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Pure-injectivity in the category of Gorenstein projective modules

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In this paper, we introduce and study (weak) pure-injective Gorenstein projective modules. Let R be an Artin algebra. We prove that the category of weak pure-injective Gorenstein projective left R-modules coincides with the intersection of the category of pure-injective left R-modules and that of Gorenstein projective left R-modules. Then, we get an equivalent characterization of virtually Gorenstein algebras (being CM-finite). Furthermore, we prove that the category of weak pure-injective Gorenstein projective left R-modules is enveloping in the category of left R-modules; and if R is virtually Gorenstein, then it is precovering in the category of pure-injective left R-modules.

Keywords: Gorenstein projective modules; G-pure exact; (weak) pure-injective; absolutely pure; virtually Gorenstein algebras; (pre)enveloping; (pre)covering.

Mathematics Subject Classification: 18G25, 16E30

1. Introduction

As a nice generalization of finitely generated projective modules over commutative Noetherian local rings, Auslander and Bridger introduced in [1] finitely generated modules of Gorenstein dimension zero; and then Enochs and Jenda generalized it in [13] to Gorenstein projective modules (not necessarily finitely generated) and introduced the dual notion — Gorenstein injective modules over general rings. Since then, Gorenstein projective and injective modules and related modules have become very important research objects in Gorenstein homological algebra and representation theory of algebras; see [2–4, 7–10, 13, 14, 19, 21, 26, 30] and references therein.

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The notion of pure-injective modules (also called algebraically compact modules in [29]) was introduced by Cohn in [11] and found many important applications in ring theory, homological algebra and representation theory of algebras. For instance, the study of generic modules (a special kind of pure-injective modules), initiated by Crawley–Boevey in [12], has proved to be essential in characterizing tameness of finite-dimensional algebras [23] and phantom maps in group algebras [5]. The geometrical and topological aspects of pure-injective modules (for instance, the Ziegler spectrum and Krull–Gabriel dimension) were also widely studied in [17, 23, 27], and so on.

In particular, in the study of the Gorenstein symmetry conjecture, Beligannis introduced in [2], the notion of virtually Gorenstein algebras as a nice generalization of Gorenstein algebras. Various equivalent characterizations of virtually Gorenstein algebras are supplied, among which the use of pure exact structures in the category of Gorenstein projective modules attracts our interest. That is, there is a natural purity theory on this locally finitely presented category with respect to its subcategory consisting of finitely presented Gorenstein projective modules.

Let R be an Artin algebra. Then the category of Gorenstein projective modules is definable by [2, Proposition 3.8]. It follows easily from [27, Theorem 3.4.8] that the pure-injective envelope of a left R-module M is Gorenstein projective if and only if M is Gorenstein projective. It indicates that the intersection of the category of pure-injective modules and that of Gorenstein projective modules provides much information on the original category.

Our aim in this paper is to study the pure-injectivity in the category of Gorenstein projective modules. The paper is organized as follows.

In Sec. 2, we give some terminology and some preliminary results.

In Sec. 3, we introduce and study G-pure exact sequences, (weak) pure-injective, pure-projective and absolutely pure Gorenstein projective modules. Let R be a ring. We give some equivalent characterizations of absolutely pure Gorenstein projective left R-modules in terms of the properties of G-pure exact sequences and the vanishing of Ext-functors in the category of contravariant functors from the category of stable finitely presented Gorentein projective left R-modules to that of abelian groups. Let R be an Artin algebra. We prove that the category of weak pure-injective Gorenstein projective left R-modules coincides with the intersection of the category of pure-injective left R-modules and that of Gorenstein projective left R-modules (Theorem 3.7). As an application, we get that R is virtually Gorenstein if and only if any weak pure-injective Gorenstein projective left R-module is a direct limit of a family of finitely presented Gorenstein projective left R-modules (Theorem 3.12). This extends a result in [4]. Moreover, we get that a virtually Gorenstein Artin algebra R is CM-finite if and only if the category of pure-injective Gorenstein projective left R-modules coincides with that of Gorenstein projective left R-modules (Theorem 3.14).

As further applications of Theorem 3.7, we study in Sec. 4, the covering and enveloping properties of the category of weak pure-injective Gorenstein projective

R-modules. Let R be an Artin algebra. We prove that this category is enveloping in the category of left R-modules; and if R is virtually Gorenstein, then, it is precovering in the category of pure-injective left R-modules.

2. Preliminaries

In this section, we give some terminology and some preliminary results.

Throughout this paper, R is an associative ring with identity, $\operatorname{Mod} R$ is the category of left R-modules and $\operatorname{mod} R$ is the category of finitely presented left R-modules. For a module $M \in \operatorname{Mod} R$, $\operatorname{Add} M$ (respectively $\operatorname{Prod} M$) is the subcategory of $\operatorname{Mod} R$ consisting of direct summands of coproducts (respectively products) of copies of M. For a subcategory $\mathcal{C}$ of $\operatorname{Mod} R$, we use $\overline{\mathcal{C}}$ and $\underline{\mathcal{C}}$ to denote the stable categories of $\mathcal{C}$ modulo projectives and injectives, respectively.

Definition 2.1 [14]. Let $\mathcal{C} \subseteq \mathcal{D}$ be subcategories of $\operatorname{Mod} R$. The homomorphism $f : C \to D$ in $\operatorname{Mod} R$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ is called a *$\mathcal{C}$-precover* of D if for any homomorphism $g : C' \to D$ in $\operatorname{Mod} R$ with $C' \in \mathcal{C}$, there exists a homomorphism $h : C' \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} & & C' \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D. \end{array}$$

The morphism $f : C \to D$ is called *right minimal*, if a homomorphism $h : C \to C$ is an automorphism, whenever $f = fh$. A $\mathcal{C}$-precover $f : C \to D$ is called a *$\mathcal{C}$-cover* if f is right minimal. $\mathcal{C}$ is called (*pre*)*covering* in $\mathcal{D}$ if each module in $\mathcal{D}$ has a $\mathcal{C}$-(pre)cover. Dually, the notions of a *$\mathcal{C}$-preenvelope*, a *left minimal homomorphism*, a *$\mathcal{C}$-envelope* and a (*pre*)*enveloping subcategory* are defined.

Let $M \in \operatorname{Mod} R$ and $N \in \operatorname{Mod} R^{op}$. Recall that a sequence in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(-, M)$-*exact* (respectively $N \otimes_R -$-exact), if it is exact after applying the functor $\operatorname{Hom}_R(-, M)$ (respectively $N \otimes_R -$); and for a subcategory $\mathcal{C}$ of $\operatorname{Mod} R$, a sequence in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(-, \mathcal{C})$-*exact*, if it is exact after applying the functor $\operatorname{Hom}_R(-, M)$ for any $M \in \mathcal{C}$. We use $\operatorname{Proj} R$, $\operatorname{Inj} R$ and $\operatorname{Flat} R$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of projective, injective and flat modules respectively.

Definition 2.2 [13, 14]. (1) A module $G \in \operatorname{Mod} R$ is called *Gorenstein projective*, if there exists a $\operatorname{Hom}_R(-, \operatorname{Proj} R)$-exact exact sequence:

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\operatorname{Mod} R$ with all P_i and P^i in $\operatorname{Proj} R$ and $G \cong \operatorname{Im}(P_0 \to P^0)$. Dually, the notion of *Gorenstein injective modules* is defined.

(2) A module $H \in \operatorname{Mod} R$ is called *Gorenstein flat*, if there exists an $\operatorname{Inj} R^{op} \otimes_R -$-exact exact sequence:

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

in $\operatorname{Mod} R$ with all F_i and F^i in $\operatorname{Flat} R$ and $H \cong \operatorname{Im}(F_0 \to F^0)$.

We use $\operatorname{GProj} R$, $\operatorname{GInj} R$ and $\operatorname{GFlat} R$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of Gorenstein projective, injective and flat modules respectively. By the adjoint isomorphism theorem, it is easy to see that $\operatorname{GProj} R = \operatorname{GFlat} R$ over an Artin algebra R. We use $\operatorname{Gproj} R$ and $\operatorname{Ginj} R$ to denote the full subcategory of $\operatorname{mod} R$ consisting of Gorenstein projective and injective modules, respectively.

We write $(-)^+ := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Lemma 2.3 [16, Lemma 1.2.13]. *Let*

$$0 \to A \to B \to C \to 0 \tag{2.1}$$

be an exact sequence in $\operatorname{Mod} R$*. Then the following statements are equivalent.*

(1) $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ *is exact for any* $M \in \operatorname{Mod} R^{op}$.
(2) $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ *is exact for any* $M \in \operatorname{mod} R^{op}$.
(3) $0 \to \operatorname{Hom}_R(N, A) \to \operatorname{Hom}_R(N, B) \to \operatorname{Hom}_R(N, C) \to 0$ *is exact for any* $N \in \operatorname{mod} R$.
(4) $0 \to C^+ \to B^+ \to A^+ \to 0$ *splits.*
(5) $0 \to A \to B \to C \to 0$ *is a direct limit of split short exact sequences.*

The exact sequence (2.1) *is called* pure exact, *if one of the above equivalent conditions is satisfied.*

Definition 2.4 [16]. A submodule A of B in $\operatorname{Mod} R$ is called *pure*, if the exact sequence

$$0 \to A \xrightarrow{\lambda} B \to B/A \to 0$$

in $\operatorname{Mod} R$ with λ the embedding is pure exact; and a quotient module C of B in $\operatorname{Mod} R$ is called *pure*, if the exact sequence

$$0 \to \operatorname{Ker} \pi \to B \xrightarrow{\pi} C \to 0$$

in $\operatorname{Mod} R$ with π the natural epimorphism is pure exact.

Definition 2.5 [16]. (1) A module $M \in \operatorname{Mod} R$ is *pure-projective*, if it is projective with respect to pure exact sequences, that is, any pure exact sequence

$$0 \to A \to B \to M \to 0$$

in $\operatorname{Mod} R$ ending at M splits. Dually, the notion of *pure-injective modules* is defined.

(2) A module $M \in \operatorname{Mod} R$ is *absolutely pure*, if any exact sequence

$$0 \to M \to B \to C \to 0$$

in $\operatorname{Mod} R$ starting from M is pure exact.

We use $\operatorname{PInj} R$ and $\mathcal{A}bsR$ to denote the full subcategories of $\operatorname{Mod} R$ consisting of pure-injective modules and absolutely pure modules, respectively. The following two lemmas are used frequently in the sequel.

Lemma 2.6 [14, Propositions 5.3.7 and 5.3.9]. *For any $M \in \operatorname{Mod} R$, we have*

(1) $M^+ \in \operatorname{PInj} R^{op}$.
(2) $\sigma_M : M \to M^{++}$ *via* $\sigma_M(m)(f) = f(m)$ *for any* $m \in M$ *and* $f \in M^+$ *induces a pure exact sequence*

$$0 \to M \xrightarrow{\sigma_M} M^{++} \to \operatorname{Coker} \sigma_M \to 0.$$

Thus M is pure-injective if and only if σ_M is a section.

Lemma 2.7 *Let R be an Artin algebra. Then $M^{++} \in \operatorname{PInj} R \cap \operatorname{GProj} R$ for any $M \in \operatorname{GProj} R$.*

Proof. Let $M \in \operatorname{GProj} R$. Then by Lemma 2.6(1), we have $M^{++} \in \operatorname{PInj} R$. By [19, Theorem 3.6] and [26, Theorem 2.3], we have $M^+ \in \operatorname{GInj} R^{op}$ and $M^{++} \in \operatorname{GFlat} R (= \operatorname{GProj} R)$. □

3. Pure-Injective Gorenstein Projective Modules

In this section, we introduce and study pure-projective, (weak) pure-injective and absolutely pure Gorenstein projective modules. We begin with the following:

Definition 3.1. (1) An exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0 \tag{3.1}$$

in $\operatorname{Mod} R$ with all terms in $\operatorname{GProj} R$ is called *G-pure exact in* $\operatorname{GProj} R$ if

$$0 \to \operatorname{Hom}_R(G, G_1) \to \operatorname{Hom}_R(G, G_2) \to \operatorname{Hom}_R(G, G_3) \to 0$$

is exact for any finitely presented Gorenstein projective left R-module G.
(2) $H \in \operatorname{GProj} R$ is called *pure-projective in* $\operatorname{GProj} R$ if

$$0 \to \operatorname{Hom}_R(H, G_1) \to \operatorname{Hom}_R(H, G_2) \to \operatorname{Hom}_R(H, G_3) \to 0$$

is an exact sequence for any G-pure exact sequence (3.1).
(3) $E \in \operatorname{GProj} R$ is called *pure-injective in* $\operatorname{GProj} R$ if

$$0 \to \operatorname{Hom}_R(G_3, E) \to \operatorname{Hom}_R(G_2, E) \to \operatorname{Hom}_R(G_1, E) \to 0$$

is an exact sequence for any G-pure exact sequence (3.1).

(4) $A \in \mathrm{GProj}\, R$ is called *absolutely pure in* $\mathrm{GProj}\, R$ if any exact sequence $0 \to A \to G_2 \to G_3 \to 0$ in $\mathrm{GProj}\, R$ is G-pure exact.

We use $\mathcal{PP}$-$\mathrm{GProj}\, R$, $\mathcal{PI}$-$\mathrm{GProj}\, R$ and $\mathcal{A}bs$-$\mathrm{GProj}\, R$ to denote the subcategories of $\mathrm{GProj}\, R$ consisting of pure-projective, pure-injective and absolutely pure Gorenstein projective modules, respectively.

For an additive category $\mathcal{C}$, we use $(\mathcal{C}^{op}, \mathcal{A}b)$ to denote the category consisting of contravariant functors from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups. The following result gives some equivalent characterizations of absolutely pure Gorenstein projective modules.

Proposition 3.2. *The following statements are equivalent for any* $A \in \mathrm{GProj}\, R$.

(1) $A \in \mathcal{A}bs$-$\mathrm{GProj}\, R$.

(2) *There exists a G-pure exact sequence*

$$0 \to A \to P \to G_1 \to 0$$

in $\mathrm{Mod}\, R$ *with* P *projective.*

(3) $\mathrm{Ext}^1_R(-, A) = 0$ *as an object of* $((\underline{\mathrm{Gproj}} R)^{op}, \mathcal{A}b)$.

Proof. (1) ⇒ (2) It follows from the definition of Gorenstein projective modules.

(2) ⇒ (1) For any exact sequence

$$0 \to A \to G_2 \to G_3 \to 0$$

in $\mathrm{GProj}\, R$, we can form the pushout of $A \to P$ and $A \to G_2$ to get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & P & \to & G_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & G_2 & \to & G_4 & \to & G_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_3 & = & G_3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since $\mathrm{Ext}^1_R(G_3, P) = 0$, the middle column in the diagram splits. Applying the functor $\mathrm{Hom}_R(G, -)$ with $G \in \mathrm{Gproj}\, R$ to this diagram, then the snake lemma implies that the leftmost column is G-pure exact. Hence $A \in \mathcal{A}bs$-$\mathrm{GProj}\, R$.

(2) ⇔ (3) For any $G \in \operatorname{Gproj} R$, the given G-pure exact sequence gives rise to a long exact sequence:

$$0 \to \operatorname{Hom}_R(G, A) \to \operatorname{Hom}_R(G, P) \to \operatorname{Hom}_R(G, G_1)$$
$$\to \operatorname{Ext}^1_R(G, A) \to \operatorname{Ext}^1_R(G, P) \to \cdots .$$

Because $\operatorname{Ext}^1_R(G, P) = 0$, we get the equivalence between (2) and (3) easily. □

A subcategory $\mathcal{X}$ of $\operatorname{GProj} R$ is called *closed under G-pure submodules*, if in a G-pure exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

$G_2 \in \mathcal{X}$ implies $G_1 \in \mathcal{X}$. Recall from [8] that an Artin algebra R is called CM-*free* if $\operatorname{Gproj} R = \operatorname{proj} R$.

Corollary 3.3. (1) *$\mathcal{A}bs$-$\operatorname{GProj} R$ is closed under extensions, direct sums and G-pure submodules.*
(2) *For an Artin algebra R, we have* $\operatorname{Gproj} R \cap \mathcal{A}bs\text{-}\operatorname{GProj} R = \operatorname{proj} R$.
(3) *If R is a* CM-*free Artin algebra, then* $\operatorname{GProj} R = \mathcal{A}bs\text{-}\operatorname{GProj} R$.

Proof. (1) It follows from Proposition 3.2 directly that $\mathcal{A}bs$-$\operatorname{GProj} R$ is closed under extensions and G-pure submodules. Note that $\operatorname{Ext}^1_R(M, \oplus N_i) \cong \oplus \operatorname{Ext}^1_R(M, N_i)$ for finitely presented module M. So the subcategory $\mathcal{A}bs$-$\operatorname{GProj} R$ of $\operatorname{GProj} R$ is closed under direct sums by Proposition 3.2.

(2) Because R is an Artin algebra, for any $G \in \operatorname{Gproj} R \cap \mathcal{A}bs\text{-}\operatorname{GProj} R$ there exists an exact sequence

$$0 \to G \to P \to G_1 \to 0$$

with $P \in \operatorname{proj} R$ and $G_1 \in \operatorname{Gproj} R$, which splits by Proposition 3.2. Thus $G \in \operatorname{proj} R$.

(3) It is an immediate consequence of Proposition 3.2. □

Recall from [2] that an Artin algebra R is called *virtually Gorenstein* if

$$(\operatorname{GProj} R)^{\perp} = {}^{\perp}(\operatorname{GInj} R),$$

where

$$(\operatorname{GProj} R)^{\perp} := \{Y \in \operatorname{Mod} R \mid \operatorname{Ext}^{\geq 1}_R(G, Y) = 0 \text{ for any } G \in \operatorname{GProj} R\} \quad \text{and}$$
$${}^{\perp}(\operatorname{GInj} R) := \{X \in \operatorname{Mod} R \mid \operatorname{Ext}^{\geq 1}_R(X, G) = 0 \text{ for any } G \in \operatorname{GInj} R\}.$$

The following result gives some properties of virtually Gorenstein Artin algebras.

Proposition 3.4. *Let R be a virtually Gorenstein Artin algebra. Then, we have*

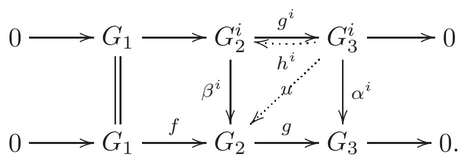
(1) *Any G-pure exact sequence is a direct limit of a system of split exact sequences in* $\operatorname{GProj} R$.

(2) $\mathcal{PP}$-GProj $R =$ Add Gproj R.
(3) PInj $R \cap$ GProj $R \subseteq \mathcal{PI}$-GProj R.
(4) $\mathcal{A}bs$-GProj $R =$ Proj R.

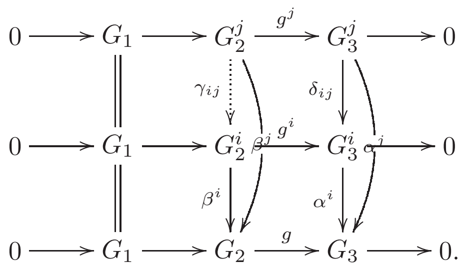
Proof. (1) Let

$$0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$$

be a G-pure exact sequence. Since R is a virtually Gorenstein Artin algebra, $G_3 \in$ GProj R can be written as $G_3 = \varinjlim G_3^i$ with all $G_3^i \in$ Gproj R by [4, Theorem 5]. Then, we can form the pullback of $G_2 \xrightarrow{g} G_3$ and the canonical homomorphism $G_3^i \xrightarrow{\alpha^i} G_3$ to get the following commutative diagram with exact rows:



By the definition of G-pure exact sequences, α^i factors through g, that is, there exits a homomorphism $u : G_3^i \to G_2$ such that $\alpha^i = gu$. Then by the universal property of pullbacks, there exists a unique homomorphism $h^i : G_3^i \to G_2^i$ such that $\beta^i h^i = u$ and $g^i h^i = 1_{G_3^i}$. So the upper exact sequence splits and $G_2^i \in$ GProj R. By the universal property of pullbacks again, there exists a unique homomorphism $\gamma_{ij} : G_2^j \to G_2^i$ for each $i < j$ such that $\delta_{ij} g^j = g^i \gamma_{ij}$ and $\beta^j = \beta^i \gamma_{ij}$.



Then clearly $G_2 \cong \varinjlim G_2^i$ and the assertion follows.

(2) It is easy to see that Add Gproj $R \subseteq \mathcal{PP}$-GProj R. So, it suffices to show $\mathcal{PP}$-GProj $R \subseteq$ Add Gproj R. Let $H \in \mathcal{PP}$-GProj R. Then, H can be written as $H = \varinjlim H_i$ with all $H_i \in$ Gproj R by [4, Theorem 5]. So the G-pure exact sequence

$$0 \to \operatorname{Ker} \pi \to \oplus H_i \xrightarrow{\pi} \varinjlim H_i \to 0$$

splits and $H \in$ Add GProj R. It follows that $\mathcal{PP}$-GProj $R \subseteq$ Add Gproj R.

(3) By (1) and Lemma 2.3, any G-pure exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is pure exact. Then for any $E \in \operatorname{PInj} R \cap \operatorname{GProj} R$, the sequence

$$0 \to \operatorname{Hom}_R(G_3, E) \to \operatorname{Hom}_R(G_2, E) \to \operatorname{Hom}_R(G_1, E) \to 0$$

is exact. So $E \in \mathcal{PI}$-$\operatorname{GProj} R$ and $\operatorname{PInj} R \cap \operatorname{GProj} R \subseteq \mathcal{PI}$-$\operatorname{GProj} R$.

(4) Note that for an Artin algebra R, $\operatorname{Ext}^1_R(-, A)|_{\operatorname{Gproj} R} = 0$ implies $\operatorname{Ext}^{\geq 1}_R(-, A)|_{\operatorname{Gproj} R} = 0$. Since $(\operatorname{Gproj} R)^{\perp} = (\operatorname{GProj} R)^{\perp}$ by [3, Remark 4.6], we have $\mathcal{A}bs$-$\operatorname{GProj} R = (\operatorname{Gproj} R)^{\perp} \cap \operatorname{GProj} R = (\operatorname{GProj} R)^{\perp} \cap \operatorname{GProj} R = \operatorname{Proj} R$ by Proposition 3.2. □

Chen conjectured in [8] that $\operatorname{GProj} R = \operatorname{Proj} R$ if R is a CM-free Artin algebra. We have the following:

Corollary 3.5. *If R is a* CM*-free Artin algebra, then* $\operatorname{GProj} R = \operatorname{Proj} R$ *if and only if R is virtually Gorenstein.*

Proof. By [3, Example 4.5.3], the necessity holds true. The sufficiency follows from Corollary 3.3(3) and Proposition 3.4(4). □

We introduce the notion of weak pure-injective Gorenstein projective modules as follows.

Definition 3.6. A module $E \in \operatorname{GProj} R$ is called *weak pure-injective in* $\operatorname{GProj} R$ if for any pure exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

in $\operatorname{GProj} R$, the sequence

$$0 \to \operatorname{Hom}_R(G_3, E) \to \operatorname{Hom}_R(G_2, E) \to \operatorname{Hom}_R(G_1, E) \to 0$$

is still exact.

We use $w\mathcal{PI}$-$\operatorname{GProj} R$ to denote the subcategory of $\operatorname{GProj} R$ consisting of weak pure-injective Gorenstein projective modules. Clearly, we have

$$\mathcal{PI}\text{-}\operatorname{GProj} R \subseteq w\mathcal{PI}\text{-}\operatorname{GProj} R.$$

For an Artin algebra R, we use $\mathbb{D}$ to denote the usual Matlis duality of R. All results in the rest of this paper are based on the following theorem.

Theorem 3.7. *If R is an Artin algebra, then*

$$\operatorname{Prod} \operatorname{Gproj} R \subseteq \operatorname{PInj} R \cap \operatorname{GProj} R = w\mathcal{PI}\text{-}\operatorname{GProj} R.$$

Proof. By [16, Corollary 1.2.22] and [7, Proposition 2.2.12(a)], we have

$$\operatorname{Prod} \operatorname{Gproj} R \subseteq \operatorname{PInj} R \cap \operatorname{GProj} R.$$

Clearly, $\operatorname{PInj} R \cap \operatorname{GProj} R \subseteq w\mathcal{PI}$-$\operatorname{GProj} R$. In the following, we prove the converse inclusion. Note that $\operatorname{PInj} R = \operatorname{Prod} \operatorname{mod} R$ and $R \in \operatorname{PInj} R$ by [16, Corollary 1.2.22]. It follows from the fact $\operatorname{Proj} R = \operatorname{Add} R = \operatorname{Prod} R$ that $\operatorname{Proj} R \subseteq \operatorname{PInj} R$.

Now, let $M \in w\mathcal{PI}\text{-}\operatorname{GProj} R (\subseteq \operatorname{GProj} R = \operatorname{GFlat} R)$. Then, we have a $\operatorname{Hom}_R(-, R)$-exact and $\mathbb{D}R \otimes_R -$-exact exact sequence

$$\mathbb{P} := \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\operatorname{Mod} R$ with all P_i and P^i in $\operatorname{Proj} R$ such that $M = \operatorname{Im}(P_0 \to P^0)$. It gives rise to the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{P} := \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \sigma_{P_1}} & & \downarrow{\scriptstyle \sigma_{P_0}} & & \downarrow{\scriptstyle \sigma_{P^0}} & & \downarrow{\scriptstyle \sigma_{P^1}} & & \\
\mathbb{P}^{++} := \cdots & \longrightarrow & P_1^{++} & \longrightarrow & P_0^{++} & \longrightarrow & (P^0)^{++} & \longrightarrow & (P^1)^{++} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{L} =: \cdots & \longrightarrow & \operatorname{Coker} \sigma_{P_1} & \longrightarrow & \operatorname{Coker} \sigma_{P_0} & \longrightarrow & \operatorname{Coker} \sigma_{P^0} & \longrightarrow & \operatorname{Coker} \sigma_{P^1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0. & &
\end{array}$$

All columns split because they are pure exact by Lemma 2.6(2). By the adjoint isomorphism theorem and [28, Lemma 3.55(ii)], we have that $\mathbb{P}^+$ is $\operatorname{Hom}_R(\mathbb{D}R, -)$-exact exact and $\mathbb{P}^{++}$ is $\mathbb{D}R \otimes_R -$-exact exact sequence. So, $\mathbb{L}$ is also $\mathbb{D}R \otimes_R -$-exact exact sequence in $\operatorname{Mod} R$ with all terms in $\operatorname{Proj} R$. Then by [21, Lemma 1.7], $\mathbb{L}$ is $\operatorname{Hom}_R(-, \operatorname{Proj} R)$-exact exact and $\operatorname{Coker} \sigma_M \cong \operatorname{Im}(\operatorname{Coker} \sigma_{P_0} \to \operatorname{Coker} \sigma_{P^0}) \in \operatorname{GProj} R$. Because $M \in w\mathcal{PI}\text{-}\operatorname{GProj} R$, the pure exact sequence

$$0 \to M \xrightarrow{\sigma_M} M^{++} \to \operatorname{Coker} \sigma_M \to 0$$

in $\operatorname{Mod} R$ with all terms in $\operatorname{GProj} R$ splits. By Lemma 2.7, we have $M^{++} \in \operatorname{PInj} R \cap \operatorname{GProj} R$. Thus, as a direct summand of M^{++}, $M \in \operatorname{PInj} R$ by Lemma 2.6(2). Therefore $M \in \operatorname{PInj} R \cap \operatorname{GProj} R$ and $w\mathcal{PI}\text{-}\operatorname{GProj} R \subseteq \operatorname{PInj} R \cap \operatorname{GProj} R$. □

As an application of Theorem 3.7, we give an equivalent characterization of pure-projective Gorenstein projective modules as follows.

Corollary 3.8. *Let R be an Artin algebra and $H \in \operatorname{GProj} R$. Then the following statements are equivalent.*

(1) $H \in \mathcal{PP}\text{-}\operatorname{GProj} R$.
(2) *H is projective relative to any G-pure exact sequence*

$$0 \to G_1 \to E \to G_3 \to 0$$

in $\operatorname{GProj} R$ *with* $E \in w\mathcal{PI}\text{-}\operatorname{GProj} R$.

Proof. (1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (1) For any G-pure exact sequence

$$0 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 0$$

in $\operatorname{GProj} R$, we can form the pushout of $G_2 \xrightarrow{f_2} G_3$ and $G_2 \xrightarrow{\sigma_{G_2}} G_2^{++}$ to get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma_{G_2}} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & G_1 & \longrightarrow & G_2^{++} & \xrightarrow{\alpha} & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Coker}\sigma_{G_2} & = & \operatorname{Coker}\sigma_{G_2} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

By Lemma 2.6(2) and Theorem 3.7, we have that

$$0 \to G_2 \xrightarrow{\sigma_{G_2}} G_2^{++} \to \operatorname{Coker}\sigma_{G_2} \to 0$$

is a pure exact sequence in $\operatorname{Mod} R$ with all terms in $\operatorname{GProj} R$ and $G_2^{++} \in w\mathcal{PI}$-$\operatorname{GProj} R$. By [19, Theorem 2.5] and the exactness of

$$0 \to G_3 \xrightarrow{g} D \to \operatorname{Coker}\sigma_{G_2} \to 0$$

we have $D \in \operatorname{GProj} R$. Then, it is easy to see that

$$0 \to G_1 \to G_2^{++} \xrightarrow{\alpha} D \to 0$$

is G-pure exact.

Let $H \in \operatorname{GProj} R$ and $u \in \operatorname{Hom}_R(H, G_3)$. Then there exists $v \in \operatorname{Hom}_R(H, G_2^{++})$ such that $gu = \alpha v$. Note that the above pushout diagram is also a pullback one. So there exists $w \in \operatorname{Hom}_R(H, G_2)$ such that $u = f_2 w$. Thus

$$0 \to \operatorname{Hom}_R(H, G_1) \xrightarrow{\operatorname{Hom}_R(H,f_1)} \operatorname{Hom}_R(H, G_2) \xrightarrow{\operatorname{Hom}_R(H,f_2)} \operatorname{Hom}_R(H, G_3) \to 0$$

is exact and $H \in \mathcal{PP}$-$\operatorname{GProj} R$. □

It is easy to see that $\mathcal{A}bsR \cap \operatorname{GProj} R \subseteq \mathcal{A}bs$-$\operatorname{GProj} R$. However, we do not know, whether there exists an inclusion relation between $\operatorname{PInj} R \cap \operatorname{GProj} R$ and $\mathcal{PI}$- $\operatorname{GProj} R$ in general. The following result shows that these two categories coincide over virtually Gorenstein Artin algebras.

Corollary 3.9. *If R is a virtually Gorenstein Artin algebra, then*

$$\operatorname{PInj} R \cap \operatorname{GProj} R = w\mathcal{PI}\text{-}\operatorname{GProj} R = \mathcal{PI}\text{-}\operatorname{GProj} R = \operatorname{Prod}(\operatorname{Ginj} R^{op})^{+}.$$

Proof. Let R be a virtually Gorenstein Artin algebra. Then G-pure exact sequences and pure exact sequences coincide by Lemma 2.3 and Proposition 3.4. So $\operatorname{PInj} R \cap \operatorname{GProj} R = w\mathcal{PI}$-$\operatorname{GProj} R = \mathcal{PI}$-$\operatorname{GProj} R$ by Theorem 3.7.

Because R^{op} is also virtually Gorenstein by [2, Theorem 8.7], we have $\operatorname{GInj} R^{op} \subseteq \varinjlim \operatorname{Ginj} R^{op}$ by [4, Theorem 5]. So

$$(\operatorname{GProj} R)^{+} \subseteq \operatorname{GInj} R^{op} \subseteq \varinjlim \operatorname{Ginj} R^{op} \text{ and } (\operatorname{Ginj} R^{op})^{+} \subseteq \operatorname{GProj} R$$

by [19, Theorem 3.6]. In addition, $\operatorname{GProj} R$ is closed under direct summands and products by [7, Proposition 2.2.12(a)]. Now it follows from [20, Corollary 5.3] that $\operatorname{PInj} R \cap \operatorname{GProj} R = \operatorname{Prod}(\operatorname{Ginj} R^{op})^{+}$. □

Recall from [3], that an Artin algebra R is called CM-*finite*, if the number of indecomposable modules in $\operatorname{Gproj} R$ is finite up to isomorphism.

Corollary 3.10. *Let R be a* CM-*finite Artin algebra. Then*

$$\varinjlim \operatorname{Gproj} R \subseteq w\mathcal{PI}\text{-}\operatorname{GProj} R.$$

Proof. Let R be a CM-finite Artin algebra. Then $\operatorname{Gproj} R = \operatorname{add} G$ for some $G \in \operatorname{Gproj} R$. So, by [24, Lemma 1.2] and Theorem 3.7, we have

$$\varinjlim \operatorname{Gproj} R = \varinjlim \operatorname{add} G = \operatorname{Prod} \operatorname{add} G \subseteq w\mathcal{PI}\text{-}\operatorname{GProj} R. \qquad \square$$

In the following, we give an equivalent characterization of virtually Gorenstein Artin algebras. We need the following:

Lemma 3.11. *Let R be an Artin algebra. Then $\varinjlim \operatorname{Gproj} R$ is closed under pure submodules.*

Proof. Let

$$0 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 0 \tag{3.2}$$

be a pure exact sequence in $\operatorname{Mod} R$ with $G_2 \in \varinjlim \operatorname{Gproj} R$. Then by [16, Lemma 1.2.9], we have that for any $u : M \to G_1$ with $M \in \operatorname{mod} R$, $f_1 u$ factors through some $G \in \operatorname{Gproj} R$, that is, there exist $s : M \to G$ and $w : G \to G_2$ such that $f_1 u = ws$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & M & \xrightarrow{s} & G & \xrightarrow{t} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle w} & & \vdots{\scriptstyle v} & & \\
0 & \longrightarrow & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & G_3 & \longrightarrow & 0,
\end{array}$$

where $N = \operatorname{Coker} s$ and v is an induced homomorphism. Since the bottom row in this diagram (that is, (3.2)) is pure exact and $N \in \operatorname{mod} R$, there exists $h : N \to G_2$

such that $v = f_2h$. Notice that $f_2(w - ht) = f_2w - f_2ht = f_2w - vt = 0$. So $w - ht$ factors through f_1, that is, there exists $k : G \to G_1$ such that $w - ht = f_1k$. Since f_1 is monic and $f_1(ks-u) = f_1ks - f_1u = (w-ht)s - f_1u = ws - hts - f_1u = ws - f_1u = 0$, we have $ks - u = 0$ and $ks = u$, that is, u factors through $G(\in \operatorname{Gproj} R)$. Thus $G_1 \in \varinjlim \operatorname{Gproj} R$ by [16, Lemma 1.2.9]. □

Beligiannis and Krause proved in [4, Theorem 5] that an Artin algebra R is virtually Gorenstein if and only if $\operatorname{GProj} R \subseteq \varinjlim \operatorname{Gproj} R$. The following theorem extends this result.

Theorem 3.12. *Let R be an Artin algebra. Then the following statements are equivalent.*

(1) *R is virtually Gorenstein.*
(2) *$w\mathcal{PI}$-$\operatorname{GProj} R \subseteq \varinjlim \operatorname{Gproj} R$.*

Proof. (1) ⇒ (2) By [4, Theorem 5].

(2) ⇒ (1) Let $G \in \operatorname{GProj} R$. Consider the pure exact sequence:

$$0 \to G \xrightarrow{\sigma_G} G^{++} \to \operatorname{Coker} \sigma_G \to 0.$$

By Lemma 2.7 and Theorem 3.7, we have $G^{++} \in \operatorname{PInj} R \cap \operatorname{GProj} R = w\mathcal{PI}$-$\operatorname{GProj} R$. So $G^{++} \in \varinjlim \operatorname{Gproj} R$ by (2), and hence $G \in \varinjlim \operatorname{Gproj} R$ by Lemma 3.11. It follows from [4, Theorem 5] that R is virtually Gorenstein. □

By Corollary 3.10 and Theorem 3.12, we have the following:

Corollary 3.13. *Let R be a virtually Gorenstein* CM-*finite Artin algebra. Then*

$$w\mathcal{PI}\text{-}\operatorname{GProj} R = \varinjlim \operatorname{Gproj} R.$$

Recall from [23], that a subcategory of $\operatorname{Mod} R$ is called *definable*, if it is closed under direct limits, direct products and pure submodules in $\operatorname{Mod} R$. The following result gives an equivalent characterization for virtually Gorenstein Artin algebras being CM-finite. It induces that both $\mathcal{PI}$-$\operatorname{GProj} R$ and $w\mathcal{PI}$-$\operatorname{GProj} R$ are proper subcategories of $\operatorname{GProj} R$ in general.

Theorem 3.14. *Let R be a virtually Gorenstein Artin algebra. Then the following statements are equivalent.*

(1) *R is* CM-*finite.*
(2) *$\mathcal{PI}$-$\operatorname{GProj} R = \operatorname{GProj} R$.*

Proof. Let R be a virtually Gorenstein Artin algebra. Then $\operatorname{GProj} R$ is definable by [2, Proposition 3.8]. It follows from [3, Theorem 4.10] and [23, Corollary 2.7] that R is CM-finite if and only if $\operatorname{GProj} R \subseteq \operatorname{PInj} R$. Now the assertion follows from Corollary 3.9. □

By [2, Theorem 3.5], we have that $\operatorname{GProj} R$ is covering and enveloping in $\operatorname{Mod} R$. For a module $M \in \operatorname{Mod} R$, we use

$$\alpha_M : G_M \to M \text{ and } \alpha^M : M \to G^M$$

to denote the $\operatorname{GProj} R$-cover and the $\operatorname{GProj} R$-envelope of M, respectively. By Theorem 3.7, we also have the following

Proposition 3.15. *Let R be an Artin algebra. If $\mathcal{PI}$-$\operatorname{GProj} R = w\mathcal{PI}$-$\operatorname{GProj} R$, then, we have*

(1) *Any G-pure exact sequence is pure exact.*
(2) $\varinjlim \operatorname{Gproj} R$ *is closed under G-pure submodules.*

Proof. (1) Let

$$0 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 0 \tag{3.3}$$

be a G-pure exact sequence in $\operatorname{Mod} R$ with all terms in $\operatorname{GProj} R$. Let $M \in \operatorname{Mod} R$ be pure-injective. By [25, Lemma 2.2], both G_M and $\operatorname{Ker} \alpha_M$ are pure-injective. So $G_M \in \operatorname{PInj} R \cap \operatorname{GProj} R = w\mathcal{PI}$-$\operatorname{GProj} R = \mathcal{PI}$-$\operatorname{GProj} R$ by Theorem 3.7 and assumption.

Consider the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}_R(G_3, \operatorname{Ker} \alpha_M) & \to & \operatorname{Hom}_R(G_2, \operatorname{Ker} \alpha_M) & \to & \operatorname{Hom}_R(G_1, \operatorname{Ker} \alpha_M) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}_R(G_3, G_M) & \to & \operatorname{Hom}_R(G_2, G_M) & \to & \operatorname{Hom}_R(G_1, G_M) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}_R(G_3, M) & \xrightarrow{\operatorname{Hom}_R(f_2, M)} & \operatorname{Hom}_R(G_2, M) & \xrightarrow{\operatorname{Hom}_R(f_1, M)} & \operatorname{Hom}_R(G_1, M) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Then

$$0 \to \operatorname{Hom}_R(G_3, M) \xrightarrow{\operatorname{Hom}_R(f_2, M)} \operatorname{Hom}_R(G_2, M) \xrightarrow{\operatorname{Hom}_R(f_1, M)} \operatorname{Hom}_R(G_1, M) \to 0$$

is exact.

Let $N \in \operatorname{mod} R^{op}$. Then $N^+ \in \operatorname{PInj} R$ by Lemma 2.6(1). Note that for any $1 \le i \le 3$, we have

$$(N \otimes_R G_i)^+ \cong \operatorname{Hom}_R(G_i, N^+)$$

by the adjoint isomorphism theorem. By the above argument, we have an exact sequence:

$$0 \to \mathrm{Hom}_R(G_3, N^+) \xrightarrow{\mathrm{Hom}_R(f_2,N^+)} \mathrm{Hom}_R(G_2, N^+)$$
$$\xrightarrow{\mathrm{Hom}_R(f_1,N^+)} \mathrm{Hom}_R(G_1, N^+) \to 0$$

which induces the following exact sequence:

$$0 \to N \otimes_R G_1 \xrightarrow{N\otimes f_1} N \otimes_R G_2 \xrightarrow{N\otimes f_2} N \otimes_R G_3 \to 0.$$

Thus (3.3) is pure exact.

(2) By (1) and Lemma 3.11. □

By Corollary 3.9, if R is a virtually Gorenstein Artin algebra, then the assumption in Proposition 3.15 is satisfied.

For a subcategory $\mathcal{C}$ of $\mathrm{Mod}\, R$ and an object $M \in \mathcal{C}$, the stable objects of M in $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are denoted by $\underline{M}$ and $\overline{M}$, respectively.

If R is an Artin algebra, then $\underline{\mathrm{GProj}}R$ is a compactly generated triangulated category by [6, Remark 4.8] and [2, Theorem 6.6]. Moreover, an Artin algebra R is virtually Gorenstein if and only if $\underline{\mathrm{Gproj}}R$ coincides with the compact subcategory of $\underline{\mathrm{GProj}}R$ by [2, Theorem 8.2].

For a compactly generated triangulated category $\mathcal{T}$, there is a theory of purity which parallels that of module categories [22]. A triangle

$$A \to B \to C \to A[1] \tag{3.4}$$

in $\mathcal{T}$ is called *pure*, if for any compact object X in $\mathcal{T}$,

$$0 \to \mathcal{T}(X, A) \to \mathcal{T}(X, B) \to \mathcal{T}(X, C) \to 0$$

is exact. An object $E \in \mathcal{T}$ is called *pure-injective*, if for any pure triangle as (3.4),

$$0 \to \mathcal{T}(C, E) \to \mathcal{T}(B, E) \to \mathcal{T}(A, E) \to 0$$

is exact.

Let $\mathcal{A}$ be an abelian category. We use $D^+(\mathcal{A})$ to denote the bounded below derived category of $\mathcal{A}$. Let $\mathrm{Inj}\,\mathcal{A}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects. We use $K^+(\mathrm{Inj}\,\mathcal{A})$ to denote the category whose objects are bounded below complexes of injective objects and whose morphisms are morphisms of complexes modulo homotopic equivalence.

It was proved in [18, Theorem 23] that the functor

$$\mathrm{Ext}^1_R(-, ?) : \overline{\mathrm{PInj}}R \to ((\underline{\mathrm{mod}}R)^{op}, \mathcal{A}b) \text{ via } \overline{N} \mapsto \mathrm{Ext}^1_R(-, N)$$

yields an equivalence between $\overline{\mathrm{PInj}}R$ and the subcategory of injective objects of $((\underline{\mathrm{mod}}R)^{op}, \mathcal{A}b)$. Then by [15, Theorem 3.5.21], there exists the following equivalence of triangulated categories:

$$K^+(\overline{\mathrm{PInj}}R) \simeq D^+((\underline{\mathrm{mod}}R)^{op}, \mathcal{A}b).$$

Put

$$\underline{\mathrm{Hom}}_R(X,Y) := \mathrm{Hom}_R(X,Y)/\{\text{the morphism } X \to Y \text{ factoring through some projective left } R\text{-module}\}.$$

We use $\mathcal{PI}$-$\underline{\mathrm{GProj}}R$ to denote the full subcategory of $\underline{\mathrm{GProj}}R$ consisting of pure-injective objects. We have the following comparable result for a virtually Gorenstein Artin algebra R, which means that $\mathcal{PI}$- GProj R plays a role in GProj R similar to PInj R does in Mod R.

Proposition 3.16. *Let R be a virtually Gorenstein Artin algebra. Then the following statements are equivalent.*

(1) $M \in \mathcal{PI}$-GProj R.
(2) $\underline{M}$ *is pure-injective in* $\underline{\mathrm{GProj}}R$.
(3) $\underline{\mathrm{Hom}}_R(-,M)(\cong \mathrm{Ext}^1_R(-,\Omega M))$ *is injective in* $((\underline{\mathrm{Gproj}}R)^{op}, \mathcal{A}b)$, *where* ΩM *is the first syzygy of* M.

Consequently, there exists the following equivalence of triangulated categories:

$$K^+(\underline{\mathcal{PI}\text{- GProj}R}) \simeq D^+((\underline{\mathrm{Gproj}}R)^{op}, \mathcal{A}b).$$

Proof. (1) ⇔ (2) follows from Corollary 3.9 and [2, Lemma 6.4].

(2) ⇔ (3) follows from [22, Corollary 1.9].

Now, we have

$$\begin{aligned}&\underline{\mathcal{PI}\text{- GProj}R}\\ &\quad = \mathcal{PI}\text{-}\underline{\mathrm{GProj}}R \text{ (by the equivalence between (1) and (2))}\\ &\quad \simeq \mathrm{Inj}((\underline{\mathrm{Gproj}}R)^{op}, \mathcal{A}b) \text{ (by [22, Corollary 1.9])}\end{aligned}$$

and hence, we have

$$\begin{aligned}&K^+(\underline{\mathcal{PI}\text{- GProj}R})\\ &\quad = K^+(\mathcal{PI}\text{-}\underline{\mathrm{GProj}}R)\\ &\quad \simeq K^+(\mathrm{Inj}((\underline{\mathrm{Gproj}}R)^{op}, \mathcal{A}b))\\ &\quad \simeq D^+((\underline{\mathrm{Gproj}}R)^{op}, \mathcal{A}b) \text{ (by [15, Theorem 3.5.21]).}\end{aligned}$$ □

4. Envelopes and Covers

As further applications of Theorem 3.7, we study in this section, the covering and enveloping properties of $w\mathcal{PI}$-GProj R. It should be remarked that all results obtained here rely on the fact that $\mathrm{GProj}\, R$ is covering and enveloping in $\mathrm{Mod}\, R$ [2, Theorem 3.5].

We first prove the following:

Proposition 4.1. *If R is an Artin algebra, then $w\mathcal{PI}$-$\mathrm{GProj}\, R$ is enveloping in $\mathrm{Mod}\, R$.*

Proof. Let $M \in \mathrm{Mod}\, R$ and $v : M \to X$ be in $\mathrm{Mod}\, R$ with $X \in w\mathcal{PI}$-$\mathrm{GProj}\, R (\subseteq \mathrm{GProj}\, R)$. We claim that $\sigma_{G^M}\alpha^M : M \to (G^M)^{++}$ is a $w\mathcal{PI}$-$\mathrm{GProj}\, R$-preenvelope of M. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^M & \xrightarrow{\sigma_{G^M}} & (G^M)^{++} & \longrightarrow & \mathrm{Coker}\, \sigma_{G^M} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha^M} & & & & & & \\ & & M & \xrightarrow{v} & X. & & & & \end{array}$$

Then there exists $\delta : G^M \to X$ such that $v = \delta\alpha^M$. Because $X \in \mathrm{PInj}\, R$ and the upper row in the above diagram is pure exact by Theorem 3.7 and Lemma 2.6(2) respectively, there exists $\gamma : (G^M)^{++} \to X$ such that $\delta = \gamma\sigma_{G^M}$. So $v = \delta\alpha^M = \gamma(\sigma_{G^M}\alpha^M)$ and $\sigma_{G^M}\alpha^M$ is a $w\mathcal{PI}$-$\mathrm{GProj}\, R$-preenvelope of M. The claim follows.

Moreover, note that $w\mathcal{PI}$-$\mathrm{GProj}\, R$ is closed under direct summands. Then from [23, Theorem 3.14], we conclude that $w\mathcal{PI}$-$\mathrm{GProj}\, R$ is enveloping in $\mathrm{Mod}\, R$. □

To give the covering property of $w\mathcal{PI}$-$\mathrm{GProj}\, R$, we need the following:

Lemma 4.2. *If R is a virtually Gorenstein Artin algebra, then the $\mathrm{GProj}\, R$-cover of any finitely generated R-module is finitely generated.*

Proof. By [2, Corollary 8.3], R is virtually Gorenstein if and only if the $\mathrm{GProj}\, R$-cover of $R/\mathfrak{r}$ is finitely generated, where $\mathfrak{r}$ is the Jacobson radical of R. Since $R/\mathfrak{r}$ is a semisimple R-module containing each simple R-module as a summand, it follows that the $\mathrm{GProj}\, R$-cover of any simple R-module is finitely generated by assumption.

Let $M \in \mathrm{mod}\, R$ and let

$$0 \to K \to G_M \xrightarrow{\alpha_M} M \to 0$$

be an exact sequence in $\mathrm{Mod}\, R$ with α_M the $\mathrm{GProj}\, R$-cover of M. We will prove that G_M is finitely generated by using induction on the length $l(M)$ of M. Let $l(M) = n$. If $n = 1$, then the assertion holds true trivially. Now suppose $n \geq 2$. Let M_1 be a maximal submodule of M. Then, we have an exact sequence

$$0 \to M_1 \to M \xrightarrow{\pi} S \to 0$$

in $\mathrm{mod}\, R$ with $S(= M/M_1)$ simple. By the Wakamatsu lemma (c.f. [14, Corollary 7.2.3]), we have $\mathrm{Ker}\, \alpha_{M_1} \in (\mathrm{GProj}\, R)^{\perp}$. Then, it is easy to get

$$\mathrm{Ext}^1_R(G_S, G_{M_1}) \cong \mathrm{Ext}^1_R(G_S, M_1). \tag{4.1}$$

Now, we construct the following commutative diagram with exact rows as follows. First take the pullback of π and α_S to get the lower commutative diagram, and then the isomorphism (4.1) gives rise to the upper commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_{M_1} & \longrightarrow & G & \longrightarrow & G_S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{M_1}} & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & L & \longrightarrow & G_S & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \alpha_S} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \xrightarrow{\pi} & S & \longrightarrow & 0.
\end{array}$$

Since both $\operatorname{Ker}\alpha_{M_1}$ and $\operatorname{Ker}\alpha_S$ are in $(\operatorname{GProj} R)^{\perp}$ by the Wakamatsu lemma again, so is $\operatorname{Ker} fg$. Then the epimorphism $fg : G \to M$ is a $\operatorname{GProj} R$-precover of M. Because both G_{M_1} and G_S are finitely generated by the induction hypothesis, so is G. Therefore, the $\operatorname{GProj} R$-cover of M, as a summand of G, is finitely generated. □

Now, we are in a position to prove the following:

Proposition 4.3. *If R is a virtually Gorenstein Artin algebra, then $w\mathcal{PI}$-$\operatorname{GProj} R$ is precovering in* $\operatorname{PInj} R$.

Proof. Because $\operatorname{PInj} R = \operatorname{Prod} \operatorname{mod} R$, it suffices to show that any $M = \prod_{i\in I} M_i$ with $M_i \in \operatorname{mod} R$ has a $w\mathcal{PI}$-$\operatorname{GProj} R$-precover by [14, Exercise 3, p. 106]. Put $\alpha := \prod_{i\in I} \alpha_{M_i}$. Since R is virtually Gorenstein, each G_{M_i} is finitely generated by Lemma 4.2. So $\prod_{i\in I} G_{M_i} \in \operatorname{Prod} \operatorname{Gproj} R \subseteq w\mathcal{PI}$-$\operatorname{GProj} R$ by Theorem 3.7.

We claim that α is a $w\mathcal{PI}$-$\operatorname{GProj} R$-precover of M. Let $u : E \to M$ in $\operatorname{Mod} R$ with $E \in w\mathcal{PI}$-$\operatorname{GProj} R (\subseteq \operatorname{GProj} R)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
 & & E \\
 & & \downarrow{\scriptstyle u} \\
\prod_{i\in I} G_{M_i} & \xrightarrow{\alpha} & \prod_{i\in I} M_i \\
\downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \pi_i} \\
G_{M_i} & \xrightarrow{\alpha_{M_i}} & M_i,
\end{array}$$

where for any $i \in I$, p_i and π_i are the ith projections of $\prod_{i\in I} G_{M_i}$ and $\prod_{i\in I} M_i$, respectively. Then there exists $\beta : E \to G_{M_i}$ such that $\pi_i u = \alpha_{M_i}\beta$. By the universal property of direct products, there exists $\gamma : E \to \prod_{i\in I} G_{M_i}$ such that $\beta = p_i\gamma$. So we have $\pi_i u = \alpha_{M_i}\beta = \alpha_{M_i} p_i \gamma = \pi_i \alpha\gamma$ for any $i \in I$. It induces that $u = \alpha\gamma$ and the claim follows. □

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