On the C-Flat Dimension of Injective Modules *†

Ya-Nan Li[1),2),‡] and Zhaoyong Huang[2),§]

[1)] *Department of Mathematics, Taizhou University, Taizhou 225300, Jiangsu Province, P. R. China;*

[2)] *School of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P. R. China*

Abstract

Let R and S be rings and ${}_RC_S$ a semidualizing bimodule. We investigate the behavior of the C-flat dimension of injective left R-modules, and establish the relation among the supremum of these C-flat dimensions and the supremum of C-injective dimensions of projective left S-modules and the supremum of C-projective dimensions of injective left R-modules.

1 Introduction

Semidualizing bimodules arise naturally in the investigation of various duality theories in commutative ring theory. The study of such modules was initiated by Foxby [9] and by Golod [10]. Then Holm and White [12] extended it to arbitrary associative rings. Many authors have studied the properties of semidualizing modules and related modules, see for example, [1, 5, 9, 10], [12]–[14], [17]–[25] and the references therein. Among various research areas on semidualizing modules, one basic theme is to extend the "absolute" classical results in homological algebra to the "relative" setting with respect to semidualizing modules. The motivation of this paper comes from Emmanouil and Talelli's work [8], in which the relations among the supremum of the projective dimensions of injective left R-modules, that of the injective dimensions of projective left R-modules, the finitistic dimension and the left self-injective dimension of a ring R were established. Our aim is to give the relative counterparts with respect to semidualizing modules of these results.

The paper is organized as follows. In Section 2, we give some terminology and some preliminary results.

Let R and S be arbitrary rings and ${}_RC_S$ a semidualizing bimodule. In Section 3, we first investigate the relationship between the supremum $\operatorname{spcli} R$ of the C-projective dimensions of injective left R-modules and the supremum $\operatorname{sfcli} R$ of the C-flat dimensions of injective left R-modules. We give some upper bounds of $\operatorname{spcli} R$ in terms of $\operatorname{sfcli} R$ together with other relative homological invariants (Proposition 3.2). Moreover, we prove the following result.

Theorem 1.1. (Theorem 3.5) *For any left R-module M and $n \geq 1$, if the C-projective dimension of M is at most n and $\operatorname{Ext}^n_R(M, C) = 0$, then the C-flat dimension of M is at most $n-1$.*

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‡E-mail address: liyanan19920929@163.com

§E-mail address: huangzy@nju.edu.cn

As a consequence, we get that for any $n \geq 1$, if $\operatorname{spcli} R \leq n$ and the injective dimension of ${}_RC$ is at most $n-1$, then $\operatorname{sfcli} R \leq n-1$ (Corollary 3.8).

The finitistic C-flat dimension $\mathcal{F}_C(R)$-FPD is defined as the supremum of the C-flat dimensions of left R-modules with finite C-flat dimension. We establish the relation among $\operatorname{spcli} R$, $\operatorname{spcli} S^{op}$ and $\mathcal{F}_C(R)$-FPD as follows: it holds that $\mathcal{F}_C(R)\text{-FPD} \leq \operatorname{sfcli} S^{op}$ and that $\operatorname{sfcli} R \leq \mathcal{F}_C(R)\text{-FPD}$ if $\operatorname{sfcli} R < \infty$ (Proposition 3.10).

In Section 4, we establish the relation among $\operatorname{spcli} R$, $\operatorname{spcli} S^{op}$, $\operatorname{sfcli} R$, $\operatorname{sfcli} S^{op}$ and other related relative homological invariants over $\aleph_0$-Noetherian rings.

Theorem 1.2. (Theorem 4.3) *If R is a right $\aleph_0$-Noetherian ring and S is a left $\aleph_0$-Noetherian ring, then* $\operatorname{spcli} R < \infty$ *and* $\operatorname{spcli} S^{op} < \infty$ *if and only if* $\operatorname{sfcli} R < \infty$ *and* $\operatorname{sfcli} S^{op} < \infty$. *In this case, we have*

$$|\operatorname{spcli} R - \operatorname{spcli} S^{op}| \leq 1.$$

For a left R-module M, we use $\operatorname{id}_R M$ to denote the injective dimension of M. The invariant $\operatorname{siclp} S$ is defined as the supremum of the C-injective dimensions of projective left S-modules.

Theorem 1.3. (Theorems 4.6 and 4.9)

(1) *If R is a left $\aleph_0$-Noetherian ring, then*

$$\operatorname{sfcli} S^{op} \leq \operatorname{id}_R C^{(\mathbb{N})} = \operatorname{siclp} S,$$

where $\mathbb{N}$ is the set of natural numbers.

(2) *If R is a left Noetherian ring, then*

$$\operatorname{sfcli} S^{op} = \operatorname{id}_R C = \operatorname{siclp} S \geq \max\{\operatorname{Fflic} S, \operatorname{Fplic} S\},$$

where $\operatorname{Fflic} S$ (*respectively,* $\operatorname{Fplic} S$) *is the supremum of the flat* (*respectively, projective*) *dimensions of C-injective left S-modules with finite flat* (*respectively, projective*) *dimension.*

By using Theorem 1.3(1), we prove that for any left $\aleph_0$-Noetherian ring R and $n \geq 1$, if $\operatorname{id}_R C \leq n$ and $\operatorname{sfcli} S^{op} \leq n-1$, then $\operatorname{siclp} S \leq n$; furthermore, if $\operatorname{id}_R C < \operatorname{siclp} S$, then $\operatorname{siclp} S = \operatorname{sfcli} S^{op} + 1$ (Proposition 4.7).

2 Preliminaries

Throughout this paper, all rings are associative rings with unit. Let R be a ring. We use $\operatorname{Mod} R$ (respectively, $\operatorname{Mod} R^{op}$) to denote the class of left (respectively, right) R-modules. We use $\mathcal{F}(R)$, $\mathcal{P}(R)$ and $\mathcal{I}(R)$ to denote the subclasses of $\operatorname{Mod} R$ consisting of flat, projective and injective modules respectively. For a module $M \in \operatorname{Mod} R$, we use $\operatorname{fd}_R M$, $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the flat, projective and injective dimensions of M respectively.

Let $\mathscr{X}$ be a subclass of $\operatorname{Mod} R$ and $M \in \operatorname{Mod} R$. The *$\mathscr{X}$-projective dimension* $\mathscr{X}\text{-pd}\, M$ of M is defined as $\inf\{n \mid$ there exists an exact sequence

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all X_i in $\mathscr{X}\}$, and set $\mathscr{X}\text{-pd}\, M = \infty$ if no such integer exists, and set $\mathscr{X}\text{-pd}\, 0 = -1$. Dually, the notion of the *$\mathscr{X}$-injective dimension* $\mathscr{X}\text{-id}\, M$ of M is defined. We write

$$\mathscr{X}\text{-FPD} := \sup\{\mathscr{X}\text{-pd}\, M \mid M \in \operatorname{Mod} R \text{ with } \mathscr{X}\text{-pd}\, M < \infty\},$$

$$\mathscr{X}\text{-FID} := \sup\{\mathscr{X}\text{-id}\, M \mid M \in \operatorname{Mod} R \text{ with } \mathscr{X}\text{-id}\, M < \infty\}.$$

Definition 2.1. ([1, 12]) Let R and S be rings. An (R,S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied.

(a1) ${}_RC$ admits a degreewise finite R-projective resolution.
(a2) C_S admits a degreewise finite S^{op}-projective resolution.
(b1) The homothety map ${}_RR_R \overset{R\gamma}{\to} \operatorname{Hom}_{S^{op}}(C,C)$ is an isomorphism.
(b2) The homothety map ${}_SS_S \overset{\gamma_S}{\to} \operatorname{Hom}_R(C,C)$ is an isomorphism.
(c1) $\operatorname{Ext}_R^{\geq 1}(C,C) = 0$.
(c2) $\operatorname{Ext}_{S^{op}}^{\geq 1}(C,C) = 0$.

Wakamatsu [23] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*, see [5, 18]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if it is Wakamatsu tilting ([25, Corollary 3.2]). Typical examples of semidualizing bimodules include the free module of rank one and the dualizing module over a Cohen-Macaulay local ring. For more examples of semidualizing bimodules, the reader is referred to [12, 21, 24].

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. For convenience, we write

$$(-)_* := \operatorname{Hom}(C,-).$$

A sequence in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(C,-)$-*exact* if it is exact after applying the functor $(-)_*$. Following [12], set

$$\mathcal{F}_C(R) := \{C \otimes_S F \mid F \text{ is flat in } \operatorname{Mod} S\},$$

$$\mathcal{P}_C(R) := \{C \otimes_S P \mid P \text{ is projective in } \operatorname{Mod} S\},$$

$$\mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in } \operatorname{Mod} R\}.$$

The modules in $\mathcal{F}_C(R)$, $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called *C-flat*, *C-projective* and *C-injective* respectively. When ${}_RC_S = {}_RR_R$, C-flat, C-projective and C-injective modules are exactly flat, projective and injective modules respectively. Symmetrically, the classes of $\mathcal{F}_C(S^{op})$, $\mathcal{P}_C(S^{op})$ and $\mathcal{I}_C(R^{op})$ are defined.

We define

$$\operatorname{spclfc} R := \sup\{\mathcal{P}_C(R)\text{-pd}\, M \mid M \in \mathcal{F}_C(R)\},$$

$$\operatorname{siclf} S := \sup\{\mathcal{I}_C(S)\text{-id}\, N \mid N \in \mathcal{F}(S)\}, \quad \operatorname{siclp} S := \sup\{\mathcal{I}_C(S)\text{-id}\, N \mid N \in \mathcal{P}(S)\},$$

$$\operatorname{sfcli} R := \sup\{\mathcal{F}_C(R)\text{-pd}\, M \mid M \in \mathcal{I}(R)\}, \quad \operatorname{spcli} R := \sup\{\mathcal{P}_C(R)\text{-pd}\, M \mid M \in \mathcal{I}(R)\},$$

$$\operatorname{silfc} R := \sup\{\operatorname{id}_R M \mid M \in \mathcal{F}_C(R)\}, \quad \operatorname{silpc} R := \sup\{\operatorname{id}_R M \mid M \in \mathcal{P}_C(R)\},$$

$$\operatorname{sflic} S := \sup\{\operatorname{fd}_S N \mid N \in \mathcal{I}_C(S)\}, \quad \operatorname{splic} S := \sup\{\operatorname{pd}_S N | N \in \mathcal{I}_C(S)\}.$$

It is trivial that $\operatorname{siclp} S \leq \operatorname{siclf} S$ and $\operatorname{sfcli} R \leq \operatorname{spcli} R$. In addition, by [22, Lemma 2.6], we have

$$\operatorname{siclf} S = \operatorname{silfc} R, \quad \operatorname{siclp} S = \operatorname{silpc} R, \quad \operatorname{sfcli} R = \operatorname{sflic} S, \quad \operatorname{spcli} R = \operatorname{splic} S.$$

Let $N \in \operatorname{Mod} S$ and $M \in \operatorname{Mod} R$. Then there exist the following two canonical evaluation homomorphisms:

$$\mu_N : N \longrightarrow (C \otimes_S N)_*$$

defined by $\mu_N(x)(c) = c \otimes x$ for any $c \in C$ and $x \in N$, and

$$\theta_M : C \otimes_S M_* \longrightarrow M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$.

Definition 2.2. ([12])

(1) The *Auslander class* $\mathcal{A}_C(S)$ with respect to C consists of all left S-modules N satisfying the following conditions.

(A1) $\operatorname{Tor}^S_{\geq 1}(C, N) = 0$.

(A2) $\operatorname{Ext}^{\geq 1}_R(C, C \otimes_S N) = 0$.

(A3) μ_N is an isomorphism.

(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all left R-modules M satisfying the following conditions.

(B1) $\operatorname{Ext}^{\geq 1}_R(C, M) = 0$.

(B2) $\operatorname{Tor}^S_{\geq 1}(C, M_*) = 0$.

(B3) θ_M is an isomorphism.

The *Auslander class* $\mathcal{A}_C(R^{op})$ in $\operatorname{Mod} R^{op}$ and the *Bass class* $\mathcal{B}_C(S^{op})$ in $\operatorname{Mod} S^{op}$ are defined symmetrically.

3 sfcli R and related invariants

3.1 The relation between sfcli R and spcli R

We begin with the following observation.

Lemma 3.1. ([22, Theorem 3.3]) $\operatorname{spclfc} R \leq \mathcal{P}_C(R)\text{-FPD} \leq \operatorname{siclp} S = \operatorname{siclf} S$.

Proof. We only give the proof of the equality $\mathcal{P}_C(R)\text{-FPD} \leq \operatorname{siclp} S$, which is different from that in [22].

Let $M \in \operatorname{Mod} R$ with $\mathcal{P}_C(R)\text{-pd}\, M = n < \infty$ and

$$0 \to C \otimes_S P_n \to \cdots \to C \otimes_S P_1 \to C \otimes_S P_0 \to M \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all P_i projective in $\operatorname{Mod} S$. By [22, Lemma 2.5(1)], applying the functor $(-)_*$ to the above exact sequence yields the following exact sequence

$$0 \to (C \otimes_S P_n)_* \to \cdots \to (C \otimes_S P_1)_* \to (C \otimes_S P_0)_* \to M_* \to 0$$

in $\operatorname{Mod} S$. By [12, Lemma 4.1], we have $(C \otimes_S P_i)_* \cong P_i$ for any $0 \leq i \leq n$. Since $M \in \mathcal{B}_C(R)$ by [12, Corollary 6.1], it follows from [22, Lemma 2.6(1)] that $\operatorname{pd}_S M_* = \mathcal{P}_C(R)\text{-pd}\, M = n$, and hence $\operatorname{Ext}^n_S(M_*, (C \otimes_S P_n)_*) \neq 0$. Then by [12, Theorem 6.4(b)], we have

$$\operatorname{Ext}^n_R(M, C \otimes_S P_n) \cong \operatorname{Ext}^n_S(M_*, (C \otimes_S P_n)_*) \neq 0,$$

which implies $\operatorname{id}_R C \otimes_S P_n \geq n$, and hence $\mathcal{I}_C(S)\text{-id}\, P_n \geq n$ by [12, Lemma 4.1] and [22, Lemma 2.6(3)]. The assertion follows. □

The second inequality in the following result was obtained in [22, Theorem 4.3(2)] when the semidualizing bimodule ${}_RC_S$ is faithful.

Proposition 3.2. *It holds that*

$$\operatorname{sfcli} R \leq \operatorname{spcli} R \leq \operatorname{sfcli} R + \operatorname{spclfc} R \leq \operatorname{sfcli} R + \mathcal{P}_C(R)\text{-FPD} \leq \operatorname{sfcli} R + \operatorname{siclp} S.$$

Proof. By Lemma 3.1, it suffices to prove the second inequality.

Suppose $\operatorname{sfcli} R = n < \infty$ and $\operatorname{spclfc} R = m < \infty$, and let $I \in \operatorname{Mod} R$ be injective. It follows from [12, Lemma 4.1 and Theorem 6.1] that $I \in \mathcal{B}_C(R)$ and there exists a $\operatorname{Hom}_R(C, -)$-exact exact sequence

$$\cdots \to C \otimes_S P_n \to C \otimes_S P_{n-1} \xrightarrow{d_{n-1}} \cdots \to C \otimes_S P_0 \to I \to 0$$

in $\operatorname{Mod} R$ with all P_i projective in $\operatorname{Mod} S$. Applying the functor $(-)_*$ to the above exact sequence yields the following exact sequence

$$\cdots \to (C \otimes_S P_n)_* \to (C \otimes_S P_{n-1})_* \xrightarrow{(d_{n-1})_*} \cdots \to (C \otimes_S P_0)_* \to I_* \to 0.$$

By [12, Lemma 4.1], we have $(C \otimes_S P_i)_* \cong P_i$ for any $i \geq 0$. So the above exact sequence is a projective resolution of I_* in $\operatorname{Mod} S$. By [22, Lemma 2.6(1)], we have

$$\operatorname{fd}_S I_* = \mathcal{F}_C(R)\text{-}\operatorname{pd} I \leq \operatorname{sfcli} R = n,$$

and hence $K_n := \operatorname{Ker}(d_{n-1})_*$ is flat, which is in $K_n \in \mathcal{A}_C(S)$ by [12, Lemma 4.1] again. Then by [12, Theorem 1] and [22, Lemma 2.6(1)], we have $C \otimes_S K_n \in \mathcal{B}_C(R)$ and

$$\operatorname{pd}_S K_n = \operatorname{pd}_S (C \otimes_S K_n)_* = \mathcal{P}_C(R)\text{-}\operatorname{pd}(C \otimes_S K_n) \leq \operatorname{spclfc} R = m,$$

and hence $\operatorname{pd}_S I_* \leq n + m$. Then by [22, Lemma 2.6(1)], we have

$$\mathcal{P}_C(R)\text{-}\operatorname{pd} I = \operatorname{pd}_S I_* \leq n + m,$$

and $\operatorname{spcli} R \leq n + m$. □

The following result was proved in [22, Corollary 4.4] when the semidualizing bimodule ${}_RC_S$ is faithful.

Corollary 3.3. *The following statements are equivalent.*

(1) $\operatorname{spcli} R = \operatorname{siclp} S < \infty$.

(2) $\operatorname{spcli} R < \infty$ *and* $\operatorname{siclp} S < \infty$.

(3) $\operatorname{sfcli} R < \infty$ *and* $\operatorname{siclp} S < \infty$.

Proof. If $\operatorname{siclp} S < \infty$, then $\operatorname{spclfc} R < \infty$ by Lemma 3.1. It follows from Proposition 3.2 that $\operatorname{spcli} R < \infty$ if and only if $\operatorname{sfcli} R < \infty$, and the assertion (2) $\Longleftrightarrow$ (3) follows.

The implication (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (1) Let $\operatorname{spcli} R = n < \infty$ and let $I \in \operatorname{Mod} R$ be injective with $\mathcal{P}_C(R)$-$\operatorname{pd} I = n$. Then there exists an exact sequence

$$0 \to C \otimes_S P_n \to C \otimes_S P_{n-1} \xrightarrow{d_{n-1}} \cdots \to C \otimes_S P_0 \to I \to 0$$

in $\operatorname{Mod} R$ with all P_i projective in $\operatorname{Mod} S$. By [22, Lemma 2.5(1)], we have $\operatorname{Ext}_R^{\geq 1}(C\otimes_S P_i, C\otimes_S P_n)=0$ for any $0\leq i\leq n$, and so

$$\operatorname{Ext}_R^n(I, C\otimes_S P_n)\cong \operatorname{Ext}_R^1(\operatorname{Im} d_{n-1}, C\otimes_S P_n)\neq 0.$$

Since $P_n\in\mathcal{A}_C(S)$ by [12, Lemma 4.1], we have

$$\operatorname{siclp} S\geq \mathcal{I}_C(S)\text{-}\operatorname{id} P_n=\operatorname{id}_R C\otimes_S P_n\geq n=\operatorname{spcli} R$$

by [22, Lemma 2.6(3)].

Conversely, let $\operatorname{siclp} S=m<\infty$ and let $P\in\operatorname{Mod} S$ be projective with $\mathcal{I}_C(S)$-pd $P=m$. Then there exists an exact sequence

$$0\to P\to {I^0}_*\to\cdots\xrightarrow{d^{m-1}} {I^{m-1}}_*\to {I^m}_*\to 0$$

in $\operatorname{Mod} S$ with all I^i injective in $\operatorname{Mod} R$. By [22, Lemma 2.5(2)], we have $\operatorname{Ext}_R^{\geq 1}({I^m}_*, {I^i}_*)=0$ for any $0\leq i\leq m$, and so

$$\operatorname{Ext}_R^m({I^m}_*, P)\cong \operatorname{Ext}_R^1({I^m}_*, \operatorname{Im} {d^{m-1}}_*)\neq 0.$$

Since $I^m\in\mathcal{B}_C(R)$ by [12, Lemma 4.1], we have

$$\operatorname{spcli} R\geq \mathcal{P}_C(R)\text{-}\operatorname{pd} I^m=\operatorname{pd}_S {I^m}_*\geq m=\operatorname{siclp} S$$

by [22, Lemma 2.6(1)]. The proof is finished. □

As a consequence, we obtain the following corollary.

Corollary 3.4. *If* $\operatorname{spcli} R<\infty$ *and* $\operatorname{siclp} S<\infty$*, then*

$$\operatorname{spcli} R=\mathcal{P}_C(R)\text{-FPD}=\operatorname{siclp} S.$$

Proof. By assumption and Lemma 3.1, we have

$$\operatorname{spcli} R\leq \mathcal{P}_C(R)\text{-FPD}\leq \operatorname{siclp} S.$$

Now the assertion follows from Corollary 3.3. □

We shall examine the extent to which the inequality $\mathcal{F}_C(R)$-pd $M\leq \mathcal{P}_C(R)$-pd M is strict.

Theorem 3.5. *Let* $M\in\operatorname{Mod} R$ *and* $n\geq 1$*. If* $\mathcal{P}_C(R)$-pd $M\leq n$ *and* $\operatorname{Ext}_R^n(M,C)=0$*, then* $\mathcal{F}_C(R)$-pd $M\leq n-1$*.*

Proof. Suppose $\mathcal{P}_C(R)$-pd $M\leq n$. Then $M\in\mathcal{B}_C(R)$ by [12, Corollary 6.1]. It follows from [22, Lemma 2.6(1)(2)] that

$$\mathcal{P}_C(R)\text{-}\operatorname{pd} M\leq n\Longleftrightarrow \operatorname{pd}_S M_*\leq n, \tag{3.1}$$

$$\mathcal{F}_C(R)\text{-}\operatorname{pd} M\leq n-1\Longleftrightarrow \operatorname{fd}_S M_*\leq n-1. \tag{3.2}$$

By [12, Theorem 6.4(b)], we have

$$\operatorname{Ext}_R^n(M,C)\cong \operatorname{Ext}_S^n(M_*,C_*)\cong \operatorname{Ext}_S^n(M_*,S). \tag{3.3}$$

It follows that

$$\begin{aligned} &\mathcal{P}_C(R)\text{-}\operatorname{pd} M \leq n \text{ and } \operatorname{Ext}_R^n(M, C) = 0 \\ \Longleftrightarrow\ &\operatorname{pd}_S M_* \leq n \text{ and } \operatorname{Ext}_S^n(M_*, S) = 0 \text{ (by (3.1) and (3.3))} \\ \Longrightarrow\ &\operatorname{fd}_S M_* \leq n-1 \text{ (by [8, Theorem 2.11])} \\ \Longleftrightarrow\ &\mathcal{F}_C(R)\text{-}\operatorname{pd} M \leq n-1 \text{ (by (3.2))}. \end{aligned}$$

□

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6. *Let $M \in \operatorname{Mod} R$ and $n \geq 1$. If $\mathcal{P}_C(R)$-$\operatorname{pd} M \leq n$ and $\operatorname{id}_R C \leq n-1$, then $\mathcal{F}_C(R)$-$\operatorname{pd} M \leq n-1$. In particular, if $\mathcal{P}_C(R)$-$\operatorname{pd} M \leq 1$ and ${}_RC$ is injective, then $M \in \mathcal{F}_C(R)$.*

Let R be an artin algebra and let ${}_RC$ be the direct sum of all representatives of indecomposable injective left R-modules which appear in the minimal injective coresolution of ${}_RR$ as direct summands of some term. Then ${}_RC_S$ is a semidualizing bimodule with $S = \operatorname{End}({}_RC)$. In this case, it is clear that ${}_RC$ is injective.

The rest of the results in this subsection are consequences of Corollary 3.6.

Corollary 3.7. *The following assertions hold.*

(1) *Let $M \in \operatorname{Mod} R$. If $\mathcal{F}_C(R)$-$\operatorname{pd} M = \mathcal{P}_C(R)$-$\operatorname{pd} M < \infty$, then*

$$\mathcal{F}_C(R)\text{-}\operatorname{pd} M = \mathcal{P}_C(R)\text{-}\operatorname{pd} M \leq \operatorname{id}_R C.$$

(2) *If S is a left perfect ring (in particular, if S is a left or right artinian ring), then*

$$\mathcal{F}_C(R)\text{-FPD} = \mathcal{P}_C(R)\text{-FPD} \leq \operatorname{id}_R C.$$

Proof. (1) Let $\mathcal{F}_C(R)$-$\operatorname{pd} M = \mathcal{P}_C(R)$-$\operatorname{pd} M = n < \infty$. If $\operatorname{id}_R C < n$, then $\mathcal{F}_C(R)$-$\operatorname{pd} M \leq n-1$ by Corollary 3.6, which is a contradiction.

(2) If S is a left perfect ring, then a left S-module is flat if and only if it is projective. So $\mathcal{F}_C(R) = \mathcal{P}_C(R)$, and hence $\mathcal{F}_C(R)$-$\operatorname{pd} M = \mathcal{P}_C(R)$-$\operatorname{pd} M$ for any $M \in \operatorname{Mod} R$. Now the assertion follows from (1). □

Corollary 3.8. *For any $n \geq 1$, if $\operatorname{spcli} R \leq n$ and $\operatorname{id}_R C \leq n-1$, then $\operatorname{sfcli} R \leq n-1$.*

Proof. Let $I \in \operatorname{Mod} R$ be injective. Then $\mathcal{P}_C(R)$-$\operatorname{pd} I \leq \operatorname{spcli} R \leq n$ by assumption. It follows from Corollary 3.6 that $\mathcal{F}_C(R)$-$\operatorname{pd} I \leq n-1$ and $\operatorname{sfcli} R \leq n-1$. □

Corollary 3.9. *The following assertions hold.*

(1) *If $\operatorname{sfcli} R = \operatorname{spcli} R < \infty$, then $\operatorname{sfcli} R = \operatorname{spcli} R \leq \operatorname{id}_R C$.*

(2) *If $\operatorname{id}_R C < \operatorname{spcli} R < \infty$, then $\operatorname{sfcli} R < \operatorname{spcli} R$.*

Proof. (1) Let $\operatorname{sfcli} R = \operatorname{spcli} R = n < \infty$. If $\operatorname{id}_R C < n$, then $\operatorname{sfcli} R \leq n-1$ by Corollary 3.8, which is a contradiction.

(2) It follows from (1). □

3.2 The relation between $\operatorname{sfcli} R$ and $\mathcal{F}_C(R)$-FPD

Proposition 3.10. *The following assertions hold.*

(1) $\mathcal{F}_C(R)\text{-FPD} \leq \operatorname{sfcli} S^{op}$.

(2) *If* $\operatorname{sfcli} R < \infty$, *then* $\operatorname{sfcli} R \leq \mathcal{F}_C(R)\text{-FPD}$.

Proof. (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)\text{-pd}\, M = n < \infty$. By [12, Corollary 6.1] and [22, Lemma 2.6(1)], we have $M \in \mathcal{B}_C(R)$ and $\operatorname{fd}_S M_* = \mathcal{F}_C(R)\text{-pd}\, M = n$. Then there exists some module $A \in \operatorname{Mod} S^{op}$ such that $\operatorname{Tor}_n^S(A, M_*) \neq 0$. Let

$$0 \to A \to I \to I/A \to 0$$

be an exact sequence in $\operatorname{Mod} S^{op}$ with I injective. It induces the following exact sequence

$$0 = \operatorname{Tor}_{n+1}^S(I/A, M_*) \to \operatorname{Tor}_n^S(A, M_*) \to \operatorname{Tor}_n^S(I, M_*),$$

which implies $\operatorname{Tor}_n^S(I, M_*) \neq 0$.

Since $M \in \mathcal{B}_C(R)$, we have $M_* \in \mathcal{A}_C(S)$ by [12, Theorem 1]. Note that $I \in \mathcal{B}_C(S^{op})$ by [12, Lemma 4.1]. It follows from [12, Theorem 6.4(c)] that

$$\operatorname{Tor}_n^R(I_*, C \otimes_S M_*) \cong \operatorname{Tor}_n^S(I, M_*) \neq 0$$

and $\operatorname{fd}_{R^{op}} I_* \geq n$. Then by [22, Lemma 2.6(3)], we have $\mathcal{F}_C(S^{op})\text{-pd}\, I = \operatorname{fd}_{R^{op}} I_* \geq n$. The assertion follows.

(2) If $\operatorname{sfcli} R = n < \infty$, then there exists an injective left R-module I such that $\mathcal{F}_C(R)\text{-pd}\, I = n$, and hence

$$\mathcal{F}_C(R)\text{-FPD} \geq \mathcal{F}_C(R)\text{-}\operatorname{pd} I = n = \operatorname{sfcli} R.$$

□

Corollary 3.11. *If* $\operatorname{sfcli} R < \infty$ *and* $\operatorname{sfcli} S^{op} < \infty$, *then*

$$\operatorname{sfcli} R = \mathcal{F}_C(R)\text{-FPD} = \operatorname{sfcli} S^{op} = \mathcal{F}_C(S^{op})\text{-FPD}.$$

Proof. By Proposition 3.10, we have

$$\operatorname{sfcli} R \leq \mathcal{F}_C(R)\text{-FPD} \leq \operatorname{sfcli} S^{op} \leq \mathcal{F}_C(S^{op})\text{-FPD} \leq \operatorname{sfcli} R.$$

□

Corollary 3.12. *If* $R \cong S^{op}$ *and* $\operatorname{sfcli} R < \infty$, *then* $\operatorname{sfcli} R = \mathcal{F}_C(R)\text{-FPD}$.

4 $\aleph_0$-Noetherian rings

Lemma 4.1. *It holds that*

(1) $\mathcal{F}_C(R)\text{-FPD} = \mathcal{F}(S)\text{-FPD}$.

(2) $\mathcal{P}_C(R)\text{-FPD} = \mathcal{P}(S)\text{-FPD}$.

(3) $\mathcal{I}_C(S)\text{-FID} = \mathcal{I}(R)\text{-FID}$.

Proof. (1) Suppose $\mathcal{F}_C(R)$-FPD $= n < \infty$. Let $N \in \operatorname{Mod} S$ with $\operatorname{fd}_S N < \infty$. Then $N \in \mathcal{A}_C(S)$ and $C \otimes_S N \in \mathcal{B}_C(R)$ by [12, Lemma 4.1 and Theorem 1]. It follows from [22, Lemma 2.6(1)] that

$$\mathcal{F}_C(R)\text{-}\operatorname{pd} C \otimes_S N = \operatorname{fd}_S (C \otimes_S N)_* = \operatorname{fd}_S N < \infty,$$

and hence $\operatorname{fd}_S N = \mathcal{F}_C(R)\text{-}\operatorname{pd} C \otimes_S N \leq n$. This yields $\mathcal{F}(S)$-FPD $\leq \mathcal{F}_C(R)$-FPD.

Conversely, suppose $\mathcal{F}(S)$-FPD $= n < \infty$. Let $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)\text{-}\operatorname{pd} M < \infty$. Then $M \in \mathcal{B}_C(R)$ by [12, Corollary 6.1]. It follows from [22, Lemma 2.6(1)] that

$$\operatorname{fd}_S M_* = \mathcal{F}_C(R)\text{-}\operatorname{pd} M < \infty,$$

and hence $\mathcal{F}_C(R)\text{-}\operatorname{pd} M = \operatorname{fd}_S M_* \leq n$. This yields $\mathcal{F}_C(R)$-FPD $\leq \mathcal{F}(S)$-FPD

Similarly, we get the assertions (2) and (3). □

Recall that a ring R is called *left* (respectively, *right*) $\aleph_0$*-Noetherian* if any left (respectively, right) ideal of R is countably generated. The class of left (respectively, right) $\aleph_0$-Noetherian rings includes countable rings and left (respectively, right) Noetherian rings.

Proposition 4.2. *If S is a left $\aleph_0$-Noetherian ring, then*

$$\mathcal{P}_C(R)\text{-FPD} \leq \mathcal{F}_C(R)\text{-FPD} + 1.$$

Proof. By [8, Propossition 2.8], we have $\mathcal{P}(S)$-FPD $\leq \mathcal{F}(S)$-FPD $+1$. Now the assertion follows from Lemma 4.1(1)(2). □

We give a sufficient condition that the finiteness of $\operatorname{spcli} R$ and $\operatorname{spcli} S^{op}$ is equivalent to the finiteness of $\operatorname{sfcli} R$ and $\operatorname{sfcli} S^{op}$.

Theorem 4.3. *If R is a right $\aleph_0$-Noetherian ring and S is a left $\aleph_0$-Noetherian ring, then the following statements are equivalent.*

(1) $\operatorname{spcli} R < \infty$ *and* $\operatorname{spcli} S^{op} < \infty$.

(2) $\operatorname{sfcli} R < \infty$ *and* $\operatorname{sfcli} S^{op} < \infty$.

If one of these two conditions is satisfied, then

$$|\operatorname{spcli} R - \operatorname{spcli} S^{op}| \leq 1.$$

Proof. (1) $\Longrightarrow$ (2) It is trivial.

(2) $\Longrightarrow$ (1) By (2) and Corollary 3.11, we have

$$\mathcal{F}_C(R)\text{-FPD} = \mathcal{F}_C(S^{op})\text{-FPD} = \operatorname{sfcli} R = \operatorname{sfcli} S^{op} < \infty.$$

Then $\mathcal{P}_C(R)$-FPD $\leq \mathcal{F}_C(R)$-FPD $+1 < \infty$ by Proposition 4.2, and hence $\operatorname{spcli} R < \infty$ by Proposition 3.2. Symmetrically, we get $\operatorname{spcli} S^{op} < \infty$.

If one of the conditions (1) and (2) is satisfied, then $\operatorname{spcli} R \leq \mathcal{P}_C(R)$-FPD and $\operatorname{spcli} S^{op} \leq \mathcal{P}_C(S^{op})$-FPD. Moreover, we may suppose

$$\mathcal{F}_C(R)\text{-FPD} = \mathcal{F}_C(S^{op})\text{-FPD} = \operatorname{sfcli} R = \operatorname{sfcli} S^{op} = n < \infty.$$

Then by Proposition 4.2, we have

$$n = \operatorname{sfcli} R \le \operatorname{spcli} R \le \mathcal{P}_C(R)\text{-FPD} \le \mathcal{F}_C(R)\text{-FPD} + 1 = n+1,$$

$$n = \operatorname{sfcli} S^{op} \le \operatorname{spcli} S^{op} \le \mathcal{P}_C(S^{op})\text{-FPD} \le \mathcal{F}_C(S^{op})\text{-FPD} + 1 = n+1.$$

The proof is finished. □

Let $M \in \operatorname{Mod} R$, and let $D \in \operatorname{Mod} S^{op}$ be injective and ${}_RN_S$ a bimodule. Then we have an additive map

$$\Phi_M : \operatorname{Hom}_{S^{op}}(N, D) \otimes_R M \longrightarrow \operatorname{Hom}_{S^{op}}(\operatorname{Hom}_R(M, N), D)$$

defined by $\Phi_M(f \otimes m)(g) = f(g(m))$ for any $m \in M, f \in \operatorname{Hom}_{S^{op}}(N, D)$ and $g \in \operatorname{Hom}_R(M, N)$. Considering a projective resolution of M and applying homology, we obtain an additive map

$$\Phi_M^{(n)} : \operatorname{Tor}_n^R(\operatorname{Hom}_{S^{op}}(N, D), M) \longrightarrow \operatorname{Hom}_{S^{op}}(\operatorname{Ext}_R^n(M, N), D)$$

for any $n \ge 0$, which does not depend on the choice of the projective resolution of M.

Lemma 4.4. *Let $M \in \operatorname{Mod} R$ and let K be an n-syzygy of M with $n \ge 0$. Then for any injective right S-module I, the additive map*

$$\Phi_M^{(n)} : \operatorname{Tor}_n^R(I_*, M) \longrightarrow \operatorname{Hom}_{S^{op}}(\operatorname{Ext}_R^n(M, C), I)$$

is monic if and only if the additive map

$$\Phi_K : I_* \otimes_R K \longrightarrow \operatorname{Hom}_{S^{op}}(\operatorname{Hom}_R(K, C), I)$$

is monic.

Proof. The case for $S^{op} = \mathbb{Z}$ (the ring of integers) was proved in [8, Proposition 1.5]. The argument there is also valid in our setting. □

We also need the following lemma.

Lemma 4.5. *For an integer $n \ge 0$, if $\operatorname{Tor}_{n+1}^R(I_*, M) = 0$ for any finitely presented left R-module M and any injective right S-module I, then $\operatorname{sfcli} S^{op} \le n$.*

Proof. Let $M \in \operatorname{Mod} R$. Then there exists a direct system $\{M_i\}_i$ consisting of finitely presented left R-modules such that $M = \varinjlim M_i$. It follows from [19, Proposition 7.8] and assumption that

$$\operatorname{Tor}_{n+1}^R(I_*, M) \cong \operatorname{Tor}_{n+1}^R(I_*, \varinjlim M_i) \cong \varinjlim \operatorname{Tor}_{n+1}^R(I_*, M_i) = 0$$

for any injective right S-module I. Thus by [22, Lemma 2.6(1)], we have $\mathcal{F}_C(S^{op})\text{-pd}\, I = \operatorname{fd}_{R^{op}} I_* \le n$ and $\operatorname{sfcli} S^{op} \le n$. □

For a countable inverse system $\{H_n, \lambda_n\}_{n \in \mathbb{N}}$, recall that $\varprojlim^1$, the *first derived functor* of the inverse limit $\varprojlim$, is defined by the exact sequence

$$0 \to \varprojlim H_n \to \prod_{n \in \mathbb{N}} H_n \xrightarrow{\Delta} \prod_{n \in \mathbb{N}} H_n \to \varprojlim{}^1 H_n \to 0,$$

where $\Delta(x_n)_{n\in\mathbb{N}} = (x_n - \lambda_n(x_{n+1}))_{n\in\mathbb{N}}$ for any $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} H_n$ (see [26, Section 3.5]).

Let R be a left $\aleph_0$-Noetherian ring, and let $M \in \operatorname{Mod} R$ be cyclic and let

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M in $\operatorname{Mod} R$ with all F_i countably generated and set $K := \operatorname{Im}(F_n \to F_{n-1})$. Then K is countably presented, and hence there exists a countable direct system $\{K_i\}_i$ of finitely presented submodules of K such that $K = \varinjlim K_i$. It is known that there exists a short exact sequence of Ext-groups

$$0 \to \varprojlim{}^1 \operatorname{Ext}_R^{j-1}(K_i, -) \to \operatorname{Ext}_R^j(K, -) \to \varprojlim \operatorname{Ext}_R^j(K_i, -) \to 0 \tag{4.1}$$

for any $j \geq 1$ (see [26, Application 3.5.10]). In particular, for any set J, we have the following exact sequence

$$0 \to \varprojlim{}^1 \operatorname{Hom}_R(K_i, C^{(J)}) \to \operatorname{Ext}_R^1(K, C^{(J)}) \to \varprojlim \operatorname{Ext}_R^1(K_i, C^{(J)}) \to 0 \tag{4.2}$$

Since K_i is finitely presented, we have that $\operatorname{Ext}_R^1(K_i, C^{(J)}) \cong \operatorname{Ext}_R^1(K_i, C)^{(J)}$ and the inverse system $\{\operatorname{Ext}_R^1(K_i, C^{(J)})\}_i$ is naturally identified with the direct sum $(\{\operatorname{Ext}_R^1(K_i, C)\}_i)^{(J)}$ of copies of the inverse system $\{\operatorname{Ext}_R^1(K_i, C)\}_i$. Then

$$\varprojlim \operatorname{Ext}_R^1(K_i, C^{(J)}) \cong \varprojlim(\operatorname{Ext}_R^1(K_i, C))^{(J)} \subseteq \varprojlim(\operatorname{Ext}_R^1(K_i, C))^J \cong (\varprojlim(\operatorname{Ext}_R^1(K_i, C)))^J.$$

Suppose $\operatorname{id}_R C \leq n$ and $\varprojlim^1 \operatorname{Hom}_R(K_i, C^{(J)}) = 0$. Then $\operatorname{Ext}_R^1(K, C) \cong \operatorname{Ext}_R^{n+1}(M, C) = 0$, and hence the exact sequence

$$0 \to \varprojlim{}^1 \operatorname{Hom}_R(K_i, C) \to \operatorname{Ext}_R^1(K, C) \to \varprojlim \operatorname{Ext}_R^1(K_i, C) \to 0. \tag{4.3}$$

implies $\varprojlim \operatorname{Ext}_R^1(K_i, C) = 0$ and $(\varprojlim(\operatorname{Ext}_R^1(K_i, C)))^J = 0$. Thus $\varprojlim \operatorname{Ext}_R^1(K_i, C^{(J)}) = 0$, and therefore

$$\operatorname{Ext}_R^{n+1}(M, C^{(J)}) \cong \operatorname{Ext}_R^1(K, C^{(J)}) = 0$$

by dimension shifting and the the exact sequence (4.2).

We are now in a position to prove the following result.

Theorem 4.6. *It holds that*

$$\operatorname{id}_R C^{(\mathbb{N})} \leq \operatorname{siclp} S.$$

Furthermore, if R is a left $\aleph_0$-Noetherian ring, then

$$\operatorname{sfcli} S^{op} \leq \operatorname{id}_R C^{(\mathbb{N})} = \operatorname{siclp} S.$$

Proof. By [12, Lemma 4.1 and Proposition 4.2(a)], we have that S and $S^{(\mathbb{N})}$ are in $\mathcal{A}_C(S)$. It follows from [22, Lemma 2.6(c)] that

$$\operatorname{id}_R C^{(\mathbb{N})} = \operatorname{id}_R(C \otimes_S S^{(\mathbb{N})}) = \mathcal{I}_C(S)\text{-}\operatorname{id} S^{(\mathbb{N})} \leq \operatorname{siclp} S.$$

Suppose that R is a left $\aleph_0$-Noetherian ring and $\operatorname{id}_R C^{(\mathbb{N})} = n < \infty$. Let $M \in \operatorname{Mod} R$ be cyclic and keep the notations as above. Since K_i is finitely presented, we have that $\operatorname{Hom}_R(K_i, C^{(J)}) \cong \operatorname{Hom}_R(K_i, C)^{(J)}$ and the inverse system $\{\operatorname{Hom}_R(K_i, C^{(J)})\}_i$ is naturally identified with the direct sum $\{\operatorname{Hom}_R(K_i, C)^{(J)}\}_i$

of copies of the inverse system $\{\mathrm{Hom}_R(K_i, C)\}_i$. Since $\mathrm{Ext}_R^1(K_i, C^{(\mathbb{N})}) \cong \mathrm{Ext}_R^{n+1}(M, C^{(\mathbb{N})}) = 0$, it follows from [7, Proposition 2.3] that the additive map

$$\Phi_K : \mathrm{Hom}_{\mathbb{Z}}(C, D) \otimes_R K \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(K, C), D)$$

is monic for any divisible abelian group D, and hence the induced additive map

$$\mathrm{Hom}_{\mathbb{Z}}(C^{(J)}, D) \otimes_R K \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(K, C^{(J)}), D)$$

is also monic by [2, Lemma 1.1 and Proposition 1.2] and [8, Theorem 1.3]. It follows from [6, Corollary 6] and [8, Theorem 1.3] that $\varprojlim^1 \mathrm{Hom}_R(K_i, C^{(J)}) = 0$.

Since $\mathrm{id}_R C \leq \mathrm{id}_R C^{(\mathbb{N})} = n$, it follows from the argument before this theorem that $\mathrm{Ext}_R^{n+1}(M, C^{(J)}) = 0$. Then by [22, Lemma 2.6(3)] and the Baer's criterion, we have

$$\mathcal{I}_C(S)\text{-}\mathrm{id}\, S^{(J)} = \mathrm{id}_R C \otimes S^{(J)} = \mathrm{id}_R C^{(J)} \leq n.$$

So $\mathcal{I}_C(S)\text{-}\mathrm{id}\, P \leq n$ for any projective left S-module P, and hence $\mathrm{siclp}\, S \leq n$.

It remains to prove $\mathrm{sfcli}\, S^{op} \leq n$. Let $M' \in \mathrm{Mod}\, R$ be countably generated (in particular, M' can be finitely presented) and let

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M' \to 0$$

be a free resolution of M' in $\mathrm{Mod}\, R$ with all F_i countably generated and set $K' := \mathrm{Im}(F_{n+1} \to F_n)$. Then $\mathrm{Ext}_R^1(K', C^{(\mathbb{N})}) \cong \mathrm{Ext}_R^{n+2}(M', C^{(\mathbb{N})}) = 0$. Since K' is countably presented, there exists a countable direct system of finitely presented submodules of K' such that K' is the direct limit of of this direct system. It follows from [2, Lemma 1.1 and Example 2.4(4)] and [3, Theorem 8.10] that the additive map

$$\Phi_{K'} : I_* \otimes_R K' \longrightarrow \mathrm{Hom}_{S^{op}}(\mathrm{Hom}_R(K', C), I)$$

is monic for any injective right S-module I. Then by Lemma 4.4, the additive map

$$\Phi_M^{(n+1)} : \mathrm{Tor}_{n+1}^R(I_*, M') \longrightarrow \mathrm{Hom}_{S^{op}}(\mathrm{Ext}_R^{n+1}(M', C), I)$$

is also monic. Since $\mathrm{id}_R C \leq \mathrm{id}_R C^{(\mathbb{N})} = n$, we have $\mathrm{Ext}_R^{n+1}(M', C) = 0$, and hence $\mathrm{Tor}_{n+1}^R(I_*, M') = 0$. It follows from Lemma 4.5 that $\mathrm{sfcli}\, S^{op} \leq n$. □

Proposition 4.7. *Let R be a left $\aleph_0$-Noetherian ring. Then the following assertions hold.*

(1) *For any $n \geq 1$, if $\mathrm{id}_R C \leq n$ and $\mathrm{sfcli}\, S^{op} \leq n-1$, then $\mathrm{siclp}\, S \leq n$.*

(2) *If $\mathrm{id}_R C < \mathrm{siclp}\, S$, then $\mathrm{siclp}\, S = \mathrm{sfcli}\, S^{op} + 1$.*

Proof. (1) Since $\mathrm{sfcli}\, S^{op} \leq n-1$, it follows from [22, Lemma 2.6(1)] that $\mathrm{fd}_{R^{op}} I_* = \mathcal{F}_C(S^{op})\text{-pd}\, I \leq n-1$ for any injective right S-module I, and hence $\mathrm{Tor}_n^R(I_*, M) = 0$ for any $M \in \mathrm{Mod}\, R$.

Let M be a cyclic left R-module and keep the notations as before Theorem 4.6. Then by (4.1), we have the following exact sequence

$$0 \to \varprojlim^1 \mathrm{Hom}_R(K_i, C^{(\mathbb{N})}) \to \mathrm{Ext}_R^1(K, C^{(\mathbb{N})}) \to \varprojlim \mathrm{Ext}_R^1(K_i, C^{(\mathbb{N})}) \to 0.$$

Notice that the additive map

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R(I_*, M) \to \mathrm{Hom}_{S^{op}}(\mathrm{Ext}_R^n(M, C), I)$$

is monic, it follows from Lemma 4.4 that the additive map

$$\Phi_K : I_* \otimes_R K \longrightarrow \mathrm{Hom}_{S^{op}}(\mathrm{Hom}_R(K, C), I)$$

is monic, and hence the induced additive map

$$\mathrm{Hom}_{S^{op}}(C^{(\mathbb{N})}, I) \otimes_R K \to \mathrm{Hom}_{S^{op}}(\mathrm{Hom}_R(K, C^{(\mathbb{N})}), I)$$

is also monic by [2, Lemma 1.1 and Proposition 1.2] and [3, Theorem 8.10]. Thus $\varprojlim^1 \mathrm{Hom}_R(K_i, C^{(\mathbb{N})}) = 0$ by [6, Corollary 6] and [3, Theorem 8.10].

Since $\mathrm{id}_R C \leq n$, it follows from the argument before Theorem 4.6 that $\mathrm{Ext}_R^{n+1}(M, C^{(\mathbb{N})}) = 0$. Then by Theorem 4.6 and the Baer's criterion, we have

$$\mathrm{siclp}\, S = \mathrm{id}_R C^{(\mathbb{N})} \leq n.$$

(2) The assumption that $\mathrm{id}_R C < \mathrm{siclp}\, S$ implies $\mathrm{id}_R C < \infty$. If $\mathrm{siclp}\, S$ is infinite, then $\mathrm{sfcli}\, S^{op}$ is also infinite and the assertion follows.

Now suppose $\mathrm{siclp}\, S = n < \infty$. Then $\mathrm{id}_R C \leq n-1$ and $\mathrm{Ext}_R^n(M, C) = 0$ for any $M \in \mathrm{Mod}\, R$. By Theorem 4.6, we have $\mathrm{sfcli}\, S^{op} \leq \mathrm{siclp}\, S = n$. If $\mathrm{sfcli}\, S^{op} \leq n-2$, then $\mathrm{siclp}\, S \leq n-1$ by (1), which is a contradiction. Thus $\mathrm{sfcli}\, S^{op} \geq n-1$.

Now it suffices to prove $\mathrm{sfcli}\, S^{op} \leq n-1$. Let $M \in \mathrm{Mod}\, R$ be countably generated (in particular, finitely presented) and let

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M in $\mathrm{Mod}\, R$ with all F_i countably generated and set $K := \mathrm{Im}(F_n \to F_{n-1})$. By Theorem 4.6, we have $\mathrm{id}_R C^{(\mathbb{N})} = \mathrm{siclp}\, S = n$, so

$$\mathrm{Ext}_R^1(K, C^{(\mathbb{N})}) \cong \mathrm{Ext}_R^{n+1}(M, C^{(\mathbb{N})}) = 0.$$

It follows from [2, Lemma 1.1 and Example 2.4(4)] and [3, Theorem 8.10] that the additive map

$$\Phi_K : I_* \otimes_R K \longrightarrow \mathrm{Hom}_{S^{op}}(\mathrm{Hom}_R(K, C), I)$$

is monic for any injective right S-module I. Then by Lemma 4.4, the additive map

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R(I_*, M) \longrightarrow \mathrm{Hom}_{S^{op}}(\mathrm{Ext}_R^n(M, C), I)$$

is also monic. Since $\mathrm{id}_R C \leq n-1$, we have $\mathrm{Ext}_R^n(M, C) = 0$, and so $\mathrm{Tor}_n^R(I_*, M) = 0$. It follows from Lemma 4.5 that $\mathrm{sfcli}\, S^{op} \leq n-1$. □

Corollary 4.8. *Let R be a left $\aleph_0$-Noetherian ring. Then the following statements are equivalent.*

(1) $\mathrm{siclp}\, S < \infty$.

(2) $\mathrm{sfcli}\, S^{op} < \infty$ *and* $\mathrm{id}_R C < \infty$.

Proof. It follows from Theorem 4.6 and Proposition 4.7(2). □

We write

$$\mathrm{Fflic}\, S := \sup\{\mathrm{fd}_S E \mid \mathrm{fd}_S E < \infty \text{ with } E \in \mathcal{I}_C(S)\},$$

$$\mathrm{Fplic}\, S := \sup\{\mathrm{pd}_S E \mid \mathrm{pd}_S E < \infty \text{ with } E \in \mathcal{I}_C(S)\}.$$

Theorem 4.9. *Let R be a left Noetherian ring. Then*

$$\operatorname{sflic} R^{op} = \operatorname{sfcli} S^{op} = \operatorname{id}_R C = \operatorname{siclp} S \geq \max\{\operatorname{Fflic} S, \operatorname{Fplic} S\}.$$

Proof. Since R is a left Noetherian ring, $\operatorname{id}_R C^{(\mathbb{N})} = \operatorname{id}_R C$ by [4, Theorem 1.1]. Then by [13, Lemma 17.2.4(2)] and Theorem 4.6, we have

$$\operatorname{sflic} R^{op} = \operatorname{id}_R C = \operatorname{siclp} S.$$

Since $\operatorname{fd}_{R^{op}} I_* = \mathcal{F}_C(S^{op})\text{-}\operatorname{pd} I$ for any injective right S-module I by [22, Lemma 2.6(1)], we have

$$\operatorname{sflic} R^{op} = \operatorname{sfcli} S^{op}.$$

Suppose that $\operatorname{id}_R C = n < \infty$ and $E \in \operatorname{Mod} S$ is C-injective with $\operatorname{fd}_S E = m < \infty$. Then $E = {I'}_*$ for some injective left R-module I' and there exists an exact sequence

$$0 \to F_m \to S^{(J_{m-1})} \to \cdots \to S^{(J_1)} \to S^{(J_0)} \to E(= {I'}_*) \to 0 \tag{4.4}$$

in Mod S with F_m flat and J_i an index set for any $0 \leq i \leq m-1$. By [12, Corollary 6.1], we have $I' \in \mathcal{B}_C(R)$ and $E \in \mathcal{A}_C(S)$, so applying the functor $C \otimes_S -$ to the exact sequence (4.4) yields the following exact sequence

$$0 \to K_m \xrightarrow{d_m} C^{(J_{m-1})} \to \cdots \to C^{(J_1)} \to C^{(J_0)} \to C \otimes_S {I'}_*(\cong I') \to 0 \tag{4.5}$$

in $\operatorname{Mod} R$ with $K_m = C \otimes_S F_m (\in \mathcal{F}_C(R))$. By [22, Lemma 2.5(1)], we have $\operatorname{Ext}_R^{\geq 1}(C^{(J_i)}, K_m) = 0$ for any $0 \leq i \leq m-1$.

Since R is a left Noetherian ring and $\operatorname{id}_R C = n$, it follows from [4, Theorem 1.1] that $\operatorname{id}_R K_m \leq n$. If $m > n$, then $\operatorname{Ext}_R^m(I', K_m) = 0$. It follows from the exact sequence (4.5) that $\operatorname{Ext}_R^1(\operatorname{Coker} d_m, K_m) = 0$. Thus the exact sequence

$$0 \to K_m \xrightarrow{d_m} C^{(J_{m-1})} \to \operatorname{Coker} d_m \to 0$$

splits and $\operatorname{Coker} d_m$ is a direct summand of $C^{(J_{m-1})}$. By [12, Proposition 5.1(b)], we have $\operatorname{Coker} d_m \in \mathcal{P}_C(R)$, and hence $\mathcal{P}_C(R)\text{-}\operatorname{pd} I' \leq m-1$. Then applying [22, Lemma 2.6(2)] yields

$$\operatorname{fd}_S E \leq \operatorname{pd}_S E = \mathcal{P}_C(R)\text{-}\operatorname{pd} I' \leq m-1,$$

which is a contradiction. Thus we conclude that $m \leq n$ and

$$\operatorname{id}_R C \geq \operatorname{Fflic} S.$$

Similarly, we have

$$\operatorname{id}_R C \geq \operatorname{Fplic} S.$$

□

Finally, we turn to countable rings.

Theorem 4.10. *Let S be a countable ring and let $M \in \operatorname{Mod} R$ be countable and $n \geq 0$. Then the following statements are equivalent.*

(1) $\mathcal{P}_C(R)\text{-pd}\, M \leq n$.
(2) $\mathcal{P}_C(R)\text{-pd}\, M \leq n+1$ *and* $\text{Ext}_R^{n+1}(M, C) = 0$.

Proof. By [22, Lemma 2.5(1)], it is easy to get (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1) Since $\mathcal{P}_C(R)\text{-pd}\, M$ is finite by (2), we have $M \in \mathcal{B}_C(R)$ by [12, Corollary 6.1]. Since ${}_RC$ is finitely generated, there exists a positive integer t such that

$$R^t \to C \to 0$$

is exact in $\text{Mod}\, R$, and hence

$$0 \to M_* \to \text{Hom}_R(R^t, M)(\cong M^t)$$

is exact. Since M is countable left R-module, we have that M^t is a countable left R-module. So the set M_* is countable, and hence M_* is a countable left S-module. It follows that

$$\begin{aligned} &\mathcal{P}_C(R)\text{- pd}\, M \leq n+1 \text{ and } \text{Ext}_R^{n+1}(M, C) = 0 \text{ (by (2))} \\ \Longleftrightarrow\ &\text{pd}_S\, M_* \leq n+1 \text{ and } \text{Ext}_S^{n+1}(M_*, S) = 0 \text{ (by (3.1) and (3.3))} \\ \Longleftrightarrow\ &\text{pd}_S\, M_* \leq n \text{ (by [8, Corollary 2.23])} \\ \Longleftrightarrow\ &\mathcal{P}_C(R)\text{- pd}\, M \leq n \text{ (by (3.1))}. \end{aligned}$$

□

Putting $n = 0$ in Theorem 4.10, we get the following result.

Corollary 4.11. *Let S be a countable ring and let $M \in \text{Mod}\, R$ be countable. Then the following statements are equivalent.*

(1) *M is C-projective.*
(2) $\mathcal{P}_C(R)\text{-pd}\, M \leq 1$ *and* $\text{Ext}_R^1(M, C) = 0$.

Corollary 4.12. *Let S be a countable ring and let $0 \neq M \in \text{Mod}\, R$ be countable with $\mathcal{P}_C(R)\text{-pd}\, M = n < \infty$. Then $\text{Ext}_R^n(M, C) \neq 0$ and hence $\text{id}_R\, C \geq n$.*

Proof. If $\text{Ext}_R^n(M, C) = 0$, then $\mathcal{P}_C(R)\text{-pd}\, M \leq n-1$ by assumption and Theorem 4.10, which is a contradiction. □

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References

[1] T. Araya, R. Takahashi and Y. Yoshino, *Homological invariants associated to semi-dualizing bimodules*, J. Math. Kyoto Univ. **45** (2005), 287–306.

[2] L. Angeleri-Hügel, S. Bazzoni and D. Herbera, *A solution to the Baer splitting problem*, Trans. Amer. Math. Soc. **360** (2008), 2409–2421.

[3] L. Angeleri-Hügel and D. Herbera, *Mittag-Leffler conditions on modules*, Indiana Univ. Math. J. **57** (2008), 2459–2517.

[4] H. Bass, *Injective dimension in noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 18–29.

[5] A. Beligiannis and I. Reiten, Homological and Homotopical Aspects of Torsion Theories, emoirs Amer. Math. Soc. **188**(883), Amer. Math. Soc., Providence, RI, 2007.

[6] I. Emmanouil, *Mittag-Leffler condition and the vanishing of* $\varprojlim^1$, Topology **35** (1996), 267–271.

[7] I. Emmanouil, *On certain cohomological invariants of groups*, Adv. Math. **225** (2010), 3446–3462.

[8] I. Emmanouil and O. Talelli, *On the flat length of injective modules*, J. London Math. Soc. **84** (2011), 408–432.

[9] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284.

[10] E. S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66.

[11] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), 167–193.

[12] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), 781–808.

[13] Z. Y. Huang, *Wakamatsu tilting modules, U-dominant dimension and k-Gorenstein modules*, Abelian Groups, Rings, Modules, and Homological Algebra, Eds. P. Goeters and O. M. G. Jenda, Lect. Notes Pure Appl. Math. **249**, Chapman and Hall/CRC, Taylor and Francis Group, New York, 2006, pp.183–202.

[14] Z. Y. Huang, *Homological dimensions relative to preresolving subcategories II*, Forum Math. **34** (2022), 507–530.

[15] Y. Iwanaga, *On rings with finite self-injective dimension II*, Tsukuba J. Math. **4**, (1980), 107–113.

[16] C. U. Jensen, *On homological dimensions of rings with countably generated ideals*, Math. Scand. **18**, (1966), 97–105.

[17] Z. F. Liu, Z. Y. Huang and A. M. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra **41** (2013), 1–18.

[18] F. Mantese and I. Reiten, *Wakamatsu tilting modules*, J. Algebra **278** (2004), 532–552.

[19] J. J. Rotman, *An Introduction to Homological Algebra*, Universitext, Springer, New York, 2009.

[20] R. Takahashi and D. White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), 5–22.

[21] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose*, Forum Math. **27** (2015), 3717–3743.

[22] X. Tang and Z. Y. Huang, *Homological invariants related to semidualizing bimodules*, Colloq. Math. **156** (2019), 135–151.

[23] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra **114** (1988), 106–114.

[24] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298–325.

[25] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra **275** (2004), 3–39.

[26] C. A. Weibel, An Introduction to Homological algebra, Cambridge Studies in Adv. Math. **38**, Cambridge Univ. Press, Cambridge, 1994.