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RESEARCH ARTICLE

Torsion pairs in recollements of abelian categories

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Abstract For a recollement $(\mathscr{A},\mathscr{B},\mathscr{C})$ of abelian categories, we show that torsion pairs in $\mathscr{A}$ and $\mathscr{C}$ can induce torsion pairs in $\mathscr{B}$; and the converse holds true under certain conditions.

Keywords Torsion pairs, recollements, abelian categories
MSC 18E40, 18G99

1 Introduction

Recollements of abelian categories and triangulated categories play an important role in geometry of singular spaces, representation theory, polynomial functors theory, and ring theory [2,3,5,6,12,14,15,19], where recollements are known as torsion torsion-free (TTF) theories. They first appeared in the construction of the category of perverse sheaves on a singular space [2]. Recollements of abelian categories and recollements of triangulated categories are closely related; for instance, Chen [4] constructed a recollement of abelian categories from a recollement of triangulated categories, generalizing a result of Lin and Wang [16]. In addition, the properties of torsion pairs and recollements of abelian categories have been studied by Psaroudakis and Vitória [22]. They established a correspondence between recollements of abelian categories up to equivalence and certain TTF-triples.

Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of triangulated categories. Chen [4] described how to glue together cotorsion pairs (which are essentially equal to torsion pairs [11]) in $\mathscr{A}$ and $\mathscr{C}$ to obtain a cotorsion pair in $\mathscr{B}$, which is a natural generalization of a similar result in [2] on gluing together t-structures of $\mathscr{A}$ and $\mathscr{C}$ to obtain a t-structure in $\mathscr{B}$. After taking the hearts $\mathscr{A}',\mathscr{B}',\mathscr{C}'$ of the glued t-structures, $(\mathscr{A}',\mathscr{B}',\mathscr{C}')$ is a recollement of abelian categories and

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a construction of gluing of torsion pairs in this recollement was given by Liu et al. [18] (also see [13]). Note that the results of Liu et al. [18, Proposition 6.5, Lemma 6.2] depend on the recollements of triangulated categories and the proofs there do not work in the general case. Our aim is to glue torsion pairs and TTF-triples in a recollement of general abelian categories.

This paper is organized as follows. In Section 2, we give some terminologies and some preliminary results. In Section 3, we study torsion pairs in a recollement of abelian categories. Letting $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories, we obtain a torsion pair in $\mathscr{B}$ from torsion pairs in $\mathscr{A}$ and $\mathscr{C}$. Conversely, we show that, under certain conditions, a torsion pair in $\mathscr{B}$ can induce torsion pairs in $\mathscr{A}$ and $\mathscr{C}$.

2 Preliminaries

Throughout this paper, all subcategories are full, additive, and closed under isomorphisms.

Definition 1 [8] A recollement, denoted by $(\mathscr{A},\mathscr{B},\mathscr{C})$, of abelian categories is a diagram

$$\mathscr{A} \;\begin{smallmatrix} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow{\;i^!\;} \end{smallmatrix}\; \mathscr{B} \;\begin{smallmatrix} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^*\;} \\ \xleftarrow{\;j_*\;} \end{smallmatrix}\; \mathscr{C}$$

of abelian categories and additive functors such that

(1) (i^*,i_*), $(i_*,i^!)$, $(j_!,j^*)$, and (j^*,j_*) are adjoint pairs,

(2) i_*, $j_!$ and j_* are fully faithful,

(3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

See [8,17,20] for examples of recollements of abelian categories. We list some properties of recollements (see [8,20–22]), which will be used in the sequel.

Lemma 1 *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories. Then we have*

(1) $i^*j_! = 0 = i^!j_*$;

(2) *the functors i_* and j^* are exact, i^* and $j_!$ are right exact, and $i^!$ and j_* are left exact*;

(3) *all the natural transformations*

$$i^*i_* \to 1_{\mathscr{A}},\quad 1_{\mathscr{A}} \to i^!i_*,\quad 1_{\mathscr{C}} \to j^*j_!,\quad j^*j_* \to 1_{\mathscr{C}},$$

are natural isomorphisms;

(4) *for any $B \in \mathscr{B}$, there exist exact sequences*

$$0 \to i_*(A) \to j_!j^*(B) \xrightarrow{\varepsilon_B} B \to i_*i^*(B) \to 0,$$
$$0 \to i_*i^!(B) \to B \xrightarrow{\eta_B} j_*j^*(B) \to i_*(A') \to 0,$$

in $\mathscr{B}$ *with* $A, A' \in \mathscr{A}$*;*

(5) *there exists an exact sequence of natural transformations:*

$$0 \to i_*i^!j_! \to j_! \to j_* \to i_*i^*j_* \to 0;$$

(6) *if* i^* *is exact, then* $i^!j_! = 0$*, and if* $i^!$ *is exact, then* $i^*j_* = 0$*.*

Definition 2 [7] A pair of subcategories $(\mathscr{X}, \mathscr{Y})$ of an abelian category $\mathscr{A}$ is called a *torsion pair* if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, \mathscr{Y}) = 0$, that is, $\mathrm{Hom}_{\mathscr{A}}(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$;

(2) for any object $M \in \mathscr{A}$, there exists an exact sequence

$$0 \to X \to M \to Y \to 0$$

in $\mathscr{A}$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Let $(\mathscr{X}, \mathscr{Y})$ be a torsion pair in an abelian category $\mathscr{A}$. Then we have

(1) $\mathscr{X}$ is closed under extensions and quotient objects,

(2) $\mathscr{Y}$ is closed under extensions and subobjects.

Moreover, we have

$$\mathscr{X} = {}^{\perp_0}\mathscr{Y} := \{M \in \mathscr{A} \mid \mathrm{Hom}_{\mathscr{A}}(M, \mathscr{Y}) = 0\},$$
$$\mathscr{Y} = \mathscr{X}^{\perp_0} := \{M \in \mathscr{A} \mid \mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, M) = 0\}.$$

Definition 3 [3,10] Let $(\mathscr{X}, \mathscr{Y})$ be a torsion pair in an abelian category $\mathscr{A}$.

(1) $(\mathscr{X}, \mathscr{Y})$ is called *tilting* (resp. *cotilting*) if any object in $\mathscr{A}$ is isomorphic to a subobject of an object in $\mathscr{X}$ (resp. a quotient object of an object in $\mathscr{Y}$).

(2) $(\mathscr{X}, \mathscr{Y})$ is called *hereditary* (resp. *cohereditary*) if $\mathscr{X}$ is closed under subobjects (resp. $\mathscr{Y}$ is closed under quotient objects).

3 Torsion pairs in a recollement

In this section, assume that $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is a recollement of abelian categories:

$$\mathscr{A} \underset{\overset{\xleftarrow{i^!}}{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i_*}}} \mathscr{B} \underset{\overset{\xleftarrow{j_*}}{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j^*}}} \mathscr{C}$$

We begin with the following result.

Lemma 2 *For any* $B \in \mathscr{B}$*,*

(1) *if* i^* *is exact, then there exists an exact sequence*

$$0 \to j_!j^*(B) \xrightarrow{\varepsilon_B} B \to i_*i^*(B) \to 0;$$

(2) *if $i^!$ is exact, then there exists an exact sequence*

$$0 \to i_*i^!(B) \to B \xrightarrow{\eta_B} j_*j^*(B) \to 0;$$

(3) *i^* and $i^!$ are exact if and only if $i^* \cong i^!$, and in this case, we have $j_* \cong j_!$.*

Proof (1) By Lemma 1 (4), it suffices to prove that ε_B is monic. Applying $i^!$ to the first exact sequence in Lemma 1 (4), we get an exact sequence

$$0 \to i^!i_*(A) \to i^!j_!j^*(B).$$

By Lemma 1 (6), we have

$$i^!j_!j^*(B) = 0.$$

So

$$A \cong i^!i_*(A) = 0$$

by Lemma 1 (3), and hence, ε_B is monic.

(2) It is similar to (1).

(3) If $i^* \cong i^!$, then i^* and $i^!$ are exact by Lemma 1 (2). Conversely, applying $i^!$ to the exact sequence in (1), we get an exact sequence of natural transformations:

$$0 \to i^!j_!j^* \to i^! \to i^!i_*i^* \to 0.$$

By Lemma 1 (6) and (3), we have

$$i^! \cong i^!i_*i^* \cong i^*.$$

The isomorphism $j_* \cong j_!$ follows from Lemma 1 (5) and (6). □

Our main result is the following theorem.

Theorem 1 *Let $(\mathscr{X}', \mathscr{Y}')$ and $(\mathscr{X}'', \mathscr{Y}'')$ be torsion pairs in $\mathscr{A}$ and $\mathscr{C}$, respectively, and let*

$$\mathscr{X} := \{B \in \mathscr{B} \mid i^*(B) \in \mathscr{X}',\ j^*(B) \in \mathscr{X}''\},$$
$$\mathscr{Y} := \{B \in \mathscr{B} \mid i^!(B) \in \mathscr{Y}',\ j^*(B) \in \mathscr{Y}''\}.$$

Then we have

(1) *$(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\mathscr{B}$;*

(2) *$(\mathscr{X}', \mathscr{Y}') = (i^*(\mathscr{X}), i^!(\mathscr{Y}))$ and $(\mathscr{X}'', \mathscr{Y}'') = (j^*(\mathscr{X}), j^*(\mathscr{Y}))$;*

(3) *if $(\mathscr{X}', \mathscr{Y}')$ and $(\mathscr{X}'', \mathscr{Y}'')$ are cohereditary (resp. hereditary), and $i^!$ (resp. i^*) is exact, then $(\mathscr{X}, \mathscr{Y})$ is cohereditary (resp. hereditary);*

(4) *if $(\mathscr{X}', \mathscr{Y}')$ and $(\mathscr{X}'', \mathscr{Y}'')$ are tilting (resp. cotilting), and $i^!$ and $j_!$ (resp. i^* and j_*) are exact, then $(\mathscr{X}, \mathscr{Y})$ is tilting (resp. cotilting).*

Proof (1) Let $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. Applying the functor $\mathrm{Hom}_{\mathscr{B}}(-, Y)$ to the exact sequence

$$j_!j^*(X) \xrightarrow{\varepsilon_X} X \to i_*i^*(X) \to 0$$

in $\mathscr{B}$, we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{B}}(i_*i^*(X), Y) \to \mathrm{Hom}_{\mathscr{B}}(X, Y) \to \mathrm{Hom}_{\mathscr{B}}(j_!j^*(X), Y).$$

By assumption, $(\mathscr{X}', \mathscr{Y}')$ and $(\mathscr{X}'', \mathscr{Y}'')$ are torsion pairs in $\mathscr{A}$ and $\mathscr{C}$, respectively. Since $i^*(X) \in \mathscr{X}'$, $i^!(Y) \in \mathscr{Y}'$, $j^*(X) \in \mathscr{X}''$, and $j^*(Y) \in \mathscr{Y}''$, we have

$$\mathrm{Hom}_{\mathscr{B}}(j_!j^*(X), Y) \cong \mathrm{Hom}_{\mathscr{C}}(j^*(X), j^*(Y)) = 0,$$
$$\mathrm{Hom}_{\mathscr{B}}(i_*i^*(X), Y) \cong \mathrm{Hom}_{\mathscr{A}}(i^*(X), i^!(Y)) = 0.$$

It follows that

$$\mathrm{Hom}_{\mathscr{B}}(X, Y) = 0, \quad \mathrm{Hom}_{\mathscr{B}}(\mathscr{X}, \mathscr{Y}) = 0.$$

Let $B \in \mathscr{B}$. There exists an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i_*i^!(B) & \longrightarrow & B & \xrightarrow{\quad\eta_B\quad} & j_*j^*(B) & \longrightarrow & i_*(A') \longrightarrow 0 \\ & & & & & \searrow \quad \nearrow & & & \\ & & & & & \mathrm{Im}\,\eta_B & & & \end{array}$$

in $\mathscr{B}$ with $A' \in \mathscr{A}$. Because $j^*(B) \in \mathscr{C}$ and $(\mathscr{X}'', \mathscr{Y}'')$ is a torsion pair in $\mathscr{C}$, there exists an exact sequence

$$0 \to X'' \to j^*(B) \xrightarrow{h} Y'' \to 0$$

in $\mathscr{C}$ with $X'' \in \mathscr{X}''$ and $Y'' \in \mathscr{Y}''$. Notice that j_* is left exact by Lemma 1 (2). Then

$$0 \to j_*(X'') \to j_*j^*(B) \xrightarrow{j_*(h)} j_*(Y'')$$

is exact and we have the following pullback diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \dashrightarrow & K & \overset{f}{\dashrightarrow} & \mathrm{Im}\,\eta_B & \dashrightarrow & \mathrm{Coker}\,f & \dashrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_*(X'') & \longrightarrow & j_*j^*(B) & \longrightarrow & \mathrm{Im}\,j_*(h) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \dashrightarrow & \mathrm{Coker}\,g & \dashrightarrow & i_*(A') & \dashrightarrow & U & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.1}$$

Then we get the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & i_*i^!(B) & = & i_*i^!(B) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & M & \dashrightarrow & B & \dashrightarrow & \operatorname{Coker} f & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & \operatorname{Im} \eta_B & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.2}
$$

Because $i^*(M) \in \mathscr{A}$ and $(\mathscr{X}', \mathscr{Y}')$ is a torsion pair in $\mathscr{A}$, there exists an exact sequence

$$0 \to X' \to i^*(M) \to Y' \to 0$$

in $\mathscr{A}$ with $X' \in \mathscr{X}'$ and $Y' \in \mathscr{Y}'$. Notice that i_* is exact by Lemma 1 (2). Then

$$0 \to i_*(X') \to i_*i^*(M) \to i_*(Y') \to 0$$

is exact and we have the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Im} \varepsilon_M & = & \operatorname{Im} \varepsilon_M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & X & \dashrightarrow & M & \dashrightarrow & i_*(Y') & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & i_*(X') & \longrightarrow & i_*i^*(M) & \longrightarrow & i_*(Y') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.3}
$$

where the exactness of the middle column follows from Lemma 1 (4). Now, we get the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X & = = = & X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \dashrightarrow & i_*(Y') & \dashrightarrow & Y & \dashrightarrow & \operatorname{Coker} f & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.4}
$$

To get the assertion, it suffices to show $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Since $i^* j_! = 0$ and i^* is right exact by Lemma 1 (1) and (2), we have

$$i^*(\operatorname{Im} \varepsilon_M) = 0.$$

Since i^* is right exact by Lemma 1 (2), applying the functor i^* to the leftmost column in diagram (3.3) yields

$$i^*(X) \cong i^* i_*(X') \cong X' \in \mathscr{X}'.$$

On the other hand, note that j^* is exact (by Lemma 1 (2)) and $\operatorname{Im} i_* = \operatorname{Ker} j^*$. Then, applying the functor j^* to the bottom row in diagram (3.1), we have

$$j^*(\operatorname{Coker} g) = 0 = j^*(U);$$

furthermore, we have

$$
\begin{aligned}
j^*(X) &\cong j^*(M) \quad \text{(by applying } j^* \text{ to middle row in diagram (3.3))} \\
&\cong j^*(K) \quad \text{(by applying } j^* \text{ to leftmost column in diagram (3.2))} \\
&\cong j^* j_*(X'') \quad \text{(by applying } j^* \text{ to leftmost column in diagram (3.1))} \\
&\cong X'' \\
&\in \mathscr{X}''.
\end{aligned}
$$

It implies $X \in \mathscr{X}$.

Applying the functor j^* to the bottom row in diagram (3.4) and the rightmost column in diagram (3.1), since j^* is exact and $\operatorname{Im} i_* = \operatorname{Ker} j^*$, we have that $j^*(Y)$ ($\cong j^*(\operatorname{Coker} f) \cong j^*(\operatorname{Im} j_*(h))$) is isomorphic to a subobject of Y'' ($\cong j^* j_*(Y'')$). Because $\mathscr{Y}''$ is closed under subobjects, it follows that $j^*(Y) \in \mathscr{Y}''$. On the other hand, applying the functor $i^!$ to the rightmost column in diagram (3.1) and the bottom row in diagram (3.4), since $i^!$ is left exact and $i^! j_* = 0$ by Lemma 1 (1) and (2), we have

$$i^!(\operatorname{Im} j_*(h)) = 0, \quad i^!(\operatorname{Coker} f) = 0.$$

So
$$i^!(Y) \cong i^!i_*(Y') \cong Y' \in \mathscr{Y}',$$
and hence, $Y \in \mathscr{Y}$.

(2) It is trivial that $i^*(\mathscr{X}) \subseteq \mathscr{X}'$. For any $X' \in \mathscr{X}'$, since
$$i^*i_*(X') \cong X' \in \mathscr{X}', \quad j^*i_*(X') = 0 \in \mathscr{X}'',$$
we have $i_*(X') \in \mathscr{X}$, and hence,
$$X' \cong i^*(i_*(X')) \in i^*(\mathscr{X}).$$
Thus,
$$\mathscr{X}' \subseteq i^*(\mathscr{X}).$$
Similarly, we get
$$\mathscr{Y}' = i^!(\mathscr{Y}), \quad \mathscr{X}'' = j^*(\mathscr{X}), \quad \mathscr{Y}'' = j^*(\mathscr{Y}).$$

(3) Assume that $(\mathscr{X}', \mathscr{Y}')$ and $(\mathscr{X}'', \mathscr{Y}'')$ are cohereditary. Then $\mathscr{Y}'$ and $\mathscr{Y}''$ are closed under quotient objects. Let $Y \in \mathscr{Y}$, and let
$$0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y_1 \longrightarrow 0$$
be an exact sequence in $\mathscr{B}$. Since j^* and $i^!$ are exact by Lemma 1 (2) and assumption, we have $j^*(Y_1)$ and $i^!(Y_1)$ are isomorphic to quotient objects of $j^*(Y)$ $(\in \mathscr{Y}'')$ and $i^!(Y)$ $(\in \mathscr{Y}')$, respectively. So
$$j^*(Y_1) \in \mathscr{Y}'', \quad i^!(Y_1) \in \mathscr{Y}'.$$
It implies that $Y_1 \in \mathscr{Y}$ and $(\mathscr{X}, \mathscr{Y})$ is cohereditary.

Dually, we get the assertion for the hereditary case.

(4) Assume that $(\mathscr{X}', \mathscr{Y}')$ and $(\mathscr{X}'', \mathscr{Y}'')$ are tilting. Let $B \in \mathscr{B}$. By Lemma 1 (4) and Lemma 2 (2), there exist exact sequences
$$\begin{array}{ccccccc} 0 \longrightarrow i_*(A) \longrightarrow & j_!j^*(B) & & \xrightarrow{\qquad\varepsilon_B\qquad} & & B & \longrightarrow i_*i^*(B) \longrightarrow 0, \\ & & \searrow & & \nearrow & & \\ & & & \operatorname{Im}\varepsilon_B & & & \end{array}$$
$$0 \to i_*i^!(B) \to B \to j_*j^*(B) \to 0,$$
in $\mathscr{B}$ with $A \in \mathscr{A}$.

Since $(\mathscr{X}'', \mathscr{Y}'')$ is tilting and $j^*(B) \in \mathscr{C}$, there exists a monomorphism
$$0 \to j^*(B) \to X''$$

in $\mathscr{C}$ with $X'' \in \mathscr{X}''$. Since $j_!$ is exact by assumption, we get the exact sequence

$$0 \to j_!j^*(B) \to j_!(X'') \to j_!(X''/j^*(B)) \to 0$$

in $\mathscr{B}$ and the pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & i_*(A) & \to & j_!j^*(B) & \to & \operatorname{Im}\varepsilon_B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & i_*(A) & \to & j_!(X'') & \to & U & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & j_!(X''/j^*(B)) & = & j_!(X''/j^*(B)) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.5}$$

Then we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \operatorname{Im}\varepsilon_B & \to & B & \to & i_*i^*(B) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & U & \to & V'' & \to & i_*i^*(B) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & j_!(X''/j^*(B)) & = & j_!(X''/j^*(B)) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.6}$$

On the other hand, since $(\mathscr{X}', \mathscr{Y}')$ is tilting and $i^!(B) \in \mathscr{A}$, there exists a monomorphism

$$0 \to i^!(B) \to X'$$

in $\mathscr{A}$ with $X' \in \mathscr{X}'$. Since i_* is exact by Lemma 1 (2), we get the exact sequence

$$0 \to i_*i^!(B) \to i_*(X') \to i_*(X'/i^!(B)) \to 0$$

in $\mathscr{B}$ and the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & i_*i^!(B) & \longrightarrow & B & \longrightarrow & j_*j^*(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \dashrightarrow & i_*(X') & \dashrightarrow & V' & \dashrightarrow & j_*j^*(B) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & i_*(X'/i^!(B)) & = & i_*(X'/i^!(B)) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.7}$$

Then we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & V'' & \longrightarrow & j_!(X''/j^*(B)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \dashrightarrow & V' & \dashrightarrow & X & \dashrightarrow & j_!(X''/j^*(B)) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & i_*(X'/i^!(B)) & = & i_*(X'/i^!(B)) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.8}$$

Since j^* is exact (by Lemma 1 (2)) and $\operatorname{Im} i_* = \operatorname{Ker} j^*$, we have

$$\begin{aligned}
j^*(X) &\cong j^*(V'') \quad \text{(by applying } j^* \text{ to middle column in diagram (3.8))} \\
&\cong j^*(U) \quad \text{(by applying } j^* \text{ to middle row in diagram (3.6))} \\
&\cong j^*j_!(X'') \quad \text{(by applying } j^* \text{ to middle row in diagram (3.5))} \\
&\cong X'' \\
&\in \mathscr{X}''.
\end{aligned}$$

Since $i^!$ is exact by assumption, we have $i^*j_* = 0$ by Lemma 1 (6). So, applying i^* to the middle row in diagram (3.7) yields that

$$i^*i_*(X') \to i^*(V') \to 0$$

is exact. Since $i^*j_! = 0$ by Lemma 1 (1), applying i^* to the middle row in diagram (3.8) yields that

$$i^*(V') \to i^*(X) \to 0$$

is exact. Thus, $i^*(X)$ is isomorphic to a quotient object of $i^*i_*(X')$ ($\cong X' \in \mathscr{X}'$). Notice that $\mathscr{X}'$ is closed under quotient objects, so $i^*(X) \in \mathscr{X}'$, and hence $X \in \mathscr{X}$. Thus, we conclude that $(\mathscr{X}, \mathscr{Y})$ is tilting.

Dually, we get the assertion for the cotilting case. □

Recall from [9] that a triple of subcategories $(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ of an abelian category is called a *TTF-triple* if $(\mathscr{X}, \mathscr{Y})$ and $(\mathscr{Y}, \mathscr{Z})$ are torsion pairs. By [22, Theorem 4.3], we know that $(\operatorname{Ker} i^*, \operatorname{Im} i_*, \operatorname{Ker} i^!)$ is a TTF-triple in $\mathscr{B}$.

Corollary 1 *Let $(\mathscr{X}', \mathscr{Y}', \mathscr{Z}')$ and $(\mathscr{X}'', \mathscr{Y}'', \mathscr{Z}'')$ are TTF-triples in $\mathscr{A}$ and $\mathscr{C}$, respectively. If i^* and $i^!$ are exact, then $(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ is a TTF-triple in $\mathscr{B}$, where $\mathscr{X}$, $\mathscr{Y}$ are as in Theorem 1 and*

$$\mathscr{Z} := \{B \in \mathscr{B} \mid i^*(B) \in \mathscr{Z}', \ j^*(B) \in \mathscr{Z}''\}.$$

Proof It follows from Lemma 2 (3) and Theorem 1. □

To study the converse of Theorem 1, we need the following easy observation.

Lemma 3 *If $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\mathscr{B}$, then we have*

(1) *$j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$ if and only if $j_!j^*(\mathscr{X}) \subseteq \mathscr{X}$;*

(2) *$i_*i^!(\mathscr{Y}) \subseteq \mathscr{Y}$ if and only if $i_*i^*(\mathscr{X}) \subseteq \mathscr{X}$.*

Proof (1) Let $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. Since

$$\operatorname{Hom}_{\mathscr{B}}(X, j_*j^*(Y)) \cong \operatorname{Hom}_{\mathscr{C}}(j^*(X), j^*(Y)) \cong \operatorname{Hom}_{\mathscr{B}}(j_!j^*(X), Y)$$

and

$$\mathscr{X} = {}^{\perp_0}\mathscr{Y}, \quad \mathscr{Y} = \mathscr{X}^{\perp_0},$$

the assertion follows.

(2) It is similar to (1). □

The following result shows that the converse of Theorem 1 (1) and (2) holds true under certain conditions.

Theorem 2 *Let $(\mathscr{X}, \mathscr{Y})$ be a torsion pair in $\mathscr{B}$. Then we have*

(1) *$(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ is a torsion pair in $\mathscr{A}$;*

(2) *$j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$ if and only if $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ is a torsion pair in $\mathscr{C}$;*

(3) *if $j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$, then*

$$\mathscr{X} = \{B \in \mathscr{B} \mid i^*(B) \in i^*(\mathscr{X}), \ j^*(B) \in j^*(\mathscr{X})\},$$
$$\mathscr{Y} = \{B \in \mathscr{B} \mid i^!(B) \in i^!(\mathscr{Y}), \ j^*(B) \in j^*(\mathscr{Y})\}.$$

Proof (1) Let $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. Applying the functor $\operatorname{Hom}_{\mathscr{B}}(-, Y)$ to the exact sequence

$$j_!j^*(X) \xrightarrow{\varepsilon_X} X \to i_*i^*(X) \to 0$$

in $\mathscr{B}$, we get an exact sequence

$$0 \to \operatorname{Hom}_{\mathscr{B}}(i_*i^*(X), Y) \to \operatorname{Hom}_{\mathscr{B}}(X, Y) \to \operatorname{Hom}_{\mathscr{B}}(j_!j^*(X), Y).$$

Since $\mathrm{Hom}_{\mathscr{B}}(X,Y)=0$, we have

$$\mathrm{Hom}_{\mathscr{B}}(i_*i^*(X),Y)=0.$$

It follows that

$$i_*i^*(X)\in {}^{\perp_0}\mathscr{Y}=\mathscr{X},\quad i_*i^*(\mathscr{X})\subseteq\mathscr{X}.$$

So $i_*i^!(\mathscr{Y})\subseteq\mathscr{Y}$ by Lemma 3 (2).

Let $X'\in i^*(\mathscr{X})$ and $Y'\in i^!(\mathscr{Y})$. Then there exist $X\in\mathscr{X}$ and $Y\in\mathscr{Y}$ such that

$$X'=i^*(X),\quad Y'=i^!(Y).$$

Because $(\mathscr{X},\mathscr{Y})$ is a torsion pair in $\mathscr{B}$ (by assumption) and $i_*i^!(Y)\in\mathscr{Y}$, we have

$$\mathrm{Hom}_{\mathscr{A}}(X',Y')=\mathrm{Hom}_{\mathscr{A}}(i^*(X),i^!(Y))\cong\mathrm{Hom}_{\mathscr{B}}(X,i_*i^!(Y))=0$$

and

$$\mathrm{Hom}_{\mathscr{A}}(i^*(\mathscr{X}),i^!(\mathscr{Y}))=0.$$

Let $A\in\mathscr{A}$. Because $i_*(A)\in\mathscr{B}$, there exists an exact sequence

$$0\to X\to i_*(A)\to Y\to 0$$

in $\mathscr{B}$ with $X\in\mathscr{X}$ and $Y\in\mathscr{Y}$. Since $i_*(\mathscr{A})$ is a Serre subcategory of $\mathscr{B}$ by [22, Proposition 2.8], there exist $X_1,Y_1\in\mathscr{A}$ such that

$$X\cong i_*(X_1),\quad Y\cong i_*(Y_1).$$

Since $i_*\colon\mathscr{A}\to i_*(\mathscr{A})$ is an equivalence, we get that

$$0\to X_1\to A\to Y_1\to 0$$

is an exact sequence in $\mathscr{A}$ with

$$X_1\cong i^*(i_*(X_1))\cong i^*(X)\in i^*(\mathscr{X}),\quad Y_1\cong i^!(i_*(Y_1))\cong i^!(Y)\in i^!(\mathscr{Y}).$$

Thus, we conclude that $(i^*(\mathscr{X}),i^!(\mathscr{Y}))$ is a torsion pair in $\mathscr{A}$.

(2) Let $j_*j^*(\mathscr{Y})\subseteq\mathscr{Y}$. For any $X'\in j^*(\mathscr{X})$ and $Y'\in j^*(\mathscr{Y})$, there exist $X\in\mathscr{X}$ and $Y\in\mathscr{Y}$ such that

$$X'=j^*(X),\quad Y'=j^*(Y).$$

Because $(\mathscr{X},\mathscr{Y})$ is a torsion pair in $\mathscr{B}$, we have

$$\mathrm{Hom}_{\mathscr{C}}(X',Y')=\mathrm{Hom}_{\mathscr{C}}(j^*(X),j^*(Y))\cong\mathrm{Hom}_{\mathscr{B}}(X,j_*j^*(Y))=0$$

and

$$\mathrm{Hom}_{\mathscr{C}}(j^*(\mathscr{X}),j^*(\mathscr{Y}))=0.$$

Let $C \in \mathscr{C}$. Because $j_*(C) \in \mathscr{B}$, there exists an exact sequence

$$0 \to X \to j_*(C) \to Y \to 0$$

in $\mathscr{B}$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. Since j^* is exact by Lemma 1 (2), we have

$$0 \to j^*(X) \to j^*j_*(C)\ (\cong C) \to j^*(Y) \to 0$$

is also exact and the assertion follows.

Conversely, if $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ is a torsion pair in $\mathscr{C}$, then we have

$$\mathrm{Hom}_{\mathscr{B}}(\mathscr{X}, j_*j^*(\mathscr{Y})) \cong \mathrm{Hom}_{\mathscr{C}}(j^*(\mathscr{X}), j^*(\mathscr{Y})) = 0,$$

which implies

$$j_*j^*(\mathscr{Y}) \subseteq \mathscr{X}^{\perp_0} = \mathscr{Y}.$$

(3) It is trivial that

$$\mathscr{X} \subseteq \{B \in \mathscr{B} \mid i^*(B) \in i^*(\mathscr{X}),\ j^*(B) \in j^*(\mathscr{X})\},$$
$$\mathscr{Y} \subseteq \{B \in \mathscr{B} \mid i^!(B) \in i^!(\mathscr{Y}),\ j^*(B) \in j^*(\mathscr{Y})\}.$$

Conversely, let $B \in \mathscr{B}$ with $i^*(B) \in i^*(\mathscr{X})$ and $j^*(B) \in j^*(\mathscr{X})$. By Lemma 1 (4), there exists an exact sequence

$$j_!j^*(B) \xrightarrow{\varepsilon_B} B \to i_*i^*(B) \to 0$$

in $\mathscr{B}$. For any $Y \in \mathscr{Y}$, applying the functor $\mathrm{Hom}_{\mathscr{B}}(-, Y)$ to the above exact sequence, we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{B}}(i_*i^*(B), Y) \to \mathrm{Hom}_{\mathscr{B}}(B, Y) \to \mathrm{Hom}_{\mathscr{B}}(j_!j^*(B), Y).$$

By (1) and (2), $(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ and $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ are torsion pairs in $\mathscr{A}$ and $\mathscr{C}$, respectively. So we have

$$\mathrm{Hom}_{\mathscr{B}}(j_!j^*(B), Y) \cong \mathrm{Hom}_{\mathscr{C}}(j^*(B), j^*(Y)) = 0,$$
$$\mathrm{Hom}_{\mathscr{B}}(i_*i^*(B), Y) \cong \mathrm{Hom}_{\mathscr{A}}(i^*(B), i^!(Y)) = 0,$$

and hence, $\mathrm{Hom}_{\mathscr{B}}(B, Y) = 0$ and $B \in {}^{\perp_0}\mathscr{Y} = \mathscr{X}$. It follows that

$$\{B \in \mathscr{B} \mid i^*(B) \in i^*(\mathscr{X}),\ j^*(B) \in j^*(\mathscr{X})\} \subseteq \mathscr{X}.$$

Dually, we have

$$\{B \in \mathscr{B} \mid i^!(B) \in i^!(\mathscr{Y}),\ j^*(B) \in j^*(\mathscr{Y})\} \subseteq \mathscr{Y}. \qquad \square$$

The following corollary is a converse of Corollary 1.

Corollary 2 *Let $(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ be a TTF-triple in $\mathscr{B}$. Then we have*

(1) *$(i^*(\mathscr{X}), i^*(\mathscr{Y}), i^!(\mathscr{Z}))$ is a TTF-triple in $\mathscr{A}$;*

(2) *if* $j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$ *and* $j_!j^*(\mathscr{Y}) \subseteq \mathscr{Y}$, *then* $(j^*(\mathscr{X}), j^*(\mathscr{Y}), j^*(\mathscr{Z}))$ *is a TTF-triple in* $\mathscr{C}$.

Proof (1) By Theorem 2 (1), we have $(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ and $(i^*(\mathscr{Y}), i^!(\mathscr{Z}))$ are torsion pairs in $\mathscr{A}$. As in the proof of Theorem 2, we have

$$i_*i^*(\mathscr{X}) \subseteq \mathscr{X}, \quad i_*i^*(\mathscr{Y}) \subseteq \mathscr{Y}.$$

By Lemma 3 (2), we have $i_*i^!(\mathscr{Y}) \subseteq \mathscr{Y}$. It follows that $i^*(\mathscr{Y}) = i^!(\mathscr{Y})$ since $i^*i_* \cong 1_{\mathscr{A}} \cong i^!i_*$ by Lemma 1 (3). Thus, $(i^*(\mathscr{X}), i^*(\mathscr{Y}), i^!(\mathscr{Z}))$ is a TTF-triple in $\mathscr{A}$.

(2) Since $j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$ and $j_!j^*(\mathscr{Y}) \subseteq \mathscr{Y}$ by assumption, it follows from Lemma 3 (1) and Theorem 2 (2) that $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ and $(j^*(\mathscr{Y}), j^*(\mathscr{Z}))$ are torsion pairs in $\mathscr{C}$. Thus, we get the assertion. □

The following result shows that the converse of Theorem 1 (3) and (4) also holds true under certain conditions.

Proposition 1 *Let* $(\mathscr{X}, \mathscr{Y})$ *be a torsion pair in* $\mathscr{B}$.

(1) *Assume that* $(\mathscr{X}, \mathscr{Y})$ *is hereditary* (*resp. cohereditary*)*. Then we have*

(a) $(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ *is a hereditary* (*resp. cohereditary*) *torsion pair*;

(b) *if* $j_!$ (*resp.* j_*) *is exact and* $j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$, *then* $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ *is a hereditary* (*resp. cohereditary*) *torsion pair.*

(2) *Assume that* $(\mathscr{X}, \mathscr{Y})$ *is tilting* (*resp. cotilting*)*. Then we have*

(a) *if* i^* (*resp.* $i^!$) *is exact, then* $(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ *is a tilting* (*resp. cotilting*) *torsion pair*;

(b) *if* $j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$, *then* $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ *is a tilting* (*resp. cotilting*) *torsion pair.*

Proof (1) (a) Let $(\mathscr{X}, \mathscr{Y})$ be hereditary, and let

$$0 \to X'_0 \to X'$$

be a monomorphism in $\mathscr{A}$ with $X' \in i^*(\mathscr{X})$. Since i_* is exact by Lemma 1 (2),

$$0 \to i_*(X'_0) \to i_*(X')$$

is a monomorphism in $\mathscr{B}$. As in the proof of Theorem 2, we have

$$i_*i^*(\mathscr{X}) \subseteq \mathscr{X}, \quad i_*(X') \in \mathscr{X}.$$

Since $(\mathscr{X}, \mathscr{Y})$ is hereditary, it follows that

$$i_*(X'_0) \in \mathscr{X}, \quad X'_0 \cong i^*i_*(X'_0) \in i^*(\mathscr{X}).$$

Thus, $(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ is a hereditary torsion pair by Theorem 2 (1).

(b) Let $(\mathscr{X}, \mathscr{Y})$ be hereditary, and let

$$0 \to X''_0 \to X''$$

be a monomorphism in $\mathscr{C}$ with $X'' \in j^*(\mathscr{X})$. Since $j_!$ is exact by assumption,

$$0 \to j_!(X''_0) \to j_!(X'')$$

is a monomorphism in $\mathscr{B}$. Since $j_*j^*(\mathscr{Y}) \subseteq \mathscr{Y}$ by assumption, by Lemma 3 (1), we have

$$j_!j^*(\mathscr{X}) \subseteq \mathscr{X}, \quad j_!(X'') \in \mathscr{X}.$$

Since $(\mathscr{X}, \mathscr{Y})$ is hereditary, it follows that

$$j_!(X''_0) \in \mathscr{X}, \quad X''_0 \cong j^*j_!(X''_0) \in j^*(\mathscr{X}).$$

Thus, $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ is a hereditary torsion pair by Theorem 2 (2).

Dually, we get the assertion for the cohereditary case.

(2) (a) Let $(\mathscr{X}, \mathscr{Y})$ be tilting and $A \in \mathscr{A}$. Then $i_*(A) \in \mathscr{B}$ and we have a monomorphism

$$0 \to i_*(A) \to X$$

in $\mathscr{B}$ with $X \in \mathscr{X}$. Since i^* is exact by assumption and $i^*i_* \cong 1_{\mathscr{A}}$ by Lemma 1 (3), we get a monomorphism

$$0 \to A \ (\cong i^*i_*(A)) \to i^*(X)$$

in $\mathscr{A}$. Thus, $(i^*(\mathscr{X}), i^!(\mathscr{Y}))$ is a tilting torsion pair by Theorem 2 (1).

(b) Let $(\mathscr{X}, \mathscr{Y})$ be tilting and $C \in \mathscr{C}$. Then $j_*(C) \in \mathscr{B}$ and we have a monomorphism

$$0 \to j_*(C) \to X$$

in $\mathscr{B}$ with $X \in \mathscr{X}$. Since j^* is exact and $j^*j_* \cong 1_{\mathscr{C}}$ by Lemma 1 (2) and (3), we get a monomorphism

$$0 \to C \ (\cong j^*j_*(C)) \to j^*(X)$$

in $\mathscr{C}$. Thus, $(j^*(\mathscr{X}), j^*(\mathscr{Y}))$ is a tilting torsion pair by Theorem 2 (2).

Dually, we get the assertion for the cotilting case. □

Finally, we give an example to illustrate the obtained results.

For an algebra A, we use $\operatorname{mod} A$ to denote the category of finitely generated left A-modules. Let A and B be artin algebras, let ${}_AM_B$ be an (A, B)-bimodule, and let

$$\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

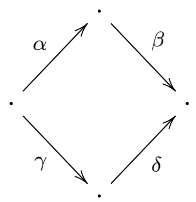
be a triangular matrix algebra. Then any module in $\operatorname{mod}\Lambda$ can be uniquely written as a triple $\begin{pmatrix} X \\ Y \end{pmatrix}_f$ with ([1, p. 76])

$$X \in \operatorname{mod} A, \quad Y \in \operatorname{mod} B, \quad f \in \operatorname{Hom}_A(M \otimes_B Y, X).$$

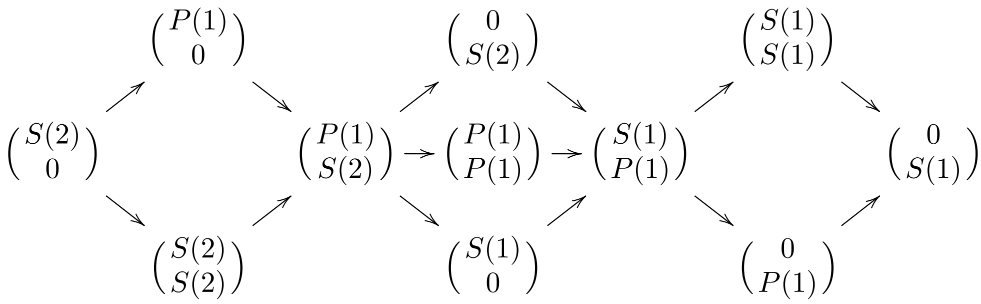
Example 1 Let A be a finite-dimensional algebra given by the quiver $1 \to 2$. Then

$$\Lambda = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$$

is a finite-dimensional algebra given by the quiver



with the relation $\beta\alpha - \delta\gamma$. The Auslander-Reiten quiver of Λ is



By [20, Example 2.12], we have

$$\operatorname{mod} A \;\underset{\xleftarrow{i^!}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i_*}}}\; \operatorname{mod}\Lambda \;\underset{\xleftarrow{j_*}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j^*}}}\; \operatorname{mod} A$$

is a recollement of abelian categories, where

$$i^*\Big(\begin{pmatrix} X \\ Y \end{pmatrix}_f\Big) = \operatorname{Coker} f, \quad i_*(X) = \begin{pmatrix} X \\ 0 \end{pmatrix}, \quad i^!\Big(\begin{pmatrix} X \\ Y \end{pmatrix}_f\Big) = X,$$
$$j_!(Y) = \begin{pmatrix} Y \\ Y \end{pmatrix}_1, \quad j^*\Big(\begin{pmatrix} X \\ Y \end{pmatrix}_f\Big) = Y, \quad j_*(Y) = \begin{pmatrix} 0 \\ Y \end{pmatrix}.$$

(1) Take torsion pairs

$$(\mathscr{X}', \mathscr{Y}') = (\operatorname{add}(P(1) \oplus S(1)), \operatorname{add} S(2)),$$
$$(\mathscr{X}'', \mathscr{Y}'') = (\operatorname{add} S(2), \operatorname{add} S(1)),$$

in $\operatorname{mod} A$. Then, by Theorem 1 (1), we get a torsion pair

$$(\mathscr{X}, \mathscr{Y}) = \Big(\operatorname{add}\Big(\begin{pmatrix} S(2) \\ S(2) \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ S(2) \end{pmatrix} \oplus \begin{pmatrix} 0 \\ S(2) \end{pmatrix} \oplus \begin{pmatrix} S(1) \\ 0 \end{pmatrix}\Big),$$
$$\operatorname{add}\Big(\begin{pmatrix} S(2) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ S(1) \end{pmatrix}\Big)\Big)$$

in $\operatorname{mod}\Lambda$.

In addition, take torsion pairs

$$(\mathscr{X}',\mathscr{Y}')=(\mathscr{X}'',\mathscr{Y}'')=(\operatorname{add}S(2),\operatorname{add}S(1))$$

in $\operatorname{mod}A$. Then by Theorem 1 (1), we get a torsion pair

$$\begin{aligned}(\mathscr{X},\mathscr{Y})=\Big(&\operatorname{add}\Big(\binom{0}{S(2)}\oplus\binom{S(2)}{S(2)}\oplus\binom{S(2)}{0}\Big),\\ &\operatorname{add}\Big(\binom{S(1)}{0}\oplus\binom{S(1)}{S(1)}\oplus\binom{0}{S(1)}\Big)\Big)\end{aligned}$$

in $\operatorname{mod}\Lambda$.

(2) Take a torsion pair

$$\begin{aligned}(\mathscr{X},\mathscr{Y})=\Big(&\operatorname{add}\Big(\binom{0}{S(2)}\oplus\binom{P(1)}{P(1)}\oplus\binom{S(1)}{0}\oplus\binom{S(1)}{P(1)}\\ &\oplus\binom{S(1)}{S(1)}\oplus\binom{0}{P(1)}\oplus\binom{0}{S(1)}\Big),\\ &\operatorname{add}\Big(\binom{S(2)}{0}\oplus\binom{S(2)}{S(2)}\oplus\binom{P(1)}{0}\oplus\binom{P(1)}{S(2)}\Big)\Big)\end{aligned}$$

in $\operatorname{mod}\Lambda$. Then by Theorem 2 (1), we have

$$(i^*(\mathscr{X}),i^!(\mathscr{Y}))=(\operatorname{add}S(1),\operatorname{add}(S(2)\oplus P(1)))$$

is a torsion pair in $\operatorname{mod}A$. Since

$$j_*j^*(\mathscr{Y})=\operatorname{add}\binom{0}{S(2)}\not\subseteq\mathscr{Y},$$

it follows from Theorem 2 (2) that

$$(j^*(\mathscr{X}),j^*(\mathscr{Y}))=(\operatorname{add}(S(2)\oplus P(1)\oplus S(1)),\operatorname{add}S(2))$$

is not a torsion pair in $\operatorname{mod}A$.

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References

1. Auslander M, Reiten I, Smalø S O. Representation Theory of Artin Algebras. Cambridge Stud Adv Math, Vol 36. Cambridge: Cambridge Univ Press, 1997
2. Beĭlinson A A, Bernstein J, Deligne P. Faisceaux pervers, analysis and topology on singular spaces, I. Astérisque, 1982, 100: 5–171
3. Beligiannis A, Reiten I. Homological and Homotopical Aspects of Torsion Theories. Mem Amer Math Soc, Vol 188, No 883. Providence: Amer Math Soc, 2007
4. Chen J M. Cotorsion pairs in a recollement of triangulated categories. Comm Algebra, 2013, 41: 2903–2915
5. Cline E, Parshall B, Scott L. Derived categories and Morita theory. J Algebra, 1986, 104: 397–409
6. Cline E, Parshall B, Scott L. Finite-dimensional algebras and highest weight categories. J Reine Angew Math, 1988, 391: 85–99
7. Dickson S E. A torsion theory for Abelian categories. Trans Amer Math Soc, 1966, 121: 223–235
8. Franjou V, Pirashvili T. Comparison of abelian categories recollements. Doc Math, 2004, 9: 41–56
9. Gentle R. T.T.F. theories in abelian categories. Comm Algebra, 1988, 16: 877–908
10. Happel D, Reiten I, Smalø S O. Tilting in Abelian Categories and Quasitilted Algebras. Mem Amer Math Soc, Vol 120, No 575. Providence: Amer Math Soc, 1996
11. Iyama O, Yoshino Y. Mutation in triangulated categories and rigid Cohen-Macaulay modules. Invent Math, 2008, 172: 117–168
12. Jans J P. Some aspects of torsion. Pacific J Math, 1965, 15: 1249–1259
13. Juteau D. Decomposition numbers for perverse sheaves. Ann Inst Fourier (Grenoble), 2009, 59: 1177–1229
14. Kuhn N J. Generic representations of the finite general linear groups and the Steenrod algebra II. K-Theory, 1994, 8: 395–428
15. Kuhn N J. A stratification of generic representation theory and generalized Schur algebras. K-Theory, 2002, 26: 15–49
16. Lin Y N, Wang M X. From recollement of triangulated categories to recollement of abelian categories. Sci China Math, 2010, 53: 1111–1116
17. Lin Z Q, Wang M X. Koenig's theorem for recollements of module categories. Acta Math Sinica (Chin Ser), 2011, 54: 461–466 (in Chinese)
18. Liu Q H, Vitória J, Yang D. Gluing silting objects. Nagoya Math J, 2014, 216: 117–151
19. Pirashvili T I. Polynomial functors. Trudy Tbiliss Mat Inst Razmadze Akad Nauk Gruzin SSR, 1988, 91: 55–66
20. Psaroudakis C. Homological theory of recollements of abelian categories. J Algebra, 2014, 398: 63–110
21. Psaroudakis C, Skartsæterhagen Ø, Solberg Ø. Gorenstein categories, singular equivalences and finite generation of cohomology rings in recollements. Trans Amer Math Soc (Ser B), 2014, 1: 45–95
22. Psaroudakis C, Vitória J. Recollements of module categories. Appl Categ Structures, 2014, 22: 579–593