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RESEARCH ARTICLE

# $G$-stable support $\tau$-tilting modules

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**Abstract** Motivated by $\tau$-tilting theory developed by T. Adachi, O. Iyama, I. Reiten, for a finite-dimensional algebra $\Lambda$ with action by a finite group $G$, we introduce the notion of $G$-stable support $\tau$-tilting modules. Then we establish bijections among $G$-stable support $\tau$-tilting modules over $\Lambda$, $G$-stable two-term silting complexes in the homotopy category of bounded complexes of finitely generated projective $\Lambda$-modules, and $G$-stable functorially finite torsion classes in the category of finitely generated left $\Lambda$-modules. In the case when $\Lambda$ is the endomorphism of a $G$-stable cluster-tilting object $T$ over a $\mathsf{Hom}$-finite 2-Calabi-Yau triangulated category $\mathscr{C}$ with a $G$-action, these are also in bijection with $G$-stable cluster-tilting objects in $\mathscr{C}$. Moreover, we investigate the relationship between stable support $\tau$-tilitng modules over $\Lambda$ and the skew group algebra $\Lambda G$.

**Keywords** $G$-stable support $\tau$-tilting modules, $G$-stable two-term silting complexes, $G$-stable functorially finite torsion classes, $G$-stable cluster-tilting objects, bijection, skew group algebras
**MSC** 16G10

## 1 Introduction

It is well known that tilting theory is a theoretical basis in the representation theory of finite-dimensional algebras, in which the notion of tilting modules is fundamental. Moreover, in the representation theory of algebras, the notion of "mutation" often plays an important role. Mutation is an operation for a certain class of objects in a fixed category to construct a new object from a given one by replacing a summand. Happel and Unger [12] gave some necessary and sufficient conditions under which mutation of tilting modules is possible; however, mutation of tilting modules is not always possible. Adachi et al. [2] introduced the notion of support $\tau$-tilting modules which generalizes that of tilting modules, and showed that mutation of support $\tau$-tilting modules is

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always possible. This is a big advantage of "support $\tau$-tilting mutation" which "tilting mutation" does not have. Note that the $\tau$-tilting theory developed in [2] has stimulated several investigations; in particular, there is a close relation between support $\tau$-tilting modules and some other important notions in the representation theory of algebras, such as torsion classes, silting complexes, cluster-tilting objects, Grothendieck groups, and $*$-modules, see [2,3,7,13–15,19] and so on. Moreover, Adachi [1] gave a classification of $\tau$-tilting modules over Nakayama algebras and an algorithm to construct the exchange quiver of support $\tau$-tilting modules. Zhang [20] studied $\tau$-rigid modules which are direct summands of support $\tau$-tilting modules over algebras with radical square zero.

On the other hand, the notion of skew group algebras was introduced in [18]. Let $\Lambda$ be a finite-dimensional algebra, and let $G$ be a finite group such that its order $|G|$ is invertible in $\Lambda$ acting on $\Lambda$. The algebra $\Lambda$ and the skew group algebra $\Lambda G$ have a lot of properties in common.

The aim of this paper is to introduce and study $G$-stable support $\tau$-tilting modules, and moreover, to establish bijections among them and $G$-stable two-term silting complexes, $G$-stable functorially finite torsion classes, and $G$-stable cluster-tilting objects. Moreover, we investigate the relationship between stable support $\tau$-tilting modules over $\Lambda$ and the skew group algebra $\Lambda G$. This paper is organized as follows.

In Section 2, we give some terminology and some known results.

In Section 3, we prove the following theorem.

**Theorem 1.1** (Theorems 3.4, 3.7, 3.13, 3.15) *Let $\Lambda$ be a finite-dimensional algebra, and let $G$ be a finite group acting on $\Lambda$. Then there exist bijections among*

(1) *the set $G$-$s\tau$-$\mathsf{tilt}\,\Lambda$ of isomorphism classes of basic $G$-stable support $\tau$-tilting modules in $\mathsf{mod}\,\Lambda$;*

(2) *the set $G$-2-$\mathsf{silt}\,\Lambda$ of isomorphism classes of basic $G$-stable two-term silting complexes in $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$;*

(3) *the set $G$-$f$-$\mathsf{tors}\,\Lambda$ of $G$-stable functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.*

*Furthermore, if $\Lambda = \mathsf{End}_{\mathscr{C}}(T)$, where $\mathscr{C}$ is a $\mathsf{Hom}$-finite 2-Calabi-Yau triangulated category with a $G$-action and $T$ is a $G$-stable cluster-tilting object, then there exists also a bijection between the following set and any one of the above sets:*

(4) *the set $G$-$c$-$\mathsf{tilt}\,\mathscr{C}$ of isomorphism classes of basic $G$-stable cluster-tilting objects in $\mathscr{C}$.*

Let $\Lambda$ be a finite-dimensional self-injective algebra, and let $G = \langle\nu\rangle$ be the subgroup of the automorphism group of $\Lambda$ generated by the Nakayama automorphism. Then $\nu$-stable support $\tau$-tilting modules introduced by Mizuno [17] are exactly $G$-stable support $\tau$-tilting modules in our sense. So Theorem 1.1 is a generalization of [17, Theorem 1.1].

In Section 4, we investigate the relationship between $G$-stable support $\tau$-tilting $\Lambda$-modules and $\mathbb{X}$-stable support $\tau$-tilting $\Lambda G$-modules, where $\mathbb{X}$, the group of characters of $G$, naturally acts on $\Lambda G$. We have the following result.

**Theorem 1.2** (Theorems 4.2 (3), 4.6) *Let $\Lambda$ be a finite-dimensional algebra, and let $G$ be a finite group acting on $\Lambda$ such that $|G|$ is invertible in $\Lambda$. Then the functor $\Lambda G \otimes_\Lambda -\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda G$ preserves stability and induces the following injection:*

$$G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda \to \mathbb{X}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G,$$
$$T \mapsto \Lambda G \otimes_\Lambda T.$$

*Moreover, if $G$ is solvable, then this map is a bijection.*

Finally, we give an example to illustrate this theorem.

## 2 Preliminaries

In this section, we give some terminology and some known results.

Let $k$ be an algebraically closed field and we denote by $D := \mathsf{Hom}_k(-, k)$. By an algebra $\Lambda$, we mean a finite-dimensional algebra over $k$. We denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated left $\Lambda$-modules, by $\mathsf{proj}\,\Lambda$ and $\mathsf{inj}\,\Lambda$ the subcategories of $\mathsf{mod}\,\Lambda$ consisting of projective modules and injective modules, respectively, and by $\tau$ the Auslander-Reiten translation of $\Lambda$. We denote by $\mathsf{K}^b(\mathsf{proj}\Lambda)$ the homotopy category of bounded complexes of $\mathsf{proj}\,\Lambda$. For $X \in \mathsf{mod}\,\Lambda$, we denote by $\mathsf{add}\,X$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of all direct summands of finite direct sums of copies of $X$, and by $\mathsf{Fac}\,X$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of all factor modules of finite direct sums of copies of $X$.

### 2.1 $\tau$-tilting theory

First, we recall the definition of support $\tau$-tilting modules from [2].

Let $(X, P)$ be a pair with $X \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$.

(1) We call $X$ in $\mathsf{mod}\,\Lambda$ *$\tau$-rigid* if $\mathsf{Hom}_\Lambda(X, \tau X) = 0$. We call $(X, P)$ a *$\tau$-rigid pair* if $X$ is $\tau$-rigid and $\mathsf{Hom}_\Lambda(P, X) = 0$.

(2) We call $X$ in $\mathsf{mod}\,\Lambda$ *$\tau$-tilting* (resp., *almost complete $\tau$-tilting*) if $X$ is $\tau$-rigid and $|X| = |\Lambda|$ (resp., $|X| = |\Lambda| - 1$), where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of $X$.

(3) We call $X$ in $\mathsf{mod}\,\Lambda$ *support $\tau$-tilting* if there exists an idempotent $e$ of $\Lambda$ such that $X$ is a $\tau$-tilting $(\Lambda/\langle e\rangle)$-module. We call $(X, P)$ a *support $\tau$-tilting pair* (resp., *almost complete support $\tau$-tilting pair*) if $(X, P)$ is $\tau$-rigid and $|X| + |P| = |\Lambda|$ (resp., $|X| + |P| = |\Lambda| - 1$).

We say that $(X, P)$ is *basic* if $X$ and $P$ are basic. Moreover, $X$ determines $P$ uniquely up to isomorphism. We denote by $s\tau\text{-}\mathsf{tilt}\,\Lambda$ the set of isomorphism classes of basic support $\tau$-tilting $\Lambda$-modules.

**Proposition 2.1** [2, Proposition 2.3] *Let $X \in \mathsf{mod}\,\Lambda$ and $P, Q \in \mathsf{proj}\,\Lambda$, and let $e$ be an idempotent of $\Lambda$ such that $\mathsf{add}\,P = \mathsf{add}\,\Lambda e$.*

(1) $(X,P)$ *is a* $\tau$*-rigid pair for* $\Lambda$ *if and only if* $X$ *is a* $\tau$*-rigid* $(\Lambda/\langle e\rangle)$*-module.*

(2) *If both* $(X,P)$ *and* $(X,Q)$ *are support* $\tau$*-tilting pairs for* $\Lambda$*, then* $\mathsf{add}\,P = \mathsf{add}\,Q$*. In other words,* $X$ *determines* $P$ *and* $e$ *uniquely up to equivalence.*

Let $\mathscr{T}$ be a full subcategory of $\mathsf{mod}\,\Lambda$. Assume that $T \in \mathscr{T}$ and $D \in \mathsf{mod}\,\Lambda$. The morphism $f\colon D \to T$ is called a *left* $\mathscr{T}$*-approximation* of $D$ if

$$\mathsf{Hom}_\Lambda(T,T') \to \mathsf{Hom}_\Lambda(D,T') \to 0$$

is exact for any $T' \in \mathscr{T}$. The subcategory $\mathscr{T}$ is called *covariantly finite* in $\mathsf{mod}\,\Lambda$ if every module in $\mathsf{mod}\,\Lambda$ has a left $\mathscr{T}$-approximation. The notions of *right* $\mathscr{T}$*-approximations* and *contravariantly finite subcategories* of $\mathsf{mod}\,\Lambda$ are defined dually. The subcategory $\mathscr{T}$ is called *functorially finite* in $\mathsf{mod}\,\Lambda$ if it is both covariantly finite and contravariantly finite in $\mathsf{mod}\,\Lambda$ ([4]).

Recall that $T \in \mathsf{mod}\,\Lambda$ is called *partial tilting* if the projective dimension of $T$ is at most one and $\mathsf{Ext}^1_\Lambda(T,T) = 0$. A partial tilting module is called *tilting* if there exists an exact sequence

$$0 \to \Lambda \to T' \to T'' \to 0$$

in $\mathsf{mod}\,\Lambda$ with $T', T'' \in \mathsf{add}\,T$ (see [6,11]). We have $|T| = |\Lambda|$ for any tilting module $T$ by [6, Theorem 2.1]. The following result gives a similar criterion for a $\tau$-rigid $\Lambda$-module to be support $\tau$-tilting.

**Proposition 2.2** [15, Proposition 2.14] *Let* $M$ *be a* $\tau$*-rigid* $\Lambda$*-module. Then* $M$ *is a support* $\tau$*-tilting* $\Lambda$*-module if and only if there exists an exact sequence*

$$\Lambda \xrightarrow{f} M' \xrightarrow{g} M'' \to 0$$

*in* $\mathsf{mod}\,\Lambda$ *with* $M', M'' \in \mathsf{add}\,M$ *and* $f$ *a left* $\mathsf{add}\,M$*-approximation of* $\Lambda$.

### 2.2 Functorially finite torsion classes

Let $\mathscr{T}$ be a full subcategory of $\mathsf{mod}\,\Lambda$. Recall that $\mathscr{T}$ is called a *torsion class* if it is closed under factor modules and extensions. We denote by $f\text{-}\mathsf{tors}\,\Lambda$ the set of functorially finite torsion classes in $\mathsf{mod}\,\Lambda$. We say that $X \in \mathscr{T}$ is $\mathsf{Ext}$*-projective* if $\mathsf{Ext}^1_\Lambda(X,\mathscr{T}) = 0$. We denote by $P(\mathscr{T})$ the direct sum of one copy of each of the indecomposable $\mathsf{Ext}$-projective objects in $\mathscr{T}$ up to isomorphism. We have $P(\mathscr{T}) \in \mathsf{mod}\,\Lambda$ if $\mathscr{T} \in f\text{-}\mathsf{tors}\,\Lambda$ ([5, Corollary 4.4]). The following result establishes a relation between $s\tau\text{-}\mathsf{tilt}\,\Lambda$ and $f\text{-}\mathsf{tors}\,\Lambda$.

**Theorem 2.3** [2, Theorem 2.7] *There exists a bijection*

$$s\tau\text{-}\mathsf{tilt}\,\Lambda \leftrightarrow f\text{-}\mathsf{tors}\,\Lambda$$

*given by* $s\tau\text{-}\mathsf{tilt}\,\Lambda \ni T \mapsto \mathsf{Fac}\,T \in f\text{-}\mathsf{tors}\,\Lambda$ *and* $f\text{-}\mathsf{tors}\,\Lambda \ni \mathscr{T} \mapsto P(\mathscr{T}) \in s\tau\text{-}\mathsf{tilt}\,\Lambda$.

### 2.3 Silting complexes

Recall from [3] that $P \in \mathsf{K}^b(\mathsf{proj}\,\Lambda)$ is called *silting* if

$$\mathsf{Hom}_{\mathsf{K}^b(\mathsf{proj}\,\Lambda)}(P, P[i]) = 0$$

for any $i > 0$ and $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ is the smallest full subcategory of $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ containing $P$ and is closed under cones, $[\pm 1]$ and direct summands; and a complex $P = (P^i, d^i)$ in $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ is called *two-term* if $P^i = 0$ for all $i \neq 0, -1$. We denote by $2\text{-}\mathsf{silt}\,\Lambda$ the set of isomorphism classes of basic two-term silting complexes in $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$. The following result establishes a relation between $2\text{-}\mathsf{silt}\,\Lambda$ and $s\tau\text{-}\mathsf{tilt}\,\Lambda$.

**Theorem 2.4** [2, Theorem 3.2] *There exists a bijection:*

$$2\text{-}\mathsf{silt}\,\Lambda \leftrightarrow s\tau\text{-}\mathsf{tilt}\,\Lambda$$

*given by*

$$2\text{-}\mathsf{silt}\,\Lambda \ni P \mapsto H^0(P) \in s\tau\text{-}\mathsf{tilt}\,\Lambda,$$

$$s\tau\text{-}\mathsf{tilt}\,\Lambda \ni (M, P) \mapsto (P_1 \oplus P \overset{(f,0)}{\to} P_0) \in 2\text{-}\mathsf{silt}\,\Lambda,$$

*where $f\colon P_1 \to P_0$ is a minimal projective presentation of $M$.*

### 2.4 Cluster tilting objects

Let $\mathscr{C}$ be a $k$-linear $\mathsf{Hom}$-finite Krull-Schmidt triangulated category. Assume that $\mathscr{C}$ is a 2-Calabi-Yau triangulated category, that is, there exists a functorial isomorphism:

$$D\,\mathsf{Ext}^1_{\mathscr{C}}(X, Y) \cong \mathsf{Ext}^1_{\mathscr{C}}(Y, X).$$

An important class of objects in such categories is that of cluster-tilting objects. Following [8], an object $T \in \mathscr{C}$ is called *cluster-tilting* if

$$\mathsf{add}\,T = \{X \in \mathscr{C} \mid \mathsf{Hom}_{\mathscr{C}}(T, X[1]) = 0\}.$$

We denote by $c\text{-}\mathsf{tilt}\,\mathscr{C}$ the set of isomorphism classes of basic cluster-tilting objects in $\mathscr{C}$. Assume that $\mathscr{C}$ has a cluster-tilting object $T$ and $\Lambda := \mathsf{End}_{\mathscr{C}}(T)^{\mathrm{op}}$. For $X \in \mathscr{C}$, we have a triangle

$$T_1 \overset{g}{\to} T_0 \overset{f}{\to} X \to T_1[1], \qquad (*)$$

where $T_1, T_0 \in \mathsf{add}\,T$ and $f$ is a minimal right $\mathsf{add}\,T$-approximation.

We have the following results, which will be used frequently in this paper.

**Theorem 2.5** [9, Theorem 2.2], [16, p. 126] *There exists an equivalence of categories*

$$\overline{(-)} := \mathsf{Hom}_{\mathscr{C}}(T, -)\colon \mathscr{C}/[T[1]] \to \mathsf{mod}\,\Lambda,$$

*where $[T[1]]$ is the ideal of $\mathscr{C}$ consisting of morphisms which factor through $\mathsf{add}\,T[1]$.*

**Theorem 2.6** [2, Theorem 4.1] *There exists a bijection*

$$c\text{-}\mathsf{tilt}\,\mathscr{C} \leftrightarrow s\tau\text{-}\mathsf{tilt}\,\Lambda$$

*given by*

$$c\text{-}\mathsf{tilt}\,\mathscr{C} \ni X = X' \oplus X'' \mapsto \widetilde{X} := (\overline{X'}, \overline{X''[-1]})) \in s\tau\text{-}\mathsf{tilt}\,\Lambda,$$

*where $X''$ is a maximal direct summand of $X$ belonging to* $\mathsf{add}\,T[1]$.

**Theorem 2.7** [2, Theorem 4.7] *There exists a bijection*

$$c\text{-}\mathsf{tilt}\,\mathscr{C} \leftrightarrow 2\text{-}\mathsf{silt}\,\Lambda$$

*given by*

$$c\text{-}\mathsf{tilt}\,\mathscr{C} \ni X \mapsto (\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}) \in 2\text{-}\mathsf{silt}\,\Lambda,$$

*where $g$ is the morphism in* $(*)$.

### 2.5 Skew group algebras

In this subsection, we recall the definition of skew group algebras and some useful results from [18].

Let $\Lambda$ be an algebra, and let $G$ be a group with identity 1. Consider an action of $G$ on $\Lambda$, that is, a map $G \times \Lambda \to \Lambda$ via $(\sigma, \lambda) \mapsto \sigma(\lambda)$ such that

(1) for any $\sigma \in G$, the map $\sigma\colon \Lambda \to \Lambda$ is an algebra automorphism;

(2) $(\sigma\sigma')(\lambda) = \sigma(\sigma'(\lambda))$ for any $\sigma, \sigma' \in G$ and $\lambda \in \Lambda$;

(3) $1(\lambda) = \lambda$ for any $\lambda \in \Lambda$.

Let $G$ be a finite group. For any $X \in \mathsf{mod}\,\Lambda$ and $\sigma \in G$, let ${}^{\sigma}X$ be a $\Lambda$-module as follows: as a $k$-vector space ${}^{\sigma}X = X$, the action on ${}^{\sigma}X$ is given by $\lambda \cdot x = \sigma^{-1}(\lambda)x$ for any $\lambda \in \Lambda$ and $x \in X$. Given a morphism of $\Lambda$-modules $f\colon X \to Y$, define ${}^{\sigma}f\colon {}^{\sigma}X \to {}^{\sigma}Y$ by ${}^{\sigma}f(x) = f(x)$ for any $x \in {}^{\sigma}X$. Then ${}^{\sigma}f$ is also a $\Lambda$-homomorphism. Indeed, for any $x \in X$ and $\lambda \in \Lambda$, we have

$${}^{\sigma}f(\lambda \cdot x) = f(\sigma^{-1}(\lambda)x) = \sigma^{-1}(\lambda)f(x) = \lambda \cdot {}^{\sigma}f(x).$$

For any $X, Y \in \mathsf{mod}\,\Lambda$ and $f \in \mathsf{Hom}_{\Lambda}(X, Y)$, we define a functor ${}^{\sigma}(-)$ by

$${}^{\sigma}(-)(X) = {}^{\sigma}X, \quad {}^{\sigma}(-)(f) = {}^{\sigma}f.$$

One can check that ${}^{\sigma}(-)\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda$ is an automorphism and the inverse is ${}^{\sigma^{-1}}(-)$. So we have $X \in \mathsf{mod}\,\Lambda$ is indecomposable (resp., projective, injective, simple) if and only if so is ${}^{\sigma}X$ in $\mathsf{mod}\,\Lambda$.

The *skew group algebra* $\Lambda G$ of $G$ over $\Lambda$ is given by the following data:

(1) as an abelian group $\Lambda G$ is a free left $\Lambda$-module with the elements of $G$ as a basis;

(2) the multiplication in $\Lambda G$ is defined by the rule

$$(\lambda_\sigma \sigma)(\lambda_\tau \tau) = (\lambda_\sigma \sigma(\lambda_\tau))\sigma\tau, \quad \forall\, \lambda_\sigma, \lambda_\tau \in \Lambda, \forall\, \sigma, \tau \in G.$$

When $G$ is a finite group such that $|G|$ is invertible in $\Lambda$, the natural inclusion $\Lambda \hookrightarrow \Lambda G$ induces the induction functor

$$F = \Lambda G \otimes_\Lambda - \colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda G$$

and the restriction functor $H \colon \mathsf{mod}\,\Lambda G \to \mathsf{mod}\,\Lambda$.

**Lemma 2.8** [18, pp. 227, 235] *Let $G$ be a finite group such that $|G|$ is invertible in $\Lambda$.*

(1) *$(F, H)$ and $(H, F)$ are adjoint pairs of functors. Consequently, $F$ and $H$ are both exact, and hence, both preserve projective modules and injective modules.*

(2) *Let $M \in \mathsf{mod}\,\Lambda$ and $\sigma \in G$. Then the subset*

$$\sigma \otimes_\Lambda M = \{\sigma \otimes_\Lambda m \mid m \in M\}$$

*of $FM$ has a structure of $\Lambda$-module given by*

$$\lambda(\sigma \otimes_\Lambda m) = \sigma\sigma^{-1}(\lambda) \otimes_\Lambda m = \sigma \otimes_\Lambda (\lambda \cdot m), \quad \forall\, \lambda \in \Lambda,$$

*so that $\sigma \otimes_\Lambda M$ and ${}^\sigma M$ are isomorphic as $\Lambda$-modules. Therefore, as $\Lambda$-modules, we have*

$$FM \cong \bigoplus_{\sigma \in G} (\sigma \otimes_\Lambda M) \cong \bigoplus_{\sigma \in G} {}^\sigma M,$$

*and then*

$$HFM \cong \bigoplus_{\sigma \in G} (\sigma \otimes_\Lambda M) \cong \bigoplus_{\sigma \in G} {}^\sigma M.$$

## 3 $G$-stable support $\tau$-tilting modules

### 3.1 Definitions

From now on, $\Lambda$ is an algebra with action by a finite group $G$.

In this subsection, we introduce the notions of $G$-stable support $\tau$-tilting modules, $G$-stable torsion classes, and $G$-stable two-term silting complexes.

Recall from [10] that a tilting $\Lambda$-module $T$ is called *$G$-stable* if ${}^\sigma T \cong T$ for any $\sigma \in G$. Motivated by this, we introduce the following definition.

**Definition 3.1** (1) We say that a support $\tau$-tilting module $X$ in $\mathsf{mod}\,\Lambda$ is *$G$-stable* if ${}^\sigma X \cong X$ for any $\sigma \in G$.

(2) We say that a support $\tau$-tilting pair (or a $\tau$-rigid pair) $(X, P)$ for $\Lambda$ is *$G$-stable* if ${}^\sigma X \cong X$ and ${}^\sigma P \cong P$ for any $\sigma \in G$.

(3) We say that a torsion class $\mathscr{T}$ is *$G$-stable* if ${}^\sigma \mathscr{T} = \mathscr{T}$ for any $\sigma \in G$.

We denote by $G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda$ the set of isomorphism classes of basic $G$-stable support $\tau$-tilting $\Lambda$-modules and $G\text{-}f\text{-}\mathsf{tors}\,\Lambda$ the set of $G$-stable functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.

The following result shows that in a support $\tau$-tilting pair $(T,P)$, the $G$-stability of $T$ implies the $G$-stability of the pair.

**Proposition 3.2** *Let $T$ be a $\Lambda$-module, and let $P$ be a projective $\Lambda$-module. Then $(T,P)\in s\tau\text{-}\mathsf{tilt}\,\Lambda$ if and only if $({}^{\sigma}T,{}^{\sigma}P)\in s\tau\text{-}\mathsf{tilt}\,\Lambda$. Moreover, if $(T,P)\in s\tau\text{-}\mathsf{tilt}\,\Lambda$ and $T$ is a $G$-stable support $\tau$-tilting module, then $P$ is $G$-stable.*

*Proof* Since ${}^{\sigma}(-)$ is an automorphism commuting with $\tau$ (see the proof of [18, Lemma 4.1]), we have $|T|+|P|=|\Lambda|$ if and only if $|{}^{\sigma}T|+|{}^{\sigma}P|=|\Lambda|$, and that $\mathsf{Hom}_{\Lambda}(T,\tau T)=0$ if and only if $\mathsf{Hom}_{\Lambda}({}^{\sigma}T,{}^{\sigma}\tau T)=0$, if and only if $\mathsf{Hom}_{\Lambda}({}^{\sigma}T,\tau{}^{\sigma}T)=0$. So $T$ is $\tau$-rigid if and only if ${}^{\sigma}T$ is $\tau$-rigid. Thus, the former assertion follows. If $T$ is a $G$-stable support $\tau$-tilting module, then by Proposition 2.1 we have $P$ is also $G$-stable. □

For any complex $M^{\bullet}=(M^i,d^i_{M^{\bullet}})_{i\in\mathbb{Z}}$ over $\mathsf{mod}\,\Lambda$ and $\sigma\in G$, let ${}^{\sigma}M^{\bullet}$ be the complex $({}^{\sigma}M^i,{}^{\sigma}d^i_{M^{\bullet}})_{i\in\mathbb{Z}}$, where $\mathbb{Z}$ is the ring of integers. Moreover, given another complex $N^{\bullet}=(N^i,d^i_{N^{\bullet}})_{i\in\mathbb{Z}}$ over $\mathsf{mod}\,\Lambda$ and a morphism of complexes $f=(f^i\colon M^i\to N^i)_{i\in\mathbb{Z}}$, let

$$ {}^{\sigma}f=({}^{\sigma}f^i\colon {}^{\sigma}M^i\to {}^{\sigma}N^i)_{i\in\mathbb{Z}}.$$

Clearly, ${}^{\sigma}f$ is a morphism of complexes.

Since ${}^{\sigma}(-)\colon \mathsf{mod}\,\Lambda\to\mathsf{mod}\,\Lambda$ is an automorphism, this construction is compatible with the homotopy relation and preserves projective modules. This allows defining an automorphism

$$ {}^{\sigma}(-)\colon \mathsf{K}^b(\mathsf{proj}\,\Lambda)\to\mathsf{K}^b(\mathsf{proj}\,\Lambda),\quad \forall\,\sigma\in G.$$

In this way, we obtain an action by $G$ on $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$.

**Definition 3.3** We call a basic two-term silting complex $P^{\bullet}\in\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ *$G$-stable* if ${}^{\sigma}P^{\bullet}\cong P^{\bullet}$ for any $\sigma\in G$.

We denote by $G\text{-}2\text{-}\mathsf{silt}\,\Lambda$ the set of isomorphism classes of basic $G$-stable two-term silting complexes for $\Lambda$.

### 3.2 Connection of $G$-$s\tau$-tilt $\Lambda$ with $G$-$f$-tors $\Lambda$ and $G$-2-silt $\Lambda$

In this subsection, we show that $G$-stable support $\tau$-tilting modules correspond bijectively to $G$-stable functorially finite torsion classes as well as $G$-stable two-term silting complexes.

The following result establishes a one-to-one correspondence between $G$-stable support $\tau$-tilting $\Lambda$-modules and $G$-stable functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.

**Theorem 3.4** *The bijection of Theorem* 2.3 *restricts to a bijection*

$$ G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda\leftrightarrow G\text{-}f\text{-}\mathsf{tors}\,\Lambda.$$

*Proof* Assume that $T$ is $G$-stable. For any $M\in\mathsf{mod}\,\Lambda$ and $\sigma\in G$, we have for any $n\geqslant 1$, $T^n\to M$ is surjective if and only if so is $({}^{\sigma}T)^n\to{}^{\sigma}M$. So we have

$$ \mathsf{Fac}\,T=\mathsf{Fac}\,{}^{\sigma}T={}^{\sigma}\,\mathsf{Fac}\,T$$

for any $\sigma \in G$, that is, $\mathsf{Fac}\, T$ is $G$-stable.

Conversely, if $\mathscr{T} \in f\text{-}\mathsf{tors}\,\Lambda$ is $G$-stable, then ${}^{\sigma}\mathscr{T} = \mathscr{T}$ for any $\sigma \in G$. Since ${}^{\sigma}(-)\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda$ is an automorphism, we have

$$\mathsf{Ext}^1_\Lambda(-, {}^{\sigma}\mathscr{T}) \cong \mathsf{Ext}^1_\Lambda({}^{\sigma^{-1}}(-), \mathscr{T}).$$

So

$$P(\mathscr{T}) = P({}^{\sigma}\mathscr{T}) \cong {}^{\sigma}P(\mathscr{T})$$

for any $\sigma \in G$, that is, $P(\mathscr{T})$ is $G$-stable. □

Recall that $M \in \mathsf{mod}\,\Lambda$ is *sincere* if every simple $\Lambda$-module appears as a composition factor in $M$. This is equivalent to the condition that $\mathsf{Hom}_\Lambda(P, M) \neq 0$ for any indecomposable projective module $P$. Also, recall that $M \in \mathsf{mod}\,\Lambda$ is *faithful* if its left annihilator

$$\mathsf{Ann}\, M := \{\lambda \in \Lambda \mid \lambda M = 0\} = 0.$$

A class of left $\Lambda$-modules $\mathscr{T}$ is *sincere* if for any indecomposable projective module $P$, we have

$$\mathsf{Hom}_\Lambda(P, \mathscr{T}) := \{\mathsf{Hom}_\Lambda(P, T) \mid T \in \mathscr{T}\} \neq 0.$$

A class of left $\Lambda$-modules $\mathscr{T}$ is *faithful* if

$$\mathsf{Ann}\, \mathscr{T} = \bigcap_{T \in \mathscr{T}} \mathsf{Ann}\, T = 0.$$

The following result shows that support $\tau$-tilting modules can be regarded as a common generalization of $\tau$-tilting modules and tilting modules.

**Proposition 3.5** [2, Proposition 2.2] (1) *$\tau$-tilting modules are precisely sincere support $\tau$-tilting modules.*

(2) *Tilting modules are precisely faithful support $\tau$-tilting modules.*

We denote by $G\text{-}sf\text{-}\mathsf{tors}\,\Lambda$ (resp., $G\text{-}ff\text{-}\mathsf{tors}\,\Lambda$) the set of $G$-stable sincere (resp., faithful) functorially finite torsion classes in $\mathsf{mod}\,\Lambda$. Using Proposition 3.5, we get the following theorem.

**Theorem 3.6** *The bijection in Theorem* 3.4 *restricts to bijections*

$$G\text{-}\tau\text{-}\mathsf{tilt}\,\Lambda \leftrightarrow G\text{-}sf\text{-}\mathsf{tors}\,\Lambda, \quad G\text{-}\mathsf{tilt}\,\Lambda \leftrightarrow G\text{-}ff\text{-}\mathsf{tors}\,\Lambda.$$

*Proof* Let $T$ be a $G$-stable support $\tau$-tilting $\Lambda$-module. It follows Proposition 3.5 that $T$ is a $\tau$-tilting $\Lambda$-module (resp., tilting $\Lambda$-module) if and only if $T$ is sincere (resp., faithful).

**Claim 1** $T$ is sincere if and only if $\mathsf{Fac}\, T$ is sincere.

If $T$ is sincere, it is obvious that $\mathsf{Fac}\, T$ is sincere by definition.

Conversely, if $\mathsf{Fac}\,T$ is sincere, then, for any indecomposable projective module $P$, we have $\mathsf{Hom}_\Lambda(P, \mathsf{Fac}\,T) \neq 0$, that is, there exists $M_P \in \mathsf{Fac}\,T$ such that $\mathsf{Hom}_\Lambda(P, M_P) \neq 0$. Since $M_P \in \mathsf{Fac}\,T$, there exist exact sequences

$$T^n \to M_P \to 0,$$
$$\mathsf{Hom}_\Lambda(P, T^n) \to \mathsf{Hom}_\Lambda(P, M_P) \to 0,$$

where $n \geqslant 1$. So we have $\mathsf{Hom}_\Lambda(P, T) \neq 0$ for any indecomposable projective module $P$. Therefore, $T$ is sincere.

**Claim 2** $T$ is faithful if and only if $\mathsf{Fac}\,T$ is faithful.

It suffices to show that $\mathsf{Ann}\,T = \mathsf{Ann}\,\mathsf{Fac}\,T$. It is obvious that $\mathsf{Ann}\,\mathsf{Fac}\,T \subseteq \mathsf{Ann}\,T$ by definition. Conversely, for any $\lambda \in \mathsf{Ann}\,T$ and $M \in \mathsf{Fac}\,T$, there exists an exact sequence $T^n \xrightarrow{f} M \to 0$ with $n \geqslant 1$ and $\lambda T = 0$. Then we have

$$\lambda M = \lambda f(T^n) = f(\lambda T^n) = 0,$$

that is, $\mathsf{Ann}\,T \subseteq \mathsf{Ann}\,\mathsf{Fac}\,T$. □

We end this subsection with the following result.

**Theorem 3.7** *The bijection of Theorem* 2.4 *restricts to a bijection*

$$G\text{-}2\text{-}\mathsf{silt}\,\Lambda \leftrightarrow G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda.$$

*Proof* If $(T, P) \in G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda$, then ${}^\sigma T \cong T$ and ${}^\sigma P \cong P$ for any $\sigma \in G$. Let $P^\bullet \to T \to 0$ be a minimal projective presentation of $T$. Then ${}^\sigma P^\bullet \to {}^\sigma T \to 0$ is a minimal projective presentation of ${}^\sigma T$ since ${}^\sigma(-)\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda$ is an automorphism. Since ${}^\sigma T \cong T$ for any $\sigma \in G$, it follows that

$${}^\sigma P^\bullet \cong P^\bullet, \quad {}^\sigma(P^\bullet \oplus P[1]) \cong P^\bullet \oplus P[1].$$

Conversely, if $P^\bullet \in G\text{-}2\text{-}\mathsf{silt}\,\Lambda$, then

$$H^0(P^\bullet) \cong H^0({}^\sigma P^\bullet) = {}^\sigma H^0(P^\bullet)$$

because $\sigma$ commutes with taking cokernel. It follows that $H^0(P^\bullet)$ is $G$-stable. □

### 3.3 Connection of $G$-$s\tau$-tilt $\Lambda$ with $G$-$c$-tilt $\mathscr{C}$

Let $\mathscr{C}$ be a $k$-linear Hom-finite Krull-Schimidt 2-Calabi-Yau triangulated category. An action of $G$ on $\mathscr{C}$ is a group homomorphism $\theta\colon G \to \mathsf{Aut}\,\mathscr{C}$ from $G$ to the group of triangulated automorphisms of $\mathscr{C}$, that is,

$${}^\sigma(-) = \theta(\sigma)\colon \mathscr{C} \to \mathscr{C}$$

is a triangle automorphism. For any $X \in \mathscr{C}$ and $\sigma \in G$, ${}^\sigma X$ denotes the image of $X$ under ${}^\sigma(-)$. An object $X$ in $\mathscr{C}$ is called *$G$-stable* if ${}^\sigma X \cong X$ for any $\sigma \in G$. We denote by $G\text{-}c\text{-}\mathsf{tilt}\,\mathscr{C}$ the set of isomorphism classes of basic $G$-stable cluster-tilting objects.

Throughout this subsection, let $T$ be a $G$-stable cluster-tilting object in $\mathscr{C}$ with a fixed isomorphism $\varphi_\sigma\colon {}^\sigma T \to T$ such that $\varphi_{\sigma\eta} = \varphi_\sigma\varphi_\eta$ and $\varphi_1 = \mathrm{id}_T$. Then $\Lambda := \mathsf{End}_{\mathscr{C}}(T)^{\mathrm{op}}$ admits a $G$-action via $\sigma(\lambda) = \varphi_\sigma \circ {}^\sigma\lambda \circ \varphi_\sigma^{-1}$ for any $\sigma \in G$ and $\lambda \in \Lambda$. The following proposition plays an important role in this subsection.

**Proposition 3.8** *The action of $G$ on $\mathscr{C}$ induces that on $\mathscr{C}/[T[1]]$.*

*Proof* It suffices to show that $\mathsf{add}\, T[1]$ is closed under the $G$-action. By the definition of triangle functors, we have the shift functor $[1]$ and the functor ${}^\sigma(-)$ commute on objects. So

$$ {}^\sigma(\mathsf{add}\, T[1]) = \mathsf{add}^\sigma(T[1]) = \mathsf{add}({}^\sigma T)[1] = \mathsf{add}\, T[1], $$

and the assertion follows. □

By Theorem 2.5, there exists an equivalence of categories between $\mathscr{C}/[T[1]]$ and $\mathsf{mod}\,\Lambda$. So the action of $G$ on $\mathscr{C}/[T[1]]$ induces an action of $G$ on $\mathsf{mod}\,\Lambda$. On the other hand, the action of $G$ on $\Lambda$ also induces that on $\mathsf{mod}\,\Lambda$. The following result shows that these two actions coincide.

**Lemma 3.9** *The functor $\mathsf{Hom}_{\mathscr{C}}(T,-)\colon \mathscr{C} \to \mathsf{mod}\,\Lambda$ commutes with $G$-action.*

*Proof* It suffices to prove that for any $M \in \mathscr{C}$, there exists a $\Lambda$-module isomorphism

$$ {}^\sigma\,\mathsf{Hom}_{\mathscr{C}}(T, M) \cong \mathsf{Hom}_{\mathscr{C}}(T, {}^\sigma M). $$

Take

$$ \begin{aligned} \Phi\colon {}^\sigma\,\mathsf{Hom}_{\mathscr{C}}(T, M) &\to \mathsf{Hom}_{\mathscr{C}}(T, {}^\sigma M), \\ g &\mapsto {}^\sigma(g \circ {}^{\sigma^{-1}}(\varphi_\sigma^{-1})) = {}^\sigma g \circ \varphi_\sigma^{-1}, \end{aligned} $$

for any $g \in {}^\sigma\,\mathsf{Hom}_{\mathscr{C}}(T, M)$. Then $\Phi$ is clearly an isomorphism as $k$-vector spaces. Because

$$ \begin{aligned} \Phi(\lambda \cdot g) &= \Phi(\sigma^{-1}(\lambda) \cdot g) \\ &= \Phi(g \circ {}^{\sigma^{-1}}(\varphi_\sigma^{-1}) \circ {}^{\sigma^{-1}}\lambda \circ {}^{\sigma^{-1}}\varphi_\sigma) \\ &= {}^\sigma g \circ \varphi_\sigma^{-1} \circ \lambda \circ \varphi_\sigma \circ \varphi_\sigma^{-1} \\ &= {}^\sigma g \circ \varphi_\sigma^{-1} \circ \lambda \\ &= \lambda \cdot ({}^\sigma g \circ \varphi_\sigma^{-1}) \\ &= \lambda \cdot \Phi(g), \end{aligned} $$

we have $\Phi$ is a $\Lambda$-module isomorphism. □

**Lemma 3.10** *If both $X_1 = X \oplus Y'$ and $X_2 = X \oplus Y''$ are basic cluster-tilting objects with $Y'$ (resp., $Y''$) a maximal direct summand of $X_1$ (resp., $X_2$), which belongs to $\mathsf{add}\, T[1]$, then $Y' \cong Y''$.*

*Proof* Let $X_1$, $X_2$, and $T$ be cluster-tilting and $Y', Y'' \in \mathsf{add}\, T[1]$. Then we have

$$ \begin{aligned} \mathsf{Hom}_{\mathscr{C}}(X_2, Y'[1]) &= \mathsf{Hom}_{\mathscr{C}}(X, Y'[1]) \oplus \mathsf{Hom}_{\mathscr{C}}(Y'', Y'[1]) = 0, \\ \mathsf{Hom}_{\mathscr{C}}(X_1, Y''[1]) &= \mathsf{Hom}_{\mathscr{C}}(X, Y''[1]) \oplus \mathsf{Hom}_{\mathscr{C}}(Y', Y''[1]) = 0. \end{aligned} $$

It is easy to get that $Y' \in \mathsf{add}\, X_2$ and $Y'' \in \mathsf{add}\, X_1$ from the definition of cluster-tilting objects. Since $Y'$ (resp., $Y''$) is a maximal direct summand of $X_1$ (resp., $X_2$), which belongs to $\mathsf{add}\, T[1]$ by assumption, we have $Y' \cong Y''$. □

**Lemma 3.11** *For any $\sigma \in G$, we have $X \in c\text{-}\mathsf{tilt}\,\mathscr{C}$ if and only if ${}^{\sigma}X \in c\text{-}\mathsf{tilt}\,\mathscr{C}$.*

*Proof* Let $X \in c\text{-}\mathsf{tilt}\,\mathscr{C}$. Then

$$\mathsf{add}\, X = \{C \in \mathscr{C} \mid \mathsf{Hom}_{\mathscr{C}}(X, C[1]) = 0\}.$$

So we have

$$\begin{aligned}\mathsf{add}\,{}^{\sigma}X &= {}^{\sigma}\,\mathsf{add}\, X\\ &= \{{}^{\sigma}C \in \mathscr{C} \mid \mathsf{Hom}_{\mathscr{C}}(X, C[1]) = 0\}\\ &= \{{}^{\sigma}C \in \mathscr{C} \mid \mathsf{Hom}_{\mathscr{C}}({}^{\sigma}X, {}^{\sigma}C[1]) = 0\}\\ &= \{C \in \mathscr{C} \mid \mathsf{Hom}_{\mathscr{C}}({}^{\sigma}X, C[1]) = 0\},\end{aligned}$$

and hence, ${}^{\sigma}X \in c\text{-}\mathsf{tilt}\,\mathscr{C}$. Dually, we get that ${}^{\sigma}X \in c\text{-}\mathsf{tilt}\,\mathscr{C}$ implies $X \in c\text{-}\mathsf{tilt}\,\mathscr{C}$. □

The following observation is useful.

**Proposition 3.12** *If $X = X' \oplus X'' \in c\text{-}\mathsf{tilt}\,\mathscr{C}$ with $X''$ a maximal direct summand of $X$, which belongs to $\mathsf{add}\, T[1]$, then $X$ is $G$-stable if and only if $X'$ is $G$-stable.*

*Proof* By Proposition 3.8, we have

$${}^{\sigma}\,\mathsf{add}\, T[1] = \mathsf{add}\,{}^{\sigma}(T[1]) = \mathsf{add}({}^{\sigma}T)[1] = \mathsf{add}\, T[1].$$

So $Y \in \mathsf{add}\, T[1]$ if and only if ${}^{\sigma}Y \in \mathsf{add}\, T[1]$. Since $X''$ is a maximal direct summand of $X$, which belongs to $\mathsf{add}\, T[1]$ by assumption, ${}^{\sigma}X''$ is also a maximal direct summand of ${}^{\sigma}X \in \mathsf{add}\, T[1]$.

If $X$ is $G$-stable, then ${}^{\sigma}X \cong X$ for any $\sigma \in G$. Since ${}^{\sigma}X''$ (resp., $X''$) is a maximal direct summand of ${}^{\sigma}X$ (resp., $X$), which belongs to $\mathsf{add}\, T[1]$, we have ${}^{\sigma}X'' \cong X''$ and ${}^{\sigma}X' \cong X'$ for any $\sigma \in G$.

By Lemma 3.11, we have ${}^{\sigma}X' \oplus {}^{\sigma}X'' \in$ c-tilt$\mathscr{C}$. If $X'$ is $G$-stable, then $X' \oplus {}^{\sigma}X''$ and $X' \oplus X''$ are basic cluster-tilting objects. By Lemma 3.10, we have ${}^{\sigma}X'' \cong X''$. Thus, $X$ is $G$-stable. □

Now, we are in a position to prove the following theorem.

**Theorem 3.13** *The bijection of Theorem* 2.6 *restricts to a bijection*

$$G\text{-}c\text{-}\mathsf{tilt}\,\mathscr{C} \leftrightarrow G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda.$$

*Proof* By Lemma 3.9, we have

$${}^{\sigma}\,\mathsf{Hom}_{\mathscr{C}}(T, M) \cong \mathsf{Hom}_{\mathscr{C}}(T, {}^{\sigma}M), \quad \forall\, M \in \mathscr{C}/[T[1]].$$

If $X \in G\text{-c-}\mathsf{tilt}\,\mathscr{C}$, then $X'$ is $G$-stable by Proposition 3.12. For any $\sigma \in G$, we have

$$^{\sigma}\overline{X'} = {}^{\sigma}\,\mathsf{Hom}_{\mathscr{C}}(T, X') \cong \mathsf{Hom}_{\mathscr{C}}(T, {}^{\sigma}X') \cong \mathsf{Hom}_{\mathscr{C}}(T, X') = \overline{X'}.$$

Thus $\overline{X'}$ is $G$-stable. Moreover, $\widetilde{X} \in G\text{-s}\tau\text{-}\,\mathsf{tilt}\,\Lambda$ is $G$-stable.

Conversely, if $\overline{X'}$ is $G$-stable, then we have $\mathsf{Hom}_{\mathscr{C}}(T, {}^{\sigma}X') \cong \mathsf{Hom}_{\mathscr{C}}(T, X')$ as before. Then by Theorem 2.5, we have ${}^{\sigma}X' \cong X'$ for any $\sigma \in G$, that is, $X'$ is $G$-stable. Thus, $X \in G\text{-}c\text{-}\,\mathsf{tilt}\,\mathscr{C}$ by Proposition 3.12. □

In the following, we establish a bijection between $G$-stable cluster-tilting objects in $\mathscr{C}$ and $G$-stable two-term silting complexes in $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$.

**Lemma 3.14** *Let $X$ be a basic object of $\mathscr{C}$ and take a triangle*

$$T_1 \xrightarrow{g} T_0 \xrightarrow{f} X \to T_1[1]$$

*with $T_1, T_0 \in \mathsf{add}\,T$ and $f$ a minimal right $\mathsf{add}\,T$-approximation. Then the following statements are equivalent*:

(1) *$X$ is $G$-stable in $\mathscr{C}$*;

(2) *$\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}$ is $G$-stable in $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$.*

*Proof* Since ${}^{\sigma}(-)\colon \mathscr{C} \to \mathscr{C}$ is a triangulated equivalence, we have the following diagram:

$$\begin{array}{ccccccc} T_1 & \xrightarrow{g} & T_0 & \xrightarrow{f} & X & \longrightarrow & T_1[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}^{\sigma}T_1 & \xrightarrow{{}^{\sigma}g} & {}^{\sigma}T_0 & \xrightarrow{{}^{\sigma}f} & {}^{\sigma}X & \longrightarrow & ({}^{\sigma}T_1)[1]. \end{array}$$

By the proof of Theorem 3.13, we have $T_1 \xrightarrow{g} T_0$ is $G$-stable if and only if $\overline{T_1} \xrightarrow{\overline{g}} \overline{T_0}$ is $G$-stable.

(2) $\Rightarrow$ (1) We already have $T_1 \xrightarrow{g} T_0$ is $G$-stable, it follows that $X$ is $G$-stable.

(1) $\Rightarrow$ (2) If $X$ is $G$-stable, then ${}^{\sigma}X \cong X$. Since $f$ is a minimal right $\mathsf{add}\,T$-approximation by assumption, ${}^{\sigma}f$ is a minimal right ${}^{\sigma}\,\mathsf{add}\,T$-approximation, and hence a minimal right $\mathsf{add}\,T$-approximation. Thus, ${}^{\sigma}T_0 \cong T_0$ and ${}^{\sigma}T_1 \cong T_1$. Therefore, $T_1 \xrightarrow{g} T_0$ is $G$-stable and the assertion follows. □

Immediately, we get the following theorem.

**Theorem 3.15** *The bijection of Theorem* 2.7 *restricts to a bijection*

$$G\text{-}c\text{-}\,\mathsf{tilt}\,\mathscr{C} \leftrightarrow G\text{-}2\text{-}\,\mathsf{silt}\,\Lambda.$$

*Proof* It follows from Theorem 2.7 and Lemma 3.14. □

## 4 Relationship between stable support $\tau$-tilting modules over $\Lambda$ and $\Lambda G$

Throughout this section, $\Lambda$ is an algebra and $G$ is a finite group acting on $\Lambda$ such that $|G|$ is invertible in $\Lambda$. We denote by $\mathbb{X}$ the group of characters on $G$, that is, the group homomorphisms

$$\chi\colon G \to k^* = k\backslash\{0\}.$$

Then $X$ acts on $\Lambda G$ via

$$\chi(\lambda g) = \chi(g)\lambda g.$$

We prove that there exists an injection from $G$-stable support $\tau$-tilting $\Lambda$-modules to $\mathbb{X}$-stable support $\tau$-tilting $\Lambda G$-modules. In the case when $G$ is a solvable group, the injection turns out to be a bijection.

We begin with the following easy observation.

**Lemma 4.1** *If $T \in \mathsf{mod}\,\Lambda$ is $G$-stable, then $FT$ is $\mathbb{X}$-stable.*

*Proof* We only need to prove ${}^{\chi}FT \cong FT$ for any $\chi \in X$. Note that ${}^{\chi}FT$ is a $\Lambda G$-module whose underlying set and the additive structure is the same as $FT$, in which $(\lambda', g') \circ ((\lambda, g) \otimes t)$ is defined to be $\chi(g')(\lambda', g')(\lambda, g) \otimes t$. Define

$$\begin{aligned} \theta\colon {}^{\chi}FT &\to FT,\\ (\lambda, g) \otimes t &\mapsto \chi^{-1}(g)(\lambda, g) \otimes t. \end{aligned}$$

Clearly, it is a bijection. Because

$$\begin{aligned} \theta((\lambda', g') \circ (\lambda, g) \otimes t) &= \theta(\chi(g')(\lambda', g')(\lambda, g) \otimes t)\\ &= \chi(g')\theta((\lambda', g')(\lambda, g) \otimes t)\\ &= \chi(g')\theta((\lambda' g'(\lambda), g'g \otimes t)\\ &= \chi(g')\chi^{-1}(g'g)(\lambda', g')(\lambda, g) \otimes t\\ &= \chi^{-1}(g)(\lambda', g')(\lambda, g) \otimes t\\ &= (\lambda', g')\theta((\lambda, g) \otimes t), \end{aligned}$$

we have $\theta$ is a $\Lambda G$-homomorphism, and hence an isomorphism. □

The first main result in this section is the following theorem.

**Theorem 4.2** *The functor*

$$F = \Lambda G \otimes_{\Lambda} -\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda G$$

*via $T \mapsto FT$ induces the following injections*:

(1) *from the set of isomorphism classes of $G$-stable $\tau$-rigid $\Lambda$-modules to the set of isomorphism classes of $\mathbb{X}$-stable $\tau$-rigid $\Lambda G$-modules*;

(2) *from the set of isomorphism classes of $G$-stable $\tau$-rigid pair in* $\mathsf{mod}\,\Lambda$ *to the set of isomorphism classes of $\mathbb{X}$-stable $\tau$-rigid pair in* $\mathsf{mod}\,\Lambda G$;

(3) $G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda \to \mathbb{X}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G$.

*Proof* We claim that the functor $F$ restricting to the set of isomorphism classes of basic $G$-stable $\Lambda$-modules is an injection. If both $T_1$ and $T_2$ are $G$-stable $\Lambda$-modules and $FT_1 \cong FT_2$, then $HFT_1 \cong HFT_2$, that is,

$$\bigoplus_{\sigma\in G} {}^{\sigma}T_1 \cong \bigoplus_{\sigma\in G} {}^{\sigma}T_2.$$

Since $T_1$ and $T_2$ are $G$-stable, we have $T_1^n \cong T_2^n$ with $n = |G|$. Thus, $T_1 \cong T_2$, and the claim follows.

(1) By definition, $T$ is $\tau$-rigid if and only if $\mathsf{Hom}_\Lambda(T, \tau T) = 0$. By the proof of [18, Lemma 4.2], we have $F$ commutes with $\tau$. So we have

$$\begin{aligned}
\mathsf{Hom}_{\Lambda G}(FT, \tau FT) &\cong \mathsf{Hom}_{\Lambda G}(FT, F\tau T)\\
&\cong \mathsf{Hom}_\Lambda(T, HF\tau T)\\
&\cong \mathsf{Hom}_\Lambda\left(T, \bigoplus_{\sigma\in G} {}^{\sigma}(\tau T)\right)\\
&\cong \bigoplus_{\sigma\in G} \mathsf{Hom}_\Lambda(T, {}^{\sigma}(\tau T))\\
&\cong \bigoplus_{\sigma\in G} \mathsf{Hom}_\Lambda({}^{\sigma^{-1}}T, \tau T)\\
&\cong \bigoplus_{\sigma\in G} \mathsf{Hom}_\Lambda(T, \tau T).
\end{aligned}$$

Since $T$ is $G$-stable $\tau$-rigid in $\mathsf{mod}\,\Lambda$, we have $FT$ is $\tau$-rigid in $\mathsf{mod}\,\Lambda G$. Now, the assertion follows from Lemma 4.1.

(2) Note that $(T, P)$ is a $G$-stable $\tau$-rigid pair if and only if $T$ is $G$-stable $\tau$-rigid, $P$ is $G$-stable projective, and $\mathsf{Hom}_\Lambda(P, T) = 0$. It follows from the injection in (1) that $FT$ is $\tau$-rigid in $\mathsf{mod}\,\Lambda G$. By Lemma 2.8, $F$ preserves projective modules and $FP$ is a projective module in $\mathsf{mod}\,\Lambda G$. We have

$$\begin{aligned}
\mathsf{Hom}_{\Lambda G}(FP, FT) &\cong \mathsf{Hom}_\Lambda(P, HFT)\\
&\cong \mathsf{Hom}_\Lambda\left(P, \bigoplus_{\sigma\in G} {}^{\sigma}T\right)\\
&\cong \bigoplus_{\sigma\in G} \mathsf{Hom}_\Lambda(P, {}^{\sigma}T)\\
&\cong \bigoplus_{\sigma\in G} \mathsf{Hom}_\Lambda(P, T).
\end{aligned}$$

Thus,

$$\mathsf{Hom}_{\Lambda G}(FP, FT) = 0,$$

and therefore, $(FT, FP)$ is an $\mathbb{X}$-stable $\tau$-rigid pair in $\mathsf{mod}\,\Lambda G$ by Lemma 4.1.

(3) By Propositions 2.1 and 2.2, $T \in s\tau\text{-}\mathsf{tilt}\,\Lambda$ if and only if $T$ is $\tau$-rigid and there exists an exact sequence

$$\Lambda \xrightarrow{f} T' \xrightarrow{g} T'' \to 0$$

in $\mathsf{mod}\,\Lambda$ with $T', T'' \in \mathsf{add}\,T$ and $f$ a left $\mathsf{add}\,T$-approximation of $\Lambda$. It follows from the injection in (1) that $FT$ is a $\tau$-rigid $\Lambda G$-module and there exists an exact sequence

$$F\Lambda\ (\cong \Lambda G) \xrightarrow{Ff} FT' \xrightarrow{Fg} FT'' \to 0$$

in $\mathsf{mod}\,\Lambda G$ with $FT', FT'' \in \mathsf{add}\,FT$. Then by Proposition 2.2, we only need to prove that $Ff$ is a left $\mathsf{add}\,FT$-approximation of $\Lambda G$, that is, $\mathsf{Hom}_{\Lambda G}(Ff, M)$ is surjective for any $M \in \mathsf{add}\,FT$. First, we prove that $\mathsf{Hom}_{\Lambda G}(Ff, FT)$ is surjective. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathsf{Hom}_{\Lambda G}(FT', FT) & \xrightarrow{\mathsf{Hom}_{\Lambda G}(Ff, FT)} & \mathsf{Hom}_{\Lambda G}(\Lambda G, FT) \\
\downarrow\cong & & \downarrow\cong \\
\mathsf{Hom}_{\Lambda}(T', HFT) & \xrightarrow{\mathsf{Hom}_{\Lambda}(f, HFT)} & \mathsf{Hom}_{\Lambda}(\Lambda, HFT) \\
\downarrow\cong & & \downarrow\cong \\
\mathsf{Hom}_{\Lambda}(T', \oplus_{\sigma\in G}{}^{\sigma}T) & \xrightarrow{\mathsf{Hom}_{\Lambda}(f, \oplus_{\sigma\in G}{}^{\sigma}T)} & \mathsf{Hom}_{\Lambda}(\Lambda, \oplus_{\sigma\in G}{}^{\sigma}T) \\
\downarrow\cong & & \downarrow\cong \\
\mathsf{Hom}_{\Lambda}(T', T^n) & \xrightarrow{\mathsf{Hom}_{\Lambda}(f, T^n)} & \mathsf{Hom}_{\Lambda}(\Lambda, T^n)
\end{array}$$

where $n = |G|$. The last row is surjective since $f$ is left $\mathsf{add}\,T$-approximation of $\Lambda$. So the first row is also surjective.

Now, let $M \in \mathsf{add}\,FT$ and $g \in \mathsf{Hom}_{\Lambda G}(\Lambda G, M)$. Then there exist $m \geqslant 1$ and $N \in \mathsf{mod}\,\Lambda G$ such that

$$M \oplus N \cong (FT)^m.$$

So we have a split exact sequence

$$0 \longrightarrow M \underset{p}{\overset{i}{\rightleftarrows}} (FT)^m \longrightarrow N \longrightarrow 0$$

in $\mathsf{mod}\,\Lambda G$ with $pi = 1_M$. By the above argument, there exists $h \in \mathsf{Hom}_{\Lambda G}(FT', (FT)^m)$ such that $hFf = ig$. So we have

$$g = pig = phFf = \mathsf{Hom}_{\Lambda G}(Ff, M)(ph),$$

and hence, $\mathsf{Hom}_{\Lambda G}(Ff, M)$ is surjective. □

As an application of Theorem 4.2, we get the following result which extends [10, Proposition 3.1.1].

**Corollary 4.3** *If $T$ is a $G$-stable (basic) tilting $\Lambda$-module, then $FT$ is an $\mathbb{X}$-stable tilting $\Lambda G$-module.*

*Proof* Let $T$ be a $G$-stable (basic) tilting $\Lambda$-module. Then by Proposition 3.5 (2), $T$ is a $G$-stable faithful support $\tau$-tilting module. So $\Lambda$ is cogenerated by $T$ and there exists an injection $0 \to \Lambda \to T^n$ in $\mathsf{mod}\,\Lambda$. Since $F$ is exact, we get an injection $0 \to \Lambda G \to (FT)^n$ in $\mathsf{mod}\,\Lambda G$. So $FT$ is an $\mathbb{X}$-stable faithful support $\tau$-tilting $\Lambda G$-module by Theorem 4.2 (3), and hence, it is a tilting $\Lambda G$-module by Proposition 3.5 (2) again. □

The following observation is standard.

**Proposition 4.4** *Any $\Lambda G$-module is a $G$-stable $\Lambda$-module.*

*Proof* Let $Y$ be a $\Lambda G$-module. For any $g \in G$ and $y \in Y$, we define a map

$$f_g \colon {}^gY \to Y$$

by $f_g(y) = gy$. Then for any $a \in \Lambda$, we have

$$f_g(ay) = g(ay) = g(g^{-1}(a)y) = ag(y) = af_g(y).$$

So $f_g$ is a $\Lambda$-module homomorphism. We also have $f_g$ is an isomorphism with the inverse $f_{g^{-1}} \colon Y \to {}^gY$ such that

$$f_{g^{-1}}(y) = g^{-1}y, \quad \forall\, y \in Y.$$ □

As an immediate consequence of Proposition 4.4, we have the following result.

**Corollary 4.5** *For any basic $G$-stable $\Lambda$-module $T$, we have*

(1) *if $T$ is $\tau$-rigid in $\mathsf{mod}\,\Lambda$, then $HFT$ is $G$-stable $\tau$-rigid in $\mathsf{mod}\,\Lambda$;*

(2) *if $T$ is support $\tau$-tilting in $\mathsf{mod}\,\Lambda$, then $HFT$ is $G$-stable support $\tau$-tilting in $\mathsf{mod}\,\Lambda$.*

*Proof* Note that

$$HFT \cong \bigoplus_{\sigma \in G} {}^{\sigma}T \cong T^n, \quad n = |G|.$$

So both assertions follow from Proposition 4.4. □

We have proved in Theorem 4.2 (3) that $F$ induces an injection from $G$-$s\tau$-$\mathsf{tilt}\,\Lambda$ to $\mathbb{X}$-$s\tau$-$\mathsf{tilt}\,\Lambda G$. It is natural to ask the following question.

**Question** When is this injection a bijection?

In the following, we give a partial answer to this question.

It follows from [18, Corollary 5.2] that $(\Lambda G)\mathbb{X}$ is Morita equivalent to $\Lambda G^{(1)}$, where $G^{(1)}$ is the commutator subgroup of $G$. Let $G$ be solvable, and let

$$G \rhd G^{(1)} \rhd G^{(2)} \rhd G^{(3)} \rhd \cdots \rhd G^{(m)} = \{1\}$$

be its derived series, that is, every subgroup is the commutator subgroup of the preceding one. Denote by $\mathbb{X}^{(i)}$ the character group of $G^{(i)}$. By [18, Proposition 5.4], we can get from $\Lambda G$ to $\Lambda$ by using a finite number of skew group algebra constructions, combined with Morita equivalences. To be more precise, there exists a chain of skew group algebras

$$\Lambda \overset{G}{\to} \Lambda G \overset{\mathbb{X}}{\to} \Lambda G^{(1)} \overset{\mathbb{X}^{(1)}}{\to} \Lambda G^{(2)} \to \cdots \overset{\mathbb{X}^{(m-1)}}{\to} \Lambda G^{(m)} \overset{\text{Morita}}{\simeq} \Lambda,$$

where each algebra $\Lambda G^{(i)}$ is the skew group algebra of the preceding algebra $\Lambda G^{(i-1)}$ under the action of the group $\mathbb{X}^{i-1}$. Then we have the induced functors

$$\mathsf{mod}\,\Lambda \overset{F}{\to} \mathsf{mod}\,\Lambda G \overset{F^{(1)}}{\to} \mathsf{mod}\,\Lambda G^{(1)} \to \cdots \overset{F^{(m)}}{\to} \mathsf{mod}\,\Lambda.$$

Under the above assumption, we give the following theorem.

**Theorem 4.6** *If $G$ is a solvable group, then the functor $F\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda G$ induces a bijection*

$$G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda \to \mathbb{X}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G.$$

*Proof* By Theorem 4.2, the functor $F$ induces an injection

$$G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda \to \mathbb{X}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G.$$

Applying Lemma 4.1 and Theorem 4.2, it is easy to see that the functors $F^{(i)}$ induce injections

$$\mathbb{X}^{(i-1)}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G^{(i-1)} \to \mathbb{X}^{(i)}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G^{(i)}.$$

Then we have the following chain of injections:

$$\begin{aligned} G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda \overset{F}{\to} \mathbb{X}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G \overset{F^{(1)}}{\to} \mathbb{X}^{(1)}\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G^{(1)} \\ \overset{F^{(2)}}{\to} \cdots \overset{F^{(m)}}{\to} G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G^{(m)} \cong G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda. \end{aligned}$$

The composition $F^{(m)}\cdots F^{(1)}F$ is a bijection. So $F$ and all $F^{(i)}$ are bijections. □

Let $G$ be an abelian group. It is well known that $\mathbb{X}$ is isomorphic to $G$. So $\Lambda G$ admits an action by $G$; and moreover, by [18], the skew group algebra $(\Lambda G)G$ is Morita equivalent to $\Lambda$. Now, the following result is an immediate consequence of Theorem 4.6.

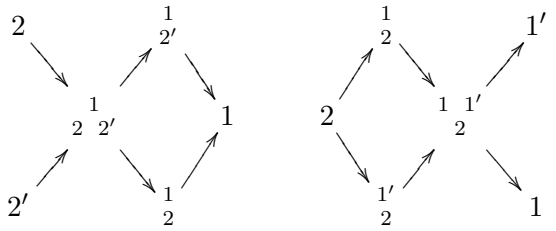
**Corollary 4.7** *If $G$ is an abelian group, then $F$ induces a bijection*

$$G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda \to G\text{-}s\tau\text{-}\mathsf{tilt}\,\Lambda G.$$

Finally, we illustrate Theorem 4.6 with the following example.

**Example 4.8** Let $\Lambda$ be the path algebra of the quiver $Q$ (see below), and let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\Lambda$ by switching 2 and $2'$, $\alpha$ and $\beta$, and fixing the vertex 1. Then the following $Q'$ is the quiver of $\Lambda G$:


The Auslander-Reiten quivers of $\mathsf{mod}\,\Lambda$ and $\mathsf{mod}\,\Lambda G$ are the following, where each module is represented by its radical filtration:




$\Gamma(\Lambda)$ $\Gamma(\Lambda G)$

We denote by $\mathsf{ind}\,\Lambda$ the set of isomorphism classes of indecomposable $\Lambda$-modules. Then we describe the map induced by $F$ between $\mathsf{ind}\,\Lambda$ and $\mathsf{mod}\,\Lambda G$. Observe that the correspondences from the Auslander-Reiten quiver of $\Lambda$ to that of $\Lambda G$:

$$\begin{aligned} F\colon \mathsf{ind}\,\Lambda &\to \mathsf{mod}\,\Lambda G,\\ 2, 2' &\mapsto 2,\\ \begin{smallmatrix}1\\2\ \ 2'\end{smallmatrix} &\mapsto \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2'\end{smallmatrix},\\ \begin{smallmatrix}1\\2'\end{smallmatrix},\ \begin{smallmatrix}1\\2\end{smallmatrix} &\mapsto \begin{smallmatrix}1\ \ 1'\\2\end{smallmatrix},\\ 1 &\mapsto 1 \oplus 1'. \end{aligned}$$

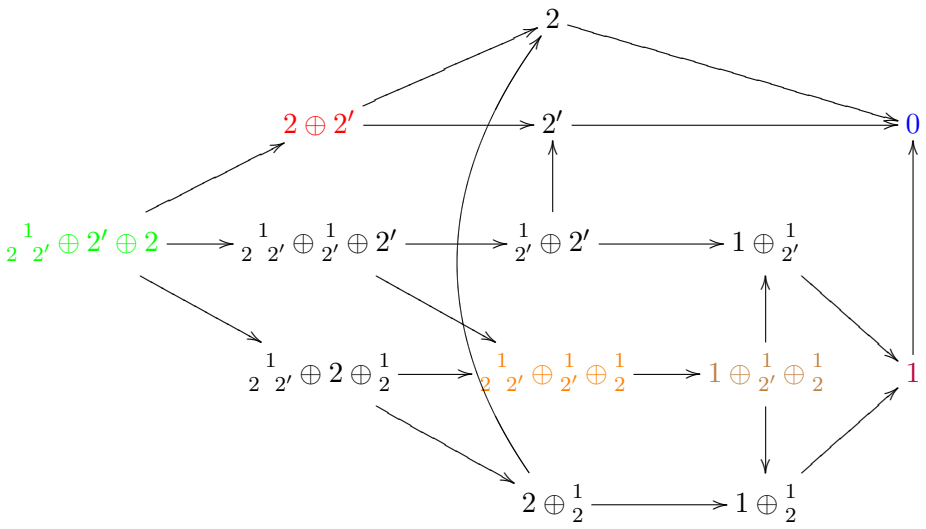
Recall from [2] the definition of the support $\tau$-tilting quiver $Q(s\tau\text{-tilt}\Lambda)$ of $\Lambda$ as follows:

(1) the set of vertices is $s\tau$-tilt$\Lambda$;

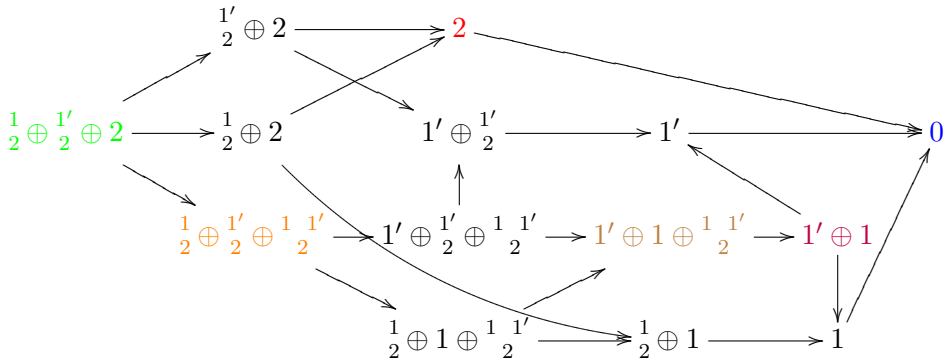
(2) we draw an arrow from $T$ to $U$ if $U$ is a left mutation of $T$ ([2, Theorem 2.30]).

One can calculate the left mutation of support $\tau$-tilting $\Lambda$-modules by exchanging sequences that are constructed from left approximations. Therefore, we can draw the support $\tau$-tilting quiver of an algebra by its Auslander-Reiten quiver. Now, we draw $Q(s\tau\text{-}\,\mathsf{tilt}\,\Lambda)$ and $Q(s\tau\text{-}\,\mathsf{tilt}\,\Lambda G)$ as follows.

$Q(s\tau\text{-tilt}\,\Lambda)$:




$Q(s\tau\text{-tilt}\,\Lambda G)$:




The colored support $\tau$-tilting modules in the graph are all the basic $G$-stable support $\tau$-tilting modules in $\mathsf{mod}\,\Lambda$ and $\mathsf{mod}\,\Lambda G$, respectively. Moreover, the bijection in Theorem 4.6 takes a $G$-stable support $\tau$-tilting module in $Q(s\tau\text{-tilt}\,\Lambda)$ to that in $Q(s\tau\text{-tilt}\,\Lambda G)$ in the same color. The $G$-stable support $\tau$-tilting modules in green, orange, and brown are $G$-stable tilting.

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