Journal of Algebra 324 (2010) 3408–3419

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Gorenstein syzygy modules

Chonghui Huang [a,b], Zhaoyong Huang [b,*]

[a] *Research Institute of Mathematics, University of South China, Hengyang 421001, PR China*
[b] *Department of Mathematics, Nanjing University, Nanjing 210093, PR China*

ARTICLE INFO

Article history:
Received 26 March 2009
Available online 23 October 2010
Communicated by Efim Zelmanov

MSC:
16E05
16E10

Keywords:
Gorenstein projective modules
Gorenstein n-syzygy modules
n-syzygy modules
Gorenstein projective dimension
Gorenstein transpose

ABSTRACT

For any ring R and any positive integer n, we prove that a left R-module is a Gorenstein n-syzygy if and only if it is an n-syzygy. Over a left and right Noetherian ring, we introduce the notion of the Gorenstein transpose of finitely generated modules. We prove that a module $M \in \operatorname{mod} R^{op}$ is a Gorenstein transpose of a module $A \in \operatorname{mod} R$ if and only if M can be embedded into a transpose of A with the cokernel Gorenstein projective. Some applications of this result are given.

1. Introduction

Throughout this paper, R is an associative ring with identity and Mod R is the category of left R-modules.

In classical homological algebra, the notion of finitely generated projective modules is an important and fundamental research object. As a generalization of this notion, Auslander and Bridger introduced in [AB] the notion of finitely generated modules of Gorenstein dimension zero over a left and right Noetherian ring. Over a general ring, Enochs and Jenda introduced in [EJ1] the notion of Gorenstein projective modules (not necessarily finitely generated). It is well known that these two notions coincide for finitely generated modules over a left and right Noetherian ring. In particular, Gorenstein projective modules share many nice properties of projective modules (e.g. [AB,C,CFH,CI,EJ1,EJ2,H]).

* Corresponding author.
E-mail address: huangzy@nju.edu.cn (Z. Huang).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.10.010

The notion of a syzygy module was defined via the projective resolution of modules as follows. For a positive integer n, a module $A \in \mathrm{Mod}\ R$ is called an *n-syzygy module* (of M) if there exists an exact sequence $0 \to A \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ in Mod R with all P_i projective. Analogously, we call A a *Gorenstein n-syzygy module* (of M) if there exists an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in Mod R with all G_i Gorenstein projective. It is trivial that an n-syzygy module is Gorenstein n-syzygy. In Section 2, our main result is that for every $n \geqslant 1$, a Gorenstein n-syzygy module is n-syzygy. The following auxiliary proposition plays a crucial role in proving this main result. Let $0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0$ be an exact sequence in Mod R with G_0 and G_1 Gorenstein projective. Then we have the following exact sequences $0 \to A \to P \to G \to M \to 0$ and $0 \to A \to H \to Q \to M \to 0$ in Mod R with P, Q projective and G, H Gorenstein projective.

In Section 3, for a left and right Noetherian ring R and a finitely generated left R-module A, we introduce the notion of the Gorenstein transpose of A, which is a Gorenstein version of that of the transpose of A. We establish a relation between a Gorenstein transpose of a module and a transpose of the same module. We prove that a finitely generated right R-module M is a Gorenstein transpose of a finitely generated left R-module A if and only if M can be embedded into a transpose of A with the cokernel Gorenstein projective. Then we give some applications of this result: (1) The direct sum of a finitely generated Gorenstein projective right R-module and a transpose of a finitely generated left R-module A is a Gorenstein transpose of A. (2) For any Gorenstein transpose and any transpose of a finitely generated left R-module, one of them is n-torsionfree if and only if so is the other. (3) A finitely generated left R-module with Gorenstein projective dimension n is a double Gorenstein transpose of a finitely generated left R-module with projective dimension n.

2. Gorenstein syzygy modules

Recall from [EJ1] a module $G \in \mathrm{Mod}\ R$ is called *Gorenstein projective* if there exists an exact sequence in Mod R:

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots,$$

such that: (1) All P_i and P^i are projective; (2) After applying the functor $\mathrm{Hom}_R(\ ,P)$ the sequence is still exact for any projective module $P \in \mathrm{Mod}\ R$; and (3) $G \cong \mathrm{Im}(P_0 \to P^0)$. Let M be a module in Mod R. The *Gorenstein projective dimension* of M, denoted by $\mathrm{Gpd}_R(M)$, is defined as $\inf\{n \mid$ for any exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in Mod R with all G_i Gorenstein projective$\}$. We have $\mathrm{Gpd}_R(M) \geqslant 0$ and we set $\mathrm{Gpd}_R(M)$ infinity if no such integer exists (see [EJ1] or [H]).

Lemma 2.1. *Let $0 \to M_3 \to M_2 \to M_1 \to 0$ be an exact sequence in* Mod R *with $M_3 \neq 0$. If M_1 is Gorenstein projective, then* $\mathrm{Gpd}_R(M_3) = \mathrm{Gpd}_R(M_2)$.

Proof. By [H, Theorems 2.24 and 2.20], it is easy to get the assertion. □

The following result plays a crucial role in this paper.

Proposition 2.2. *Let $0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0$ be an exact sequence in* Mod R *with G_0 and G_1 Gorenstein projective. Then we have the following exact sequences*:

$$0 \to A \to P \to G \to M \to 0,$$

and

$$0 \to A \to H \to Q \to M \to 0,$$

in Mod R *with P, Q projective and G, H Gorenstein projective.*

Proof. Because G_1 is Gorenstein projective, there exists an exact sequence $0 \to G_1 \to P \to G_2 \to 0$ in Mod R with P projective and G_2 Gorenstein projective. Then we have the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d] & \operatorname{Im} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{Im} f \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G_2 \arrow[r, equal] \arrow[d] & G_2 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Because both G_0 and G_2 are Gorenstein projective, G is also Gorenstein projective by Lemma 2.1. Connecting the middle rows in the above two diagrams, then we get the first desired exact sequence. Since G_0 is Gorenstein projective, there exists an exact sequence $0 \to G_3 \to Q \to G_0 \to 0$ in Mod R with Q projective and G_3 Gorenstein projective. Dually, taking pull-back, one gets the second desired exact sequence. $\square$

For a positive integer n, recall that a module $A \in \operatorname{Mod} R$ is called an *n-syzygy module* (of M) if there exists an exact sequence $0 \to A \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ in Mod R with all P_i projective. Analogously, we give the following

Definition 2.3. For a positive integer n, a module $A \in \operatorname{Mod} R$ is called a *Gorenstein n-syzygy module* (of M) if there exists an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in Mod R with all G_i Gorenstein projective.

The following theorem is the main result in this section.

Theorem 2.4. *Let n be a positive integer and $0 \to A \to G_{n-1} \to G_{n-2} \to \cdots \to G_0 \to M \to 0$ an exact sequence in* Mod R *with all G_i Gorenstein projective. Then we have the following*:

(1) *There exist exact sequences $0 \to A \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to N \to 0$ and $0 \to M \to N \to G \to 0$ in* Mod R *with all P_i projective and G Gorenstein projective. In particular, a module in* Mod R *is an n-syzygy if and only if it is a Gorenstein n-syzygy.*
(2) *There exist exact sequences $0 \to B \to Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to M \to 0$ and $0 \to H \to B \to A \to 0$ in* Mod R *with all Q_i projective and H Gorenstein projective.*

Proof. (1) We proceed by induction on n. When $n = 1$, it has been proved in the proof of Proposition 2.2. Now suppose that $n \geqslant 2$ and we have an exact sequence:

$$0 \to A \to G_{n-1} \to G_{n-2} \to \cdots \to G_0 \to M \to 0$$

in Mod R with all G_i Gorenstein projective. Put $K = \mathrm{Coker}(G_{n-1} \to G_{n-2})$. By Proposition 2.2, we get an exact sequence:

$$0 \to A \to P_{n-1} \to G'_{n-2} \to K \to 0$$

in Mod R with P_{n-1} projective and G'_{n-2} Gorenstein projective. Put $A' = \mathrm{Im}(P_{n-1} \to G'_{n-2})$. Then we get an exact sequence:

$$0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_0 \to M \to 0$$

in Mod R. So, by the induction hypothesis, we get the assertion.

(2) The proof is dual to that of (1), so we omit it. □

For a module $M \in$ Mod R, we use $\mathrm{pd}_R(M)$ to denote the projective dimension of M.

Corollary 2.5. *(See [CFH, Lemma 2.17].) Let $M \in$* Mod R *and n be a non-negative integer. If* $\mathrm{Gpd}_R(M) = n$, *then there exists an exact sequence $0 \to M \to N \to G \to 0$ in* Mod R *with* $\mathrm{pd}_R(N) = n$ *and G Gorenstein projective.*

Proof. Let $M \in$ Mod R with $\mathrm{Gpd}_R(M) = n$. Then one uses Theorem 2.4(1) with $A = 0$ to get an exact sequence $0 \to M \to N \to G \to 0$ in Mod R with $\mathrm{pd}_R(N) \leqslant n$ and G Gorenstein projective. By Lemma 2.1, $\mathrm{Gpd}_R(N) = n$, and thus $\mathrm{pd}_R(N) = n$. □

By [H, Theorem 2.20], we have that $\mathrm{Gpd}_R(M) \leqslant n$ if and only if there exists an exact sequence $0 \to G_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ in Mod R with all P_i projective and G_n Gorenstein projective. The following theorem generalizes this result. In particular, the following theorem was proved by Christensen and Iyengar in [CI, Theorem 3.1] when R is a commutative Noetherian ring.

Theorem 2.6. *Let $M \in$* Mod R *and n be a non-negative integer. Then the following statements are equivalent.*

(1) $\mathrm{Gpd}_R(M) \leqslant n$.
(2) *For every non-negative integer t such that $0 \leqslant t \leqslant n$, there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in* Mod R *such that X_t is Gorenstein projective and X_i is projective for $i \neq t$.*

Proof. (2) ⇒ (1) It is trivial.

(1) ⇒ (2) We proceed by induction on n. Suppose $\mathrm{Gpd}_R(M) \leqslant 1$. Then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in Mod R with G_0 and G_1 Gorenstein projective. By Proposition 2.2

with $A = 0$, we get the exact sequences $0 \to P_1 \to G'_0 \to M \to 0$ and $0 \to G'_1 \to P_0 \to M \to 0$ in Mod R with P_0, P_1 projective and G'_0 and G'_1 Gorenstein projective.

Now suppose $n \geqslant 2$. Then there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in Mod R with G_i Gorenstein projective for any $1 \leqslant i \leqslant n$. Set $A = \mathrm{Coker}(G_3 \to G_2)$. By applying Proposition 2.2 to the exact sequence $0 \to A \to G_1 \to G_0 \to M \to 0$, we get an exact sequence $0 \to G_n \to \cdots \to G_2 \to G'_1 \to P_0 \to M \to 0$ in Mod R with G'_1 Gorenstein projective and P_0 projective. Set $N = \mathrm{Coker}(G_2 \to G'_1)$. Then we have $\mathrm{Gpd}_R(N) \leqslant n-1$. By the induction hypothesis, there exists an exact sequence $0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to P_0 \to M \to 0$ in Mod R such that P_0 is projective and X_t is Gorenstein projective and X_i is projective for $i \neq t$ and $1 \leqslant t \leqslant n$.

Now we need only to prove (2) for $t = 0$. Set $B = \mathrm{Coker}(G_2 \to G_1)$. By the induction hypothesis, we get an exact sequence $0 \to P_n \to \cdots \to P_3 \to P_2 \to G'_1 \to B \to 0$ in Mod R with G'_1 Gorenstein projective and P_i projective for any $2 \leqslant i \leqslant n$. Set $C = \mathrm{Coker}(P_3 \to P_2)$. Then by applying Proposition 2.2 to the exact sequence $0 \to C \to G'_1 \to G_0 \to M \to 0$, we get an exact sequence $0 \to C \to P_1 \to G'_0 \to M \to 0$ in Mod R with P_0 projective and G'_0 Gorenstein projective. Thus we obtain the desired exact sequence $0 \to P_n \to \cdots \to P_2 \to P_1 \to G'_0 \to M \to 0$. $\square$

Let $\mathscr{X}$ be a full subcategory of Mod R. Recall from [EJ2] that a morphism $f : X \to M$ in Mod R with $X \in \mathscr{X}$ is called an *$\mathscr{X}$-precover* of M if $\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X', f)} \mathrm{Hom}_R(X', M) \to 0$ is exact for any $X' \in \mathscr{X}$. We use $\mathscr{GP}(R)$ to denote the full subcategory of Mod R consisting of Gorenstein projective modules. Let $M \in \mathrm{Mod}\ R$ with $\mathrm{Gpd}_R(M) = n < \infty$. Taking $t = 0$ in Theorem 2.6, one gets an exact sequence $0 \to N \to G \to M \to 0$ in Mod R with G Gorenstein projective and $\mathrm{pd}_R(N) \leqslant n-1$. It is easy to see that this exact sequence is a surjective $\mathscr{GP}(R)$-precover of M (see [H, Theorem 2.10]).

Remark 2.7. It is known that a module $A \in \mathrm{Mod}\ R$ is called an *n-cosyzygy module* (of M) if there exists an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to A \to 0$ in Mod R with all I^i injective. Recall from [EJ1] that a module $E \in \mathrm{Mod}\ R$ is called *Gorenstein injective* if there exists an exact sequence in Mod R:

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots,$$

such that: (1) All I_i and I^i are injective; (2) After applying the functor $\mathrm{Hom}_R(I,\ )$ the sequence is still exact for any injective module $I \in \mathrm{Mod}\ R$; and (3) $E \cong \mathrm{Im}(I_0 \to I^0)$. We call A a *Gorenstein n-cosyzygy module* (of M) if there exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to A \to 0$ in Mod R with all E^i Gorenstein injective. We point out the dual versions on Gorenstein injectivity and (Gorenstein) n-cosyzygy of all of the above results also hold true by using completely dual arguments.

3. Gorenstein transpose

In this section, R is a left and right Noetherian ring and mod R is the category of finitely generated left R-modules. For any $A \in \mathrm{mod}\ R$, there exists a projective presentation in mod R:

$$P_1 \xrightarrow{f} P_0 \to A \to 0.$$

Then we get an exact sequence

$$0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \mathrm{Coker}\, f^* \to 0$$

in mod R^{op}, where $(\ )^* = \mathrm{Hom}(\ , R)$. Recall from [AB] that $\mathrm{Coker}\, f^*$ is called a *transpose* of A, and denoted by $\mathrm{Tr}\, A$. We remark that the transpose of A depends on the choice of the projective presentation of A, but it is unique up to projective equivalence (see [AB]).

Analogously, we introduce the notion of Gorenstein transpose of modules as follows. Let $A \in \operatorname{mod} R$. Then there exists a Gorenstein projective presentation in mod R:

$$\pi : X_1 \xrightarrow{g} X_0 \to A \to 0,$$

and we get an exact sequence:

$$0 \to A^* \to X_0^* \xrightarrow{g^*} X_1^* \to \operatorname{Coker} g^* \to 0$$

in mod R^{op}. We call $\operatorname{Coker} g^*$ a *Gorenstein transpose* of A, and denote it by $\operatorname{Tr}_G^{\pi} A$. It is trivial that a transpose of A is a Gorenstein transpose of A, but the converse does not hold true in general. For example, for a module A in mod R, if A is Gorenstein projective but not projective, then some Gorenstein transpose of A is zero, and any transpose of A is Gorenstein projective (see Proposition 3.4(3) below) but non-zero (otherwise, if a transpose of A is zero, then A is projective, which is a contradiction).

Let $A \in \operatorname{mod} R$. Recall from [AB] that A is said to have *Gorenstein dimension zero* if $\operatorname{Ext}_R^i(A, R) = 0 = \operatorname{Ext}_{R^{op}}^i(\operatorname{Tr} A, R)$ for any $i \geqslant 1$. It is easy to see that if A has Gorenstein dimension zero, then so does A^*. In addition, it is well known that A has Gorenstein dimension zero if and only if it is Gorenstein projective. Let $\sigma_A : A \to A^{**}$ defined via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. Recall that a module $A \in \operatorname{mod} R$ is called *torsionless* (resp. *reflexive*) if σ_A is a monomorphism (resp. an isomorphism)

The following result establishes a relation between a Gorenstein transpose of a module with a transpose of the same module.

Theorem 3.1. *Let $M \in \operatorname{mod} R^{op}$ and $A \in \operatorname{mod} R$. Then M is a Gorenstein transpose of A if and only if M can be embedded into a transpose $\operatorname{Tr} A$ of A with the cokernel Gorenstein projective, that is, there exists an exact sequence $0 \to M \to \operatorname{Tr} A \to H \to 0$ in $\operatorname{mod} R^{op}$ with H Gorenstein projective.*

Proof. We first prove the necessity. Assume that $M(\cong \operatorname{Tr}_G^{\pi} A)$ is a Gorentein transpose of A. Then there exists an exact sequence $\pi : X_1 \xrightarrow{g} X_0 \to A \to 0$ in mod R with X_0 and X_1 Gorenstein projective such that $\operatorname{Tr}_G^{\pi} A = \operatorname{Coker} g^*$. So there exists an exact sequence $0 \to H_1' \to P_0' \to X_0 \to 0$ in mod R with P_0' projective and H_1' Gorenstein projective. Let $K_1 = \operatorname{Im} g$ and $g = i\alpha$ be the natural epic-monic decomposition of g. Then we have the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H_1' \arrow[r, equal] \arrow[d] & H_1' \arrow[d] & & \\
0 \arrow[r] & K_1' \arrow[r] \arrow[d] & P_0' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1 \arrow[r, "i"] \arrow[d] & X_0 \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Now consider the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H_1' \arrow[r, equal] \arrow[d] & H_1' \arrow[d] & \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & K_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & X_1 \arrow[r, "\alpha"] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $K_2 = \operatorname{Ker} g$. Because both X_1 and H_1' are Gorenstein projective, G is Gorenstein projective by Lemma 2.1. So there exists an exact sequence $0 \to G_1 \to P_0 \to G \to 0$ in mod R with P_0 projective and G_1 Gorenstein projective. Consider the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G_1 \arrow[r, equal] \arrow[d] & G_1 \arrow[d] & & \\
0 \arrow[r] & K_2' \arrow[r] \arrow[d, "\beta"] & P_0 \arrow[r] \arrow[d] & K_1' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & K_1' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

So we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K_2' \arrow[r] \arrow[d, "\beta"] & P_0 \arrow[r] \arrow[d] & K_1' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & K_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & X_1 \arrow[r, "\alpha"] & K_1 \arrow[r] & 0
\end{tikzcd}$$

It yields the following commutative diagram with exact columns and rows:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \operatorname{Ker}\beta \arrow[d] & H_1 \arrow[d] & H_1' \arrow[d] & \\
0 \arrow[r] & K_2' \arrow[r] \arrow[d, "\beta"] & P_0 \arrow[r] \arrow[d] & K_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d] & X_1 \arrow[r, "\alpha"] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

where $H_1 = \operatorname{Ker}(P_0 \to X_1)$. By the snake lemma, we get the exact sequence $0 \to \operatorname{Ker}\beta \to H_1 \xrightarrow{h} H_1' \to 0$. By Lemma 2.1, H_1 is Gorenstein projective and hence $\operatorname{Ker}\beta$ is also Gorenstein projective. Combining the above diagram with the first one in this proof, we get the following commutative diagram with exact columns and rows:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] \arrow[d] & H_1 \arrow[r, "h"] \arrow[d] & H_1' \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & K_2' \arrow[r] \arrow[d, "\beta"] & P_0 \arrow[r] \arrow[d] & P_0' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d] & X_1 \arrow[r, "g"] \arrow[d] & X_0 \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

By applying the functor $(\ )^*$ to the above diagram, we get the following commutative diagram with exact columns and rows:

$$
\begin{tikzcd}
0 & 0 & & \\
H_1^* \arrow[u] & H_1'^{\,*} \arrow[l, "h^*"'] \arrow[u] & 0 \arrow[l] & \\
P_0^* \arrow[u] & P_0'^{\,*} \arrow[l] \arrow[u] & A^* \arrow[l] & 0 \arrow[l] \\
X_1^* \arrow[u] & X_0^* \arrow[l, "g^*"'] \arrow[u] & A^* \arrow[l] \arrow[u, equal] & 0 \arrow[l] \\
0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

By the snake lemma, we get an exact sequence:

$$0 \to \mathrm{Tr}_G^{\pi} A\big(= \mathrm{Coker}\, g^*\big) \to \mathrm{Tr}\, A \to \mathrm{Coker}\, h^* \to 0$$

in mod R^{op} with $\mathrm{Coker}\, h^* (\cong (\mathrm{Ker}\, h)^* \cong (\mathrm{Ker}\, \beta)^*)$ Gorenstein projective.

We next prove the sufficiency. Let $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a projective presentation of A in mod R. Then we have the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A^* \arrow[r, equal] \arrow[d] & A^* \arrow[d] & & \\
 & P_0^* \arrow[r, equal] \arrow[d, "h"] & P_0^* \arrow[d, "f^*"] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d, "g"] & P_1^* \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathrm{Tr}\, A \arrow[r] \arrow[d] & H \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Because H is Gorenstein projective and P_1^* is projective, K is Gorenstein projective by Lemma 2.1. Again because H is Gorenstein projective, by applying the functor $(\ )^*$ to the above commutative diagram, we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^* \arrow[r] \arrow[d, equal] & (\mathrm{Tr}\, A)^* \arrow[r] \arrow[d] & M^* \arrow[r] \arrow[d, "g^*"] & 0 \\
0 \arrow[r] & H^* \arrow[r] & P_1^{**} \arrow[r] \arrow[d, "f^{**}"] & K^* \arrow[r] \arrow[d, "h^*"] & 0 \\
 & & P_0^{**} \arrow[r, equal] \arrow[d] & P_0^{**} & \\
 & & A \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

By the snake lemma, we have $\mathrm{Im}\, h^* \cong \mathrm{Im}\, f^{**}$. Thus we get $\mathrm{Coker}\, h^* = P_0^{**}/\mathrm{Im}\, h^* \cong P_0^{**}/\mathrm{Im}\, f^{**} \cong A$, and therefore we get a Gorenstein projective presentation of A in mod R:

$$K^* \xrightarrow{h^*} P_0^{**} \to A \to 0.$$

Because both K and P_0^* are reflexive, we get an exact sequence $0 \to A^* \to P_0^{***} \xrightarrow{h^{**}} K^{**} \to M \to 0$ in mod R^{op} and M is a Gorenstein transpose of A. $\square$

As a consequence of Theorem 3.1, we get the following

Corollary 3.2. *Let $A \in \operatorname{mod} R$. Then for any Gorenstein projective module $H \in \operatorname{mod} R^{op}$ and any transpose $\operatorname{Tr} A$ of A, $H \oplus \operatorname{Tr} A$ is a Gorenstein transpose of A.*

Proof. Assume that $H \in \operatorname{mod} R^{op}$ is a Gorenstein projective module. Then there exists an exact sequence $0 \to H \to P \to H' \to 0$ in mod R^{op} with P projective and H' Gorenstein projective, which induces an exact sequence $0 \to H \oplus \operatorname{Tr} A \to P \oplus \operatorname{Tr} A \to H' \to 0$. Because $P \oplus \operatorname{Tr} A$ is again a transpose of A, $H \oplus \operatorname{Tr} A$ is a Gorenstein transpose of A by Theorem 3.1. $\square$

It is clear that the Gorenstein transpose of a module A in mod R depends on the choice of the Gorenstein projective presentation of A. Corollary 3.2 provides a method to construct a Gorenstein transpose of a module from a transpose of the same module. It is interesting to ask the following

Question 3.3. Is any Gorenstein transpose obtained in this way?

If the answer to this question is positive, then we can conclude that the Gorenstein transpose of a module is unique up to Gorenstein projective equivalence.

Let $A \in \operatorname{mod} R$. By [A, Proposition 6.3] (or [AB, Proposition 2.6]), there exists an exact sequence:

$$0 \to \operatorname{Ext}^1_{R^{op}}(\operatorname{Tr} A, R) \to A \xrightarrow{\sigma_A} A^{**} \to \operatorname{Ext}^2_{R^{op}}(\operatorname{Tr} A, R) \to 0 \qquad (*)$$

in mod R. For a positive integer n, recall from [AB] that A is called *n-torsionfree* if $\operatorname{Ext}^i_{R^{op}}(\operatorname{Tr} A, R) = 0$ for any $1 \leqslant i \leqslant n$. From the exact sequence $(*)$, it is easy to see that A is torsionless (resp. reflexive) if and only if it is 1-torsionfree (resp. 2-torsionfree).

The following result shows that some homological properties of any Gorenstein transpose and any transpose of a given module are identical.

Proposition 3.4. *Let $A \in \operatorname{mod} R$. Then for any Gorenstein transpose $\operatorname{Tr}^{\pi}_G A$ and any transpose $\operatorname{Tr} A$ of A, we have*

(1) $\operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\pi}_G A, R) \cong \operatorname{Ext}^i_{R^{op}}(\operatorname{Tr} A, R)$ *for any* $i \geqslant 1$.
(2) *For any $n \geqslant 1$, $\operatorname{Tr}^{\pi}_G A$ is n-torsionfree if and only if so is* $\operatorname{Tr} A$.
(3) *Some Gorenstein transpose of A is zero if and only if A is Gorenstein projective, if and only if any (Gorenstein) transpose of A is Gorenstein projective.*
(4) $\operatorname{Gpd}_{R^{op}}(\operatorname{Tr}^{\pi}_G A) = \operatorname{Gpd}_{R^{op}}(\operatorname{Tr} A)$.

Proof. (1) It is an immediate consequence of Theorem 3.1.

(2) Let $\operatorname{Tr}^{\pi}_G A$ be any Gorenstein transpose of A. By Theorem 3.1, there exists a transpose $\operatorname{Tr} A$ of A satisfying the exact sequence $0 \to \operatorname{Tr}^{\pi}_G A \to \operatorname{Tr} A \to H \to 0$ in mod R^{op} with H Gorenstein projective.

If $\operatorname{Ext}^1_R(\operatorname{Tr}(\operatorname{Tr} A), R) = 0$, then $\operatorname{Tr} A$ is torsionless. So $\operatorname{Tr}^{\pi}_G A$ is also torsionless and $\operatorname{Ext}^1_R(\operatorname{Tr}(\operatorname{Tr}^{\pi}_G A), R) = 0$. Because H is Gorenstein projective, we get an exact sequence $0 \to \operatorname{Tr} H \to \operatorname{Tr}(\operatorname{Tr} A) \to \operatorname{Tr}(\operatorname{Tr}^{\pi}_G A) \to 0$ in mod R with $\operatorname{Tr} H$ Gorenstein projective. So we have that $\operatorname{Ext}^i_R(\operatorname{Tr}(\operatorname{Tr}^{\pi}_G A), R) \cong \operatorname{Ext}^i_R(\operatorname{Tr}(\operatorname{Tr} A), R)$ for any $i \geqslant 2$, and $\operatorname{Ext}^1_R(\operatorname{Tr}(\operatorname{Tr}^{\pi}_G A), R) \to \operatorname{Ext}^1_R(\operatorname{Tr}(\operatorname{Tr} A), R) \to 0$ is exact. So for any $i \geqslant 1$, $\operatorname{Ext}^i_R(\operatorname{Tr}(\operatorname{Tr}^{\pi}_G A), R) = 0$ if and only if $\operatorname{Ext}^i_R(\operatorname{Tr}(\operatorname{Tr} A), R) = 0$, and thus we conclude that for any $n \geqslant 1$, $\operatorname{Tr}^{\pi}_G A$ is n-torsionfree if and only if so is $\operatorname{Tr} A$.

(3) Because A is a (Gorenstein) transpose of any (Gorenstein) transpose of A, it is not difficult to verify the assertion by (1) and (2).

(4) Let $\mathrm{Tr}_G^\pi A$ be any Gorenstein transpose of A. If $\mathrm{Tr}_G^\pi A = 0$, then the assertion follows from (3). Now suppose $\mathrm{Tr}_G^\pi A \neq 0$. By Theorem 3.1, there exists a transpose $\mathrm{Tr}\, A$ of A satisfying the exact sequence $0 \to \mathrm{Tr}_G^\pi A \to \mathrm{Tr}\, A \to H \to 0$ in mod R^{op} with H Gorenstein projective. Then we have that $\mathrm{Gpd}_{R^{op}}(\mathrm{Tr}_G^\pi A) = \mathrm{Gpd}_{R^{op}}(\mathrm{Tr}\, A)$ by Lemma 2.1. □

Let $A \in \mathrm{mod}\ R$. By Proposition 3.4(1), we have that A is n-torsionfree if and only if $\mathrm{Ext}^i_{R^{op}}(\mathrm{Tr}_G^\pi A, R) = 0$ for any (or some) Gorenstein transpose $\mathrm{Tr}_G^\pi A$ of A and $1 \leqslant i \leqslant n$. On the other hand, also by Proposition 3.4(1), we get a Gorenstein version of the formula $(*)$ as follows. For any Gorenstein transpose $\mathrm{Tr}_G^\pi A$ of A, we have the following exact sequence:

$$0 \to \mathrm{Ext}^1_{R^{op}}\big(\mathrm{Tr}_G^\pi A, R\big) \to A \xrightarrow{\sigma_A} A^{**} \to \mathrm{Ext}^2_{R^{op}}\big(\mathrm{Tr}_G^\pi A, R\big) \to 0$$

in mod R. It is easy to see that A is a Gorenstein transpose of $\mathrm{Tr}_G^\pi A$. So we also get the following exact sequence:

$$0 \to \mathrm{Ext}^1_R(A, R) \to \mathrm{Tr}_G^\pi A \xrightarrow{\sigma_{\mathrm{Tr}_G^\pi A}} \big(\mathrm{Tr}_G^\pi A\big)^{**} \to \mathrm{Ext}^2_R(A, R) \to 0$$

in mod R^{op}.

The following result shows that any double Gorenstein transpose of A shares some homological properties of A.

Corollary 3.5. *Let $A \in \mathrm{mod}\ R$. Then for any Gorenstein transpose $\mathrm{Tr}_G^\pi A$ of A and any Gorenstein transpose $\mathrm{Tr}_G^{\pi'}(\mathrm{Tr}_G^\pi A)$ of $\mathrm{Tr}_G^\pi A$, we have*

(1) $\mathrm{Ext}^i_R(\mathrm{Tr}_G^{\pi'}(\mathrm{Tr}_G^\pi A), R) \cong \mathrm{Ext}^i_R(A, R)$ *for any $i \geqslant 1$.*
(2) *For any $n \geqslant 1$, $\mathrm{Tr}_G^{\pi'}(\mathrm{Tr}_G^\pi A)$ is n-torsionfree if and only if so is A.*
(3) $\mathrm{Gpd}_R(\mathrm{Tr}_G^{\pi'}(\mathrm{Tr}_G^\pi A)) = \mathrm{Gpd}_R(A)$.

Proof. Note that A is a Gorenstein transpose of any Gorenstein transpose $\mathrm{Tr}_G^\pi A$ of A. So all of the assertions follow from Proposition 3.4. □

Note that a transpose of a module is a special Gorenstein transpose of the same module. The following result shows that a module with Gorenstein projective dimension n is a double Gorenstein transpose of a module with projective dimension n.

Proposition 3.6. *Let $A \in \mathrm{mod}\ R$ and n be a non-negative integer. Then $\mathrm{Gpd}_R(A) = n$ if and only if there exists a module $B \in \mathrm{mod}\ R$ with $\mathrm{pd}_R(B) = n$ such that A is a Gorenstein transpose of some transpose $\mathrm{Tr}\, B$ of B (that is, $A = \mathrm{Tr}_G^\pi(\mathrm{Tr}\, B)$, where $\mathrm{Tr}_G^\pi(\mathrm{Tr}\, B)$ is a Gorenstein transpose of some transpose $\mathrm{Tr}\, B$ of B).*

Proof. Assume that $A \in \mathrm{mod}\ R$ with $\mathrm{Gpd}_R(A) = n$. By Corollary 2.5, there exists an exact sequence $0 \to A \to B \to H \to 0$ in mod R with $\mathrm{pd}_R(B) = n$ and H Gorenstein projective. Note that B is a transpose of some transpose $\mathrm{Tr}\, B$ of B. By Theorem 3.1, A is a Gorenstein transpose of $\mathrm{Tr}\, B$.

Conversely, if A is a Gorenstein transpose of some transpose $\mathrm{Tr}\, B$ of a module $B \in \mathrm{mod}\ R$ with $\mathrm{pd}_R(B) = n$, then $\mathrm{Gpd}_R(A) = \mathrm{Gpd}_R(B) = \mathrm{pd}_R(B) = n$ by Corollary 3.5. □

Acknowledgments

This research was partially supported by the Specialized Research Fund for the Doctoral Program of Higher Education, NSFC (Grant No. 10771095) and NSF of Jiangsu Province of China (Grant Nos. BK2010047, BK2010007). The authors thank Lars Winther Christensen and the referee for the useful suggestions.

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