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Relative FP-gr-injective and gr-flat modules

Tiwei Zhao*,§, Zenghui Gao†,¶ and Zhaoyong Huang‡,‖

*School of Mathematical Sciences, Qufu Normal University
Qufu, Shandong 273165, P. R. China

†College of Applied Mathematics, Chengdu University
of Information Technology, Chengdu 610225, P. R. China

‡Department of Mathematics, Nanjing University
Nanjing 210093, P. R. China

§tiweizhao@hotmail.com
¶gaozenghui@cuit.edu.cn
‖huangzy@nju.edu.cn

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Let $n \geq 1$ be an integer. We introduce the notions of n-FP-gr-injective and n-gr-flat modules. Then we investigate the properties of these modules by using the properties of special finitely presented graded modules and obtain some equivalent characterizations of n-gr-coherent rings in terms of n-FP-gr-injective and n-gr-flat modules. Moreover, we prove that the pairs (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) and (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) are duality pairs over left n-coherent rings, where gr-$\mathcal{FI}_n$ and gr-$\mathcal{F}_n$ denote the subcategories of n-FP-gr-injective left R-modules and n-gr-flat right R-modules respectively. As applications, we obtain that any graded left (respectively, right) R-module admits an n-FP-gr-injective (respectively, n-gr-flat) cover and preenvelope.

Keywords: n-presented graded modules; n-FP-gr-injective modules; n-gr-flat modules; n-gr-coherent rings; covers, (pre)envelopes.

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1. Introduction

Graded rings and modules are a classical topic in algebra, and the homological theory of graded rings has very important applications in algebraic geometry [14]. In recent years, relative homological theory for graded rings has been studied by many authors and has become a vigorously active area of research (cf. [2, 3, 11, 12, 19, 20]). In particular, García Rozas *et al.* in [12] proved the existence of flat covers in the category of graded modules over a graded ring. In [2, 22], the homological properties of FP-gr-injective modules were investigated and the results were applied to characterize the gr-coherent rings.

As we know, coherent rings have been characterized in various ways, and many nice properties were obtained for such rings in [4, 10, 13, 18, 21] and so on. For a non-negative integer n, Costa in [8] introduced the notion of n-coherent rings. Following [8], a left R-module M is said to be *n-presented* if it has a finite n-presentation, that is, there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated projective; and a ring R is called *left n-coherent* if every n-presented left R-module is $(n+1)$-presented. In [7], Chen and Ding introduced the notions of n-FP-injective and n-flat modules, and they showed that there are many similarities between coherent rings and n-coherent rings. Along the same lines, it seems to be natural to extend the ideas of [7, 8] and studied the relative homological theory associated to the notions of n-gr-coherent rings and n-presented graded modules. The aim of this paper is to introduce and study n-FP-gr-injective and n-gr-flat modules and show that these modules share many nice properties of FP-gr-injective and gr-flat modules in [2, 22].

This paper is organized as follows. In Sec. 2, we give some terminology and some preliminary results. In Sec. 3, we introduce and study n-FP-gr-injective and n-gr-flat modules for an integer $n \geq 1$. In our study, the properties of special finitely presented graded modules, defined via projective resolutions of n-presented graded modules, play a crucial role. Then we obtain some equivalent characterizations of n-gr-coherent rings in terms of n-FP-gr-injective and n-gr-flat modules. Section 4 is devoted to investigating duality pairs relative to n-FP-gr-injective and n-gr-flat modules. It is shown that the pairs (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) and (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) are duality pairs over left n-coherent rings, where gr-$\mathcal{FI}_n$ and gr-$\mathcal{F}_n$ denote the subcategories of n-FP-gr-injective left R-modules and n-gr-flat right R-modules respectively. As applications, we get that any graded left (respectively, right) R-module admits an n-FP-gr-injective (respectively, n-gr-flat) cover and preenvelope. In addition, cotorsion pairs associated to n-FP-gr-injective and n-gr-flat modules are considered.

2. Preliminaries

Throughout this paper, all rings considered are associative with identity element and the R-modules are unital. By R-Mod we will denote the Grothendieck category of all left R-modules. Let G be a multiplicative group with neutral element e. A *graded ring* R is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus, R_e is a subring of R, $1 \in R_e$ and R_σ is an R_e-bimodule for every $\sigma \in G$. A *graded left R-module* is a left R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For any graded left R-modules M and N, set

$$\mathrm{Hom}_{R\text{-gr}}(M, N) := \{f : M \to N \mid f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \text{ for any } \sigma \in G\}$$

which is the group of all morphisms from M to N in the category R-gr of all graded left R-modules (gr-R will denote the category of all graded right R-modules). It is well known that R-gr is a Grothendieck category. An R-linear map $f : M \to N$ is said to be a *graded morphism of degree* τ with $\tau \in G$ if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\mathrm{HOM}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote $\mathrm{Ext}^i_{R\text{-gr}}$ and EXT^i_R as the right derived functors of $\mathrm{Hom}_{R\text{-gr}}$ and HOM_R, respectively. Given a graded left R-module M, the *graded character module* of M is defined as $M^+ := \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the rational numbers field and $\mathbb{Z}$ is the integers ring. It is easy to see that $M^+ = \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma$ with $\sigma \in G$ to be the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$ and $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the *graded tensor product* of M and N.

If M is a graded left R-module and $\sigma \in G$, then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for any $\tau \in G$. The graded module $M(\sigma)$ is called the *σ-suspension* of M. We may regard the σ-suspension as an isomorphism of categories $T_\sigma : R\text{-gr} \to R\text{-gr}$, given on objects as $T_\sigma(M) = M(\sigma)$ for any $M \in R$-gr.

For any element $m = \sum_{\sigma \in G} m_\sigma$ of R, set $\mathrm{Supp}(m) := \{\sigma \in G \,|\, m_\sigma \neq 0\}$. Consider a set $\{M_i \,|\, i \in I\}$ of graded left R-modules and let $\{\prod_{i \in I} M_i, \pi_i\}$ be the direct product in R-Mod of the underlying left R-modules M_i, where $\pi_j : \prod_{i \in I} M_i \to M_j$ denotes the jth canonical projection for each $j \in I$. Given $m \in \prod_{i \in I} M_i$, define $\mathrm{SUPP}(m) := \bigcup_{i \in I} \mathrm{Supp}(\pi_i(m)) \subset G$; and define $\prod^{R\text{-gr}}_{i \in I} M_i := \{m \in \prod_{i \in I} M_i \,|\, \mathrm{SUPP}(m) \text{ is finite}\}$. Then $\{\prod^{R\text{-gr}}_{i \in I} M_i, \pi_i\}$ is the direct product of the graded left R-modules $\{M_i \,|\, i \in I\}$. It is a graded left R-module, where $(\prod^{R\text{-gr}}_{i \in I} M_i)_\sigma = \{m \in \prod^{R\text{-gr}}_{i \in I} M_i \,|\, \mathrm{SUPP}(m) \subset \{\sigma\}\}$. Observe that, as R_e-modules, $(\prod^{R\text{-gr}}_{i \in I} M_i)_\sigma \cong \prod_{i \in I} (M_i)_\sigma$ for any $\sigma \in G$.

The injective objects of R-gr will be called *gr-injective modules*. Projective (respectively, flat) objects of R-gr will be called *projective* (respectively, *flat*) graded modules because M is gr-projective (respectively, gr-flat) if and only if it is a projective (respectively, flat) graded module. By $\text{gr-id}_R M$, $\mathrm{pd}_R M$ and $\mathrm{fd}_R M$ we will denote the gr-injective, projective and flat dimension of a graded module M, respectively. The gr-injective envelope of M is denoted by $E^g(M)$. A graded R-module M is called *FP-gr-injective* if $\mathrm{EXT}^1_R(N, M) = 0$ for any finitely presented graded R-module N. It can be proved that if R is gr-noetherian, then M is gr-injective if and only if M is FP-gr-injective, and that if R is gr-coherent (that is, a graded ring R such that given a family of graded flat R-modules $\{F_i\}_{i \in I}$, the graded R-module $\prod^{R\text{-gr}}_{i \in I} F_i$ is flat), then M is FP-gr-injective if and only if M^+ is flat.

The forgetful functor $U : R\text{-gr} \to R\text{-Mod}$ associates M to the underlying ungraded R-module. This functor has a right adjoint F which associates M in

R-Mod to the graded R-module $F(M) = \bigoplus_{\sigma\in G}({}^{\sigma}M)$, where each ${}^{\sigma}M$ is a copy of M written $\{{}^{\sigma}x \,|\, x \in M\}$ with R-module structure defined by $r * {}^{\tau}x = {}^{\sigma\tau}(rx)$ for any $r \in R_\sigma$. If $f : M \to N$ is R-linear, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)({}^{\sigma}x) = {}^{\sigma}f(x)$.

For a graded ring R, let $\mathcal{F}$ be a class of graded left R-modules and M a graded left R-module. Following [3], we say that a graded morphism $f : F \to M$ is an *$\mathcal{F}$-precover* of M if $F \in \mathcal{F}$ and $\mathrm{Hom}_{R\text{-gr}}(F', F) \to \mathrm{Hom}_{R\text{-gr}}(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$. Moreover, if whenever a graded morphism $g : F \to F$ such that $f \circ g = f$ is an automorphism of F, then $f : F \to M$ is called an *$\mathcal{F}$-cover* of M. The class $\mathcal{F}$ is called *(pre)covering* if each object in R-gr has an $\mathcal{F}$-(pre)cover. Dually, the notions of *$\mathcal{F}$-preenvelopes*, *$\mathcal{F}$-envelopes* and *(pre)enveloping* are defined.

3. n-FP-gr-Injective and n-gr-Flat Modules

In this section, we give a treatment of n-FP-gr-injective and n-gr-flat modules. Some general properties of these modules are discussed.

Definition 3.1. Let $n \geq 0$ be an integer. A graded left R-module F is called *n-presented* if there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

in R-gr with each P_i finitely generated projective.

A graded ring R is called *left n-gr-coherent* if each n-presented graded left R-modules is $(n+1)$-presented.

If F is an n-presented graded left R-module, then there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

in R-gr with each P_i finitely generated projective. Set $K_n := \mathrm{Im}(P_n \to P_{n-1})$ and $K_{n-1} := \mathrm{Im}(P_{n-1} \to P_{n-2})$. Then we get a short exact sequence

$$(\Delta) : 0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$$

in R-gr with P_{n-1} finitely generated projective. It follows that $\mathrm{EXT}^n_R(F, A) \cong \mathrm{EXT}^1_R(K_{n-1}, A)$ for any graded left R-module A.

Note that K_{n-1} and K_n obtained as above are finitely presented and finitely generated in R-gr respectively. We call the objects K_{n-1} and K_n *special finitely presented* and *special finitely generated* graded left R-modules, respectively, and we shall say the sequence $(\Delta) : 0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ in R-gr is a *special short exact sequence*. Moreover, a short exact sequence $0 \to A \to B \to C \to 0$ in R-gr is called *special gr-pure* if the induced sequence

$$0 \to \mathrm{HOM}_R(K_{n-1}, A) \to \mathrm{HOM}_R(K_{n-1}, B) \to \mathrm{HOM}_R(K_{n-1}, C) \to 0$$

is exact for all special finitely presented graded left R-modules K_{n-1}. In this case, A is said to be *special gr-pure* in B.

Remark 3.2. (1) Obviously, 0-presented (respectively, 1-presented) graded left R-modules are exactly finitely generated (respectively, finitely presented) graded left R-modules. If $m \geq n$, then m-presented graded left R-modules are n-presented. Also, left 0-gr-coherent (respectively, 1-gr-coherent) rings are just left gr-Noetherian (respectively, gr-coherent) rings.

(2) One checks readily that every finitely presented graded left R-module is n-presented ($n \geq 2$) if and only if R is a left gr-coherent ring. Moreover, a graded ring R is left n-gr-coherent if and only if each special finitely generated graded left R-module is finitely presented, and if and only if each special finitely presented graded left R-module is 2-presented.

The following lemma is the graded version of [17, Lemma 2.3].

Lemma 3.3. *Let* $0 \to F_1 \to F_2 \to F_3 \to 0$ *be an exact sequence in* R-gr. *Then the following statements hold for any* $n \geq 0$.

(1) *If* F_1, F_3 *are* n*-presented, then so is* F_2.
(2) *If* F_1 *is* n*-presented and* F_2 *is* $(n+1)$*-presented, then* F_3 *is* $(n+1)$*-presented.*
(3) *If* F_2, F_3 *is* $(n+1)$*-presented, then* F_1 *is* n*-presented.*

Ungraded n-presented modules have been investigated by many authors. For example, Bravo, Gillespie and Hovey in [5] introduced and investigated FP_∞-injective modules in terms of modules of type FP_∞, and Bravo and Pérez in [6] introduced and investigated the FP_n-injective modules in terms of n-presented modules for any $n \geq 0$. More precisely, let R be an associative ring and M a left R-module. Then M is called *FP_n-injective* if $\mathrm{Ext}^1_R(L, M) = 0$ for all n-presented left R-modules L.

We claim that if we similarly use the derived functor EXT^1 to define the FP_n-gr-injective and FP_∞-gr-injective modules, then they are just the FP_n-injective and FP_∞-injective objects in the category of graded modules respectively. In fact, if $M \in R$-gr and L is an n-presented graded left R-module with $n \geq 2$, then there is an exact sequence

$$P_2 \to P_1 \to P_0 \to L \to 0$$

in R-gr with each P_i finitely generated projective. By [20, Corollary 2.4.4], we have the following commutative diagram with the rows being complexes

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{HOM}_R(P_0, M) & \longrightarrow & \mathrm{HOM}_R(P_1, M) & \longrightarrow & \mathrm{HOM}_R(P_2, M) \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & \mathrm{Hom}_R(P_0, M) & \longrightarrow & \mathrm{Hom}_R(P_1, M) & \longrightarrow & \mathrm{Hom}_R(P_2, M) \end{array}$$

which shows that $\mathrm{EXT}^1_R(L, M) \cong \mathrm{Ext}^1_R(L, M)$.

Based on the above observation, we introduce the following.

Definition 3.4. Let $n \geq 1$ be an integer. A module M in R-gr is called *n-FP-gr-injective* if $\mathrm{EXT}^n_R(F, M) = 0$ for any n-presented graded left R-module F. A module N in gr-R is called *n-gr-flat* if $\mathrm{Tor}_n^R(N, F) = 0$ for any n-presented graded left R-module F.

We denote by gr-$\mathcal{FI}_n$ (respectively, gr-$\mathcal{F}_n$) the subcategory of all n-FP-gr-injective (respectively, n-gr-flat) graded left (respectively, right) R-modules.

Remark 3.5. (1) By definition, we have that a module $M \in R$-gr is 1-FP-gr-injective if and only if M is FP-gr-injective; and a module $N \in$ gr-R is 1-gr-flat if and only if N is a flat graded right R-module.

(2) In general, whenever $1 \leq m \leq n$, every m-FP-gr-injective left R-module is n-FP-gr-injective, and every m-gr-flat right R-module is n-gr-flat. Indeed, there are the following implications:

$$\text{gr-}\mathcal{FI} = \text{gr-}\mathcal{FI}_1 \subseteq \text{gr-}\mathcal{FI}_2 \subseteq \cdots \subseteq \text{gr-}\mathcal{FI}_m \subseteq \text{gr-}\mathcal{FI}_{m+1} \subseteq \cdots$$

$$\text{gr-}\mathcal{F} = \text{gr-}\mathcal{F}_1 \subseteq \text{gr-}\mathcal{F}_2 \subseteq \cdots \subseteq \text{gr-}\mathcal{F}_m \subseteq \text{gr-}\mathcal{F}_{m+1} \subseteq \cdots.$$

To see it, it suffices to show that every m-FP-gr-injective left R-module is $(m+1)$-FP-gr-injective. If M is an m-FP-gr-injective module and F is an $(m+1)$-presented graded left R-module, then there exists an exact sequence $0 \to K \to Q \to F \to 0$ in R-gr, where Q is finitely generated projective and K is m-presented by Lemma 3.3. So we get the exactness of $0 \to \mathrm{EXT}_R^m(K, M) \to \mathrm{EXT}_R^{m+1}(F, M) \to 0$. Note that $\mathrm{EXT}_R^m(K, M) = 0$ since M is m-FP-gr-injective, it follows that $\mathrm{EXT}_R^{m+1}(F, M) = 0$ for any $(m+1)$-presented graded left R-module F. Thus M is $(m+1)$-FP-gr-injective. Similarly, we can deduce that every m-gr-flat right R-module is n-gr-flat.

(3) If N is an n-presented graded left R-module and $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ is a special short exact sequence in R-gr with respect to N, then from the isomorphism $\mathrm{EXT}_R^n(N, A) \cong \mathrm{EXT}_R^1(K_{n-1}, A)$ for any graded left R-module A, we conclude that a module $M \in R$-gr is n-FP-gr-injective if and only if $\mathrm{EXT}_R^1(K_{n-1}, M) = 0$ for any special finitely presented graded left R-module K_{n-1}.

The following example illustrates that, in general, n-FP-gr-injective (respectively, n-gr-flat) modules need not be m-FP-gr-injective (respectively, m-gr-flat) whenever $m < n$. Before that, recall that the *FP-gr-injective dimension* of a graded left R-module M, denoted by FP-gr-id$_R M$, is defined to be the least integer n such that $\mathrm{EXT}_R^{n+1}(N, M) = 0$ for any finitely presented graded left R-module N, and l.FP-gr-dim $R = \sup\{\text{FP-gr-id}_R M \mid M \text{ is a graded left } R\text{-module}\}$.

Example 3.6. Let R be a graded ring with l.FP-gr-dim$R \leq 1$ but not gr-regular, for example, let $R = k[X]$ where k is a field. Then there exists a graded left R-module which is not FP-gr-injective by [22, Proposition 3.11]. We claim that every graded left R-module is 2-FP-gr-injective. In fact, let M be a graded left R-module and F a 2-presented graded left R-module. Then there exists an exact sequence

$0 \to M \to E \to L \to 0$ in R-gr with E gr-injective. It follows from [22, Lemma 3.7] that L is FP-gr-injective and $0 \to \mathrm{EXT}^1_R(F, L) \to \mathrm{EXT}^2_R(F, M) \to 0$ is exact. Notice that F is finitely presented, so we have $\mathrm{EXT}^1_R(F, L) = 0$. Thus, $\mathrm{EXT}^2_R(F, M) = 0$ for any 2-presented graded left R-modules F. Further, one can deduce that there exists a 2-gr-flat module, but not gr-flat by Theorem 3.17.

We have the following easy observations.

Proposition 3.7. (1) *Let $\{M_i\}_{i\in I}$ be a family of graded left R-modules. Then $\prod^{R\text{-gr}}_{i\in I} M_i$ is n-FP-gr-injective if and only if each M_i is n-FP-gr-injective.*

(2) *Let $\{N_i\}_{i\in I}$ be a family of graded right R-modules. Then $\bigoplus_{i\in I} N_i$ is n-gr-flat if and only if each N_i is n-gr-flat.*

Proposition 3.8. *Let M be a graded right R-module. Then M is n-gr-flat if and only if M^+ is n-FP-gr-injective.*

Proof. By [12, Lemma 2.1], we have $\mathrm{EXT}^1_R(N, M^+) \cong \mathrm{Tor}^R_1(M, N)^+$ for any graded left R-module N. Let F be any graded left R-module. Then there exists an exact sequence $0 \to K \to P \to F \to 0$ in R-gr with P projective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{EXT}^1_R(K, M^+) & \longrightarrow & \mathrm{EXT}^2_R(F, M^+) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Tor}^R_1(M, K)^+ & \longrightarrow & \mathrm{Tor}^R_2(M, F)^+ & \longrightarrow & 0. \end{array}$$

It follows that $\mathrm{EXT}^2_R(F, M^+) \cong \mathrm{Tor}^R_2(M, F)^+$ for any graded left R-module F. By using induction on n, one easily gets that $\mathrm{EXT}^n_R(N, M^+) \cong \mathrm{Tor}^R_n(M, N)^+$ for any graded left R-module N, and so the assertion follows. □

Proposition 3.9. *The category of n-gr-flat right R-modules is closed under gr-pure submodules and gr-pure quotients.*

Proof. Let B be an n-gr-flat right R-module and A a gr-pure submodule of B. Then there exists a gr-pure exact sequence $0 \to A \to B \to B/A \to 0$ in gr-R, which gives rise to a split exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$ in R-gr. By Proposition 3.8, B^+ is n-FP-gr-injective, and hence A^+ and $(B/A)^+$ is n-FP-gr-injective as a direct summand of B^+ by Proposition 3.7. Therefore, A and B/A is n-gr-flat by Proposition 3.8. □

Proposition 3.10. *Let R be a ring graded by a group G and $n \geq 1$ an integer. Then the following statements are equivalent for a graded left R-module M.*

(1) *M is n-FP-gr-injective.*
(2) *M is gr-injective with respect to all special short exact sequences in R-gr.*
(3) *$M(\sigma)$ is n-FP-gr-injective for any $\sigma \in G$.*
(4) *M is special gr-pure in any graded left R-module containing it.*

(5) *M is special gr-pure in any gr-injective left R-module containing it.*
(6) *M is special gr-pure in $E^g(M)$.*
(7) *$M(\sigma)$ is gr-injective with respect to all special short exact sequences in R-gr for any $\sigma \in G$.*

Proof. (1) $\Rightarrow$ (2) is trivial by Remark 3.5(3).

(2) $\Rightarrow$ (1) Let N be an n-presented graded left R-module. Taking a special short exact sequence $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ in R-gr with P_{n-1} finitely generated projective, which induces an exact sequence

$$\mathrm{HOM}_R(P_{n-1}, M) \to \mathrm{HOM}_R(K_n, M) \to \mathrm{EXT}_R^1(K_{n-1}, M) \to 0.$$

Note that $\mathrm{HOM}_R(P_{n-1}, M) \to \mathrm{HOM}_R(K_n, M) \to 0$ is exact by assumption, it follows that $\mathrm{EXT}_R^n(N, M) \cong \mathrm{EXT}_R^1(K_{n-1}, M) = 0$. Thus M is n-FP-gr-injective, as desired.

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(2) $\Rightarrow$ (3) Assume that N is an n-presented graded left R-module and $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ is a special short exact sequence in R-gr with P_{n-1} finitely generated projective. Then we get the exactness of

$$0 \to \mathrm{HOM}_R(K_{n-1}, M)_\sigma \to \mathrm{HOM}_R(P_{n-1}, M)_\sigma \to \mathrm{HOM}_R(K_n, M)_\sigma \to 0$$

for any $\sigma \in G$. Now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{HOM}_R(K_{n-1}, M)_{\sigma\tau} & \longrightarrow & \mathrm{HOM}_R(P_{n-1}, M)_{\sigma\tau} & \longrightarrow & \mathrm{HOM}_R(K_n, M)_{\sigma\tau} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(K_{n-1}, M(\sigma\tau)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(P_{n-1}, M(\sigma\tau)) & \longrightarrow & \mathrm{Hom}_{R\text{-gr}}(K_n, M(\sigma\tau)) & & \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{HOM}_R(K_{n-1}, M(\sigma))_\tau & \longrightarrow & \mathrm{HOM}_R(P_{n-1}, M(\sigma))_\tau & \longrightarrow & \mathrm{HOM}_R(K_n, M(\sigma))_\tau & &
\end{array}$$

with the upper row exact for any $\tau \in G$. It follows that

$$0 \to \mathrm{HOM}_R(K_{n-1}, M(\sigma))_\tau \to \mathrm{HOM}_R(P_{n-1}, M(\sigma))_\tau \to \mathrm{HOM}_R(K_n, M(\sigma))_\tau \to 0$$

is exact, which gives rise to the exactness of

$$0 \to \mathrm{HOM}_R(K_{n-1}, M(\sigma)) \to \mathrm{HOM}_R(P_{n-1}, M(\sigma)) \to \mathrm{HOM}_R(K_n, M(\sigma)) \to 0.$$

It follows that $\mathrm{EXT}_R^n(N, M(\sigma)) \cong \mathrm{EXT}_R^1(K_{n-1}, M(\sigma)) = 0$ and $M(\sigma)$ is n-FP-gr-injective for any $\sigma \in G$.

(1) $\Rightarrow$ (4) Let N be an n-presented graded left R-module and K_{n-1} a special finitely presented graded left R-module. Then $\mathrm{EXT}_R^1(K_{n-1}, M) \cong \mathrm{EXT}_R^n(N, M) = 0$ by assumption. Now suppose $(\dagger) : 0 \to M \to Q \to Q/M \to 0$ is an exact sequence in R-gr. Applying the functor $\mathrm{HOM}_R(K_{n-1}, -)$ to the sequence $(\dagger)$, one gets the following exact sequence

$$0 \to \mathrm{HOM}_R(K_{n-1}, M) \to \mathrm{HOM}_R(K_{n-1}, Q) \to \mathrm{HOM}_R(K_{n-1}, Q/M) \to 0.$$

Thus M is special gr-pure in Q and (4) follows.

(6) ⇒ (1) Let N be an n-presented grade left R-module and K_{n-1} a special finitely presented graded left R-module. Then

$$0 \to \mathrm{HOM}_R(K_{n-1}, M) \to \mathrm{HOM}_R(K_{n-1}, E^g(M)) \to \mathrm{HOM}_R(K_{n-1}, E^g(M)/M) \to 0$$

is exact by (6). So $\mathrm{EXT}^n_R(N, M) \cong \mathrm{EXT}^1_R(K_{n-1}, M) = 0$ and M is n-FP-gr-injective.

(2) ⇔ (7) It follows directly from the isomorphism: $\mathrm{HOM}_R(-, M)_\sigma \cong \mathrm{Hom}_{R\text{-gr}}(-, M(\sigma))$ for any $\sigma \in G$. □

Proposition 3.11. *The category of n-FP-gr-injective left R-modules is closed under gr-pure submodules.*

Proof. Let A be an n-FP-gr-injective module and A_1 a gr-pure submodule of A. Then we have a gr-pure short exact sequence $0 \to A_1 \to A \to A/A_1 \to 0$. Moreover, for any n-presented graded left R-module N, take a special short exact sequence

$$0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$$

in R-gr with P_{n-1} finitely generated projective. Consider the following commutative diagram with rows and columns exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{HOM}_R(K_{n-1}, A_1) & \to & \mathrm{HOM}_R(K_{n-1}, A) & \to & \mathrm{HOM}_R(K_{n-1}, A/A_1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{HOM}_R(P_{n-1}, A_1) & \to & \mathrm{HOM}_R(P_{n-1}, A) & \to & \mathrm{HOM}_R(P_{n-1}, A/A_1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{HOM}_R(K_n, A_1) & \to & \mathrm{HOM}_R(K_n, A) & \to & \mathrm{HOM}_R(K_n, A/A_1) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0. & & & &
\end{array}$$

Here, the upper row is exact since K_{n-1} is finitely presented, and the middle column is exact since A is an n-FP-gr-injective module. By the snake lemma, we have that the sequence

$$0 \to \mathrm{HOM}_R(K_{n-1}, A_1) \to \mathrm{HOM}_R(P_{n-1}, A_1) \to \mathrm{HOM}_R(K_n, A_1) \to 0$$

is exact. It follows that $\mathrm{EXT}^1_R(K_{n-1}, A_1) = 0$. Thus A_1 is an n-FP-gr-injective module by Remark 3.5(3). □

Let R and S be graded rings of type G. If $A \in R$-gr, $B \in R$-gr-S and $C \in$ gr-S with A finitely presented and C gr-injective, then there exists a natural isomorphism ([1, Lemma 2.3]):

$$\mathrm{HOM}_S(B, C) \otimes_R A \cong \mathrm{HOM}_S(\mathrm{HOM}_R(A, B), C) \tag{3.1}$$

Thus, if A is an n-presented graded left R-module and taking an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to A \to 0$$

in R-gr with each P_i finitely generated projective, then we get the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{HOM}_S(B,C) \otimes_R P_n & \longrightarrow & \mathrm{HOM}_S(B,C) \otimes_R P_{n-1} & \longrightarrow \cdots \longrightarrow & \mathrm{HOM}_S(B,C) \otimes_R P_0 & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\mathrm{HOM}_S(\mathrm{HOM}_R(P_n,B),C) & \longrightarrow & \mathrm{HOM}_S(\mathrm{HOM}_R(P_{n-1},B),C) & \longrightarrow \cdots \longrightarrow & \mathrm{HOM}_S(\mathrm{HOM}_R(P_0,B),C) & \longrightarrow & 0.
\end{array}$$

Moreover, if C is gr-injective, then the functor $\mathrm{HOM}_S(-,C)$ is exact, and hence we get the isomorphism

$$\mathrm{Tor}_i^R(\mathrm{HOM}_S(B,C),A) \cong \mathrm{HOM}_S(\mathrm{EXT}_R^i(A,B),C) \tag{3.2}$$

for any $0 \leq i \leq n-1$, and an exact sequence

$$\mathrm{Tor}_n^R(\mathrm{HOM}_S(B,C),A) \to \mathrm{HOM}_S(\mathrm{EXT}_R^n(A,B),C) \to 0.$$

Proposition 3.12. *Let $M \in R$-gr. If M^+ is n-gr-flat, then M is n-FP-gr-injective.*

Proof. Let N be an n-presented graded left R-module. By the above argument, one gets the exact sequence $\mathrm{Tor}_n^R(M^+,N) \to \mathrm{EXT}_R^n(N,M)^+ \to 0$. Since M^+ is n-gr-flat, we have $\mathrm{Tor}_n^R(M^+,N) = 0$, and so $\mathrm{EXT}_R^n(N,M) = 0$, which shows that M is n-FP-gr-injective. □

Proposition 3.13. *Let F be an n-present graded left R-module and $n \geq 1$ an integer, and let $\{M_i\}_{i\in I}$ be a direct system of graded left R-modules with I directed. Then we have*

(1) *There is the isomorphism $\varinjlim \mathrm{EXT}_R^j(F,M_i) \cong \mathrm{EXT}_R^j(F,\varinjlim M_i)$ for any $0 \leq j \leq n-1$.*
(2) *There is an exact sequence $0 \to \varinjlim \mathrm{EXT}_R^n(F,M_i) \to \mathrm{EXT}_R^n(F,\varinjlim M_i)$.*

Proof. If F is an n-presented graded left R-module, then there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

in R-gr with each P_i finitely generated projective. Thus, we get the following commutative diagram:

$$\begin{array}{ccccc}
0 \longrightarrow \varinjlim \mathrm{HOM}_R(P_0,M_i) & \longrightarrow \cdots \longrightarrow & \varinjlim \mathrm{HOM}_R(P_{n-1},M_i) & \longrightarrow & \varinjlim \mathrm{HOM}_R(P_n,M_i) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 \longrightarrow \mathrm{HOM}_R(P_0,\varinjlim M_i) & \longrightarrow \cdots \longrightarrow & \mathrm{HOM}_R(P_{n-1},\varinjlim M_i) & \longrightarrow & \mathrm{HOM}_R(P_n,\varinjlim M_i.)
\end{array}$$

Note that $\{M_i\}_{i\in I}$ is a direct system of graded left R-modules with I directed and the functor $\varinjlim$ is exact. So the assertion follows. □

Similarly, we have the following.

Proposition 3.14. *Let F be an n-present graded left R-module and $n \geq 1$ an integer, and let $\{M_i\}_{i\in I}$ be a family of graded right R-modules. Then we have*

(1) *There is the isomorphism* $\mathrm{Tor}_j^R(\prod_{i\in I}^{\text{gr-}R} M_i, F) \cong \prod_{i\in I} \mathrm{Tor}_j^R(M_i, F)$ *for any* $0 \leq j \leq n-1$.
(2) *There is an exact sequence* $\mathrm{Tor}_n^R(\prod_{i\in I}^{\text{gr-}R} M_i, F) \to \prod_{i\in I} \mathrm{Tor}_n^R(M_i, F) \to 0$.

By using an argument similar to the ungraded case, we have

Lemma 3.15. *The following statements are equivalent for a finitely generated graded left R-module A.*

(1) *A is finitely presented.*
(2) $(\prod_{i\in I}^{R\text{-gr}} L_i) \otimes_R A \cong \prod_{i\in I}(L_i \otimes_R A)$ *for any class of graded right R-modules* $\{L_i\}_{i\in I}$.
(3) $(\prod_{i\in I}^{R\text{-gr}} R) \otimes_R A \cong A^I$ *for any set I.*

Proposition 3.16. *Let R be left n-gr-coherent. Then the category of n-FP-gr-injective left R-modules is closed under direct sums.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of n-FP-gr-injective left R-modules and F an n-presented graded left R-module. Then there exists a special short exact sequence

$$0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$$

in R-gr. Since R is left n-gr-coherent, we have that F is $(n+1)$-presented and K_n is also special finitely presented. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{HOM}_R(K_{n-1}, \bigoplus_{i\in I} M_i) & \to & \mathrm{HOM}_R(P_{n-1}, \bigoplus_{i\in I} M_i) & \to & \mathrm{HOM}_R(K_n, \bigoplus_{i\in I} M_i) & \\
 & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \to & \bigoplus_{i\in I} \mathrm{HOM}_R(K_{n-1}, M_i) & \to & \bigoplus_{i\in I} \mathrm{HOM}_R(P_{n-1}, M_i) & \to & \bigoplus_{i\in I} \mathrm{HOM}_R(K_n, M_i) & \to 0
\end{array}$$

with the lower row exact. So the upper row exact, and hence $\mathrm{EXT}_R^1(K_{n-1}, \bigoplus_{i\in I} M_i) = 0$. It follows that $\bigoplus_{i\in I} M_i$ is n-FP-gr-injective by Remark 3.5(3). □

We now are in a position to give the main result in this section.

Theorem 3.17. *Let $n \geq 1$ be an integer. Then the following statements are equivalent for a graded ring R.*

(1) *R is left n-gr-coherent.*
(2) *Any graded direct product of R is an n-gr-flat right R-module.*
(3) *Any graded direct product of n-gr-flat right R-modules is n-gr-flat.*
(4) *Any direct limit of n-FP-gr-injective left R-modules is n-FP-gr-injective.*

(5) $\mathrm{Tor}_n^R(\mathrm{HOM}_S(B,C),A) \to \mathrm{HOM}_S(\mathrm{EXT}_R^n(A,B),C)$ *is an isomorphism for any graded ring* S *with* $A \in R\text{-gr}$ *n-presented,* $B \in R\text{-gr-}S$ *and* $C \in \text{gr-}S$ *gr-injective.*

(6) $\varinjlim \mathrm{EXT}_R^n(F,M_i) \to \mathrm{EXT}_R^n(F,\varinjlim M_i)$ *is an isomorphism for any n-present graded left R-module F and any direct system* $\{M_i\}_{i\in I}$ *of graded left R-modules with I directed.*

(7) $\mathrm{Tor}_n^R(\prod_{i\in I}^{\text{gr-}R} M_i, F) \to \prod_{i\in I} \mathrm{Tor}_n^R(M_i,F)$ *is an isomorphism for any n-present graded left R-module F and any family of graded right R-modules* $\{M_i\}_{i\in I}$.

(8) $M \in R\text{-gr}$ *is n-FP-gr-injective if and only if* M^+ *is n-gr-flat.*

(9) $M \in R\text{-gr}$ *is n-FP-gr-injective if and only if* M^{++} *is n-FP-gr-injective.*

(10) $M \in \text{gr-}R$ *is n-gr-flat if and only if* M^{++} *is n-gr-flat.*

Proof. (1) $\Rightarrow$ (5) Let $A \in R\text{-gr}$ be n-presented. Then A is $(n+1)$-presented since R is left n-gr-coherent. It follows that $\mathrm{Tor}_n^R(\mathrm{HOM}_S(B,C),A) \cong \mathrm{HOM}_S(\mathrm{EXT}_R^n(A,B),C)$ by (3.2).

(5) $\Rightarrow$ (8) If M^+ is n-gr-flat in gr-R, then M is n-FP-gr-injective by Proposition 3.12. Conversely, let $S=\mathbb{Z}$, $C=\mathbb{Q}/\mathbb{Z}$ and $B=M$. Then by assumption, we have $\mathrm{Tor}_n^R(M^+,A) \cong \mathrm{EXT}_R^n(A,M)^+$ for any n-present graded left R-module A. Thus, if M is n-FP-gr-injective, then $\mathrm{EXT}_R^n(A,M)=0$. So $\mathrm{Tor}_n^R(M^+,A)=0$ and M^+ is n-gr-flat, as desired.

(8) $\Rightarrow$ (9) If $M \in R\text{-gr}$ is n-FP-gr-injective, then M^+ is n-gr-flat by assumption, and so M^{++} is n-FP-gr-injective by Proposition 3.8. Conversely, if M^{++} is n-FP-gr-injective, then we have a gr-pure exact sequence $0 \to M \to M^{++} \to M^{++}/M \to 0$ by [2, Lemma 2.3]. Thus M is n-FP-gr-injective by Proposition 3.11.

(9) $\Rightarrow$ (10) Let $M \in \text{gr-}R$ be n-gr-flat. Then M^+ is n-FP-gr-injective by Proposition 3.8. So M^{+++} is n-FP-gr-injective by assumption. By Proposition 3.8 again, we get that M^{++} is n-gr-flat. Conversely, if M^{++} is n-gr-flat, then M, as a gr-pure submodule of M^{++}, is n-gr-flat by Proposition 3.9.

(10) $\Rightarrow$ (3) Let $\{M_i\}_{i\in I}$ be a family of n-gr-flat right R-modules. By Proposition 3.7(2), we have that $\bigoplus_{i\in I} M_i$ is n-gr-flat. Then $(\bigoplus_{i\in I} M_i)^{++}$ is n-gr-flat by assumption. Now by the isomorphism $(\bigoplus_{i\in I} M_i)^+ \cong \prod_{i\in I}^{R\text{-gr}} M_i^+$, we have that $(\prod_{i\in I}^{R\text{-gr}} M_i^+)^+ \cong (\bigoplus_{i\in I} M_i)^{++}$ is n-gr-flat. On the other hand, since $\bigoplus_{i\in I} M_i{}^+$ is gr-pure in $\prod_{i\in I}^{R\text{-gr}} M_i{}^+$, it follows that $(\bigoplus_{i\in I} M_i{}^+)^+$ is a graded direct summand of $(\prod_{i\in I}^{R\text{-gr}} M_i{}^+)^+$, and so $(\bigoplus_{i\in I} M_i{}^+)^+$ is n-gr-flat. From the isomorphism $\prod_{i\in I}^{\text{gr-}R} M_i^{++} \cong (\bigoplus_{i\in I} M_i^+)^+$, we have that $\prod_{i\in I}^{\text{gr-}R} M_i^{++}$ is n-gr-flat. By [2, Lemma 2.3], we have that M_i is gr-pure in M_i^{++} and $\prod_{i\in I}^{\text{gr-}R} M_i$ is gr-pure in $\prod_{i\in I}^{\text{gr-}R} M_i^{++}$. Therefore $\prod_{i\in I}^{\text{gr-}R} M_i$ is n-gr-flat by Proposition 3.9. The assertion follows.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) We will show that any n-presented graded left R-module is $(n+1)$-presented. Let F be an n-presented graded left R-module and $0 \to K_n \to P_{n-1} \to$

$K_{n-1} \to 0$ a special short exact sequence in R-gr. Then $\mathrm{Tor}_1^R(\prod_{i\in I}^{R\text{-gr}} R, K_{n-1}) \cong \mathrm{Tor}_n^R(\prod_{i\in I}^{R\text{-gr}} R, F) = 0$ by the dimension shifting. Note that $\prod_{i\in I}^{R\text{-gr}} R$ is an n-gr-flat right R-module by assumption. So, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\prod_{i\in I}^{R\text{-gr}} R)\otimes_R K_n & \longrightarrow & (\prod_{i\in I}^{R\text{-gr}} R)\otimes_R P_{n-1} & \longrightarrow & (\prod_{i\in I}^{R\text{-gr}} R)\otimes_R K_{n-1} & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & K_n^I & \longrightarrow & P_{n-1}^I & \longrightarrow & K_{n-1}^I & \longrightarrow & 0. \end{array}$$

Because K_{n-1} and P_{n-1} are both finitely presented, we have that g and h are isomorphisms by Lemma 3.15. So f is an isomorphism by the five lemma. By Lemma 3.15 again, we get that K_n is finitely presented. The assertion follows:

(1) $\Rightarrow$ (6) Let $A \in R$-gr be n-presented. Since R is left n-gr-coherent, we have that A is $(n+1)$-presented. Now the assertion follows from Proposition 3.13(1).

(6) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Let F be an n-presented left R-module and $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ a special short exact sequence. It suffices to show that the special finitely generated graded left R-module K_n is finitely presented. Suppose that $\{E_i\}_{i\in I}$ is a direct system of gr-injective left R-modules. Then each E_i is n-FP-gr-injective, and hence $\varinjlim E_i$ is n-FP-gr-injective by (4). Thus $\mathrm{EXT}_R^1(K_{n-1}, \varinjlim E_i) \cong \mathrm{EXT}_R^n(F, \varinjlim E_i) = 0$ and we get a commutative diagram as follows:

$$\begin{array}{ccc} \mathrm{HOM}_R(K_{n-1}, \varinjlim E_i) & \xrightarrow{f} & \varinjlim \mathrm{HOM}_R(K_{n-1}, E_i) \\ \downarrow & & \downarrow \\ \mathrm{HOM}_R(P_{n-1}, \varinjlim E_i) & \xrightarrow{g} & \varinjlim \mathrm{HOM}_R(P_{n-1}, E_i) \\ \downarrow & & \downarrow \\ \mathrm{HOM}_R(K_n, \varinjlim E_i) & \xrightarrow{h} & \varinjlim \mathrm{HOM}_R(K_n, E_i) \\ \downarrow & & \downarrow \\ 0 & & 0, \end{array}$$

where all columns are exact. Since K_{n-1} and P_{n-1} are both finitely presented and each E_i is gr-injective, we have that f and g are isomorphisms by [22, Lemma 3.1]. Thus h is an isomorphism, and therefore K_n is finitely presented by [22, Lemma 3.1] again.

(1) $\Rightarrow$ (7) Since any n-presented graded left R-module is $(n+1)$-presented over a left n-gr-coherent ring R, the assertion follows directly from Proposition 3.14(1).

(7) $\Rightarrow$ (3) is trivial. $\square$

4. Covers and Preenvelopes by n-FP-gr-Injective and n-gr-Flat Modules

In this section, all rings are assumed to be left n-gr-coherent rings. Holm and Jørgensen [15] introduced the notion of duality pairs for the category of ungraded modules, and the duality pairs play an important role in the aspect of showing the existence of covers and preenvelopes. Now we generalize this notion to the category of graded modules as follows.

Definition 4.1. A *duality pair* over a graded ring R is a pair $(\mathcal{M}, \mathcal{C})$, where $\mathcal{M}$ is a class of graded left (respectively, right) R-modules and $\mathcal{C}$ is a class of graded right (respectively, left) R-modules, subject to the following conditions:

(1) For any graded module M, one has $M \in \mathcal{M}$ if and only if $M^+ \in \mathcal{C}$.
(2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{C})$ is called *(co)product-closed* if the class of $\mathcal{M}$ is closed under graded direct (co)products; and a duality pair $(\mathcal{M}, \mathcal{C})$ is called *perfect* if it is coproduct-closed, $\mathcal{M}$ is closed under extensions and R belongs to $\mathcal{M}$.

The following theorem is the graded version of [15, Theorem 3.1].

Theorem 4.2. *Let $(\mathcal{M}, \mathcal{C})$ be a duality pair. Then $\mathcal{M}$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. Furthermore, the following hold:*

(1) *If $(\mathcal{M}, \mathcal{C})$ is product-closed, then $\mathcal{M}$ is preenveloping.*
(2) *If $(\mathcal{M}, \mathcal{C})$ is coproduct-closed, then $\mathcal{M}$ is covering.*
(3) *If $(\mathcal{M}, \mathcal{C})$ is perfect, then $(\mathcal{M}, \mathcal{M}^+)$ is a perfect cotorsion pair.*

Proof. First we will prove that $\mathcal{M}$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions, that is, given a gr-pure exact sequence $0 \to M' \to M \to M'' \to 0$ of graded R-modules, then $M \in \mathcal{M}$ if and only if $M', M'' \in \mathcal{M}$. Applying the functor $\mathrm{HOM}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the above sequence, we get a split exact sequence $0 \to M''^+ \to M^+ \to M'^+ \to 0$ by [2, Proposition 2.2]. By Definition 4.1(2), it follows that $M^+ \in \mathcal{C}$ if and only if $M'^+, M''^+ \in \mathcal{C}$. So $M \in \mathcal{M}$ if and only if $M', M'' \in \mathcal{M}$ by Definition 4.1(1).

Similar to the proofs of the parts (1), (2) and (3) in [15, Theorem 3.1], we obtain easily the rest of the desired results. □

Proposition 4.3. *The pair* (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) *is a duality pair.*

Proof. For any $M \in$ gr-R, we have that $M \in$ gr-$\mathcal{F}_n$ if and only if $M^+ \in$ gr-$\mathcal{FI}_n$ by Proposition 3.8. Also, the category gr-$\mathcal{FI}_n$ is closed under direct summands by Proposition 3.7(1), and is closed under direct sums by Proposition 3.16. So the assertion follows. □

Theorem 4.4. *The category* gr-$\mathcal{F}_n$ *is covering and preenveloping.*

Proof. Since the category gr-$\mathcal{F}_n$ of all n-gr-flat right R-modules is closed under direct sums by Proposition 3.7(2), the duality pair (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) is coproduct-closed. By Theorem 4.2(2), we have that gr-$\mathcal{F}_n$ is covering.

Since any graded direct product of n-gr-flat modules is n-gr-flat by Theorem 3.17, it follows that the duality pair (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) is product-closed. Thus, the category gr-$\mathcal{F}_n$ is preenveloping by Theorem 4.2(1). □

Now we turn to discuss the pair (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$).

Proposition 4.5. *The pair* (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) *is a duality pair.*

Proof. By Theorem 3.17, we have that $M \in$ gr-$\mathcal{FI}_n$ if and only if $M^+ \in$ gr-$\mathcal{F}_n$ for any $M \in R$-gr. Also, the category gr-$\mathcal{F}_n$ is closed under direct summands and direct sums by Proposition 3.7(2). Thus, the pair (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) is a duality pair. □

Theorem 4.6. *The category* gr-$\mathcal{FI}_n$ *is covering and preenveloping.*

Proof. By Proposition 3.16, the category gr-$\mathcal{FI}_n$ of all n-FP-gr-injective left R-modules is closed under direct sums. It follows that the duality pair (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) is coproduct-closed. So gr-$\mathcal{FI}_n$ is covering by Theorem 4.2(2).

Since any graded direct product of n-FP-gr-injective left R-modules is n-FP-gr-injective, the duality pair (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) is product-closed. By Theorem 4.2(1), we have that gr-$\mathcal{FI}_n$ is preenveloping. □

Remark 4.7. By Propositions 3.9 and 3.11, we have that gr-$\mathcal{F}_n$ is closed under gr-pure submodules and gr-pure quotients and gr-$\mathcal{FI}_n$ is closed under gr-pure submodules. We point out that if R is a left n-gr-coherent ring, then we may directly obtain the same results by Propositions 4.3, 4.5 and Theorem 4.2; moreover, gr-$\mathcal{FI}_n$ is also closed under gr-pure quotients.

Now we give some equivalent characterizations for ${}_RR$ being n-FP-gr-injective in terms of the properties of n-FP-gr-injective and n-gr-flat modules.

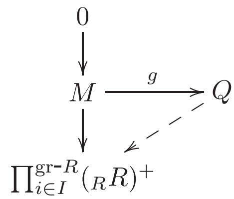
Theorem 4.8. *The following statements are equivalent*:

(1) *${}_RR$ is n-FP-gr-injective.*
(2) *Every graded module in* gr-R *has a monic n-gr-flat preenvelope.*
(3) *Every gr-injective module in* gr-R *is n-gr-flat.*
(4) *Every graded flat module in* R-gr *is n-FP-gr-injective.*
(5) *Every graded projective module in* R-gr *is n-FP-gr-injective.*
(6) *Every graded module in* R-gr *has an epic n-FP-gr-injective cover.*

Proof. (4) ⇒ (5) ⇒ (1) and (6) ⇒ (1) are trivial.

(1) ⇒ (2) Let M be a graded right R-module. Then M has an n-gr-flat preenvelope $g : M \to Q$ by Theorem 4.4. Since $({}_RR)^+$ is a cogenerator in gr-R, there exists

an exact sequence $0 \to M \to \prod_{i\in I}^{\text{gr-}R}({}_RR)^+$. Since ${}_RR$ is n-FP-gr-injective in R-gr by assumption, we have that $({}_RR)^+$ is n-gr-flat in gr-R by Theorem 3.17 since R is left n-gr-coherent. It follows that $\prod_{i\in I}^{\text{gr-}R}({}_RR)^+$ is n-gr-flat. Consider the following commutative diagram:



One gets that the n-gr-flat preenvelope $g : M \to Q$ is monic.

(2) $\Rightarrow$ (3) If E is a gr-injective right R-module, then E has a monic n-gr-flat preenvelope $0 \to E \to Q$ by assumption. Since E is gr-injective, we have that E, as a direct summand of Q, is n-gr-flat by Proposition 3.7(2).

(3) $\Rightarrow$ (4) Let M be a flat graded left R-module. Then M^+ is gr-injective, and hence M^+ is n-gr-flat by assumption. It follows that M is n-FP-gr-injective by Proposition 3.12, as desired.

(1) $\Rightarrow$ (6) Let M be a graded left R-module. Then M has an n-FP-gr-injective cover $f : E \to M$ by Theorem 4.4. On the other hand, there exists an exact sequence $\bigoplus_{\sigma\in S} R(\sigma) \to M \to 0$ for some $S \subseteq G$. Since $R(\sigma)$ is n-FP-gr-injective by assumption, we have that $\bigoplus_{\sigma\in S} R(\sigma)$ is n-FP-gr-injective by Proposition 3.16. Thus f is epic. □

In the following, we consider cotorsion pairs associated to n-FP-gr-injective and n-gr-flat modules.

Definition 4.9 ([12]). A pair $(\mathcal{F}, \mathcal{C})$ of classes of graded left R-modules is called a *cotorsion pair* in R-gr if the following properties are satisfied.

(1) $\operatorname{Ext}^1_{R\text{-gr}}(F, C) = 0$ for every $F \in \mathcal{F}, C \in \mathcal{C}$.
(2) $\operatorname{Ext}^1_{R\text{-gr}}(F, C) = 0$ for every $F \in \mathcal{F}$, implies $C \in \mathcal{C}$.
(3) $\operatorname{Ext}^1_{R\text{-gr}}(F, C) = 0$ for every $C \in \mathcal{C}$, implies $F \in \mathcal{F}$.

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be *perfect* if any graded left R-module has an $\mathcal{F}$-cover and a $\mathcal{C}$-envelope; a cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called *hereditary* if whenever $0 \to F' \to F \to F'' \to 0$ is exact in R-gr with $F, F'' \in \mathcal{F}$, then F' is also in $\mathcal{F}$; and a cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called *complete* provided that for any graded R-module M, there exist exact sequences $0 \to M \to C \to D \to 0$ and $0 \to C' \to D' \to M \to 0$ of graded R-modules with $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{F}$.

In [9], Eklof and Trlifaj proved that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod is complete when it is cogenerated by a set. This result actually holds in any Grothendieck category with enough projectives, as Hovey proved in [16].

Proposition 4.10. (1) *If R is a self n-FP-gr-injective ring, then the pair* $(\mathrm{gr}\text{-}\mathcal{FI}_n, (\mathrm{gr}\text{-}\mathcal{FI}_n)^+)$ *is a perfect cotorsion pair.*

(2) *The pair* $({}^{\perp}\mathrm{gr}\text{-}\mathcal{FI}_n, \mathrm{gr}\text{-}\mathcal{FI}_n)$ *is a hereditary cotorsion pair.*

Proof. (1) By Proposition 4.5, we have that (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) is a duality pair. Note that R belongs to gr-$\mathcal{FI}_n$ by assumption. Since the category gr-$\mathcal{FI}_n$ is closed under extensions by definition, and is closed under direct sums by Proposition 3.16, we have that (gr-$\mathcal{FI}_n$, gr-$\mathcal{F}_n$) is a perfect duality pair. So (gr-$\mathcal{FI}_n$, $(\text{gr-}\mathcal{FI}_n)^+$) is a perfect cotorsion pair by Theorem 4.2(3).

(2) For any $X \in {}^{\perp}(\text{gr-}\mathcal{FI}_n)$, we have $X(\sigma) \in {}^{\perp}(\text{gr-}\mathcal{FI}_n)$ for any $\sigma \in G$. Suppose that $M \in ({}^{\perp}(\text{gr-}\mathcal{FI}_n))^{\perp}$ and $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ is a special short exact sequence in R-gr with respect to an n-presented graded left R-module N. Then $K_{n-1} \in {}^{\perp}(\text{gr-}\mathcal{FI}_n)$ and $M(\sigma) \in ({}^{\perp}(\text{gr-}\mathcal{FI}_n))^{\perp}$ for any $\sigma \in G$. It follows that

$$\mathrm{EXT}^1_R(K_{n-1}, M)_\sigma \cong \mathrm{Ext}^1_{R\text{-}gr}(K_{n-1}, M(\sigma)) = 0.$$

So $\mathrm{EXT}^1_R(K_{n-1}, M) = 0$, and hence $M \in \text{gr-}\mathcal{FI}_n$ by Remark 3.2(3).

Now suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in R-gr with $M, M' \in \text{gr-}\mathcal{FI}_n$ and F is an n-presented graded left R-module. Then F is $(n+1)$-presented since R is left n-gr-coherent. So we have the following exact sequence:

$$\mathrm{EXT}^n_R(F, M) \to \mathrm{EXT}^n_R(F, M'') \to \mathrm{EXT}^{n+1}_R(F, M').$$

It is clear that $\mathrm{EXT}^n_R(F, M) = 0$, and $\mathrm{EXT}^{n+1}_R(F, M') = 0$ since M' is $(n+1)$-FP-gr-injective by Remark 3.5(2). So we have that $\mathrm{EXT}^n_R(F, M'') = 0$ for any n-presented graded left R-module F and M'' is n-FP-gr-injective. Consequently, $({}^{\perp}(\text{gr-}\mathcal{FI}_n), \text{gr-}\mathcal{FI}_n)$ is a hereditary cotorsion pair. □

Proposition 4.11. *The pair* $(gr\text{-}\mathcal{F}_n,\ (gr\text{-}\mathcal{F}_n)^{\perp})$ *is a hereditary perfect cotorsion pair.*

Proof. By Proposition 4.3, we have that (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) is a duality pair. Note that the category gr-$\mathcal{F}_n$ is closed under extensions and direct sums and R belongs to gr-$\mathcal{F}_n$. So (gr-$\mathcal{F}_n$, gr-$\mathcal{FI}_n$) is a perfect duality pair, and hence (gr-$\mathcal{F}_n$, $(\text{gr-}\mathcal{F}_n)^{\perp}$) is a perfect cotorsion pair by Theorem 4.2(3).

On the other hand, one checks readily that if there exists an exact sequence $0 \to A \to B \to C \to 0$ in gr-R with B, C n-gr-flat, then A is also n-gr-flat. Thus, the pair (gr-$\mathcal{F}_n$, $(\text{gr-}\mathcal{F}_n)^{\perp}$) is a hereditary cotorsion pair. □

Remark 4.12. (1) Proposition 4.10(2) shows that the pair $({}^{\perp}(\text{gr-}\mathcal{FI}_n), \text{gr-}\mathcal{FI}_n)$ is a hereditary cotorsion pair. By $\mathcal{SFP}^g$ we denote the subcategory of all the special finitely presented graded left R-modules. We point out that the pair $({}^{\perp}(\text{gr-}\mathcal{FI}_n), \text{gr-}\mathcal{FI}_n)$ is also a complete cotorsion pair since it is cogenerated by a set of representatives for $\mathcal{SFP}^g$ in R-gr.

(2) Let $1 \leq n \leq m$ be integers. If we denote by gr-Pres_n the class of all n-presented graded left R-modules, then clearly

$$\text{gr-Pres}_n \subseteq \text{gr-Pres}_{n+1} \Rightarrow \text{gr-Pres}_m \subseteq \text{gr-Pres}_{m+1},$$

that is, each left n-gr-coherent ring is left m-gr-coherent. In addition, given two cotorsion pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{C}, \mathcal{D})$, we write $(\mathcal{A}, \mathcal{B}) \preceq (\mathcal{C}, \mathcal{D})$ if $\mathcal{B} \subseteq \mathcal{D}$. Therefore, in our setting, we get a series of cotorsion pairs as follows:

(i) there are the following hereditary complete cotorsion pairs:

$$\begin{aligned}({}^{\perp}(\text{gr-}\mathcal{FI}_n), \text{gr-}\mathcal{FI}_n) \preceq \cdots \preceq ({}^{\perp}(\text{gr-}\mathcal{FI}_m), \text{gr-}\mathcal{FI}_m)\\ \preceq ({}^{\perp}(\text{gr-}\mathcal{FI}_{m+1}), \text{gr-}\mathcal{FI}_{m+1}) \preceq \cdots\end{aligned}$$

(ii) there are the following hereditary perfect cotorsion pairs:

$$\begin{aligned}(\text{gr-}\mathcal{F}_n, (\text{gr-}\mathcal{F}_n)^{\perp}) \succeq \cdots \succeq (\text{gr-}\mathcal{F}_m, (\text{gr-}\mathcal{F}_m)^{\perp})\\ \succeq (\text{gr-}\mathcal{F}_{m+1}, (\text{gr-}\mathcal{F}_{m+1})^{\perp}) \succeq \cdots.\end{aligned}$$

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References

[1] M. J. Asensio, J. A. López-Ramos and B. Torrecillas, Gorenstein gr-injective and gr-projective modules, *Comm. Algebra* **26** (1998) 225–240.

[2] M. J. Asensio, J. A. López-Ramos and B. Torrecillas, FP-gr-injective modules and gr-FC rings, *Algebra and Number Theory* (*Fez*), Lecture Notes in *Pure and Appl. Math.* Vol. 208 (Dekker, New York, 2000), pp. 1–11.

[3] M. J. Asensio, J. A. López-Ramos and B. Torrecillas, Covers and envelopes over gr-Gorenstein rings, *J. Algebra* **215** (1999) 437–459.

[4] D. Bennis and N. Mahdou, First, second, and third change of rings theorems for Gorenstein homological dimensions, *Comm. Algebra* **38**(10) (2010) 3837–3850.

[5] D. Bravo, J. Gillespie and M. Hovey, The stable module category of a general ring, arXiv:1405.5768v1, 2014.

[6] D. Bravo and M. A. Pérez, Finiteness conditions and cotorsion pairs, *J. Pure Appl. Algebra* **221**(6) (2017) 1249–1267.

[7] J. L. Chen and N. Q. Ding, On n-coherent rings, *Comm. Algebra* **24** (1996) 3211–3216.

[8] D. L. Costa, Parameterizing families of non-noetherian rings, *Comm. Algebra* **22** (1994) 3997–4011.

[9] P. C. Eklof and J. Trlifaj, How to make Ext vanish, *Bull. London Math. Soc.* **33** (2001) 41–51.
[10] E. E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981) 189–209.
[11] E. E. Enochs and J. A. López-Ramos, Graded matlis duality and applications to covers, *Quaest. Math.* **24**(4) (2001) 555–564.
[12] J. R. García Rozas, J. A. López-Ramos and B. Torrecillas, On the existence of flat covers in R-gr, *Comm. Algebra* **29** (2001) 3341–3349.
[13] S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Math. Vol. 1371 (Springer-Verlag, Berlin, 1989).
[14] M. Hermann, S. Ikeda and U. Orbanz, *Equimultiplicity and Blowing Up* (Springer, New York, 1988).
[15] H. Holm and P. Jørgensen, Cotorsion pairs induced by duality pairs, *J. Commut. Algebra* **1** (2009) 621–633.
[16] M. Hovey, Cotorsion theories, model category structures and representation theory, *Math. Z.* **241** (2002) 553–592.
[17] L. Hummel and T. Marley, The Auslander–Bridger formula and the Gorenstein property for coherent rings, *J. Commut. Algebra* **1** (2009) 283–314.
[18] L. X. Mao, Relative homological groups over coherent rings, *Quaest. Math.* **35** (2012) 331–352.
[19] L. X. Mao, Strongly Gorenstein graded modules, *Front. Math. China* **12** (2017) 157–176.
[20] C. Năstăsescu and F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Math. Vol. 1836 (Springer-Verlag, Berlin, 2004).
[21] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.* **2** (1970) 323–329.
[22] X. Y. Yang and Z. K. Liu, FP-gr-injective modules, *Math. J. Okayama Univ.* **53** (2011) 83–100.