AUSLANDER-TYPE CONDITIONS

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We survey some recent results on Noetherian rings satisfying the Auslander-type conditions, with emphasis on the homological behavior of such rings.

1. Introduction

Throughout this article, Λ is a left and right Noetherian ring (unless stated otherwise), $\operatorname{mod}\Lambda$ is the category of finitely generated left Λ-modules and

$$0 \to \Lambda \to I_0(\Lambda) \to I_1(\Lambda) \to \cdots \to I_i(\Lambda) \to \cdots$$

is the minimal injective resolution of Λ as a left Λ-module.

For a module $M \in \operatorname{mod}\Lambda$ and a non-negative integer n, recall that the *grade* of M, denoted by $\operatorname{grade} M$, is said to be at least n if $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ for any $0 \leq i < n$; and the *strong grade* of M, denoted by $\operatorname{s.grade} M$, is said to be at least n if $\operatorname{grade} X \geq n$ for any submodule X of M (see [3] and [7]). Bass in [8] proved the following result.

Theorem 1.1. *For a commutative Noetherian ring Λ, the following statements are equivalent:*

(1) The self-injective dimension of Λ is finite.
(2) The flat dimension of $I_i(\Lambda)$ is at most i for any $i \geq 0$.
(3) $\operatorname{grade} \operatorname{Ext}^i_\Lambda(M, \Lambda) \geq i$ for any $M \in \operatorname{mod}\Lambda$ and $i \geq 1$.

A commutative Noetherian ring Λ is called *Gorenstein* if it satisfies one of the above equivalent conditions. For the non-commutative case, Λ is also called *Gorenstein* if the left and right self-injective dimensions of Λ are finite. It was proved by [44, Lemma A] that the left and right self-injective dimensions of a Gorenstein ring are identical.

Theorem 1.2. *[15, Auslander's Theorem 3.7] The following statements are equivalent:*

(1) The flat dimension of $I_i(\Lambda)$ is at most i for any $0 \leq i \leq n-1$.
(2) $\text{s.grade}\,\text{Ext}^i_\Lambda(M,\Lambda) \geq i$ *for any* $M \in \text{mod}\,\Lambda$ *and* $1 \leq i \leq n$.
(3) The flat dimension of $I_i(\Lambda^{op})$ is at most i for any $0 \leq i \leq n-1$.
(4) $\text{s.grade}\,\text{Ext}^i_\Lambda(N,\Lambda) \geq i$ *for any* $N \in \text{mod}\,\Lambda^{op}$ *and* $1 \leq i \leq n$.

Λ is called *n-Gorenstein* if it satisfies one of the above equivalent conditions, and Λ is said to satisfy the *Auslander conditio*n if it is n-Gorenstein for all n. Theorem 1.2 means that the notion of n-Gorenstein rings (and hence that of the Auslander condition) is left-right symmetric. Motivated by Theorem 1.2, the notion of the Auslander-type conditions was introduced in [24] as follows.

Definition 1.1. [24] Let $n, k \geq 0$. We say that Λ is $G_n(k)$ if $\text{s.grade}\,\text{Ext}^{i+k}_\Lambda(M,\Lambda) \geq i$ for any $M \in \text{mod}\,\Lambda$ and $1 \leq i \leq n$. Similarly, we say that Λ is $g_n(k)$ if $\text{grade}\,\text{Ext}^{i+k}_\Lambda(M,\Lambda) \geq i$ for any $M \in \text{mod}\,\Lambda$ and $1 \leq i \leq n$. We say that Λ is $G_n(k)^{op}$ (resp. $g_n(k)^{op}$) if Λ^{op} is $G_n(k)$ (resp. $g_n(k)$). We call both $G_n(k)$ and $g_n(k)$ the *Auslander-type conditions*.

The following relations are obvious for any $n \geq n'$ and $k \leq k'$:

$$\begin{array}{ccc} G_n(k) & \Rightarrow & G_{n'}(k') \\ \Downarrow & & \Downarrow \\ g_n(k) & \Longrightarrow & g_{n'}(k') \end{array}$$

The Auslander-type conditions can be regarded as certain non-commutative analogs of commutative Gorenstein rings. Such conditions, especially dominant dimension and the n-Gorenstein ring, play a crucial role in representation theory and non-commutative algebraic geometry (e.g. [2, 6, 7, 10, 12, 13, 15, 16, 18, 24, 28, 31, 32, 33, 34, 35, 36, 38, 40, 41, 42]). They are also interesting from the viewpoint of some unsolved homological conjectures, e.g. the finitistic dimension conjecture, Nakayama conjecture, Gorenstein symmetry conjecture, and so on. In this article, we survey some recent results on Noetherian rings satisfying the Auslander-type conditions, with emphasis on the homological behavior of such rings. In Section 2, we give some equivalent characterizations of the conditions $G_n(k)$ and $g_n(k)$, respectively. In Section 3, we investigate the properties of rings being $G_n(k)$ or $g_n(k)$, especially for the case $k = 0, 1$. In Section 4, we give the definition of the (weak) (m,n)-condition, which is closely related to the conditions $G_n(k)$ and $g_n(k)$. Then we investigate the properties of rings satisfying certain (m,n)-condition.

2. Characterizations of the Auslander-type conditions

In this section, we give some equivalent characterizations of the Auslander-type conditions $G_n(k)$ and $g_n(k)$, respectively.

Put $(\ )^* = \mathrm{Hom}_\Lambda(\ ,\Lambda)$ and $\mathbb{E}_n = \mathrm{Ext}^n_\Lambda(\ ,\Lambda)$ for any $n \geq 0$. Let $M \in \mathrm{mod}\,\Lambda$ and $P_1 \to P_0 \to M \to 0$ be a projective resolution of M in $\mathrm{mod}\,\Lambda$. Then $\mathrm{Coker}(P_0^* \to P_1^*)$ is called the *transpose* of M, and denoted by $\mathrm{Tr}\,M$ (see [3]).

Definition 2.1. [3] Let $k \geq 1$ and M be a module in $\mathrm{mod}\,\Lambda$. M is called *k-torsionfree* if $\mathbb{E}_i(\mathrm{Tr}\,M) = 0$ for any $1 \leq i \leq k$; and M is called *k-syzygy* if there exists an exact sequence $0 \to M \to P_{k-1} \to \cdots \to P_1 \to P_0$ in $\mathrm{mod}\,\Lambda$ with all P_i projective.

We use $\mathcal{F}_k$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of k-torsionfree modules, and $\Omega^k(\mathrm{mod}\,\Lambda)$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of k-syzygy modules. It is well known that that $\mathcal{F}_k \subseteq \Omega^k(\mathrm{mod}\,\Lambda)$ for any $k \geq 1$ (see [3, Theorem 2.17]).

For subcategories $\mathcal{C}_i$ $(i = 1, 2)$ of $\mathrm{mod}\,\Lambda$, we use $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of $C \in \mathrm{mod}\,\Lambda$ such that there exists an exact sequence $0 \to C_2 \to C \to C_1 \to 0$ with $C_i \in \mathcal{C}_i$ $(i = 1, 2)$. For a left Λ-module M, we use $\mathrm{pd}\,M$, $\mathrm{fd}\,M$ and $\mathrm{id}\,M$ to denote the projective dimension, flat dimension and injective dimension of M, respectively.

The following result gives some equivalent characterizations of $G_n(k)$.

Theorem 2.1. *[24, 2.4(7) and Theorem 3.5] The conditions (1)–(3) are equivalent for any $n, k \geq 0$. If $k \geq 1$, then (1)–(4) are equivalent:*

(1) Λ is $G_n(k)$.
(2) $\mathrm{fd}\,I_i(\Lambda^{op}) \leq i + k$ for any $0 \leq i \leq n-1$.
(3) For any exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ with $C \in \Omega^k(\mathrm{mod}\,\Lambda)$, $\mathbb{E}_i\mathbb{E}_i(f)$ is a monomorphism for any $0 \leq i \leq n-1$.
(4) $\mathcal{E}(\Omega^{i+k}(\mathrm{mod}\,\Lambda), \Omega^{i+k}(\mathrm{mod}\,\Lambda)) \subseteq \mathcal{F}_{i+1}$ for any $0 \leq i \leq n-1$.

Remark 2.1. $G_n(0)$ (resp. $G_\infty(0)$) is just the n-Gorenstein ring (resp. the Auslander condition).

Let $k \geq 1$. We denote by

$$\mathcal{W}_k = \{M \in \mathrm{mod}\,\Lambda \mid \mathbb{E}_i(M) = 0 \text{ for any } 1 \leq i \leq k\}$$

and

$$\mathcal{P}_k = \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{pd}\,M < k\}.$$

For a module $M \in \operatorname{mod}\Lambda$, $\Omega^k(M)$ denotes a k-syzygy module of M.

The following result gives some equivalent characterizations of $g_n(k)$, where (1) $\Leftrightarrow$ (3) for the case $k = 1$ is [3, Proposition 2.26].

Theorem 2.2. *[24, Theorem 3.4] The conditions (1) and (2) are equivalent for any $n, k \geq 0$. If $k \geq 1$, then (1)–(5) are equivalent:*

(1) Λ is $g_n(k)$.
(2) For any monomorphism $A \xrightarrow{f} B$ with $A, B \in \Omega^{k+1}(\operatorname{mod}\Lambda)$, $\mathbb{E}_i\mathbb{E}_i(f)$ is a monomorphism for any $0 \leq i \leq n-1$.
(3) $\Omega^{i+k}(\operatorname{mod}\Lambda) \subseteq \mathcal{F}_{i+1}$ for any $1 \leq i \leq n$.
(4) For any $C \in \operatorname{mod}\Lambda$ and $0 \leq i \leq n$, there exists an exact sequence $0 \to Y \to X \to \Omega^{k-1}(C) \to 0$ with $X \in \mathcal{W}_{i+1}$ and $Y \in \mathcal{P}_{i+1}$.
(5) For any $C \in \operatorname{mod} R$ and $0 \leq i \leq n$, there exists an exact sequence $0 \to \Omega^k(C) \to Y' \to X' \to 0$ with $X' \in \mathcal{W}_{i+1}$ and $Y' \in \mathcal{P}_{i+1}$.

Now we concentrate on the conditions $G_n(k)$ and $g_n(k)$ for the case $k = 0, 1$.

In [24, Theorem 4.1], we gave a quick proof of the following remarkable left-right symmetry of $G_n(k)$ and $g_n(k)$ for the case $k = 0, 1$, where (1) is in Theorem 1.2, (2) is in [18, Theorem 4.7] and [23, Theorem 2.4], and (3) is in [7, Theorem 0.1] and [18, Theorem 4.1].

Theorem 2.3. *(Symmetry)*

(1) $G_n(0) \Leftrightarrow G_n(0)^{op}$.
(2) $g_n(1) \Leftrightarrow g_n(1)^{op}$.
(3) $G_n(1) \Leftrightarrow g_n(0)^{op}$.

Question 2.1. [24, Question 4.1.1] It is natural to ask for the existence of a common generalization of the conditions $G_n(k)$ and $g_n(k)$ satisfying certain "left-right symmetry". For example, does there exist some natural condition $G_n(k, l)$ for each triple (n, k, l) of non-negative integers with the following properties?
(i) $G_n(k, 0) = G_n(k)$, and $G_n(k, 1) = g_n(k)$.
(ii) $G_n(k, l) \Leftrightarrow G_n(l, k)^{op}$.
(iii) $G_n(k, l) \Rightarrow G_{n'}(k', l')$ if $n \geq n'$, $k \leq k'$ and $l \leq l'$.

Combining [6, Proposition 3.4] with Theorems 1.2 and 2.1, we have the following equivalent characterizations of $G_n(0)$.

Theorem 2.4. *The conditions (1)–(3) and their opposite versions are equivalent. If Λ is an Artinian algebra, then all conditions are equivalent:*

(1) Λ *is* $G_n(0)$.
(2) $\mathrm{fd}\, I_i(\Lambda) \leq i$ *for any* $0 \leq i \leq n-1$
(3) $\mathbb{E}_i\mathbb{E}_i$ *preserves monomorphisms in* $\mathrm{mod}\,\Lambda$ *for any* $0 \leq i \leq n-1$.
(4) All simple composition factors of $\mathbb{E}_i(S)$ *have grade at least* i *for any simple* Λ*-module* S *and* $1 \leq i \leq n$.
(5) Opposite side version of (i) *(*$1 \leq i \leq 4$*).*

Remark 2.2. It was proved in [29, Theorem 8] that if Λ is a left and right Artinian ring and k is a positive integer, then Λ is $G_n(0)$ if and only if a (lower) triangular matrix ring of degree k over Λ is also $G_n(0)$. Note that this is a generalization of [15, Theorem 3.10] where the case $k = 2$ was established.

By Theorems 2.2 and 2.3, we have the following equivalent characterizations of $g_n(1)$.

Theorem 2.5. *The following statements are equivalent:*

(1) Λ *is* $g_n(1)$.
(2) For any monomorphism $A \xrightarrow{f} B$ *with* $A, B \in \Omega^2(\mathrm{mod}\,\Lambda)$, $\mathbb{E}_i\mathbb{E}_i(f)$ *is a monomorphism for any* $0 \leq i \leq n-1$.
(3) $\Omega^i(\mathrm{mod}\,\Lambda) = \mathcal{F}_i$ *holds for any* $1 \leq i \leq n+1$.
(4) For any $C \in \mathrm{mod}\,\Lambda$ *and* $0 \leq i \leq n$*, there exists an exact sequence* $0 \to Y \to X \to C \to 0$ *with* $X \in \mathcal{W}_{i+1}$ *and* $Y \in \mathcal{P}_{i+1}$.
(5) For any $C \in \mathrm{mod}\,\Lambda$ *and* $0 \leq i \leq n$*, there exists an exact sequence* $0 \to \Omega^1(C) \to Y \to X \to 0$ *with* $X \in \mathcal{W}_{i+1}$ *and* $Y \in \mathcal{P}_{i+1}$.
(6) Opposite side version of (i) *(*$1 \leq i \leq 5$*).*

Let $\mathcal{D}$ be a full subcategory of mod Λ. Recall that $\mathcal{D}$ is said to be *closed under extensions* if the middle term B of any short sequence $0 \to A \to B \to C \to 0$ is in $\mathcal{D}$ provided that the end terms A and C are in $\mathcal{D}$. We use $\mathrm{add}\,\mathcal{D}$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of all Λ-modules isomorphic to direct summands of finite direct sums of modules in $\mathcal{D}$. For any $k \geq 1$, we denote by $\mathcal{I}_k = \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{id}\, M < k\}$.

The following result gives some equivalent characterizations of $G_n(1)$, where (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are in Theorem 2.1, (1) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) follow from Theorems 2.3 and 2.2, (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are in [7, Theorem 0.1] and [19, Theorem 3.3], and (2) + (5) $\Rightarrow$ (10) $\Rightarrow$ (6) are in [24, Theorem 4.4].

Theorem 2.6. *The following conditions (1)–(9) are equivalent. If* Λ *is an Artinian algebra, then (10) is also equivalent:*

(1) Λ *is* $G_n(1)$.
(2) $\mathrm{fd}\, I_i(\Lambda^{op}) \leq i+1$ *holds for any* $0 \leq i \leq n-1$.
(3) *For any exact sequence* $0 \to A \xrightarrow{f} B \to C \to 0$ *with* $C \in \Omega^1(\mathrm{mod}\,\Lambda)$, $\mathbb{E}_i\mathbb{E}_i(f)$ *is a monomorphism for any* $0 \leq i \leq n-1$.
(4) $\mathcal{E}(\Omega^i(\mathrm{mod}\,\Lambda), \Omega^i(\mathrm{mod}\,\Lambda)) \subseteq \mathcal{F}_i$ *holds for any* $1 \leq i \leq n$.
(5) $\Omega^i(\mathrm{mod}\,\Lambda)$ *is closed under extensions for any* $1 \leq i \leq n$.
(6) $\mathrm{add}\,\Omega^i(\mathrm{mod}\,\Lambda)$ *is closed under extensions for any* $1 \leq i \leq n$.
(7) $\mathcal{F}_i$ *is closed under extensions for any* $1 \leq i \leq n$.
(8) Λ *is* $g_n(0)^{op}$.
(9) *For any monomorphism* $A \xrightarrow{f} B$ *with* $A, B \in \Omega^1(\mathrm{mod}\,\Lambda^{op})$, $\mathbb{E}_i\mathbb{E}_i(f)$ *is a monomorphism for any* $0 \leq i \leq n-1$.
(10) *For any* $C \in \mathrm{mod}\,\Lambda$ *and* $1 \leq i \leq n$, *there exist exact sequences* $0 \to Y \to X \to C \to 0$ *and* $0 \to C \to Y' \to X' \to 0$ *with* $X, X' \in \Omega^i(\mathrm{mod}\,\Lambda)$ *and* $Y, Y' \in \mathcal{I}_{i+1}$.

Example 2.1. Contrary to the condition $G_n(0)$, the condition $G_n(1)$ is not left-right symmetric. Consider the following example. Let K be a field and Δ the quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3.$$

(1) If $\Lambda = K\Delta/(\alpha\beta\alpha)$, then $\mathrm{fd}\, I_0(\Lambda) = 1$ and $\mathrm{fd}\, I_0(\Lambda^{op}) \geq 2$. (2) If $\Lambda = K\Delta/(\gamma\alpha, \beta\alpha)$, then $\mathrm{fd}\, I_0(\Lambda) = 2$ and $\mathrm{fd}\, I_0(\Lambda^{op}) = 1$.

By Theorems 1.1, 2.4 and 2.6, we have the following result about commutative Gorenstein rings.

Corollary 2.1. *If* Λ *is commutative, then the following statements are equivalent:*

(1) Λ *is Gorenstein.*
(2) Λ *is* $G_\infty(0)$.
(3) Λ *is* $G_\infty(1)$.

3. Properties of rings satisfying the Auslander-type conditions

In this section, we investigate the properties of rings satisfying the Auslander-type conditions. These properties involve duality theory, the socle of modules, homological dimensions, homological finiteness of certain subcategories, cotorsion pairs, Evans-Griffith presentations, and so on.

For a non-negative integer l, put $\mathcal{C}_l(\Lambda) = \{X \in \operatorname{mod} \Lambda \mid X = \mathbb{E}_l(Y)$ for some $Y \in \operatorname{mod} \Lambda^{op}$ and $\operatorname{grade} Y \geq l\}$. The following result is a duality between $\mathcal{C}_l(\Lambda)$ and $\mathcal{C}_l(\Lambda^{op})$, which generalizes results in [32, 6.2] and [26, Theorem 4].

Theorem 3.1. *[30, Theorem 1.2] Let Λ be $G_n(0)$ and $0 \leq l \leq n-1$. Then $\mathbb{E}_l$ gives a duality between $\mathcal{C}_l(\Lambda)$ and $\mathcal{C}_l(\Lambda^{op})$, and $\mathbb{E}_l\mathbb{E}_l$ is isomorphic to the identity functor.*

The following result is a duality between simple Λ-modules and simple Λ^{op}-modules.

Theorem 3.2. *[30, Theorem 1.3] Let Λ be a Noetherian algebra which is $G_n(0)$ and $0 \leq l \leq n-1$. Then $\mathbb{F}_l := \operatorname{Soc} \mathbb{E}_l$ gives a duality between simple Λ-modules X with $\operatorname{grade} X = l$ and that of Λ^{op}, and $\mathbb{F}_l\mathbb{F}_l$ is isomorphic to the identity functor. Moreover, $\operatorname{grade} \mathbb{E}_l(X)/\mathbb{F}_l(X) > l$.*

Let Λ be an Artinian algebra. Recall that the Nakayama conjecture states that Λ is self-injective if $I_i(\Lambda)$ is projective for any $i \geq 0$ (see [37] or [43]), and the generalized Nakayama conjecture states that each indecomposable injective Λ-module occurs as the direct summand of some $I_i(\Lambda)$ (see [4]). In view of Theorems 1.1 and 1.2, Auslander and Reiten conjectured in [6] that Λ is Gorenstein if it is $G_\infty(0)$. This conjecture is situated between the Nakayama conjecture and the generalized Nakayama conjecture. The following result is related to this conjecture. It means that the Gorenstein symmetry conjecture holds true for a left and right Artinian ring which is $G_\infty(1)$. Recall from [9] that the Gorenstein symmetry conjecture states that the left and right self-injective dimensions of any Artinian algebra are identical.

Proposition 3.1. *[21, Proposition 4.6] If a left and right Artinian ring Λ is $G_\infty(1)$, then $\operatorname{id} \Lambda = \operatorname{id} \Lambda^{op}$.*

Observe that Auslander and Reiten proved in [6, Corollary 5.5(b)] that if an Artinian algebra Λ is $G_\infty(0)$, then $\operatorname{id} \Lambda = \operatorname{id} \Lambda^{op}$. Proposition 3.1 is a generalization of this result.

Theorem 3.3. *(1) [16, Proposition 1.1] If Λ is $G_n(0)$ with $\operatorname{id} \Lambda = \operatorname{id} \Lambda^{op} = n$, then $\operatorname{pd} I_n(\Lambda) = \operatorname{fd} I_n(\Lambda) = n$ and so Λ is $G_\infty(0)$.*

(2) [27, Theorem 2] If Λ is $G_\infty(0)$ with $\operatorname{id} \Lambda = \operatorname{id} \Lambda^{op} = n$, then any injective indecomposable Λ-module E with $\operatorname{fd} E = n$ is isomorphic to a direct summand of $I_n(\Lambda)$ and is isomorphic to the injective envelope of a simple

Λ-module. Thus if M is a Λ-module with $\mathrm{id}\, M = n$, then $I_n(M)$ has an essential socle, where $I_n(M)$ is the $(n+1)$-st term in a minimal injective resolution of M.

As an immediate consequence of Theorem 3.3, we have the following result.

Corollary 3.1. *[28, Theorem 6] If Λ is $G_\infty(0)$ with $\mathrm{id}\,\Lambda = \mathrm{id}\,\Lambda^{op} = n$, then $I_n(\Lambda)$ has an essential socle.*

Recall that the *finitistic dimension* of Λ, denoted by $\mathrm{fin.dim}\,\Lambda$, is defined as $\sup\{\mathrm{pd}\, X \mid X \in \mathrm{mod}\,\Lambda \text{ and } \mathrm{pd}\, X < \infty\}$. By using Theorem 2.2, it is not difficult to get the following result.

Lemma 3.1. *[24, Lemma 5.1] Assume that Λ is $g_{n+1}(k)$ with $n \geq 0$ and $k \geq 1$. If $\mathrm{fin.dim}\,\Lambda = n$, then $n \leq \mathrm{id}\,\Lambda \leq n + k$.*

In the following result, the case for $k \geq 1$ follows from Lemma 3.1, and the case for $k = 0$ is in [23, Corollary 2.15].

Theorem 3.4. *[24, Theorem 5.2] If Λ is $g_\infty(k)$ with $k \geq 0$, then $\mathrm{fin.dim}\,\Lambda \leq \mathrm{id}\,\Lambda \leq \mathrm{fin.dim}\,\Lambda + k$.*

As an application of Theorem 3.4, we have the following result, where (1) and (2) follow from the symmetry of $G_\infty(0)$ and the fact that $G_\infty(1) \Leftrightarrow g_\infty(0)^{op}$ (see Theorem 2.3), respectively.

Corollary 3.2. *(1) If Λ is $G_\infty(0)$, then $\mathrm{fin.dim}\,\Lambda = \mathrm{id}\,\Lambda$ and $\mathrm{fin.dim}\,\Lambda^{op} = \mathrm{id}\,\Lambda^{op}$.*

(2) If Λ is $G_\infty(1)$, then $\mathrm{fin.dim}\,\Lambda \leq \mathrm{id}\,\Lambda \leq \mathrm{fin.dim}\,\Lambda + 1$ and $\mathrm{fin.dim}\,\Lambda^{op} = \mathrm{id}\,\Lambda^{op}$.

Definition 3.1. [5] Assume that $\mathcal{D}$ is a full subcategory of $\mathrm{mod}\,\Lambda$ and $C \in \mathrm{mod}\,\Lambda$, $D \in \mathcal{D}$. A morphism $f : D \to C$ is said to be a *right $\mathcal{D}$-approximation* of C if $\mathrm{Hom}_\Lambda(X, f) : \mathrm{Hom}_\Lambda(X, D) \to \mathrm{Hom}_\Lambda(X, C) \to 0$ is exact for any $X \in \mathcal{D}$. A right $\mathcal{D}$-approximation $f : D \to C$ is called *minimal* if an endomorphism $g : D \to D$ is an automorphism whenever $f = fg$. The subcategory $\mathcal{D}$ is said to be *contravariantly finite* in $\mathrm{mod}\,\Lambda$ if any module in $\mathrm{mod}\,\Lambda$ has a right $\mathcal{D}$-approximation. Dually, we define the notions of *(minimal) left $\mathcal{D}$-approximations* and *covariantly finite subcategories*. The subcategory of $\mathrm{mod}\,\Lambda$ is said to be *functorially finite* in $\mathrm{mod}\,\Lambda$ if it is both contravariantly finite and covariantly finite in $\mathrm{mod}\,\Lambda$.

As applications of Theorems 2.5 and 2.6, we have the following result about homological finiteness of $\mathcal{P}_k$.

Corollary 3.3. *[24, Corollary 4.7]*

(1) If Λ is $g_n(1)$, then $\mathcal{P}_{i+1}$ is covariantly finite in $\operatorname{mod}\Lambda$ *for any $0 \leq i \leq n$.*
(2) [22, Theorem 3.6] If an Artinian algebra Λ is $g_n(0)$, then $\mathcal{P}_{i+1}$ is functorially finite in $\operatorname{mod}\Lambda$ *for any $0 \leq i \leq n$.*

Definition 3.2. (1) [39] A pair $(\mathcal{C}, \mathcal{D})$ of full subcategories of $\operatorname{mod}\Lambda$ is called a *cotorsion pair* if

$$\mathcal{C} = \{C \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(C, \mathcal{D}) = 0\} \text{ and } \mathcal{D} = \{D \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\mathcal{C}, D) = 0\}.$$

(2) [24] For an Artinian algebra Λ, a cotorsion pair $(\mathcal{C}, \mathcal{D})$ is called *functorially finite* if the following equivalent conditions are satisfied:
(i) $\mathcal{C}$ is contravariantly finite in $\operatorname{mod}\Lambda$.
(ii) $\mathcal{D}$ is covariantly finite in $\operatorname{mod}\Lambda$.
(iii) For any $C \in \operatorname{mod}\Lambda$, there exists an exact sequence $0 \to Y \to X \xrightarrow{f} C \to 0$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.
(iv) For any $C \in \operatorname{mod}\Lambda$, there exists an exact sequence $0 \to C \xrightarrow{g} Y' \to X' \to 0$ with $X' \in \mathcal{C}$ and $Y' \in \mathcal{D}$.

For any $m, n \geq 0$, put $\mathcal{X}_{n,m} = \mathcal{W}_n \cap \mathcal{F}_m$ and $\mathcal{Y}_{n,m} = \operatorname{add}\mathcal{E}(\mathcal{I}_m, \mathcal{P}_n)$. When Λ is an Artinian algebra over R, we denote by $\mathbb{D} : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ the duality induced by the Matlis duality of R.

Theorem 3.5. *[24, Corollary 4.9 and Theorem 1.3] Let Λ be an Artinian algebra which is $G_\infty(1)$ and $G_\infty(1)^{op}$ (In particular, let Λ be an Artinian algebra which is $G_\infty(0)$). Then $(\mathcal{X}_{i,j-1}, \mathcal{Y}_{i,j})$ $(i \geq 0,\ j \geq 1)$ and $(\mathcal{Y}_{i,j}, \mathbb{D}\,\mathcal{X}^{op}_{j,i-1})$ $(i \geq 1,\ j \geq 0)$ form functorially finite cotorsion pairs.*

By Theorem 3.5, $(\mathcal{W}_i, \mathcal{Y}_{i,1})$ $(j := 1)$ and $(\mathcal{F}_{j-1}, \mathcal{I}_j)$ $(i := 0)$ form functorially finite cotorsion pairs. In addition, as an immediate consequence of Theorem 3.5, we have the following result.

Corollary 3.4. *Under the assumption of Theorem 3.5, $\mathcal{W}_1 \supseteq \mathcal{W}_2 \supseteq \cdots \supseteq \mathcal{W}_i \supseteq \cdots$ is a chain of contravariantly finite subcategories of* $\operatorname{mod}\Lambda$.

Let Λ be a commutative Noetherian ring and let n be a non-negative integer and $M \in \Omega^n(\operatorname{mod}\Lambda)$. Recall from [14] that an *Evans-Griffith presentation* of M is an exact sequence in $\operatorname{mod}\Lambda$:

$$0 \to S \to B \to M \to 0,$$

where B is an n-syzygy module of $\mathbb{E}_{n+1}(\operatorname{Tr} M)$ and $S \in \Omega^{n+2}(\operatorname{mod} \Lambda)$. In the case Λ is not necessarily commutative, we also call such an exact sequence an *Evans-Griffith presentation* of M.

Theorem 3.6. *[20, Proposition 4.4] Let Λ be $G_n(1)$. Then, for any $0 \leq d \leq n-1$, each module in $\Omega^d(\operatorname{mod} \Lambda)$ has an Evans-Griffith presentation.*

By Theorem 3.6, we have the following result.

Corollary 3.5. *[20, Corollary 4.5] If Λ is $G_\infty(1)$, then for any non-negative integer d, each module in $\Omega^d(\operatorname{mod} \Lambda)$ has an Evans-Griffith presentation.*

Combining Corollaries 3.5 and 2.1, we immediately have the following result.

Corollary 3.6. *[20, Corollary 4.6] If Λ is a commutative Gorenstein ring, then for any non-negative integer d, each module in $\Omega^d(\operatorname{mod} \Lambda)$ has an Evans-Griffith presentation.*

Observe that a special instance of Corollary 3.6 was already considered by Evans and Griffith in [14, Theorem 2.1]. They showed that if Λ is a commutative Noetherian local ring with finite global dimension and contains a field then each non-free d-syzygy of rank d has an Evans-Griffith presentation. Corollary 3.6 generalizes this result to much more general setting.

4. (m, n)-conditions

In this section, we give the definition of the (weak) (m, n)-condition, and then investigate the properties of rings satisfying certain (m, n)-condition.

Definition 4.1. [30] Let $m, n \geq 1$. Λ is said to satisfy the *(m, n)-condition* (or Λ^{op} satisfies the *$(m, n)^{op}$-condition*) if $\operatorname{s.grade} \mathbb{E}_m(N) \geq n$ for any $N \in \operatorname{mod} \Lambda^{op}$. Similarly, Λ is said to satisfy the *weak (m, n)-condition* (or Λ^{op} satisfies the *weak $(m, n)^{op}$-condition*) if $\operatorname{grade} \mathbb{E}_m(N) \geq n$ for any $N \in \operatorname{mod} \Lambda^{op}$.

Remark 4.1. (1) By [32, 6.1], Λ satisfies the (m, n)-condition if and only if $\operatorname{fd} I_i(\Lambda) \leq m-1$ for any $0 \leq i \leq n-1$.

(2) It is easy to see that Λ is $G_n(k)$ if and only if Λ satisfies the $(k+i, i)^{op}$-condition for any $1 \leq i \leq n$, and Λ is $g_n(k)$ if and only if Λ satisfies the weak $(k+i, i)^{op}$-condition for any $1 \leq i \leq n$.

(3) For a module $M \in \operatorname{mod}\Lambda$ and $n \geq 1$, recall that the *dominant dimension* of M, denoted by $\operatorname{dom.dim} M$, is said to be at least n if the first n terms in a minimal injective resolution of M are flat. So, $\operatorname{dom.dim}\Lambda \geq n$ if and only if Λ satisfies the $(1, n)$-condition. It was proved in [17, Theorem] that $\operatorname{dom.dim}\Lambda = \operatorname{dom.dim}\Lambda^{op}$.

The following result gives some relations between different (weak) (m, n)-conditions.

Proposition 4.1. *[24, Lemma 5.4] and [30, Lemma 2.3]*

(1) (m, l) + *weak* $(l, n) \Rightarrow (m, n)$.
(2) weak (m, l) + *weak* $(l, n) \Rightarrow$ *weak* (m, n).
(3) (m, l) + *weak* $(l, n)^{op} \Rightarrow (m, n)$.
(4) weak (m, l) + *weak* $(l, n)^{op} \Rightarrow$ *weak* (m, n).

It is known that Λ is $G_n(0)$ if and only if so is Λ^{op} (see Theorem 2.3). However, the (i, i)-condition does not possess such a symmetric property in general. For example, the finite dimensional algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \leftarrow 4$ modulo the ideal $\beta\alpha$ satisfies exactly one of the $(2, 2)$ and $(2, 2)^{op}$-conditions.

The following result gives a sufficient condition that the (i, i)-condition implies the $(i, i)^{op}$-condition.

Proposition 4.2. *[24, Corollary 5.7]* $G_{n-1}(1) + (n, n) \Rightarrow (n, n)^{op}$.

In particular, putting $n = 3$ in Proposition 4.2, we get the following result.

Corollary 4.1. $(2, 2)^{op} + (3, 3) \Rightarrow (3, 3)^{op}$.

In [24], we gave an example satisfying the conditions in Corollary 4.1 as follows.

Example 4.1. Let Λ be a finite dimensional algebra given by the quiver:

$$\begin{tikzcd}
 & 1 \arrow[d] & \\
2 \arrow[r, "\alpha"] & 3 \arrow[r] \arrow[d, "\beta"] & 4 \\
 & 5 &
\end{tikzcd}$$

modulo the ideal $\beta\alpha$. Then $\operatorname{fd} I_0(\Lambda) = \operatorname{fd} I_1(\Lambda) = \operatorname{fd} I_0(\Lambda^{op}) = \operatorname{fd} I_1(\Lambda^{op}) = 1$, and $\operatorname{fd} I_2(\Lambda) = \operatorname{fd} I_2(\Lambda^{op}) = 2$.

Recall from [10] that a module $M \in \operatorname{mod}\Lambda$ is called *pure* if $\operatorname{grade} X = \operatorname{grade} M$ for any non-zero submodule X of M. Björk in [10, p.144] raised a question: If Λ is $G_\infty(0)$ with finite left and right self-injective dimensions, is $\mathbb{E}_{\operatorname{grade} M}(M)$ pure for any $M \in \operatorname{mod}\Lambda$? Björk and Ekström in [11, Proposition 2.11] gave a positive answer to this question, and then Iyama proved in [30] that the answer to it is positive in more general case.

Proposition 4.3. *[30, Proposition 2.9] If Λ satisfies the $(n,n)^{op}$-condition, then for any $M \in \operatorname{mod}\Lambda$ with $\operatorname{grade} M = n$, $\mathbb{E}_n(M)$ is pure with $\operatorname{grade} \mathbb{E}_n(M) = n$.*

For a positive integer n, we denote $\mathcal{E}_n(\Lambda_\Lambda) = \{M \in \operatorname{mod}\Lambda \mid M = \mathbb{E}_n(N)$ for some $N \in \operatorname{mod}\Lambda^{op}\}$. Auslander showed in [1, Proposition 3.3] that any direct summand of a module in $\mathcal{E}_1(\Lambda_\Lambda)$ is still in $\mathcal{E}_1(\Lambda_\Lambda)$. He then asked whether any submodule of a module in $\mathcal{E}_1(\Lambda_\Lambda)$ is still in $\mathcal{E}_1(\Lambda_\Lambda)$. Recall that a full subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ is said to be *submodule closed* if any non-zero submodule of a module in $\mathcal{X}$ is also in $\mathcal{X}$. Then the above Auslander's question is equivalent to the following question: Is $\mathcal{E}_1(\Lambda_\Lambda)$ submodule closed?

Proposition 4.4. *[21, Corollaries 3.9 and 3.14]*

(1) If Λ is $G_\infty(0)$ with $\operatorname{id}\Lambda = \operatorname{id}\Lambda^{op} = n$, then $\mathcal{E}_n(\Lambda_\Lambda)$ is submodule closed.
(2) If $\operatorname{id}\Lambda = \operatorname{id}\Lambda^{op} = 1$, then $\mathcal{E}_1(\Lambda_\Lambda)$ is submodule closed if and only if Λ satisfies the $(1,1)$-condition.
(3) If $\operatorname{id}\Lambda = \operatorname{id}\Lambda^{op} = 2$, then $\mathcal{E}_2(\Lambda_\Lambda)$ is submodule closed if and only if Λ satisfies the $(2,2)$-condition.

As an application of Proposition 4.4, the following examples were constructed in [21] to illustrate that neither $\mathcal{E}_1(\Lambda_\Lambda)$ nor $\mathcal{E}_2(\Lambda_\Lambda)$ are submodule closed in general, by which the above Auslander's question is answered negatively.

Example 4.2. (1) Let Λ be a finite dimensional algebra given by the quiver:

$$2 \longleftarrow 1 \longrightarrow 3.$$

Then $\operatorname{id}\Lambda = \operatorname{id}\Lambda^{op} = 1$ and $\operatorname{fd} I_0(\Lambda) = 1$. By Proposition 4.4(2), $\mathcal{E}_1(\Lambda_\Lambda)$ is not submodule closed.

(2) Let Λ be a finite dimensional algebra given by the quiver:

$$\begin{tikzcd} 1 \arrow[r, "\gamma"] \arrow[d, "\alpha"] & 3 \arrow[d, "\delta"] \\ 2 \arrow[r, "\beta"] & 4 \end{tikzcd}$$

modulo the ideal $\beta\alpha$. Then $\mathrm{id}\,\Lambda = \mathrm{id}\,\Lambda^{op} = 2$ and $\mathrm{fd}\,I_0(\Lambda) = 2$. By Proposition 4.4(3), $\mathcal{E}_2(\Lambda_\Lambda)$ is not submodule closed.

It is clear that $\mathrm{mod}\,\Lambda \supseteq \mathcal{E}_1(\Lambda_\Lambda) \supseteq \mathcal{E}_2(\Lambda_\Lambda) \supseteq \cdots \supseteq \mathcal{E}_i(\Lambda_\Lambda) \supseteq \cdots$. For any positive integer n, $\mathcal{E}_n(\Lambda_\Lambda)$ is submodule closed for Λ being $G_\infty(0)$ with $\mathrm{id}\,\Lambda = \mathrm{id}\,\Lambda^{op} = n$ by Proposition 4.4(1), and neither $\mathcal{E}_1(\Lambda_\Lambda)$ nor $\mathcal{E}_2(\Lambda_\Lambda)$ are submodule closed in general by Example 4.2. It is interesting to know whether $\mathcal{E}_n(\Lambda_\Lambda)$ (where $n \geq 3$) is submodule closed in general.

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