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τ-Tilting modules over triangular matrix artin algebras

Yeyang Peng
Department of Mathematics, Nanjing University
Nanjing 210093, Jiangsu Province, P. R. China
pengyy@smail.nju.edu.cn

Xin Ma
College of Science, Henan University of Engineering
Zhengzhou 451191, Henan Province, P. R. China
maxin0719@126.com

Zhaoyong Huang*
Department of Mathematics, Nanjing University
Nanjing 210093, Jiangsu Province, P. R. China
huangzy@nju.edu.cn

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Let Λ and Γ be artin algebras and $T := \left(\begin{smallmatrix}\Lambda & M\\ 0 & \Gamma\end{smallmatrix}\right)$ the triangular matrix algebra with M a finitely generated (Λ,Γ)-bimodule. We construct support τ-tilting modules and (τ-)tilting modules in $\mathrm{mod}\, T$ from that in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$, and give the converse constructions under some condition.

Keywords: Triangular matrix algebras; (support) τ-tilting modules; tilting modules; torsion pairs; functorially finite subcategories.

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1. Introduction

Tilting theory is very important in representation theory of artin algebras, in which tilting modules are fundamental, see [3, 8–11, 13, 14, 16, 17, 20] and references therein. A natural question is how to construct a new tilting module from a given one. Mutation of tilting modules is an effective way to do it. Happel and Unger gave some necessary and sufficient conditions such that mutation of tilting modules is

*Corresponding author.

possible [14]; however, mutation of tilting modules is impossible in general. Recently, Adachi *et al.* [1] introduced (support) τ-tilting modules which are a generalization of tilting modules. They showed that mutation of support τ-tilting modules is always possible; and also established a bijection between functorially finite torsion classes and support τ-tilting modules as well as a bijection between sincere functorially finite torsion classes and τ-tilting modules.

Let Λ and Γ be artin algebras and M a finitely generated (Λ, Γ)-bimodule, and let $T := \left(\begin{smallmatrix} \Lambda & M \\ 0 & \Gamma \end{smallmatrix}\right)$ be the triangular matrix algebra. Smalø [18] showed that a subcategory $\left(\begin{smallmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{smallmatrix}\right)$ of $\operatorname{mod} T$ is functorially finite if and only if $\mathcal{T}_1$ and $\mathcal{T}_2$ are functorially finite in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. Let T be the one-point extension of a finite-dimensional algebra Λ by a projective Λ-module, Assem *et al.* [2] showed how to construct in a natural way a tilting T-module from a tilting Λ-module; conversely, given a tilting T-module they constructed a tilting Λ-module. Motivated by this work, Suarez [19] showed that the extension of a given support τ-tilting Λ-module by some simple module is a support τ-tilting T-module; conversely, the restriction of a given support τ-tilting T-module is a support τ-tilting Λ-module. In addition, Chen *et al.* [11] studied how to lift tilting modules over an arbitrary ring to its trivial extension ring, and how to construct tilting modules over triangular matrix rings. Based on these works, we will investigate the relationship between (support) τ-tilting modules in $\operatorname{mod} T$ and that in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$. The paper is organized as follows.

In Sec. 2, we give some terminology and preliminary results. In Sec. 3, we construct torsion pairs in $\operatorname{mod} T$ from that in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ as well as the converse construction. In Sec. 4, we construct support τ-tilting modules and (τ-)tilting modules in $\operatorname{mod} T$ from that in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ (Theorem 4.2), and give the converse constructions under some condition (Theorems 4.4 and 4.6). In particular, we give many examples to illustrate our results.

2. Preliminaries

For an artin algebra Λ, we use $\operatorname{mod}\Lambda$ to denote the category of finitely generated left Λ-modules. For a module $A \in \operatorname{mod}\Lambda$, we use $\operatorname{add}_\Lambda A$ to denote the subcategory of $\operatorname{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of A. Let Λ and Γ be artin algebras and M a (Λ, Γ)-bimodule such that ${}_\Lambda M$ and M_Γ are finitely generated, and let

$$T := \begin{pmatrix} \Lambda & M \\ 0 & \Gamma \end{pmatrix}$$

be the triangular matrix algebra with addition and multiplication given by the ones of matrices. All modules considered are finitely generated and all subcategories are full and closed under isomorphisms.

A module in $\operatorname{mod} T$ is identified with a triple $\binom{Y}{X}_\phi$, or simply $\binom{Y}{X}$ if ϕ is clear, where $Y \in \operatorname{mod}\Lambda$, $X \in \operatorname{mod}\Gamma$ and $\phi : M \otimes_\Gamma X \to Y$ is a Λ-map. A T-map $\binom{Y}{X}_\phi \to$

$\binom{Y'}{X'}_{\phi'}$ is identified with a pair $\binom{f}{g}$ where $f \in \mathrm{Hom}_\Lambda(Y, Y')$ and $g \in \mathrm{Hom}_\Gamma(X, X')$ such that the diagram

$$\begin{array}{ccc} M \otimes_\Gamma X & \xrightarrow{\phi} & Y \\ \downarrow{\scriptstyle 1\otimes g} & & \downarrow{\scriptstyle f} \\ M \otimes_\Gamma X' & \xrightarrow{\phi'} & Y' \end{array}$$

commutes. A sequence

$$0 \to \begin{pmatrix} Y_1 \\ X_1 \end{pmatrix} \xrightarrow{\binom{f_1}{g_1}} \begin{pmatrix} Y_2 \\ X_2 \end{pmatrix} \xrightarrow{\binom{f_2}{g_2}} \begin{pmatrix} Y_3 \\ X_3 \end{pmatrix} \to 0$$

in $\mathrm{mod}\, T$ is exact if and only if the following two sequences:

$$0 \to Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \to 0$$

in $\mathrm{mod}\, \Lambda$ and

$$0 \to X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3 \to 0$$

in $\mathrm{mod}\, \Gamma$ are exact. The indecomposable injective T-modules are exactly $\binom{I}{\mathrm{Hom}_\Lambda(M,I)}$ and $\binom{0}{J}$ where I runs over indecomposable injective Λ-modules and J runs over indecomposable injective Γ-modules. The indecomposable projective T-modules are exactly $\binom{M\otimes_\Gamma Q}{Q}$ and $\binom{P}{0}$ where P runs over indecomposable projective Λ-modules and Q runs over indecomposable projective Γ-modules. By the adjunction isomorphism, a left T module is also identified with a triple $\binom{Y}{X}_\varphi$, or simply $\binom{Y}{X}$ if φ is clear, where $Y \in \mathrm{mod}\, \Lambda$, $X \in \mathrm{mod}\, \Gamma$ and $\varphi : X \to \mathrm{Hom}_\Lambda(M, Y)$ is a Γ-map [5, p. 76].

Definition 2.1 ([4, 6]). Let $\mathcal{C}$ be a subcategory of an abelian category $\mathcal{A}$. A morphism $f_N : C_N \to N$ in $\mathcal{A}$ with $C_N \in \mathcal{C}$ is called a *right $\mathcal{C}$-approximation* of N if the induced map $\mathrm{Hom}_\mathcal{A}(C, f_N)$ is surjective for any $C \in \mathcal{C}$. The subcategory $\mathcal{C}$ is called *contravariantly finite* in $\mathcal{A}$ if any object in $\mathcal{A}$ admits a right $\mathcal{C}$-approximation. Dually, the notions of *left $\mathcal{C}$-approximations* and *covariantly finite subcategories* are defined. If $\mathcal{C}$ is both contravariantly and covariantly finite in $\mathcal{A}$, then it is called *functorially finite* in $\mathcal{A}$.

Let $\mathcal{T}_1$ be a subcategory of $\mathrm{mod}\, \Lambda$ and $\mathcal{T}_2$ a subcategory of $\mathrm{mod}\, \Gamma$. Let $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ denote the full subcategory of $\mathrm{mod}\, T$ consisting of the modules of the form $\binom{U}{V}_f$ with $U \in \mathcal{T}_1$, $V \in \mathcal{T}_2$ and $f \in \mathrm{Hom}_\Lambda(M \otimes_\Gamma V, U)$. Then we have the following.

Proposition 2.2 ([18, Theorem 2.1]). *Let $\mathcal{T}_1$ be a subcategory of $\mathrm{mod}\, \Lambda$ and $\mathcal{T}_2$ a subcategory of $\mathrm{mod}\, \Gamma$. The subcategory $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is functorially finite in $\mathrm{mod}\, T$ if and only if $\mathcal{T}_1$ is functorially finite in $\mathrm{mod}\, \Lambda$ and $\mathcal{T}_2$ is functorially finite in $\mathrm{mod}\, \Gamma$.*

For a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, we write

$$\mathcal{C}^{\perp} := \{X \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_{\Lambda}(\mathcal{C}, X) = 0\}, \quad {}^{\perp}\mathcal{C} := \{X \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_{\Lambda}(X, \mathcal{C}) = 0\}.$$

Definition 2.3 ([12]). A pair of subcategories $(\mathcal{X}, \mathcal{Y})$ of $\operatorname{mod}\Lambda$ is called a *torsion pair* if $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. In the situation, $\mathcal{X}$ is called the *torsion class* and $\mathcal{Y}$ is called the *torsion-free class*.

For a module $N \in \operatorname{mod}\Lambda$, $\operatorname{Fac}(N)$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of all factor modules of finite direct sums of copies of N and $\operatorname{Sub}(N)$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of all submodules of finite direct sums of copies of N.

Proposition 2.4 (17, Theorem; 1, Proposition 1.1). *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod}\Lambda$. Then the following statements are equivalent*:

(1) *$\mathcal{T}$ is functorially finite.*
(2) *$\mathcal{F}$ is functorially finite.*
(3) *$\mathcal{T} = \operatorname{Fac}(X)$ for some X in $\operatorname{mod}\Lambda$.*
(4) *$\mathcal{F} = \operatorname{Sub}(Y)$ for some Y in $\operatorname{mod}\Lambda$.*

In this case, the torsion pair $(\mathcal{T}, \mathcal{F})$ is said to be functorially finite.

For a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$, $\mathcal{X}$ is the torsion class of some torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{mod}\Lambda$ if and only if $\mathcal{X}$ is closed under images and extensions. Dually, for a subcategory $\mathcal{Y}$ of $\operatorname{mod}\Lambda$, $\mathcal{Y}$ is the torsion-free class of some torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{mod}\Lambda$ if and only if $\mathcal{Y}$ is closed under submodules and extensions. For a module $N \in \operatorname{mod}\Lambda$, $\operatorname{pd}_{\Lambda} N$ and $|N|$ are the projective dimension and the number of pairwise non-isomorphic indecomposable direct summands of N, respectively.

Definition 2.5 ([1]). Let $N \in \operatorname{mod}\Lambda$.

(1) N is called *τ-rigid* if $\operatorname{Hom}_{\Lambda}(N, \tau N) = 0$, where τ is the Auslander–Reiten translation.
(2) N is called *τ-tilting* if N is τ-rigid and $|N| = |\Lambda|$.
(3) N is called *support τ-tilting* if there exists an idempotent e of Λ such that N is a τ-tilting $(\Lambda/\langle e\rangle)$-module.
(4) N is called *tilting* if $\operatorname{pd}_{\Lambda} N \leq 1$, $\operatorname{Ext}^1_{\Lambda}(N, N) = 0$ and $|N| = |\Lambda|$.

We denote by $s\tau\text{-tilt}\,\Lambda$ (respectively, $\tau\text{-tilt}\,\Lambda$, $\text{tilt}\,\Lambda$) the set of isomorphism classes of basic support τ-tilting (respectively, τ-tilting, tilting) Λ-modules.

Let $\mathcal{T}$ be a subcategory of $\operatorname{mod}\Lambda$. We say that $X \in \mathcal{T}$ is Ext-*projective* if $\operatorname{Ext}^1_{\Lambda}(X, \mathcal{T}) = 0$. We denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-projective objects in $\mathcal{T}$ up to isomorphism. Dually, the notion of Ext-*injective objects* is defined, and we denote by $I(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-injective objects in $\mathcal{T}$ up to isomorphism.

Recall that a module $N \in \operatorname{mod}\Lambda$ is *sincere* if any simple Λ-module appears as a composition factor in N. This is equivalent to the fact that $\operatorname{Hom}_{\Lambda}(P, N) \neq 0$ for any non-zero projective Λ-module P. Also recall that a module $N \in \operatorname{mod}\Lambda$

is *faithful* if its annihilator $\{a \in \Lambda \mid aN = 0\}$ vanishes. This is equivalent to the fact that $D\Lambda$ is generated by N, where D is the usual duality of Λ. We call a functorially finite torsion class *sincere* (respectively, *faithful*) if it contains a sincere (respectively, faithful) module. We denote by f-tors Λ the set of functorially finite torsion classes in mod Λ and by sf-tors Λ (respectively, ff-tors Λ) the set of sincere (respectively, faithful) functorially finite torsion classes in mod Λ.

Proposition 2.6 ([1, Theorem 2.7 and Corollary 2.8]). *There is a bijection*

$$s\tau\text{-}\operatorname{tilt}\Lambda \leftrightarrow f\text{-}\operatorname{tors}\Lambda$$

given by $N \in s\tau\text{-}\operatorname{tilt}\Lambda \mapsto \operatorname{Fac}(N) \in f\text{-}\operatorname{tors}\Lambda$ *and* $\mathcal{T} \in f\text{-}\operatorname{tors}\Lambda \mapsto P(\mathcal{T}) \in s\tau\text{-}\operatorname{tilt}\Lambda$. *Moreover, the bijection restricts to bijections*

$$\tau\text{-}\operatorname{tilt}\Lambda \leftrightarrow sf\text{-}\operatorname{tors}\Lambda \quad \text{and} \quad \operatorname{tilt}\Lambda \leftrightarrow ff\text{-}\operatorname{tors}\Lambda.$$

3. Torsion Pairs

Let $\binom{Y}{X}_f \in \operatorname{mod} T$. Then we have the following exact sequence:

$$M \otimes_\Gamma X \xrightarrow{f} Y \to \operatorname{Coker} f \to 0$$

in mod Λ. We denote Coker$f$ by $\operatorname{Coker}\binom{Y}{X}_f$, or simply $\operatorname{Coker}\binom{Y}{X}$ if f is clear. Let $\binom{Y}{X}_g \in \operatorname{mod} T$. Then we have the following exact sequence:

$$0 \to \operatorname{Ker} g \to X \xrightarrow{g} \operatorname{Hom}_\Lambda(M, Y)$$

in mod Γ. We denote Ker g by $\operatorname{Ker}\binom{Y}{X}_g$, or simply $\operatorname{Ker}\binom{Y}{X}$ if g is clear. For a subcategory $\mathcal{T}$ of mod T, we set

$$\operatorname{Coker}(\mathcal{T}) := \left\{ \operatorname{Coker}\begin{pmatrix} Y \\ X \end{pmatrix}_f \,\middle|\, \begin{pmatrix} Y \\ X \end{pmatrix}_f \in \mathcal{T} \right\},$$

$$\operatorname{Ker}(\mathcal{T}) := \left\{ \operatorname{Ker}\begin{pmatrix} Y \\ X \end{pmatrix}_g \,\middle|\, \begin{pmatrix} Y \\ X \end{pmatrix}_g \in \mathcal{T} \right\}.$$

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in mod T. Then $\mathcal{T}$ is not necessarily of the form $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ with $\mathcal{T}_1$ and $\mathcal{T}_2$ torsion classes in mod Λ and mod Γ, respectively. We set

$$\mathfrak{T}_1 := \left\{ Y \in \operatorname{mod}\Lambda \mid \text{there exists } X \in \operatorname{mod}\Gamma \text{ such that } \begin{pmatrix} Y \\ X \end{pmatrix} \in \mathcal{T} \right\},$$

$$\mathfrak{T}_2 := \left\{ X \in \operatorname{mod}\Gamma \mid \text{there exists } Y \in \operatorname{mod}\Lambda \text{ such that } \begin{pmatrix} Y \\ X \end{pmatrix} \in \mathcal{T} \right\},$$

$$\mathfrak{F}_1 := \left\{ Y \in \operatorname{mod}\Lambda \mid \text{there exists } X \in \operatorname{mod}\Gamma \text{ such that } \begin{pmatrix} Y \\ X \end{pmatrix} \in \mathcal{F} \right\},$$

$$\mathfrak{F}_2 := \left\{ X \in \operatorname{mod}\Gamma \mid \text{there exists } Y \in \operatorname{mod}\Lambda \text{ such that } \begin{pmatrix} Y \\ X \end{pmatrix} \in \mathcal{F} \right\}.$$

The following proposition can be obtained by applying [15, Theorems 1 and 2] to the recollement of triangular matrix algebras. Here we give a direct and different proof.

Proposition 3.1. (1) *Let* $(\mathcal{T}_1, \mathcal{F}_1)$ *and* $(\mathcal{T}_2, \mathcal{F}_2)$ *be torsion pairs in* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$, *respectively. Then we have two torsion pairs*

$$\left(\begin{pmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{pmatrix}, \widetilde{\mathcal{F}}\right) \quad and \quad \left(\widetilde{\mathcal{T}}, \begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\end{pmatrix}\right),$$

where $\widetilde{\mathcal{F}} = \{\binom{Y}{X}_g \mid Y \in \mathcal{F}_1$ *and* $\operatorname{Ker}\binom{Y}{X}_g \in \mathcal{F}_2\}$ *and* $\widetilde{\mathcal{T}} = \{\binom{Y}{X}_f \mid X \in \mathcal{T}_2$ *and* $\operatorname{Coker}\binom{Y}{X}_f \in \mathcal{T}_1\}$ *in* $\operatorname{mod} T$.

(2) *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\operatorname{mod} T$ *and* $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{F}_1, \mathfrak{F}_2$ *as above. Then*

$$(\mathfrak{T}_2, \operatorname{Ker}(\mathcal{F})) \quad and \quad (\operatorname{Coker}(\mathcal{T}), \mathfrak{F}_1)$$

are torsion pairs in $\operatorname{mod}\Gamma$ *and* $\operatorname{mod}\Lambda$, *respectively. Moreover,* $\binom{0}{\mathfrak{F}_2} \subseteq \mathcal{F}$ *if and only if* $(\mathfrak{T}_2, \mathfrak{F}_2)$ *is a torsion pair in* $\operatorname{mod}\Gamma$, *and* $\binom{\mathfrak{T}_1}{0} \subseteq \mathcal{T}$ *if and only if* $(\mathfrak{T}_1, \mathfrak{F}_1)$ *is a torsion pair in* $\operatorname{mod}\Lambda$.

Proof. (1) Let

$$0 \to \begin{pmatrix}T_1\\ T_2\end{pmatrix}_f \to \begin{pmatrix}Y\\ X\end{pmatrix}_g \to \begin{pmatrix}T_1'\\ T_2'\end{pmatrix}_h \to 0$$

be an exact sequence in $\operatorname{mod} T$ with $\binom{T_1}{T_2}_f, \binom{T_1'}{T_2'}_h \in \binom{\mathcal{T}_1}{\mathcal{T}_2}$. Then

$$0 \to T_1 \to Y \to T_1' \to 0$$

is exact in $\operatorname{mod}\Lambda$ and

$$0 \to T_2 \to X \to T_2' \to 0$$

is exact in $\operatorname{mod}\Gamma$. So we have $Y \in \mathcal{T}_1$ and $X \in \mathcal{T}_2$ by the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are closed under extensions. Hence $\binom{Y}{X}_g \in \binom{\mathcal{T}_1}{\mathcal{T}_2}$ and $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is closed under extensions. Similarly, $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is closed under images by the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are closed under images. Consequently we have that $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is a torsion class in $\operatorname{mod} T$.

Let $\binom{T_1}{T_2}_f \in \binom{\mathcal{T}_1}{\mathcal{T}_2}$ and $\binom{Y}{X}_g \in \operatorname{mod} T$. Then we have the following two exact sequences:

$$0 \to \begin{pmatrix}T_1\\ 0\end{pmatrix} \to \begin{pmatrix}T_1\\ T_2\end{pmatrix}_f \to \begin{pmatrix}0\\ T_2\end{pmatrix} \to 0 \tag{3.1}$$

and

$$0 \to \begin{pmatrix}Y\\ 0\end{pmatrix} \to \begin{pmatrix}Y\\ X\end{pmatrix}_g \to \begin{pmatrix}0\\ X\end{pmatrix} \to 0 \tag{3.2}$$

in $\operatorname{mod} T$. Applying the functors $\operatorname{Hom}_T(-, \binom{Y}{X}_g)$ and $\operatorname{Hom}_T(\binom{T_1}{T_2}_f, -)$ to (3.1) and (3.2), respectively, yields the following two exact sequences:

$$0 \to \operatorname{Hom}_T\left(\begin{pmatrix} 0 \\ T_2 \end{pmatrix}, \begin{pmatrix} Y \\ X \end{pmatrix}_g\right) \to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}_f, \begin{pmatrix} Y \\ X \end{pmatrix}_g\right) \to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ 0 \end{pmatrix}, \begin{pmatrix} Y \\ X \end{pmatrix}_g\right) \tag{3.3}$$

and

$$0 \to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}_f, \begin{pmatrix} Y \\ 0 \end{pmatrix}\right) \to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}_f, \begin{pmatrix} Y \\ X \end{pmatrix}_g\right) \to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}_f, \begin{pmatrix} 0 \\ X \end{pmatrix}\right).$$

So, if $\operatorname{Hom}_T(\binom{\mathcal{T}_1}{\mathcal{T}_2}, \binom{Y}{X}_g) = 0$, then

$$\operatorname{Hom}_T\left(\begin{pmatrix} 0 \\ T_2 \end{pmatrix}, \begin{pmatrix} Y \\ X \end{pmatrix}_g\right) = 0 = \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}_f, \begin{pmatrix} Y \\ 0 \end{pmatrix}\right),$$

and hence $\operatorname{Ker}\binom{Y}{X}_g \in \mathcal{F}_2$. Since $\binom{\mathcal{T}_1}{0} \subseteq \binom{\mathcal{T}_1}{\mathcal{T}_2}$, we have $Y \in \mathcal{F}_1$. It follows that $\binom{\mathcal{T}_1}{\mathcal{T}_2}^{\perp} \subseteq \widetilde{\mathcal{F}}$.

Conversely, if $\binom{Y}{X}_g \in \widetilde{\mathcal{F}}$, that is, $Y \in \mathcal{F}_1$ and $\operatorname{Ker}\binom{Y}{X}_g \in \mathcal{F}_2$, then $\operatorname{Hom}_T(\binom{\mathcal{T}_1}{\mathcal{T}_2}, \binom{Y}{X}_g) = 0$ by (3.3). So $\binom{Y}{X}_g \in \binom{\mathcal{T}_1}{\mathcal{T}_2}^{\perp}$ and $\widetilde{\mathcal{F}} \subseteq \binom{\mathcal{T}_1}{\mathcal{T}_2}^{\perp}$. Thus $\widetilde{\mathcal{F}} = \binom{\mathcal{T}_1}{\mathcal{T}_2}^{\perp}$ and $(\binom{\mathcal{T}_1}{\mathcal{T}_2}, \widetilde{\mathcal{F}})$ is a torsion pair in $\operatorname{mod} T$.

Similarly, $(\widetilde{\mathcal{T}}, \binom{\mathcal{F}_1}{\mathcal{F}_2})$ is also a torsion pair in $\operatorname{mod} T$.

(2) Let

$$0 \to X_1 \to X \to X_2 \to 0$$

be an exact sequence in $\operatorname{mod}\Gamma$ with $X_1, X_2 \in \mathfrak{T}_2$. By the definition of $\mathfrak{T}_2$, there exists an epimorphism $\binom{Y_i}{X_i} \twoheadrightarrow \binom{0}{X_i}$ in $\operatorname{mod} T$ with $\binom{Y_i}{X_i} \in \mathcal{T}$. Since $\mathcal{T}$ is closed under images, we have $\binom{0}{X_i} \in \mathcal{T}$ for $i = 1, 2$. Then we have an exact sequence

$$0 \to \begin{pmatrix} 0 \\ X_1 \end{pmatrix} \to \begin{pmatrix} 0 \\ X \end{pmatrix} \to \begin{pmatrix} 0 \\ X_2 \end{pmatrix} \to 0$$

in $\operatorname{mod} T$. It follows that $\binom{0}{X} \in \mathcal{T}$ by the fact that $\mathcal{T}$ is closed under extensions. So $X \in \mathfrak{T}_2$ and $\mathfrak{T}_2$ is closed under extensions. Similarly, $\mathfrak{T}_2$ is closed under images. Thus $\mathfrak{T}_2$ is a torsion class in $\operatorname{mod}\Gamma$.

Let $X \in \operatorname{mod}\Gamma$. If $\binom{0}{X} \in \mathcal{F}$, then it is clear that $X \in \operatorname{Ker}(\mathcal{F})$; conversely, if $X \in \operatorname{Ker}(\mathcal{F})$, then there exists $\binom{Y_1}{X_1}_{\varphi} \in \mathcal{F}$ such that

$$0 \to X \to X_1 \xrightarrow{\varphi} \operatorname{Hom}_\Lambda(M, Y_1)$$

is an exact sequence in $\operatorname{mod}\Gamma$. It induces an exact sequence

$$0 \to \begin{pmatrix} 0 \\ X \end{pmatrix} \to \begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}_{\varphi}$$

in $\operatorname{mod} T$, which implies $\binom{0}{X} \in \mathcal{F}$. Thus we have

$$X \in \mathfrak{T}_2^{\perp} \Leftrightarrow \operatorname{Hom}_\Gamma(\mathfrak{T}_2, X) = 0 \Leftrightarrow \operatorname{Hom}_T\left(\mathcal{T}, \begin{pmatrix} 0 \\ X \end{pmatrix}\right) = 0$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ X \end{pmatrix} \in \mathcal{F} \Leftrightarrow X \in \operatorname{Ker}(\mathcal{F}).$$

Hence $(\mathfrak{T}_2, \operatorname{Ker}(\mathcal{F}))$ is a torsion pair in $\operatorname{mod}\Gamma$.

Similarly, $(\operatorname{Coker}(\mathcal{T}), \mathfrak{F}_1)$ is a torsion pair in $\operatorname{mod}\Lambda$.

Moreover, if $\binom{0}{\mathfrak{F}_2} \subseteq \mathcal{F}$, then $\operatorname{Hom}_T(\binom{0}{\mathfrak{T}_2}, \binom{0}{\mathfrak{F}_2}) = 0$ and $\operatorname{Hom}_\Gamma(\mathfrak{T}_2, \mathfrak{F}_2) = 0$. Thus we have

$$X \in \mathfrak{T}_2^{\perp} \Leftrightarrow \begin{pmatrix} 0 \\ X \end{pmatrix} \in \mathcal{F} \Leftrightarrow X \in \mathfrak{F}_2,$$

and therefore $(\mathfrak{T}_2, \mathfrak{F}_2)$ is a torsion pair in $\operatorname{mod}\Gamma$. Conversely, if $(\mathfrak{T}_2, \mathfrak{F}_2)$ is a torsion pair in $\operatorname{mod}\Gamma$, then $\operatorname{Hom}_\Gamma(\mathfrak{T}_2, \mathfrak{F}_2) = 0$ and $\operatorname{Hom}_T(\binom{0}{\mathfrak{T}_2}, \binom{0}{\mathfrak{F}_2})$=0. For any $\binom{T_1}{T_2}_g \in \mathcal{T}$, we have the following exact sequence:

$$0 \to \operatorname{Hom}_T\left(\begin{pmatrix} 0 \\ T_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathfrak{F}_2 \end{pmatrix}\right) \to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}_g, \begin{pmatrix} 0 \\ \mathfrak{F}_2 \end{pmatrix}\right)$$

$$\to \operatorname{Hom}_T\left(\begin{pmatrix} T_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathfrak{F}_2 \end{pmatrix}\right) (= 0),$$

so $\operatorname{Hom}_T(\binom{T_1}{T_2}_g, \binom{0}{\mathfrak{F}_2}) = 0$, and hence $\binom{0}{\mathfrak{F}_2} \subseteq \mathcal{F}$.

Similarly, we have that $\binom{\mathfrak{T}_1}{0} \subseteq \mathcal{T}$ if and only if $(\mathfrak{T}_1, \mathfrak{F}_1)$ is a torsion pair in $\operatorname{mod}\Lambda$. □

In the following result, we give some sufficient and necessary conditions to ensure that the two torsion pairs in Proposition 3.1(1) are identical.

Proposition 3.2. *Let* $(\mathcal{T}_1, \mathcal{F}_1)$ *and* $(\mathcal{T}_2, \mathcal{F}_2)$ *be torsion pairs in* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$, *respectively. Then the following statements are equivalent*:

(1) $M \otimes_\Gamma \mathcal{T}_2 \subseteq \mathcal{T}_1$.
(2) $\operatorname{Hom}_\Lambda(M, \mathcal{F}_1) \subseteq \mathcal{F}_2$.
(3) $(\binom{\mathcal{T}_1}{\mathcal{T}_2}, \widetilde{\mathcal{F}}) = (\widetilde{\mathcal{T}}, \binom{\mathcal{F}_1}{\mathcal{F}_2})$, *where* $\widetilde{\mathcal{F}}$ *and* $\widetilde{\mathcal{T}}$ *are as in Proposition* 3.1(1).

Proof. (1) $\Leftrightarrow$ (2) We have

$$M \otimes_\Gamma \mathcal{T}_2 \subseteq \mathcal{T}_1 \Leftrightarrow \mathrm{Hom}_\Lambda(M \otimes_\Gamma \mathcal{T}_2, \mathcal{F}_1) = 0 \Leftrightarrow \mathrm{Hom}_\Gamma(\mathcal{T}_2, \mathrm{Hom}_\Lambda(M, \mathcal{F}_1)) = 0$$
$$\Leftrightarrow \mathrm{Hom}_\Lambda(M, \mathcal{F}_1) \subseteq \mathcal{F}_2.$$

(2) $\Rightarrow$ (3) Obviously, we have $\binom{\mathcal{F}_1}{\mathcal{F}_2} \subseteq \widetilde{\mathcal{F}}$. Let $\binom{Y}{X}_g \in \widetilde{\mathcal{F}}$. Then we have an exact sequence

$$0 \to \mathrm{Ker}\, g \to X \xrightarrow{g} \mathrm{Hom}_\Lambda(M, Y)$$

in $\mathrm{mod}\,\Gamma$ with $\mathrm{Ker}\, g \in \mathcal{F}_2$. Since $\mathrm{Hom}_\Lambda(M, Y) \in \mathcal{F}_2$ by (2), we have $X \in \mathcal{F}_2$ by the fact that $\mathcal{F}_2$ is closed under extensions and subobjects. So $\binom{Y}{X}_g \in \binom{\mathcal{F}_1}{\mathcal{F}_2}$ and $\widetilde{\mathcal{F}} \subseteq \binom{\mathcal{F}_1}{\mathcal{F}_2}$. It follows that $\widetilde{\mathcal{F}} = \binom{\mathcal{F}_1}{\mathcal{F}_2}$

(3) $\Rightarrow$ (1) Let $T_2 \in \mathcal{T}_2$. Since $\binom{M \otimes T_2}{T_2} \in \widetilde{\mathcal{T}} = \binom{\mathcal{T}_1}{\mathcal{T}_2}$ by (3), we have $M \otimes T_2 \in \mathcal{T}_1$ and $M \otimes \mathcal{T}_2 \subseteq \mathcal{T}_1$. □

Corollary 3.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathrm{mod}\, T$. Then we have two torsion pairs*

$$\left(\begin{pmatrix} \mathrm{Coker}(\mathcal{T}) \\ \mathfrak{T}_2 \end{pmatrix}, \mathcal{F}'\right) \quad \textit{and} \quad \left(\mathcal{T}', \begin{pmatrix} \mathfrak{F}_1 \\ \mathrm{Ker}(\mathcal{F}) \end{pmatrix}\right) \tag{3.4}$$

in $\mathrm{mod}\, T$ with $\binom{\mathrm{Coker}(\mathcal{T})}{\mathfrak{T}_2} \subseteq \mathcal{T} \subseteq \mathcal{T}'$, where

$$\mathcal{F}' = \left\{ \begin{pmatrix} Y \\ X \end{pmatrix}_g \middle| Y \in \mathfrak{F}_1 \textit{ and } \mathrm{Ker} \begin{pmatrix} Y \\ X \end{pmatrix}_g \in \mathrm{Ker}(\mathcal{F}) \right\},$$

$$\mathcal{T}' = \left\{ \begin{pmatrix} Y \\ X \end{pmatrix}_f | \, X \in \mathfrak{T}_2 \textit{ and } \mathrm{Coker} \begin{pmatrix} Y \\ X \end{pmatrix}_f \in \mathrm{Coker}(\mathcal{T}) \right\}.$$

Proof. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\, T$, by Proposition 3.1(2) we have two torsion pairs $(\mathfrak{T}_2, \mathrm{Ker}(\mathcal{F}))$ and $(\mathrm{Coker}(\mathcal{T}), \mathfrak{F}_1)$ in $\mathrm{mod}\,\Gamma$ and $\mathrm{mod}\,\Lambda$, respectively. Then by Proposition 3.1(1) we have two torsion pairs as in (3.4).

For any $\binom{F_1}{F_2}_g \in \mathcal{F}$, we have $\mathrm{Ker}\binom{F_1}{F_2}_g \in \mathrm{Ker}(\mathcal{F})$. Thus $\mathcal{F} \subseteq \mathcal{F}'$, which yields $\binom{\mathrm{Coker}(\mathcal{T})}{\mathfrak{T}_2} \subseteq \mathcal{T}$. On the other hand, for any $\binom{T_1}{T_2}_f \in \mathcal{T}$, we have $\mathrm{Coker}\binom{T_1}{T_2}_f \in \mathrm{Coker}(\mathcal{T})$, thus $\mathcal{T} \subseteq \mathcal{T}'$. □

Corollary 3.4. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathrm{mod}\, T$. If $\mathrm{Hom}_\Lambda(M, \mathfrak{F}_1) \subseteq \mathrm{Ker}(\mathcal{F})$, then we can recover this torsion pair from torsion pairs in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$.*

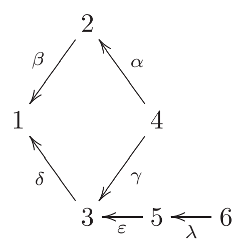
Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathrm{mod}\, T$. As in Corollary 3.3,

$$\left(\begin{pmatrix} \mathrm{Coker}(\mathcal{T}) \\ \mathfrak{T}_2 \end{pmatrix}, \mathcal{F}'\right) \quad \text{and} \quad \left(\mathcal{T}', \begin{pmatrix} \mathfrak{F}_1 \\ \mathrm{Ker}(\mathcal{F}) \end{pmatrix}\right)$$

are two torsion pairs in $\operatorname{mod} T$. Since $\operatorname{Hom}_\Lambda(M, \mathfrak{F}_1) \subseteq \operatorname{Ker}(\mathcal{F})$ by assumption, we have $\binom{\operatorname{Coker}(\mathcal{T})}{\mathfrak{T}_2} = \mathcal{T}'$ by Proposition 3.2. It follows from Corollary 3.3 that $\binom{\operatorname{Coker}(\mathcal{T})}{\mathfrak{T}_2} = \mathcal{T} = \mathcal{T}'$ and $(\mathcal{T}, \mathcal{F}) = (\binom{\operatorname{Coker}(\mathcal{T})}{\mathfrak{T}_2}, \binom{\mathfrak{F}_1}{\operatorname{Ker}(\mathcal{F})})$. □

Throughout the paper, the explicitly given subcategories in the examples are always additively generated by the indecomposable modules listed. The following example shows that, in general, the two torsion pairs in Proposition 3.1(1) are different, and that $\mathcal{T} \subsetneq \binom{\mathfrak{T}_1}{\mathfrak{T}_2}$ and $\mathcal{F} \subsetneq \binom{\mathfrak{F}_1}{\mathfrak{F}_2}$.

Example 3.5. Let k be a field and T a finite-dimensional k-algebra given by the quiver



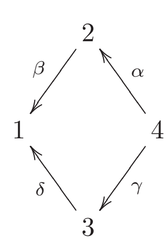
with the relation $\delta\varepsilon = 0$ and $\beta\alpha = \delta\gamma$. Then

$$T = \begin{pmatrix} (e_1+e_2+e_3+e_4)T(e_1+e_2+e_3+e_4) & (e_1+e_2+e_3+e_4)T(e_5+e_6) \\ 0 & (e_5+e_6)T(e_5+e_6) \end{pmatrix},$$

where e_i is the idempotent corresponding to the vertex i for any $1 \leq i \leq 6$. We have that $\Gamma := (e_5+e_6)T(e_5+e_6)$ is a finite-dimensional k-algebra given by the quiver

$$5 \longleftarrow 6$$

and $\Lambda := (e_1+e_2+e_3+e_4)T(e_1+e_2+e_3+e_4)$ is a finite-dimensional k-algebra given by the quiver



with the relation $\beta\alpha = \delta\gamma$.

(1) Take a torsion pair $(\mathcal{T}_1, \mathcal{F}_1)$ in $\operatorname{mod}\Lambda$ where

$$\mathcal{T}_1 = \left\{ \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix} \right\} \quad \text{and} \quad \mathcal{F}_1 = \left\{ \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix} \right\},$$

and take a torsion pair $(\mathcal{T}_2, \mathcal{F}_2)$ in $\operatorname{mod}\Gamma$ where $\mathcal{T}_2 = \{P(6), S(6)\}$ and $\mathcal{F}_2 = \{P(5)\}$. Then we have two torsion pairs $(\binom{\mathcal{T}_1}{\mathcal{T}_2}, \widetilde{\mathcal{F}})$ and $(\widetilde{\mathcal{T}}, \binom{\mathcal{F}_1}{\mathcal{F}_2})$ in $\operatorname{mod} T$ by Proposition 3.1(1), where

$$\widetilde{\mathcal{F}} = \left\{ \begin{array}{c}0\\0\quad 0\\1\,0\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\1\,1\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\1\,1\,1\end{array},\ \begin{array}{c}0\\0\quad 0\\0\,1\,0\end{array} \right\}$$

and

$$\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\end{pmatrix} = \left\{ \begin{array}{c}0\\0\quad 0\\1\,0\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\1\,1\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\0\,1\,0\end{array} \right\}.$$

Obviously, these two torsion pairs are different. Moreover, we have $M \otimes_\Gamma \mathcal{T}_2 = \{ \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix} \} \not\subseteq \mathcal{T}_1$.

(2) Take a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod} T$, where

$$\mathcal{T} = \left\{ \begin{array}{c}0\\1\quad 0\\0\,0\,0\end{array},\ \begin{array}{c}1\\1\quad 0\\0\,0\,0\end{array},\ \begin{array}{c}1\\1\quad 1\\1\,0\,0\end{array},\ \begin{array}{c}1\\0\quad 0\\0\,0\,0\end{array},\ \begin{array}{c}1\\0\quad 1\\1\,0\,0\end{array},\ \begin{array}{c}0\\0\quad 1\\1\,0\,0\end{array},\ \begin{array}{c}1\\0\quad 1\\1\,1\,1\end{array},\right.$$
$$\left. \begin{array}{c}0\\0\quad 0\\1\,1\,1\end{array}\ \ \begin{array}{c}1\\0\quad 1\\0\,0\,0\end{array},\ \begin{array}{c}0\\0\quad 1\\1\,1\,1\end{array},\ \begin{array}{c}0\\0\quad 1\\0\,0\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\0\,1\,1\end{array},\ \begin{array}{c}0\\0\quad 0\\0\,0\,1\end{array} \right\}$$

and

$$\mathcal{F} = \left\{ \begin{array}{c}0\\0\quad 0\\1\,0\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\1\,1\,0\end{array},\ \begin{array}{c}0\\0\quad 0\\0\,1\,0\end{array} \right\}.$$

Then we have

$$\mathfrak{T}_1 = \left\{ \begin{array}{c}0\\1\quad 0\\0\end{array},\ \begin{array}{c}1\\1\quad 0\\0\end{array},\ \begin{array}{c}1\\0\quad 0\\0\end{array},\ \begin{array}{c}0\\0\quad 0\\1\end{array},\ \begin{array}{c}1\\1\quad 1\\1\end{array},\ \begin{array}{c}1\\0\quad 1\\1\end{array},\ \begin{array}{c}1\\0\quad 1\\0\end{array},\ \begin{array}{c}0\\0\quad 1\\1\end{array},\ \begin{array}{c}0\\0\quad 1\\0\end{array} \right\}$$

in $\operatorname{mod}\Lambda$ and $\mathfrak{T}_2 = \{P(6), S(6)\}$ in $\operatorname{mod}\Gamma$. Notice that $\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1\,0\,0 & \end{smallmatrix} \in \binom{\mathfrak{T}_1}{\mathfrak{T}_2}$ but not in $\mathcal{T}$, so $\mathcal{T} \subsetneq \binom{\mathfrak{T}_1}{\mathfrak{T}_2}$.

We have $\mathfrak{F}_1 = \{ \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix} \}$ in $\operatorname{mod}\Lambda$ and $\mathfrak{F}_2 = \{P(5)\}$ in $\operatorname{mod}\Gamma$. Then we have that $(\mathfrak{T}_2, \mathfrak{F}_2)$ is a torsion pair with $M \otimes_\Gamma \mathfrak{T}_2 \subseteq \mathfrak{T}_1$ and $\binom{0}{\mathfrak{F}_2} \subseteq \mathcal{F}$. But $\binom{\mathfrak{T}_1}{0} \not\subseteq \mathcal{T}$, $\operatorname{Hom}_\Lambda(M, \mathfrak{F}_1) \not\subseteq \mathfrak{F}_2$, so $(\mathfrak{T}_1, \mathfrak{F}_1)$ is not a torsion pair. The torsion class corresponding to $\mathfrak{F}_1$ is $\operatorname{Coker}(\mathcal{T})$ by Proposition 3.1(2).

(3) Take a torsion pair $(\mathcal{T},\mathcal{F})$ in $\operatorname{mod} T$, where

$$\mathcal{T}=\left\{\begin{smallmatrix}&0&\\1&&0\\0&0&0\end{smallmatrix},\ \begin{smallmatrix}&1&\\1&&0\\0&0&0\end{smallmatrix},\ \begin{smallmatrix}&1&\\1&&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}&1&\\0&&0\\0&0&0\end{smallmatrix},\ \begin{smallmatrix}&1&\\0&&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}&1&\\0&&1\\1&1&1\end{smallmatrix},\ \begin{smallmatrix}&1&\\0&&1\\0&0&0\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&1\\1&1&1\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&1\\0&0&0\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&0\\0&1&1\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&0\\0&0&1\end{smallmatrix}\right\}$$

and

$$\mathcal{F}=\left\{\begin{smallmatrix}&0&\\0&&0\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&0\\1&1&0\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&0\\1&1&1\end{smallmatrix},\ \begin{smallmatrix}&0&\\0&&0\\0&1&0\end{smallmatrix}\right\}.$$

Then $\mathfrak{F}_1=\{\begin{smallmatrix}&0&\\0&&0\\&1&\end{smallmatrix}\}$ in $\operatorname{mod}\Lambda$ and $\mathfrak{F}_2=\{P(5),P(6)\}$ in $\operatorname{mod}\Gamma$. Notice that $\begin{smallmatrix}&0&\\0&&0\\0&1&1\end{smallmatrix}\in\binom{\mathfrak{F}_1}{\mathfrak{F}_2}$ but not in $\mathcal{F}$, so $\mathcal{F}\subsetneq\binom{\mathfrak{F}_1}{\mathfrak{F}_2}$.

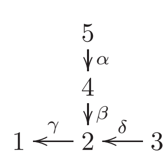
We have

$$\mathfrak{T}_1=\left\{\begin{smallmatrix}&0&\\1&&0\\&0&\end{smallmatrix},\begin{smallmatrix}&1&\\1&&0\\&0&\end{smallmatrix},\begin{smallmatrix}&1&\\0&&0\\&0&\end{smallmatrix},\begin{smallmatrix}&1&\\1&&1\\&1&\end{smallmatrix},\begin{smallmatrix}&1&\\0&&1\\&1&\end{smallmatrix},\begin{smallmatrix}&1&\\0&&1\\&0&\end{smallmatrix},\begin{smallmatrix}&0&\\0&&1\\&1&\end{smallmatrix},\begin{smallmatrix}&0&\\0&&1\\&0&\end{smallmatrix}\right\}$$

in $\operatorname{mod}\Lambda$ and $\mathfrak{T}_2=\{P(6),S(6)\}$ in $\operatorname{mod}\Gamma$. Then we have $(\mathfrak{T}_1,\mathfrak{F}_1)$ is a torsion pair with $\operatorname{Hom}_\Lambda(M,\mathfrak{F}_1)\subseteq\mathfrak{F}_2$ and $\binom{\mathfrak{T}_1}{0}\subseteq\mathcal{T}$. But $\binom{0}{\mathfrak{F}_2}\nsubseteq\mathcal{F}$, so $M\otimes_\Gamma\mathfrak{T}_2\nsubseteq\mathfrak{T}_1$ and $(\mathfrak{T}_2,\mathfrak{F}_2)$ is not a torsion pair. The torsion-free class corresponding to $\mathfrak{T}_2$ is $\operatorname{Ker}(\mathcal{F})=\{P(5)\}$ by Proposition 3.1(2).

Now we give an example to illustrate Proposition 3.2.

Example 3.6. Let k be a field and T a finite-dimensional k-algebra given by the quiver



with the relation $\gamma\beta=0$. Then

$$T=\begin{pmatrix}(e_1+e_2+e_3)T(e_1+e_2+e_3) & (e_1+e_2+e_3)T(e_4+e_5)\\ 0 & (e_4+e_5)T(e_4+e_5)\end{pmatrix},$$

where e_i is the idempotent corresponding to the vertex i for any $1\le i\le 5$. We have that $\Gamma:=(e_4+e_5)T(e_4+e_5)$ is a finite-dimensional k-algebra given by the quiver

$$4 \longleftarrow 5,$$

and $\Lambda := (e_1 + e_2 + e_3)T(e_1 + e_2 + e_3)$ is a finite-dimensional k-algebra given by the quiver

$$1 \longleftarrow 2 \longleftarrow 3.$$

Take a torsion pair $(\mathcal{T}_1, \mathcal{F}_1)$ in $\operatorname{mod}\Lambda$ where $\mathcal{T}_1 = \{\, {}_{110}, {}_{111}, {}_{010}, {}_{011}, {}_{001} \,\}$ and $\mathcal{F}_1 = \{\, {}_{100} \,\}$, and take a torsion pair $(\mathcal{T}_2, \mathcal{F}_2)$ in $\operatorname{mod}\Gamma$ where $\mathcal{T}_2 = \{P(5), S(5)\}$ and $\mathcal{F}_2 = \{P(4)\}$. Since $M \otimes_\Gamma \mathcal{T}_2 = \{\, {}_{010} \,\} \subseteq \mathcal{T}_1$, we have $\widetilde{\mathcal{F}} = \left\{ \begin{smallmatrix} & 0 & \\ & 1 & \\ 0 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ & 0 & \\ 1 & 0 & 0 \end{smallmatrix} \right\} = \binom{\mathcal{F}_1}{\mathcal{F}_2}$ by Proposition 3.2.

4. Main Results

Lemma 4.1. *Let* $(\mathcal{T}_1, \mathcal{F}_1)$ *and* $(\mathcal{T}_2, \mathcal{F}_2)$ *be torsion pairs in* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$, *respectively. Then we have*

(1) *If* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are sincere functorially finite, then* $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ *is sincere functorially finite in* $\operatorname{mod} T$.

(2) *If* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are faithful functorially finite and* $\operatorname{Hom}_\Lambda(M, \mathcal{T}_1) \subseteq \mathcal{T}_2$, *then* $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ *is faithful functorially finite in* $\operatorname{mod} T$.

Proof. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are functorially finite, then $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is functorially finite in $\operatorname{mod} T$ by Proposition 2.2.

(1) Note that the projective T-modules are in the form of $\binom{P_i}{0}$ and $\binom{M \otimes Q_i}{Q_i}$ where P_i runs all projective Λ-modules and Q_i runs all projective Γ-modules. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are sincere by assumption, we have

$$\operatorname{Hom}_T\left(\begin{pmatrix} P_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}\right) \cong \operatorname{Hom}_\Lambda(P_i, \mathcal{T}_1) \neq 0$$

and

$$\operatorname{Hom}_T\left(\begin{pmatrix} M \otimes Q_i \\ Q_i \end{pmatrix}, \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}\right) \cong \operatorname{Hom}_\Gamma(Q_i, \mathcal{T}_2) \neq 0,$$

and thus $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is sincere.

(2) By [17, Theorem], we have $\binom{\mathcal{T}_1}{\mathcal{T}_2} = \operatorname{Fac}(N)$ for some module N in $\operatorname{mod} T$. Then $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is faithful if and only if N is faithful by [1, Corollary 2.8]. Since N is faithful if and only if DT is generated by N by [7, VI Lemma 2.2], we have that $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is faithful if and only if all injective T-modules are in $\binom{\mathcal{T}_1}{\mathcal{T}_2}$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are faithful, $\mathcal{T}_1$ and $\mathcal{T}_2$ contain all indecomposable injective modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. Note that the indecomposable injective T-modules are exactly $\binom{I}{\operatorname{Hom}_\Lambda(M,I)}$ and $\binom{0}{J}$ where I runs over indecomposable injective Λ-modules and J runs over indecomposable injective Γ-modules. Thus $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ contains all indecomposable injective T-modules and $\binom{\mathcal{T}_1}{\mathcal{T}_2}$ is faithful. $\square$

Now we construct (support) τ-tilting modules and tilting modules in $\operatorname{mod} T$ from that in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$.

Theorem 4.2. *Let $N_1 \in \operatorname{mod}\Lambda$ and $N_2 \in \operatorname{mod}\Gamma$.*

(1) *If N_1 and N_2 are support τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, then $P\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ and $P(\widetilde{\mathcal{T}})$ are support τ-tilting modules in $\operatorname{mod} T$, where*

$$\widetilde{\mathcal{T}} = \left\{ \begin{pmatrix} Y \\ X \end{pmatrix} \mid X \in \mathrm{Fac}(N_2) \text{ and } \mathrm{Coker}\begin{pmatrix} Y \\ X \end{pmatrix} \in \mathrm{Fac}(N_1) \right\}.$$

(2) *If N_1 and N_2 are τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, then $P\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ is a τ-tilting module in $\operatorname{mod} T$.*

(3) *If N_1 and N_2 are tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, with $\mathrm{Hom}_\Lambda(M, \mathrm{Fac}(N_1)) \subseteq \mathrm{Fac}(N_2)$, then $P\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ is a tilting module in $\operatorname{mod} T$.*

Proof. (1) If N_1 and N_2 are support τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, then $\mathrm{Fac}(N_1)$ and $\mathrm{Fac}(N_2)$ are functorially finite torsion classes by Proposition 2.6. It follows from Propositions 2.2 and 3.1(1) that $\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ and $\widetilde{\mathcal{T}} = \{\binom{Y}{X} \mid X \in \mathrm{Fac}(N_2)$ and $\mathrm{Coker}\binom{Y}{X} \in \mathrm{Fac}(N_1)\}$ are functorially finite torsion classes in $\operatorname{mod} T$. Thus $P\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ and $P(\widetilde{\mathcal{T}})$ are support τ-tilting modules in $\operatorname{mod} T$ by Proposition 2.6 again.

(2) If N_1 and N_2 are τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, then $\mathrm{Fac}(N_1)$ and $\mathrm{Fac}(N_2)$ are sincere functorially finite torsion classes by Proposition 2.6. Then $\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ is a sincere functorially finite torsion class in $\operatorname{mod} T$ by Lemma 4.1. Again by Proposition 2.6, we have that $P\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ is a τ-tilting module in $\operatorname{mod} T$.

(3) If N_1 and N_2 are tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, then $\mathrm{Fac}(N_1)$ and $\mathrm{Fac}(N_2)$ are faithful functorially finite torsion classes by Proposition 2.6. If $\mathrm{Hom}_\Lambda(M, \mathrm{Fac}(N_1)) \subseteq \mathrm{Fac}(N_2)$, then $\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ is a faithful functorially finite torsion class by Lemma 4.1. Thus $P\binom{\mathrm{Fac}(N_1)}{\mathrm{Fac}(N_2)}$ is a tilting module in $\operatorname{mod} T$ by Proposition 2.6 again. □

In the following, we will give the converse construction of Theorem 4.2; that is, we will construct (support) τ-tilting modules and tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ from that in $\operatorname{mod} T$.

Lemma 4.3. *If $(\mathcal{T}, \mathcal{F})$ is a functorially finite torsion pair in $\operatorname{mod} T$, then $\mathfrak{F}_1, \mathfrak{T}_2$ are functorially finite in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a functorially finite torsion pair in $\operatorname{mod} T$. Then for any $Y \in \operatorname{mod}\Lambda$, there exist a left $\mathcal{F}$-approximation $\binom{f_1}{0} : \binom{Y}{0} \to \binom{F_1}{F_2}_f$ of $\binom{Y}{0}$ and a right $\mathcal{F}$-approximation $\binom{f_2}{g_2} : \binom{\widetilde{F}_1}{\widetilde{F}_2}_g \to \binom{Y}{\mathrm{Hom}_\Lambda(M,Y)}$ of $\binom{Y}{\mathrm{Hom}_\Lambda(M,Y)}$ in $\operatorname{mod} T$. So, for any $h_1 : Y \to F_1'$ and $h_2 : \widetilde{F}_1' \to Y$ in $\operatorname{mod}\Lambda$ with $F_1', \widetilde{F}_1' \in \mathfrak{F}_1$, there exist $\binom{\alpha_1}{0} : \binom{F_1}{F_2}_f \to \binom{F_1'}{0}$ and $\binom{\widetilde{\alpha}_1}{0} : \binom{\widetilde{F}_1'}{0} \to \binom{\widetilde{F}_1}{\widetilde{F}_2}_g$ in $\operatorname{mod} T$, such that the following two diagrams:

$$\begin{array}{ccc} \binom{Y}{0} & \xrightarrow{\binom{f_1}{0}} & \binom{F_1}{F_2}_f \\ {\scriptstyle \binom{h_1}{0}}\downarrow & \swarrow {\scriptstyle \binom{\alpha_1}{0}} & \\ \binom{F_1'}{0}, & & \end{array} \qquad \begin{array}{ccc} & & \binom{\widetilde{F}_1'}{0} \\ & \swarrow {\scriptstyle \binom{\widetilde{\alpha}_1}{0}} & \downarrow {\scriptstyle \binom{h_2}{0}} \\ \binom{\widetilde{F}_1}{\widetilde{F}_2}_g & \xrightarrow{\binom{f_2}{g_2}} & \binom{Y}{\operatorname{Hom}_\Lambda(M,Y)} \end{array}$$

commute, which induce the following commutative diagrams:

$$\begin{array}{ccc} Y & \xrightarrow{f_1} & F_1 \\ {\scriptstyle h_1}\downarrow & \swarrow {\scriptstyle \alpha_1} & \\ F_1', & & \end{array} \qquad \begin{array}{ccc} & & \widetilde{F}_1' \\ & \swarrow {\scriptstyle \widetilde{\alpha}_1} & \downarrow {\scriptstyle h_2} \\ \widetilde{F}_1 & \xrightarrow{f_2} & Y. \end{array}$$

It follows that $f_1 : Y \to F_1$ and $f_2 : \widetilde{F}_1 \to Y$ are left and right $\mathfrak{F}_1$-approximations of Y, respectively. Thus $\mathfrak{F}_1$ is functorially finite. Similarly, $\mathfrak{T}_2$ is functorially finite. $\square$

For a module $N \in \operatorname{mod} T$, we write

$$\mathfrak{Fac}(N)_1 := \left\{ Y \in \operatorname{mod}\Lambda \mid \text{there exists } X \in \operatorname{mod}\Gamma \text{ such that } \begin{pmatrix} Y \\ X \end{pmatrix} \in \operatorname{Fac}(N) \right\},$$

$$\mathfrak{Fac}(N)_2 = \left\{ X \in \operatorname{mod}\Gamma \mid \text{there exists } Y \in \operatorname{mod}\Lambda \text{ such that } \begin{pmatrix} Y \\ X \end{pmatrix} \in \operatorname{Fac}(N) \right\}.$$

In the following result, we construct support τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ from that in $\operatorname{mod} T$.

Theorem 4.4. *If N is a support τ-tilting module in $\operatorname{mod} T$, then $P(\operatorname{Coker}(\operatorname{Fac}(N)))$ and $P(\mathfrak{Fac}(N)_2)$ are support τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively.*

Proof. Let N be a support τ-tilting module in $\operatorname{mod} T$. Then we have a functorially finite torsion class $\operatorname{Fac}(N)$ by Proposition 2.6, which implies that $\operatorname{Fac}(N)^\perp$ is functorially finite. By Lemma 4.3, both $\mathfrak{Fac}(N)_1^\perp$ and $\mathfrak{Fac}(N)_2$ are also functorially finite. It follows from Proposition 3.1(2) that

$$(\operatorname{Coker}(\operatorname{Fac}(N)), \mathfrak{Fac}(N)_1^\perp) \quad \text{and} \quad (\mathfrak{Fac}(N)_2, \operatorname{Ker}(\operatorname{Fac}(N)^\perp))$$

are functorially finite torsion pairs in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. Again by Proposition 2.6, we have that $P(\operatorname{Coker}(\operatorname{Fac}(N)))$ and $P(\mathfrak{Fac}(N)_2)$ are support τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. $\square$

We need the following lemma.

Lemma 4.5. *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion pair in* $\operatorname{mod} T$ *and* $\binom{\mathfrak{T}_1}{0} \subseteq \mathcal{T}$. *Then we have*

(1) *If* $\mathcal{T}$ *is sincere functorially finite, then* $\operatorname{Coker}(\mathcal{T})$ *and* $\mathfrak{T}_2$ *are sincere functorially finite torsion classes in* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$, *respectively.*
(2) *If* $\mathcal{T}$ *is faithful functorially finite, then* $\operatorname{Coker}(\mathcal{T})$ *and* $\mathfrak{T}_2$ *are faithful functorially finite torsion classes in* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$, *respectively.*

Proof. Let $\mathcal{T}$ be a functorially finite torsion class. Then $\operatorname{Coker}(\mathcal{T})$ and $\mathfrak{T}_2$ are functorially finite torsion classes by Proposition 3.1(2) and Lemma 4.3. Also, if $\binom{\mathfrak{T}_1}{0} \subseteq \mathcal{T}$, then $\operatorname{Coker}(\mathcal{T}) = \mathfrak{T}_1$ by Proposition 3.1(2).

(1) Let $\mathcal{T}$ be sincere. Because the projective objects in $\operatorname{mod} T$ are in the form of $\binom{P_i}{0}$ and $\binom{M\otimes_\Gamma Q_i}{Q_i}$ where P_i runs all projective objects in $\operatorname{mod}\Lambda$ and Q_i runs all projective objects in $\operatorname{mod}\Gamma$, we have

$$\operatorname{Hom}_\Lambda(P_i,\mathfrak{T}_1) \cong \operatorname{Hom}_T\left(\begin{pmatrix} P_i \\ 0 \end{pmatrix},\mathcal{T}\right) \neq 0 \quad \text{and}$$

$$\operatorname{Hom}_\Gamma(Q_i,\mathfrak{T}_2) \cong \operatorname{Hom}_T\left(\begin{pmatrix} M\otimes_\Gamma Q_i \\ Q_i \end{pmatrix},\mathcal{T}\right) \neq 0,$$

and so $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are sincere.

(2) Let $a\in\Lambda$ such that $a\mathfrak{T}_1 = 0$. Then $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\mathcal{T} = \binom{a\mathfrak{T}_1}{0} = 0$. Because $(\mathcal{T},\mathcal{F})$ is a faithful torsion pair, we have that $a = 0$ and $\mathfrak{T}_1$ is faithful. Similarly, $\mathfrak{T}_2$ is also faithful. □

In the following result, we construct (τ-)tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ from that in $\operatorname{mod} T$ under some condition.

Theorem 4.6. *Let* $N \in \operatorname{mod} T$ *satisfying* $\binom{\mathfrak{Fac}(N)_1}{0} \subseteq \operatorname{Fac}(N)$. *If* N *is a* (τ-)*tilting* T-*module, then* $P(\operatorname{Coker}(\operatorname{Fac}(N)))$ *and* $P(\mathfrak{Fac}(N)_2)$ *are* (τ-)*tilting modules in* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$, *respectively.*

Proof. Let N be a (τ-)tilting module in $\operatorname{mod} T$. Then we have a faithful (respectively, sincere) functorially finite torsion pair $(\operatorname{Fac}(N),\operatorname{Fac}(N)^\perp)$ by Proposition 2.6. It follows from Proposition 3.1(2) and Lemma 4.3 that

$$(\operatorname{Coker}(\operatorname{Fac}(N)),\mathfrak{Fac}(N)_1^\perp) \quad \text{and} \quad (\mathfrak{Fac}(N)_2,\operatorname{Ker}(\operatorname{Fac}(N)^\perp))$$

are functorially finite torsion pairs. Moreover, both $\operatorname{Coker}(\operatorname{Fac}(N))$ and $\mathfrak{Fac}(N)_2$ are faithful (respectively, sincere) by Lemma 4.5. Hence $P(\operatorname{Coker}(\operatorname{Fac}(N)))$ and $P(\mathfrak{Fac}(N)_2)$ are (τ-)tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively, by Proposition 2.6. □

As a consequence, we have the following.

Corollary 4.7. *If* $N = \binom{N_1}{N_2}_\phi$ *is a support* τ-*tilting* (*respectively,* (τ-)*tilting*) T-*module, then* N_2 *is a support* τ-*tilting* (*respectively,* (τ-)*tilting*) Γ-*module.*

Proof. By Theorems 4.4 and 4.6, it suffices to prove $\operatorname{add}_\Gamma P(\mathfrak{Fac}(N)_2) = \operatorname{add}_\Gamma N_2$. Since $\operatorname{Ext}^1_T(N, \operatorname{Fac}(N)) = 0$, we have $\operatorname{Ext}^1_T(N, \binom{0}{\mathfrak{Fac}(N)_2}) = 0$. Then applying the functor $\operatorname{Hom}_T(-, \binom{0}{\mathfrak{Fac}(N)_2})$ to the exact sequence

$$0 \to \begin{pmatrix} N_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_\phi \to \begin{pmatrix} 0 \\ N_2 \end{pmatrix} \to 0$$

yields the following exact sequence:

$$\begin{aligned}(0=)\operatorname{Hom}_T\left(\begin{pmatrix} N_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathfrak{Fac}(N)_2 \end{pmatrix}\right) &\to \operatorname{Ext}^1_T\left(\begin{pmatrix} 0 \\ N_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathfrak{Fac}(N)_2 \end{pmatrix}\right) \\ &\to \operatorname{Ext}^1_T\left(\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_\phi, \begin{pmatrix} 0 \\ \mathfrak{Fac}(N)_2 \end{pmatrix}\right)(=0),\end{aligned}$$

which implies $\operatorname{Ext}^1_T(\binom{0}{N_2}, \binom{0}{\mathfrak{Fac}(N)_2}) = 0$. It follows that $\operatorname{Ext}^1_\Gamma(N_2, \mathfrak{Fac}(N)_2) = 0$ and $N_2 \in \operatorname{add}_\Gamma P(\mathfrak{Fac}(N)_2)$.

Conversely, suppose $L \in \operatorname{add}_\Gamma P(\mathfrak{Fac}(N)_2)$. Then $\operatorname{Ext}^1_\Gamma(L, \mathfrak{Fac}(N)_2) = 0$. Let $\binom{Y}{X}_f \in \operatorname{Fac}(N)$. Then $X \in \mathfrak{Fac}(N)_2$, and so $\operatorname{Ext}^1_T(\binom{M\otimes_\Gamma L}{L}, \binom{0}{X}) \cong \operatorname{Ext}^1_\Gamma(L, X) = 0$ by [21, Lemma 1.2(ii)]. We claim that $\operatorname{Ext}^1_T(\binom{M\otimes_\Gamma L}{L}, \binom{Y}{0}) = 0$. Otherwise, if $\operatorname{Ext}^1_T(\binom{M\otimes_\Gamma L}{L}, \binom{Y}{0}) \neq 0$, then we have a non-split exact sequence

$$0 \to \begin{pmatrix} Y \\ 0 \end{pmatrix} \to \begin{pmatrix} Z \\ L \end{pmatrix}_g \xrightarrow{\binom{h_1}{1}} \begin{pmatrix} M\otimes_\Gamma L \\ L \end{pmatrix} \to 0,$$

which implies that the following diagram:

$$\begin{array}{ccc} M\otimes_\Gamma L & \xrightarrow{\quad g \quad} & Z \\ {\scriptstyle M\otimes 1_L}\downarrow & & \downarrow{\scriptstyle h_1} \\ M\otimes_\Gamma L & \xrightarrow{1_{M\otimes_\Gamma L}} & M\otimes_\Gamma L \end{array}$$

commutes and $h_1 g = 1_{M\otimes_\Gamma L}$. So $\binom{h_1}{1}\binom{g}{1} = 1_{\binom{M\otimes_\Gamma L}{L}}$ and $\binom{h_1}{1}$ is a retraction, a contradiction. The claim is proved. Applying the functor $\operatorname{Hom}_T(\binom{M\otimes_\Gamma L}{L}, -)$ to the exact sequence

$$0 \to \begin{pmatrix} Y \\ 0 \end{pmatrix} \to \begin{pmatrix} Y \\ X \end{pmatrix}_f \to \begin{pmatrix} 0 \\ X \end{pmatrix} \to 0$$

yields the following exact sequence:

$$\begin{aligned}(0=)\operatorname{Ext}^1_T\left(\begin{pmatrix} M\otimes_\Gamma L \\ L \end{pmatrix}, \begin{pmatrix} Y \\ 0 \end{pmatrix}\right) &\to \operatorname{Ext}^1_T\left(\begin{pmatrix} M\otimes_\Gamma L \\ L \end{pmatrix}, \begin{pmatrix} Y \\ X \end{pmatrix}_f\right) \\ &\to \operatorname{Ext}^1_T\left(\begin{pmatrix} M\otimes_\Gamma L \\ L \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}\right)(=0).\end{aligned}$$

It follows that $\operatorname{Ext}^1_T(\binom{M\otimes_\Gamma L}{L}, \binom{Y}{X}_f) = 0$ and $\binom{M\otimes_\Gamma L}{L} \in \operatorname{add}_T N$. Thus $L \in \operatorname{add}_\Gamma N_2$ and $\operatorname{add}_\Gamma P(\mathfrak{Fac}(N)_2) \subseteq \operatorname{add}_\Gamma N_2$. □

In the following, we give some examples to illustrate our results.

Example 4.8. Let T, Λ and Γ be as in Example 3.5.

(1) Take support τ-tilting modules $N_1 = \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}$ and $N_2 = S(5)$ in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. Then $\operatorname{Fac}(N_1) = \{\begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}\}$ and $\operatorname{Fac}(N_2) = \{S(5)\}$. We have

$$\begin{pmatrix} \operatorname{Fac}(N_1) \\ \operatorname{Fac}(N_2) \end{pmatrix} = \left\{ \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 1 & \\ 0 & & 0 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 1 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0\,1\,0 & \end{matrix} \right\},$$

$$\widetilde{\mathcal{T}} = \left\{ \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 1 & \\ 0 & & 0 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 1 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0\,1\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 0 \\ & 1\,1\,0 & \end{matrix}, \begin{matrix} & 1 & \\ 0 & & 1 \\ & 1\,1\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 1 \\ & 1\,1\,0 & \end{matrix} \right\},$$

and so both

$$P\begin{pmatrix} \operatorname{Fac}(N_1) \\ \operatorname{Fac}(N_2) \end{pmatrix} = \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0\,0\,0 & \end{matrix} \oplus \begin{matrix} & 1 & \\ 0 & & 0 \\ & 0\,0\,0 & \end{matrix} \oplus \begin{matrix} & 1 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix} \oplus \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0\,1\,0 & \end{matrix}$$

and

$$P(\widetilde{\mathcal{T}}) = \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0\,0\,0 & \end{matrix} \oplus \begin{matrix} & 1 & \\ 0 & & 0 \\ & 0\,0\,0 & \end{matrix} \oplus \begin{matrix} & 1 & \\ 0 & & 1 \\ & 1\,1\,0 & \end{matrix} \oplus \begin{matrix} & 0 & \\ 0 & & 0 \\ & 1\,1\,0 & \end{matrix} \oplus \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0\,1\,0 & \end{matrix}$$

are support τ-tilting T-modules by Theorem 4.2(1).

(2) Take τ-tilting modules $N_1 = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}$ and $N_2 = P(6) \oplus S(6)$ in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. Then $\operatorname{Fac}(N_1) = \{\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}\}$ and $\operatorname{Fac}(N_2) = \{P(6), S(6)\}$. We have

$$\begin{pmatrix} \operatorname{Fac}(N_1) \\ \operatorname{Fac}(N_2) \end{pmatrix} = \left\{ \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 1 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 1 \\ & 1\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 1 \\ & 0\,0\,0 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 1 \\ & 1\,1\,1 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0\,1\,1 & \end{matrix}, \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0\,0\,1 & \end{matrix} \right\},$$

and so

$$P\begin{pmatrix}\mathrm{Fac}(N_1)\\ \mathrm{Fac}(N_2)\end{pmatrix} = \begin{smallmatrix}0\\1\quad 0\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\\0\quad 1\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\\0\quad 1\\1\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\\0\quad 1\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\\0\quad 1\\1\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}0\\0\quad 0\\0\,0\,1\end{smallmatrix}$$

is a τ-tilting T-module by Theorem 4.2(2).

(3) Take tilting modules $N_1 = \begin{smallmatrix}0\\1\;0\\0\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\;0\\0\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\;1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\0\;1\\0\end{smallmatrix}$ and $N_2 = P(6) \oplus S(6)$ in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$, respectively. Then

$$\mathrm{Fac}(N_1) = \left\{ \begin{smallmatrix}0\\1\quad 0\\0\end{smallmatrix}, \begin{smallmatrix}1\\1\quad 0\\0\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 0\\0\end{smallmatrix}, \begin{smallmatrix}1\\1\quad 1\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 1\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 1\\0\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 1\\1\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 1\\0\end{smallmatrix} \right\}$$

and $\mathrm{Fac}(N_2) = \{P(6), S(6)\}$. We have $\mathrm{Hom}_\Lambda(M, \mathrm{Fac}(N_1)) = \{P(6)\} \subseteq \mathrm{Fac}(N_2)$ and

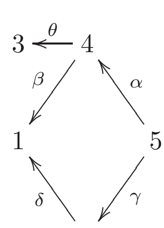
$$\begin{pmatrix}\mathrm{Fac}(N_1)\\ \mathrm{Fac}(N_2)\end{pmatrix} = \left\{ \begin{smallmatrix}0\\1\quad 0\\0\,0\,0\end{smallmatrix}, \begin{smallmatrix}1\\1\quad 0\\0\,0\,0\end{smallmatrix}, \begin{smallmatrix}1\\1\quad 1\\1\,0\,0\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 0\\0\,0\,0\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 1\\1\,0\,0\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 1\\1\,0\,0\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 1\\1\,1\,1\end{smallmatrix}, \begin{smallmatrix}1\\0\quad 1\\0\,0\,0\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 1\\1\,1\,1\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 1\\0\,0\,0\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 0\\0\,1\,1\end{smallmatrix}, \begin{smallmatrix}0\\0\quad 0\\0\,0\,1\end{smallmatrix} \right\},$$

and so

$$P\begin{pmatrix}\mathrm{Fac}(N_1)\\ \mathrm{Fac}(N_2)\end{pmatrix} = \begin{smallmatrix}0\\1\quad 0\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\quad 0\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\\1\quad 1\\1\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\\0\quad 1\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\\0\quad 1\\1\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}0\\0\quad 0\\0\,0\,1\end{smallmatrix}$$

is a tilting T-module by Theorem 4.2(3).

Example 4.9. Let k be a field and T a finite-dimensional k-algebra given by the quiver



with the relation $\beta\alpha = \delta\gamma$ and $\theta\alpha = 0$. Then

$$T = \begin{pmatrix} (e_1+e_2)T(e_1+e_2) & (e_1+e_2)T(e_3+e_4+e_5) \\ 0 & (e_3+e_4+e_5)T(e_3+e_4+e_5) \end{pmatrix},$$

where e_i is the idempotent corresponding to the vertex i for any $1 \le i \le 5$. We have that $\Gamma := (e_3+e_4+e_5)T(e_3+e_4+e_5)$ is a finite-dimensional k-algebra given by the quiver

$$3 \xleftarrow{\theta} 4 \xleftarrow{\alpha} 5$$

with relation $\theta\alpha = 0$ and $\Lambda := (e_1+e_2)T(e_1+e_2)$ is a finite-dimensional k-algebra given by the quiver

$$1 \xleftarrow{\delta} 2.$$

(1) Take a support τ-tilting module $N = \begin{smallmatrix}&1\,0&\\0&&0\\&0&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,0&\\0&&0\\&1&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,0&\\0&&1\\&1&\end{smallmatrix}$ in $\operatorname{mod} T$. Then $\operatorname{Fac}(N) = \{\begin{smallmatrix}&1\,0&\\0&&0\\&0&\end{smallmatrix}, \begin{smallmatrix}&0\,0&\\0&&0\\&1&\end{smallmatrix}, \begin{smallmatrix}&0\,0&\\0&&1\\&1&\end{smallmatrix}, \begin{smallmatrix}&0\,0&\\0&&1\\&0&\end{smallmatrix}\}$, $\operatorname{Coker}(\operatorname{Fac}(N)) = \{S(2)\}$ and $\mathfrak{Fac}(N)_2 = \{S(3), S(5)\}$. Then by Theorem 4.4, we have that $S(2)(= P(\operatorname{Coker}(\operatorname{Fac}(N))))$ and $S(3) \oplus S(5)(= P(\mathfrak{Fac}(N)_2))$ are support τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively.

(2) Take a τ-tilting module $N = \begin{smallmatrix}&1\,0&\\0&&0\\&0&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,0&\\1&&0\\&1&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,0&\\0&&0\\&1&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,1&\\1&&1\\&1&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,0&\\0&&1\\&1&\end{smallmatrix}$ in $\operatorname{mod} T$. Then

$$\operatorname{Fac}(N) = \left\{ \begin{matrix}&1\,0&\\0&&0\\&0&\end{matrix}, \begin{matrix}&0\,0&\\1&&0\\&1&\end{matrix}, \begin{matrix}&0\,0&\\0&&0\\&1&\end{matrix}, \begin{matrix}&0\,1&\\1&&1\\&1&\end{matrix}, \begin{matrix}&0\,0&\\0&&1\\&1&\end{matrix}, \begin{matrix}&0\,1&\\0&&1\\&1&\end{matrix}, \begin{matrix}&0\,1&\\0&&1\\&0&\end{matrix}, \begin{matrix}&0\,0&\\0&&1\\&0&\end{matrix} \right\},$$

$$\operatorname{Coker}(\operatorname{Fac}(N)) = \mathfrak{Fac}(N)_1 = \{S(2), P(2)\} \quad \text{and} \quad \mathfrak{Fac}(N)_2 = \{S(3), S(5), P(5)\}.$$

Since $\binom{\mathfrak{Fac}(N)_1}{0} \subseteq \operatorname{Fac}(N)$, it follows from Theorem 4.6 that $S(2) \oplus P(2)(= P(\operatorname{Coker}(\operatorname{Fac}(N))))$ and $S(3) \oplus S(5) \oplus P(5)(= P(\mathfrak{Fac}(N)_2))$ are τ-tilting modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively.

(3) Take a tilting module $N = \begin{smallmatrix}&0\,0&\\1&&0\\&0&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,0&\\1&&0\\&1&\end{smallmatrix} \oplus \begin{smallmatrix}&1\,1&\\1&&0\\&0&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,1&\\1&&0\\&0&\end{smallmatrix} \oplus \begin{smallmatrix}&0\,1&\\1&&1\\&1&\end{smallmatrix}$ in $\operatorname{mod} T$. Then

$$\operatorname{Fac}(N) = \left\{ \begin{matrix}&0\,1&\\1&&0\\&0&\end{matrix}, \begin{matrix}&0\,0&\\0&&0\\&1&\end{matrix}, \begin{matrix}&0\,1&\\0&&1\\&0&\end{matrix}, \begin{matrix}&1\,1&\\1&&0\\&0&\end{matrix}, \begin{matrix}&0\,1&\\1&&0\\&1&\end{matrix}, \begin{matrix}&0\,1&\\1&&1\\&1&\end{matrix}, \begin{matrix}&0\,1&\\0&&1\\&1&\end{matrix}, \begin{matrix}&0\,0&\\0&&1\\&0&\end{matrix}, \right.$$

$$\left. \begin{matrix}&0\,0&\\1&&0\\&0&\end{matrix}, \begin{matrix}&1\,1&\\1&&0\\&1&\end{matrix}, \begin{matrix}&0\,1&\\0&&0\\&0&\end{matrix}, \begin{matrix}&0\,0&\\0&&1\\&1&\end{matrix}, \begin{matrix}&0\,0&\\1&&0\\&1&\end{matrix}, \begin{matrix}&1\,1&\\0&&0\\&0&\end{matrix} \right\},$$

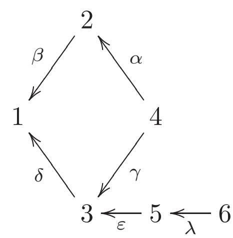
$\operatorname{Coker}(\operatorname{Fac}(N)) = \mathfrak{Fac}(N)_1 = \{P(1), P(2), S(2)\}$ and

$\mathfrak{Fac}(N)_2 = \{P(4), S(4), P(5), S(5)\}$.

Since $\begin{pmatrix} \mathfrak{Fac}(N)_1 \\ 0 \end{pmatrix} \subseteq \mathrm{Fac}(N)$, it follows from Theorem 4.6 that $P(1) \oplus P(2) (= P(\mathrm{Coker}(\mathrm{Fac(N)})))$ and $P(4) \oplus S(4) \oplus P(5) (= P(\mathfrak{Fac}(N)_2))$ are tilting modules in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$, respectively.

The following example illustrates that the condition $\mathrm{Hom}_\Lambda(M, \mathcal{T}_1) \nsubseteq \mathcal{T}_2$ in Lemma 4.1 is necessary.

Example 4.10. Let k be a field and T a finite-dimensional k-algebra given by the quiver



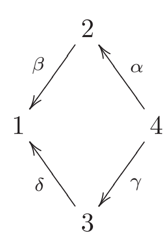
with the relation $\delta\varepsilon = \varepsilon\lambda = 0$ and $\beta\alpha = \delta\gamma$. Then

$$T = \begin{pmatrix} (e_1 + e_2 + e_3 + e_4)T(e_1 + e_2 + e_3 + e_4) & (e_1 + e_2 + e_3 + e_4)T(e_5 + e_6) \\ 0 & (e_5 + e_6)T(e_5 + e_6) \end{pmatrix},$$

where e_i is the idempotent corresponding to the vertex i for any $1 \leq i \leq 6$. We have that $\Gamma := (e_5 + e_6)T(e_5 + e_6)$ is a finite-dimensional k-algebra given by the quiver

$$5 \longleftarrow 6$$

and $\Lambda := (e_1 + e_2 + e_3 + e_4)T(e_1 + e_2 + e_3 + e_4)$ is a finite-dimensional k-algebra given by the quiver



with the relation $\beta\alpha = \delta\gamma$.

Take

$$\mathcal{T}_1 = \left\{ \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix} \right\} \quad \text{and}$$

$$\mathcal{T}_2 = \{0\,1, 1\,1\},$$

which are faithful functorially finite torsion classes in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$, respectively. We have that $\operatorname{Hom}_\Lambda(M, \mathcal{T}_1) = \{ {\scriptstyle 1\ 0} \} \nsubseteq \mathcal{T}_2$ and the torsion class

$$\begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix} = \left\{ \begin{array}{c} 0 \\ 1\quad 0 \\ 0\,0\,0 \end{array}, \begin{array}{c} 1 \\ 1\quad 0 \\ 0\,0\,0 \end{array}, \begin{array}{c} 1 \\ 1\quad 1 \\ 1\,0\,0 \end{array}, \begin{array}{c} 1 \\ 0\quad 0 \\ 0\,0\,0 \end{array}, \begin{array}{c} 1 \\ 0\quad 1 \\ 1\,0\,0 \end{array}, \begin{array}{c} 0 \\ 0\quad 1 \\ 1\,0\,0 \end{array}, \right.$$

$$\left. \begin{array}{c} 1 \\ 0\quad 1 \\ 0\,0\,0 \end{array}, \begin{array}{c} 0 \\ 0\quad 1 \\ 0\,0\,0 \end{array}, \begin{array}{c} 0 \\ 0\quad 0 \\ 0\,1\,1 \end{array}, \begin{array}{c} 0 \\ 0\quad 0 \\ 0\,0\,1 \end{array} \right\}$$

is not faithful.

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