}\widetilde{\mathcal{P}(R^{\mathrm{op}})}$-pd $N \le n$ *for any* $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} R^{\mathrm{op}}$.
(5) $\mathcal{SGF}(R)$-pd $M \le n$ *for any* $M \in \operatorname{Mod} R$.
$(5)^{\mathrm{op}}$ $\mathcal{SGF}(R)$-pd $N \le n$ *for any* $N \in \operatorname{Mod} R^{\mathrm{op}}$.
(6) ${}^{\perp}\mathcal{F}(R)$-pd $M \le n$ *and* ${}^{\perp}\mathcal{F}(R^{\mathrm{op}})$-pd $N \le n$ *for any* $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} R^{\mathrm{op}}$.
(7) $\operatorname{cores}_{\mathcal{F}(R)}\widetilde{\mathcal{P}(R)}$-pd $M \le n$ *and* $\operatorname{cores}_{\mathcal{F}(R^{\mathrm{op}})}\widetilde{\mathcal{P}(R^{\mathrm{op}})}$-pd $N \le n$ *for any* $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} R^{\mathrm{op}}$.

Proof. First, the implications $(2) + (2)^{\mathrm{op}} \Longrightarrow (3) + (4)$, $(5) + (5)^{\mathrm{op}} \Longrightarrow (6) + (7)$, $(5) \Longrightarrow (2)$, $(5)^{\mathrm{op}} \Longrightarrow (2)^{\mathrm{op}}$, $(6) \Longrightarrow (3)$ and $(7) \Longrightarrow (4)$ are trivial. By [13, Theorem 11.5.1], we have $(1) \Longrightarrow (2) + (2)^{\mathrm{op}}$.

If R is n-Gorenstein, then $\mathcal{GP}(R) = \mathcal{SGF}(R)$ and $\mathcal{GP}(R^{\mathrm{op}}) = \mathcal{SGF}(R^{\mathrm{op}})$ by [12, Corollary 2.8], and thus $(1) \Longrightarrow (5) + (5)^{\mathrm{op}}$ holds true.

"$(3) \Longrightarrow (1)$": By (3) and dimension shifting, it is easy to see that

$$\operatorname{Ext}_R^{\ge n+1}(M, R) = 0 = \operatorname{Ext}_{R^{\mathrm{op}}}^{\ge n+1}(N, R)$$

for any $M \in \operatorname{Mod} R$ and $N \in \operatorname{Mod} R^{\mathrm{op}}$. This implies $\operatorname{id}_R R \le n$ and $\operatorname{id}_{R^{\mathrm{op}}} R \le n$.

"$(2) \Longrightarrow (1)$": By (2) and dimension shifting, it is easy to get $\operatorname{Ext}_R^{\ge n+1}(M, R) = 0$ for any $M \in \operatorname{Mod} R$, and so $\operatorname{id}_R R \le n$. By [18, Theorem 2.5], we have that $\mathcal{GP}(R)$ is resolving in $\operatorname{Mod} R$ admitting a $\mathcal{P}(R)$-coproper cogenerator $\mathcal{P}(R)$ $(\subseteq \mathcal{F}(R))$. Thus, $\operatorname{id}_{R^{\mathrm{op}}} R \le n$ by (2) and Lemma 4.8.

Symmetrically, we get $(2)^{\mathrm{op}} \Longrightarrow (1)$.

"$(4) \Longrightarrow (1)$": By the dual version of [13, Lemma 8.2.1] (cf. [18, Horseshoe Lemma 1.7]), we have that $\operatorname{cores}\widetilde{\mathcal{P}(R)}$ is closed under $\mathcal{P}(R)$-coproper extensions. Thus $\operatorname{cores}\widetilde{\mathcal{P}(R)}$ is a $\mathcal{P}(R)$-precoresolving subcategory of $\operatorname{Mod} R$ admitting a $\mathcal{P}(R)$-coproper cogenerator $\mathcal{P}(R)$ $(\subseteq \mathcal{F}(R))$. Thus, $\operatorname{id}_{R^{\mathrm{op}}} R \le n$ by (4) and Lemma 4.8. Symmetrically, we have $\operatorname{id}_R R \le n$. □

The following result is a dual version of Lemma 4.8.

Lemma 4.10. *Let $\mathscr{T}$ be an $\mathscr{E}$-preresolving subcategory of* $\operatorname{Mod} R$ *admitting an $\mathscr{E}$-proper generator $\mathscr{C}$, where $\mathscr{E}$ is a subcategory of* $\operatorname{Mod} R$ *and* $\mathscr{C} \subseteq \mathcal{I}(R)$. *If* $\mathscr{T}$-id $M \le n$ *for any* $M \in \operatorname{Mod} R$, *then* $\operatorname{id}_R R \le n$.

Proof. Let $N \in \operatorname{mod} R^{\mathrm{op}}$. Then $N^+ \in \operatorname{Mod} R$ and $\mathscr{T}$-id $N^+ \le n$ by assumption. It follows from Corollary 3.11 (1) that there exists an exact sequence

$$0 \to T' \to K' \xrightarrow{f} N^+ \to 0$$

in $\operatorname{Mod} R$ with $T' \in \mathscr{T}$ and $\mathscr{C}$-id $K' \le n$. Since $\mathscr{C} \subseteq \mathcal{I}(R)$, we have $\operatorname{id}_R K' \le n$. It follows from [16, Theorem 2.2] that $\operatorname{fd}_{R^{\mathrm{op}}} K'^{+} \le n$.

On the other hand, by [13, Proposition 5.3.9], there exists a monomorphism $\lambda : N \rightarrowtail N^{++}$ in $\operatorname{Mod} R^{\mathrm{op}}$, and hence $\lambda f^+ : N \rightarrowtail K'^{+}$ is also a monomorphism in $\operatorname{Mod} R^{\mathrm{op}}$. Thus, $\operatorname{id}_R R \le n$ by [22, Lemma 3.8]. □

Recall from Remark 4.4 (3) that

$$\mathcal{GI}(R) = \mathcal{I}(R)^{\perp} \cap \operatorname{res}\widetilde{\mathcal{I}(R)}.$$

In terms of the injective dimensions relative to all three subcategories of $\operatorname{Mod} R$ that appear in this equality, we give some equivalent characterizations of n-Gorenstein rings as follows.

Theorem 4.11. *The following statements are equivalent:*
(1) *R is n-Gorenstein.*
(2) G-$\operatorname{id}_R M \le n$ *for any* $M \in \operatorname{Mod} R$.
$(2)^{\mathrm{op}}$ G-$\operatorname{id}_{R^{\mathrm{op}}} N \le n$ *for any* $N \in \operatorname{Mod} R^{\mathrm{op}}$.
(3) $\mathcal{I}(R)^{\perp}$-id $M \le n$ *and* $\mathcal{I}(R^{\mathrm{op}})^{\perp}$-id $N \le n$ *for any* $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} R^{\mathrm{op}}$.
(4) $\operatorname{res}\widetilde{\mathcal{I}(R)}$-id $M \le n$ *and* $\operatorname{res}\widetilde{\mathcal{I}(R^{\mathrm{op}})}$-id $N \le n$ *for any* $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} R^{\mathrm{op}}$.

Proof. The implications $(2) + (2)^{\mathrm{op}} \Longrightarrow (3) + (4)$ are trivial. On account of [13, Theorem 11.2.1], we have $(1) \Longrightarrow (2) + (2)^{\mathrm{op}}$.

"$(3) \Longrightarrow (1)$": By [16, Theorem 2.1], we have $(R_R)^+ \in \mathcal{I}(R)$ and $({}_RR)^+ \in \mathcal{I}(R^{\mathrm{op}})$. Then, by (3) and dimension shifting, it is easy to see that

$$\operatorname{Ext}_R^{\ge n+1}((R_R)^+, M) = 0 = \operatorname{Ext}_{R^{\mathrm{op}}}^{\ge n+1}(({}_RR)^+, N)$$

for any $M \in \operatorname{Mod} R$ and $N \in \operatorname{Mod} R^{\mathrm{op}}$. This implies

$$\mathrm{fd}_R(R_R)^+ \leq \mathrm{pd}_R(R_R)^+ \leq n \quad \text{and} \quad \mathrm{fd}_{R^{\mathrm{op}}}({}_RR)^+ \leq \mathrm{pd}_{R^{\mathrm{op}}}({}_RR)^+ \leq n.$$

It follows from [16, Theorem 2.2] that $\mathrm{id}_{R^{\mathrm{op}}} R \leq n$ and $\mathrm{id}_R R \leq n$.

"(2) $\Longrightarrow$ (1)": Similar to the proof of (3) $\Longrightarrow$ (1), we have $\mathrm{id}_{R^{\mathrm{op}}} R \leq n$. By [18, Theorem 2.6], we have that $\mathcal{GI}(R)$ is coresolving in $\operatorname{Mod} R$ admitting an $\mathcal{I}(R)$-proper generator $\mathcal{I}(R)$. Thus, $\mathrm{id}_R R \leq n$ by (2) and Lemma 4.10.

Symmetrically, we get $(2)^{\mathrm{op}} \Longrightarrow (1)$.

"(4) $\Longrightarrow$ (1)": By [13, Lemma 8.2.1], we have that $\widetilde{\mathrm{res}\,\mathcal{I}(R)}$ is closed under $\mathcal{I}(R)$-proper extensions. Thus $\widetilde{\mathrm{res}\,\mathcal{I}(R)}$ is an $\mathcal{I}(R)$-preresolving subcategory of $\operatorname{Mod} R^{\mathrm{op}}$ admitting an $\mathcal{I}(R)$-proper generator $\mathcal{I}(R^{\mathrm{op}})$. Thus, $\mathrm{id}_R R \leq n$ by (4) and Lemma 4.10. Symmetrically, we have $\mathrm{id}_{R^{\mathrm{op}}} R \leq n$. □

Recall from [13] that a module $M \in \operatorname{Mod} R$ is called *cotorsion* if $\mathrm{Ext}_R^1(F, M) = 0$ for any $F \in \mathcal{F}(R)$ (equivalently, $M \in \mathcal{F}(R)^{\perp}$). We write

$$\mathcal{FC}(R) := \{\text{flat and cotorsion modules in } \operatorname{Mod} R\}.$$

Lemma 4.12. (1) $\mathcal{I}(R^{\mathrm{op}})^+$ *is an* $\mathcal{I}(R^{\mathrm{op}})^+$*-coproper cogenerator and* $\mathcal{FC}(R)$ *is an* $\mathcal{FC}(R)$*-coproper cogenerator for* $\mathcal{F}(R)$.

(2) *We have*

$$\begin{aligned}\widetilde{\mathrm{cores}\,\mathcal{I}(R^{\mathrm{op}})^+} &= \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{FC}(R) = \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{F}(R)\\ &= \widetilde{\mathrm{cores}\,\mathcal{FC}}(R) = \mathrm{cores}_{\widetilde{\mathcal{FC}(R)}}\,\mathcal{F}(R) \supseteq \widetilde{\mathrm{cores}\,\mathcal{F}}(R).\end{aligned}$$

Moreover, all of these subcategories except $\widetilde{\mathrm{cores}\,\mathcal{F}}(R)$ *are closed under* $\mathcal{I}(R^{\mathrm{op}})^+$*-coproper extensions.*

Proof. (1) It essentially follows from [32, Proposition 4.4] and its proof. However, we still give the proof in details.

Let $Q \in \mathcal{F}(R)$. By [17, Corollary 2.21 (b)], there exists the following pure exact sequence:

$$0 \to Q \to Q^{++} \to Q^{++}/Q \to 0 \tag{4.3}$$

in $\operatorname{Mod} R$. Since $Q^+ \in \mathcal{I}(R^{\mathrm{op}})$ and $Q^{++} \in \mathcal{I}(R^{\mathrm{op}})^+ \cap \mathcal{F}(R)$ by [16, Theorems 2.1 and 2.2], we have $Q^{++}/Q \in \mathcal{F}(R)$ by [20, Lemma 5.2(a)], and so (4.3) is a $\mathrm{Hom}_R(-, \mathcal{I}(R^{\mathrm{op}})^+)$-exact exact sequence by [32, Lemma 4.13]. It follows that $\mathcal{I}(R^{\mathrm{op}})^+$ is an $\mathcal{I}(R^{\mathrm{op}})^+$-coproper cogenerator for $\mathcal{F}(R)$.

Since Q^{++} is pure injective by [13, Proposition 5.3.7], we have $Q^{++} \in \mathcal{FC}(R)$ by [32, Proposition 4.4 (1)]. Notice that (4.3) is a $\mathrm{Hom}_R(-, \mathcal{FC}(R))$-exact exact sequence, so $\mathcal{FC}(R)$ is an $\mathcal{FC}(R)$-coproper cogenerator for $\mathcal{F}(R)$.

(2) Since $\mathcal{I}(R^{\mathrm{op}})^+ \subseteq \mathcal{FC}(R) \subseteq \mathcal{F}(R)$ by [13, Proposition 5.3.7] and [32, Lemma 4.13], we have

$$\widetilde{\mathrm{cores}\,\mathcal{I}(R^{\mathrm{op}})^+} \subseteq \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{FC}(R) \subseteq \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{F}(R) \supseteq \mathrm{cores}_{\widetilde{\mathcal{FC}(R)}}\,\mathcal{F}(R) \supseteq \widetilde{\mathrm{cores}\,\mathcal{F}}(R).$$

By (1) and Proposition 3.7, we have

$$\widetilde{\mathrm{cores}\,\mathcal{I}(R^{\mathrm{op}})^+} = \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{F}(R) \quad \text{and} \quad \widetilde{\mathrm{cores}\,\mathcal{FC}}(R) = \mathrm{cores}_{\widetilde{\mathcal{FC}(R)}}\,\mathcal{F}(R).$$

Suppose that $M \in \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{F}(R)$ and

$$0 \to M \to F^0 \to F^1 \to \cdots \to F^i \to \cdots \tag{4.4}$$

is a $\mathrm{Hom}_R(-, \mathcal{I}(R^{\mathrm{op}})^+)$-exact exact sequence in $\operatorname{Mod} R$ with all F^i flat. Let $D \in \mathcal{FC}(R)$. Then $D^{++} \in \mathcal{I}(R^{\mathrm{op}})^+$ by [16, Theorem 2.1]. Since D is pure injective by [32, Proposition 4.4 (1)], D is a direct summand of D^{++} by [17, Theorem 2.27]. Notice that (4.4) is $\mathrm{Hom}_R(-, D^{++})$-exact, so it is also $\mathrm{Hom}_R(-, D)$-exact. Thus,

$$M \in \mathrm{cores}_{\widetilde{\mathcal{FC}(R)}}\,\mathcal{F}(R) \quad \text{and} \quad \mathrm{cores}_{\widetilde{\mathcal{I}(R^{\mathrm{op}})^+}}\,\mathcal{F}(R) \subseteq \mathrm{cores}_{\widetilde{\mathcal{FC}(R)}}\,\mathcal{F}(R).$$

Since $\mathcal{I}(R^{\mathrm{op}})^+$ is closed under $\mathcal{I}(R^{\mathrm{op}})^+$-coproper extensions by [18, Horseshoe Lemma 1.7], the latter assertion follows. □

Recall from Remark 4.4 (7) and (8) and [32, Theorem 4.6] that

$$\begin{aligned}
{}^{\perp}(\mathcal{I}(R^{\mathrm{op}})^{+}) \cap \widetilde{\mathrm{cores}_{\mathcal{I}(R^{\mathrm{op}})^{+}}\,\mathcal{F}}(R) &= {}^{\perp}\mathcal{FC}(R) \cap \widetilde{\mathrm{cores}_{\mathcal{FC}(R)}\,\mathcal{F}}(R)\\
&= {}^{\perp}\mathcal{FC}(R) \cap \widetilde{\mathrm{cores}\,\mathcal{FC}}(R) = \mathcal{GF}(R)\\
&\supseteq \mathcal{PGF}(R)\\
&= {}^{\perp}(\mathcal{I}(R^{\mathrm{op}})^{+}) \cap \widetilde{\mathrm{cores}_{\mathcal{I}(R^{\mathrm{op}})^{+}}\,\mathcal{P}}(R).
\end{aligned}$$

In terms of the projective dimensions relative to $\widetilde{\mathrm{cores}\,\mathcal{F}}(R)$ and all eight subcategories of $\mathrm{Mod}\,R$ that appear in the above relation, we give some equivalent characterizations of n-Gorenstein rings as follows.

Theorem 4.13. *The following statements are equivalent:*

(1) *R is n-Gorenstein.*

(2) $\text{G-fd}_R\,M \leq n$ *for any* $M \in \mathrm{Mod}\,R$.

$(2)^{\mathrm{op}}$ $\text{G-fd}_{R^{\mathrm{op}}}\,N \leq n$ *for any* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(3) ${}^{\perp}(\mathcal{I}(R^{\mathrm{op}})^{+})\text{-pd}\,M \leq n$ *and* ${}^{\perp}(\mathcal{I}(R)^{+})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(4) ${}^{\perp}\mathcal{FC}(R)\text{-pd}\,M \leq n$ *and* ${}^{\perp}\mathcal{FC}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(5) $\widetilde{\mathrm{cores}_{\mathcal{I}(R^{\mathrm{op}})^{+}}\,\mathcal{F}}(R)\text{-pd}\,M \leq n$ *and* $\widetilde{\mathrm{cores}_{\mathcal{I}(R)^{+}}\,\mathcal{F}}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(6) $\widetilde{\mathrm{cores}_{\mathcal{FC}(R)}\,\mathcal{F}}(R)\text{-pd}\,M \leq n$ *and* $\widetilde{\mathrm{cores}_{\mathcal{FC}(R^{\mathrm{op}})}\,\mathcal{F}}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(7) $\widetilde{\mathrm{cores}\,\mathcal{FC}}(R)\text{-pd}\,M \leq n$ *and* $\widetilde{\mathrm{cores}\,\mathcal{FC}}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(8) $\widetilde{\mathrm{cores}\,\mathcal{F}}(R)\text{-pd}\,M \leq n$ *and* $\widetilde{\mathrm{cores}\,\mathcal{F}}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(9) $\mathcal{PGF}(R)\text{-pd}\,M \leq n$ *for any* $M \in \mathrm{Mod}\,R$.

$(9)^{\mathrm{op}}$ $\mathcal{PGF}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

(10) $\widetilde{\mathrm{cores}_{\mathcal{I}(R^{\mathrm{op}})^{+}}\,\mathcal{P}}(R)\text{-pd}\,M \leq n$ *and* $\widetilde{\mathrm{cores}_{\mathcal{I}(R)^{+}}\,\mathcal{P}}(R^{\mathrm{op}})\text{-pd}\,N \leq n$ *for any* $M \in \mathrm{Mod}\,R$ *and* $N \in \mathrm{Mod}\,R^{\mathrm{op}}$.

Proof. The implications $(2)+(2)^{\mathrm{op}} \Longrightarrow (3)+(4)$, $(9) \Longrightarrow (2)$, $(9)^{\mathrm{op}} \Longrightarrow (2)^{\mathrm{op}}$ and $(9)+(9)^{\mathrm{op}} \Longrightarrow (10) \Longrightarrow (5)$ are trivial. By Lemma 4.12, we have $(5) \Longleftrightarrow (6) \Longleftrightarrow (7) \Longleftarrow (8)$.

Since

$$\widetilde{\mathrm{cores}\,\mathcal{F}}(R) \supseteq \widetilde{\mathrm{cores}_{\mathcal{F}(R)}\,\mathcal{P}}(R) \quad \text{and} \quad \widetilde{\mathrm{cores}\,\mathcal{F}}(R^{\mathrm{op}}) \supseteq \widetilde{\mathrm{cores}_{\mathcal{F}(R^{\mathrm{op}})}\,\mathcal{P}}(R^{\mathrm{op}}),$$

we have $(1) \Longrightarrow (8)$ by Theorem 4.9.

By [16, Theorem 2.2] and [32, Lemma 4.13], we have $\mathcal{I}(R^{\mathrm{op}})^{+} \subseteq \mathcal{FC}(R)$ and $\mathcal{I}(R)^{+} \subseteq \mathcal{FC}(R^{\mathrm{op}})$. Thus,

$${}^{\perp}(\mathcal{I}(R^{\mathrm{op}})^{+}) \supseteq {}^{\perp}\mathcal{FC}(R) \quad \text{and} \quad {}^{\perp}(\mathcal{I}(R)^{+}) \supseteq {}^{\perp}\mathcal{FC}(R^{\mathrm{op}}),$$

and the implication $(4) \Longrightarrow (3)$ follows.

"$(1) \Longrightarrow (9) + (9)^{\mathrm{op}}$": By (1) and [26, Theorem 2], we have

$$\mathcal{SGF}(R) = \mathcal{PGF}(R) \quad \text{and} \quad \mathcal{SGF}(R^{\mathrm{op}}) = \mathcal{PGF}(R^{\mathrm{op}}).$$

Now the assertion follows from Theorem 4.9.

"$(3) \Longrightarrow (1)$": By [16, Theorem 2.1], we have $({}_RR)^{+} \in \mathcal{I}(R^{\mathrm{op}})$ and $(R_R)^{+} \in \mathcal{I}(R)$. Then, by (3) and dimension shifting, it is easy to see that

$$\mathrm{Ext}_R^{\geq n+1}(M, ({}_RR)^{++}) = 0 = \mathrm{Ext}_{R^{\mathrm{op}}}^{\geq n+1}(N, (R_R)^{++})$$

for any $M \in \mathrm{Mod}\,R$ and $N \in \mathrm{Mod}\,R^{\mathrm{op}}$. This now implies $\mathrm{id}_R({}_RR)^{++} \leq n$ and $\mathrm{id}_{R^{\mathrm{op}}}(R_R)^{++} \leq n$. It follows from [16, Theorems 2.1 and 2.2] that

$$\mathrm{id}_R\,R = \mathrm{fd}_{R^{\mathrm{op}}}({}_RR)^{+} \leq n \quad \text{and} \quad \mathrm{id}_{R^{\mathrm{op}}}\,R = \mathrm{fd}_R(R_R)^{+} \leq n.$$

"$(2) \Longrightarrow (1)$": Similar to the proof of $(3) \Longrightarrow (1)$, we have $\mathrm{id}_R\,R \leq n$. By Remark 4.4 (7), we have that $\mathcal{GF}(R)$ is resolving and admits an $\mathcal{I}_C(R^{\mathrm{op}})^{+}$-coproper cogenerator $\mathcal{F}(R)$. Thus, $\mathrm{id}_{R^{\mathrm{op}}}\,R \leq n$ by (2) and Lemma 4.8.

Symmetrically, we get $(2)^{\mathrm{op}} \Longrightarrow (1)$.

"$(5) \Longrightarrow (1)$": It follows from Lemma 4.12 (2) that $\widetilde{\mathrm{cores}_{\mathcal{I}(R^{\mathrm{op}})^{+}}\,\mathcal{F}}(R)$ is an $\mathcal{I}(R^{\mathrm{op}})^{+}$-precoresolving subcategory of $\mathrm{Mod}\,R$ admitting an $\mathcal{I}(R^{\mathrm{op}})^{+}$-coproper cogenerator $\mathcal{F}(R)$. Thus, $\mathrm{id}_{R^{\mathrm{op}}}\,R \leq n$ by (5) and Lemma 4.8. Symmetrically, we have $\mathrm{id}_R\,R \leq n$. □

4.3 C-Gorenstein flat modules

In this subsection, R, S are arbitrary rings and ${}_RC_S$ is a semidualizing bimodule.

Lemma 4.14. *For any $M \in \operatorname{Mod} R$, we have* $\operatorname{fd}_S M_* = \operatorname{id}_{S^{\mathrm{op}}} M^+ \otimes_R C$.

Proof. By [17, Lemma 2.16 (c)], we have

$$(M_*)^+ \cong M^+ \otimes_R C.$$

It follows from [16, Theorem 2.1] that

$$\operatorname{fd}_S M_* = \operatorname{id}_{S^{\mathrm{op}}}(M_*)^+ = \operatorname{id}_{S^{\mathrm{op}}} M^+ \otimes_R C,$$

as desired. □

We also need the following observation.

Lemma 4.15. *Let $n \geq 0$. Then the following assertions hold:*
(1) *For any $M \in \operatorname{Mod} R$, we have*

$$\mathcal{F}_C(R)\text{-pd}\, M \leq n \quad \textit{if and only if} \quad M \in \mathcal{B}_C(R) \textit{ and } \operatorname{fd}_S M_* \leq n.$$

(2) *For any $N \in \operatorname{Mod} R^{\mathrm{op}}$, we have*

$$\mathcal{I}_C(R^{\mathrm{op}})\text{-id}\, N \leq n \quad \textit{if and only if} \quad N \in \mathcal{A}_C(R^{\mathrm{op}}) \textit{ and } \operatorname{id}_{S^{\mathrm{op}}} N \otimes_R C \leq n.$$

Proof. By [20, Corollary 6.1], we have

$$\mathcal{F}_C(R)\text{-pd}^{<\infty} \subseteq \mathcal{B}_C(R) \quad \text{and} \quad \mathcal{I}_C(R^{\mathrm{op}})\text{-id}^{<\infty} \subseteq \mathcal{A}_C(R^{\mathrm{op}}).$$

Then the assertions follow from [36, Lemma 2.6 (1) and (3)]. □

For any $M \in \operatorname{Mod} R$, we have the following canonical evaluation homomorphism:

$$\sigma_M : M \to M^{++}$$

defined by $\sigma_M(x)(\alpha) = \alpha(x)$ for any $x \in M$ and $\alpha \in M^+$.

Lemma 4.16. (1) *Let I be an injective right S-module. Then $(I_*)^{++} \cong (I^{++})_*$. Moreover, $(I_*)^+ \in \mathcal{F}_C(R)$ if S is a right coherent ring.*
(2) *Let $f : M_1^+ \to M_2^+$ be a homomorphism in* $\operatorname{Mod} R^{\mathrm{op}}$ *with $M_1, M_2 \in \operatorname{Mod} R$. If M_1 is pure injective, then there exists a homomorphism $g : M_2 \to M_1$ in* $\operatorname{Mod} R$ *such that $f = g^+$.*

Proof. (1) Let I be an injective right S-module. Then $(I_*)^+ \cong C \otimes_S I^+$ by [17, Lemma 2.16 (c)], and hence

$$(I_*)^{++} \cong (C \otimes_S I^+)^+ \cong (I^{++})_*$$

by [17, Lemma 2.16 (a)]. If S is a right coherent ring, then I^+ is a flat left S-module by [11, Theorem 1], and hence $(I_*)^+ \cong C \otimes_S I^+ \in \mathcal{F}_C(R)$.

(2) Let $f : M_1^+ \to M_2^+$ be a homomorphism in $\operatorname{Mod} R^{\mathrm{op}}$ with $M_1, M_2 \in \operatorname{Mod} R$. If M_1 is pure injective, then $\sigma_{M_1} : M_1 \to M_1^{++}$ is a split monomorphism in $\operatorname{Mod} R$ by [17, Proposition 2.27]. So there exists a split epimorphism $\beta : M_1^{++} \to M_1$ in $\operatorname{Mod} R$ such that $\beta\sigma_{M_1} = 1_{M_1}$, and hence $(\sigma_{M_1})^+\beta^+ = 1_{M_1^+}$. On the other hand, we also have $(\sigma_{M_1})^+\sigma_{M_1^+} = 1_{M_1^+}$ by [1, Proposition 20.14 (1)]. It follows that

$$\beta^+ = \sigma_{M_1^+}. \tag{4.5}$$

Since the diagram

$$\begin{tikzcd} M_1^+ \arrow[r, "f"] \arrow[d, "\sigma_{M_1^+}"'] & M_2^+ \arrow[d, "\sigma_{M_2^+}"] \\ M_1^{+++} \arrow[r, "f^{++}"] & M_2^{+++} \end{tikzcd}$$

is commutative, we have $\sigma_{M_2^+}f = f^{++}\sigma_{M_1^+}$. Then, by [1, Proposition 20.14 (1)] and (4.5), we have

$$f = 1_{M_2^+}f = (\sigma_{M_2})^+\sigma_{M_2^+}f = (\sigma_{M_2})^+f^{++}\sigma_{M_1^+} = (\sigma_{M_2})^+f^{++}\beta^+ = (\beta f^+\sigma_{M_2})^+.$$

Set $g := \beta f^+\sigma_{M_2}$. Then $f = g^+$. □

The assertions in the following result are the C-versions of [16, Theorem 2.1] and [18, Theorem 3.6], respectively.

Theorem 4.17. *For any $M \in \operatorname{Mod} R$, the following assertions hold:*
(1) $\mathcal{F}_C(R)$-pd $M = \mathcal{I}_C(R^{op})$-id M^+.
(2) G_C-$\mathrm{fd}_R M \geq \mathrm{G}_C$-$\mathrm{id}_{R^{op}} M^+$ *with equality if S is a right coherent ring.*

Proof. (1) For any $n \geq 0$, we have, by Lemma 4.15 (1),

$$\mathcal{F}_C(R)\text{-pd}\, M \leq n \quad \text{if and only if} \quad M \in \mathcal{B}_C(R) \text{ and } \operatorname{fd}_S M_* \leq n.$$

The latter is equivalent, by [25, Proposition 3.2(b)] and Lemma 4.14, to

$$M^+ \in \mathcal{A}_C(R^{op}) \text{ and } \operatorname{id}_{S^{op}} M^+ \otimes_R C \leq n,$$

which in turn, by Lemma 4.15 (2), is equivalent to

$$\mathcal{I}_C(R^{op})\text{-id}\, M^+ \leq n.$$

(2) Let $E \in \mathcal{I}_C(R^{op})$ and $n \geq 1$. By [17, Lemma 2.16 (a) and (b)], we have

$$(E \otimes_R -)^+ \cong \operatorname{Hom}_{R^{op}}(E, (-)^+), \tag{4.6}$$
$$[\operatorname{Tor}_n^R(E, -)]^+ \cong \operatorname{Ext}_{R^{op}}^n(E, (-)^+). \tag{4.7}$$

If $G \in \mathcal{GF}_C(R)$, then $G \in \mathcal{I}_C(R^{op})^\top$ and there exists an $(\mathcal{I}_C(R^{op}) \otimes_R -)$-exact exact sequence

$$0 \to G \to Q^0 \to Q^1 \to \cdots \to Q^i \to \cdots$$

in $\operatorname{Mod} R$ with all Q^i in $\mathcal{F}_C(R)$. It follows from (1) and the above two isomorphisms that

$$G^+ \in \mathcal{I}_C(R^{op})^\perp \cap \operatorname{res}\widetilde{\mathcal{I}_C(R^{op})},$$

and thus $G^+ \in \mathcal{GI}_C(R^{op})$ by Remark 4.4 (3) (b). Then it is easy to get G_C-$\mathrm{fd}_R M \geq \mathrm{G}_C$-$\mathrm{id}_{R^{op}} M^+$ for any $M \in \operatorname{Mod} R$.

Now, let S be a right coherent ring and let $G \in \operatorname{Mod} R$.

Claim. If $G^+ \in \mathcal{GI}_C(R^{op})$, then $G \in \mathcal{GF}_C(R)$.

By Remark 4.4 (3) (b), we have

$$G^+ \in \mathcal{I}_C(R^{op})^\perp \cap \operatorname{res}\widetilde{\mathcal{I}_C(R^{op})}.$$

It follows from (4.7) that $G \in \mathcal{I}_C(R^{op})^\top$. In addition, there exists the following $\operatorname{Hom}_{R^{op}}(\mathcal{I}_C(R^{op}), -)$-exact exact sequence:

$$\cdots \to (I_i)_* \to \cdots \to (I_1)_* \to (I_0)_* \to G^+ \to 0 \tag{4.8}$$

in $\operatorname{Mod} R^{op}$ with all I_i injective right S-modules. Set $K_i := \operatorname{Im}((I_i)_* \to (I_{i-1})_*)$ for any $i \geq 1$. Since $I_0 \oplus I_0' \cong I_0^{++}$ for some injective right S-module I_0', from Lemma 4.16 (1) and the exact sequence (4.8), we get the following $\operatorname{Hom}_{R^{op}}(\mathcal{I}_C(R^{op}), -)$-exact short exact sequence:

$$0 \to K_1 \oplus (I_0')_* \to (I_0)_* \oplus (I_0')_* (\cong ((I_0)_*)^{++}) \to G^+ \to 0$$

in $\operatorname{Mod} R^{op}$. Similarly, since $(I_1 \oplus I_0') \oplus I_1' \cong (I_1 \oplus I_0')^{++}$ for some injective right S-module I_1', from Lemma 4.16 (1) and the exact sequence (4.8), we get the following $\operatorname{Hom}_{R^{op}}(\mathcal{I}_C(R^{op}), -)$-exact short exact sequence:

$$0 \to K_2 \oplus (I_1')_* \to (I_1)_* \oplus (I_0')_* \oplus (I_1')_* (\cong ((I_1 \oplus I_0')_*)^{++}) \to K_1 \oplus (I_0')_* \to 0$$

in Mod R^{op}. Continuing this process and splicing these obtained short exact sequences, we get the following $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathcal{I}_C(R^{\mathrm{op}}), -)$-exact exact sequence:

$$\cdots \to ((I_i \oplus I'_{i-1})_*)^{++} \to \cdots \to ((I_1 \oplus I'_0)_*)^{++} \to ((I_0)_*)^{++} \to G^+ \to 0 \tag{4.9}$$

in Mod R^{op} with all I'_i injective right S-modules. Since $((I_0)_*)^+$ and all $((I_i \oplus I'_{i-1})_*)^+$ are pure injective by [13, Proposition 5.3.7], according to Lemma 4.16 (2), we can rewrite (4.9) as follows:

$$\cdots \longrightarrow ((I_i \oplus I'_{i-1})_*)^{++} \xrightarrow{(g_i)^+} \cdots \to ((I_1 \oplus I'_0)_*)^{++} \xrightarrow{(g_1)^+} ((I_0)_*)^{++} \xrightarrow{(g_0)^+} G^+ \longrightarrow 0.$$

Then, by (4.6), we get the following $(\mathcal{I}_C(R^{\mathrm{op}}) \otimes_R -)$-exact exact sequence:

$$0 \longrightarrow G \xrightarrow{g_0} ((I_0)_*)^+ \xrightarrow{g_1} ((I_1 \oplus I'_0)_*)^+ \longrightarrow \cdots \xrightarrow{g_i} ((I_i \oplus I'_{i-1})_*)^+ \longrightarrow \cdots$$

in Mod R. By Lemma 4.16 (1), we have that $((I_0)_*)^+$ and all $((I_i \oplus I'_{i-1})_*)^+$ are in $\mathcal{F}_C(R)$. Consequently, we conclude that $G \in \mathcal{GF}_C(R)$. The claim is proved.

Let $M \in \mathrm{Mod}\, R$ with $\mathrm{G}_C\text{-id}_{R^{\mathrm{op}}} M^+ = n < \infty$, and let

$$0 \to K_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

be an exact sequence in Mod R with all G_i in $\mathcal{GF}_C(R)$. Then we get the following exact sequence:

$$0 \to M^+ \to G_0^+ \to G_1^+ \cdots \to G_{n-1}^+ \to K_n^+ \to 0$$

in Mod R^{op}. By the former assertion, all G_i^+ are in $\mathcal{GI}_C(R^{\mathrm{op}})$. It follows from Remark 4.4 (3) (b) and Lemma 3.8 (1) that $K_n^+ \in \mathcal{GI}_C(R^{\mathrm{op}})$. Then $K_n \in \mathcal{GF}_C(R)$ by the above claim, and thus $\mathrm{G}_C\text{-fd}_R M \le n$. □

As a consequence, we get the following result, in which assertion (1) generalizes [20, Lemma 5.2 (a)].

Corollary 4.18. *For any $n \ge 0$, the following assertions hold:*
(1) *The class of left R-modules with $\mathcal{F}_C(R)$-projective dimension at most n is closed under pure submodules and pure quotients; in particular, the class $\mathcal{F}_C(R)$ is closed under pure submodules and pure quotients.*
(2) *If S is a right coherent ring, then the class of left R-modules with $\mathcal{GF}_C(R)$-projective dimension at most n is closed under pure submodules and pure quotients; in particular, the class $\mathcal{GF}_C(R)$ is closed under pure submodules and pure quotients.*

Proof. (1) Let

$$0 \to K \to G \to L \to 0$$

be a pure exact sequence in Mod R with $\mathcal{F}_C(R)\text{-pd}\, G \le n$. Then, by [13, Proposition 5.3.8], the induced exact sequence

$$0 \to L^+ \to G^+ \to K^+ \to 0$$

splits and both K^+ and L^+ are direct summands of G^+. By Theorem 4.17 (1), we have $\mathcal{I}_C(R^{\mathrm{op}})\text{-id}\, G^+ \le n$. Since $\mathcal{I}_C(R^{\mathrm{op}})$ is closed under direct summands by [20, Proposition 5.1 (c)], the class of right R-modules with $\mathcal{I}_C(R^{\mathrm{op}})$-injective dimension at most n is closed under direct summands by [24, Corollary 4.9]. It follows that $\mathcal{I}_C(R^{\mathrm{op}})\text{-id}\, K^+ \le n$ and $\mathcal{I}_C(R^{\mathrm{op}})\text{-id}\, L^+ \le n$. Thus, $\mathcal{F}_C(R)\text{-pd}\, K \le n$ and $\mathcal{F}_C(R)\text{-pd}\, L \le n$ by Theorem 4.17 (1) again.

(2) It is trivial that $\mathcal{I}_C(R^{\mathrm{op}})^{\perp}$ is closed under direct summands. By [23, Theorem 4.6 (1)], the class $\mathrm{res}\,\widetilde{\mathcal{I}_C(R^{\mathrm{op}})}$ is closed under direct summands. Notice that

$$\mathcal{GI}_C(R^{\mathrm{op}}) = \mathcal{I}_C(R^{\mathrm{op}})^{\perp} \cap \mathrm{res}\,\widetilde{\mathcal{I}_C(R^{\mathrm{op}})}$$

by Remark 4.4 (3) (b), thus $\mathcal{GI}_C(R^{\mathrm{op}})$ is also closed under direct summands. We also know from Remark 4.4 (3) (b) that $\mathcal{GI}_C(R^{\mathrm{op}})$ is coresolving in Mod R^{op}. Thus, the class of right R-modules with $\mathcal{GI}_C(R^{\mathrm{op}})$-injective dimension at most n is closed under direct summands by [24, Corollary 4.9]. Now applying Theorem 4.17 (2), we obtain the assertion by using an argument similar to that in the proof of (1). □

In the following result, assertion (1) is the C-version of [8, Theorem 2.2]. Assertion (3) means that the assumption "R is a right coherent ring" in [18, Theorem 3.24] is superfluous; compare it with Corollaries 4.6 (2) and 4.7 (2).

Theorem 4.19. (1) *For any* $M \in \operatorname{Mod} R$, *it holds*

$$\mathrm{G}_C\text{-}\mathrm{fd}_R M \leq \mathcal{F}_C(R)\text{-pd}\, M,$$

with equality if $\mathcal{F}_C(R)\text{-pd}\, M < \infty$.

(2) $\mathcal{F}_C(R)\text{-FPD} \leq \mathcal{GF}_C(R)\text{-FPD}$, *with equality if* $\mathcal{GF}_C(R)$ *is closed under extensions.*

(3) $\mathcal{F}(R)\text{-FPD} = \mathcal{GF}(R)\text{-FPD}$.

Proof. (1) Since $\mathcal{GF}_C(R) \subseteq \mathcal{F}_C(R)$, we have

$$\mathrm{G}_C\text{-}\mathrm{fd}_R M \leq \mathcal{F}_C(R)\text{-pd}\, M \quad \text{for any } M \in \operatorname{Mod} R.$$

Now let $\mathcal{F}_C(R)\text{-pd}\, M < \infty$. By Theorem 4.17 (1),

$$\mathcal{I}_C(R^{\mathrm{op}})\text{-id}\, M^+ < \infty.$$

This implies, by Corollary 4.7 (1),

$$\mathrm{G}_C\text{-}\mathrm{id}_{R^{\mathrm{op}}} M^+ = \mathcal{I}_C(R^{\mathrm{op}})\text{-id}\, M^+.$$

This in turn implies, by Theorem 4.17,

$$\mathrm{G}_C\text{-}\mathrm{fd}_R M \geq \mathcal{F}_C(R)\text{-pd}\, M,$$

which finally implies

$$\mathrm{G}_C\text{-}\mathrm{fd}_R M = \mathcal{F}_C(R)\text{-pd}\, M.$$

(2) The assertion that $\mathcal{F}_C(R)\text{-FPD} \leq \mathcal{GF}_C(R)\text{-FPD}$ follows from (1).

It is trivial that $\mathcal{P}(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{GF}_C(R)$. By Remark 4.4 (8), we have that

$$\mathcal{GF}_C(R) = {}^{\perp}(\mathcal{I}_C(R^{\mathrm{op}})^+) \cap \operatorname{cores} \widetilde{\mathcal{I}_C(R^{\mathrm{op}})^+}\, \mathcal{F}_C(R)$$

and it admits an $\mathcal{I}_C(R^{\mathrm{op}})^+$-coproper cogenerator $\mathcal{F}_C(R)$. If $\mathcal{GF}_C(R)$ is closed under extensions, then $\mathcal{GF}_C(R)$ is resolving in $\operatorname{Mod} R$ by Proposition 3.6. Now let $M \in \operatorname{Mod} R$ with $\mathrm{G}_C\text{-}\mathrm{fd}_R M = n < \infty$. By Corollary 3.4 (2), there exists an exact sequence

$$0 \to M \to K' \to T' \to 0$$

in $\operatorname{Mod} R$ with $\mathcal{F}_C(R)\text{-pd}\, K' = n$. It follows that $\mathcal{GF}_C(R)\text{-FPD} \leq \mathcal{F}_C(R)\text{-FPD}$.

(3) Since $\mathcal{GF}(R)$ is closed under extensions by [31, Theorem 4.11], the assertion follows from (2) by putting ${}_RC_S = {}_RR_R$. □

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