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Invariant properties of representations under excellent extensions

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ABSTRACT

In this paper, we introduce the notion of weak excellent extensions of rings as a generalization of that of excellent extensions of rings. Let Γ be a weak excellent extension of an Artinian algebra Λ. We prove that if Λ is of finite representation type (resp. CM-finite, CM-free), then so is Γ; furthermore, if Γ is an excellent extension of Λ, then the converse also holds true. We also study when the representation dimension of an Artinian algebra is invariant under excellent extensions.

1. Introduction

In studying the algebraic structure of group rings, Passman introduced in [26] the notion of the excellent extensions of rings, which was named in [8]. Such extensions of rings are vital since they include two important classes of extensions of rings, that is, finite matrix rings and skew group rings $\Lambda * G$ where the finite group G satisfies the condition $|G|^{-1} \in \Lambda$ (see Example 2.2 below for the details). Many authors have studied the invariant properties of rings under excellent extensions [8,15, 23,25,26,29,33]. It has been known that many important homological properties, such as the (weak) global dimension of rings, the projectivity, injectivity and flatness of modules and so on, are invariant under excellent extensions [23,29].

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Recall that an Artinian algebra Λ is said to be of *finite representation type* if there exist only finitely many isomorphism classes of finitely generated indecomposable Λ-modules. It is well known that determining the representation type of algebras is fundamental and important in representation theory of Artinian algebras. Auslander proved in [2] that there exists a bijective correspondence between the Morita equivalent classes of Artinian algebras of finite representation type and that of Artinian algebras with global dimension at most 2 and with dominant dimension at least 2. Motivated by this correspondence, Auslander introduced the notion of the representation dimension of Artinian algebras, and proved that an Artinian algebra is of finite representation type if and only if its representation dimension is at most 2. In this sense, the representation dimension of an Artinian algebra is regarded as a trial to give a reasonable way of measuring homologically how far an Artinian algebra is from being of finite representation type. Recently, the interest in the representation dimension was revived, and many interesting connections were established with different problems in representation theory, as well as with other areas (see [1,14,16,18,19,28,30–32] for the details). In particular, Iyama proved in [19] that the representation dimension of any Artinian algebra is finite, and Rouquier proved in [28] that the representation dimension of the exterior algebra $\bigwedge^n K$ is $n+1$. However, in general, it is quite hard to compute the representation dimension or even to control it. One possible method is to study the relationship between the representation dimensions of "nicely" related algebras. For instance, Guo proved in [16] that the representation dimension of an Artinian algebra is invariant under stable equivalences.

As an analogy of Artinian algebras of finite representation type, recall that an Artinian algebra Λ is called *CM-finite* if there exist only finitely many isomorphism classes of finitely generated indecomposable Gorenstein projective Λ-modules. This notion was introduced by Beligiannis in [6]. Since then CM-finite Artinian algebras have attracted considerable attentions [6,7,9,21,22].

In this paper, we will study the invariance of the representation type, the CM-finite type and the representation dimension of Artinian algebras under excellent extensions. This paper is organized as follows.

In Section 2, we give some notations in our terminology and some preliminary results which are often used in this paper; in particular, we introduce the notion of weak excellent extensions of rings as a generalization of that of the excellent extensions of rings.

Recall from [10] that an Artinian algebra Λ is called *CM-free* if any finitely generated Gorenstein projective Λ-module is projective. Note that CM-free algebras are an extreme case of CM-finite algebras. In Section 3, we prove the following

Theorem 1.1. *Let Γ be a weak excellent extension of an Artinian algebra Λ. If Λ is of finite representation type* (*resp. CM-finite, CM-free*)*, then so is Γ; furthermore, if Γ is an excellent extension of Λ, then the converse also holds true.*

Let Γ be an excellent extension of an Artinian algebra Λ. By the above theorem, we have that the representation dimensions of Λ and Γ are identical provided either of them is at most two. We conjecture that these two representation dimensions are always identical. In Section 4, we prove that the answer to this conjecture is positive when Λ is commutative; that is, we have the following

Theorem 1.2. *Let R be a commutative Artinian ring and Γ an R-algebra. If Γ is an excellent extension of R, then the representation dimensions of R and Γ are identical.*

2. Preliminaries

Throughout this paper, all rings are associative rings with identity and all modules are finitely generated right modules unless stated otherwise.

We first introduce the notion of weak excellent extensions of rings as follows.

Definition 2.1. Let Λ be a subring of a ring Γ such that Λ and Γ have the same identity. Then Γ is called a *ring extension* of Λ, and denoted by $\Gamma \geqslant \Lambda$. A ring extension $\Gamma \geqslant \Lambda$ is called a *weak excellent extension* if:

(1) Γ is *right Λ-projective* [26, p. 273], that is, if N_Γ is a submodule of M_Γ and if N_Λ is a direct summand of M_Λ, denoted by $N_\Lambda \mid M_\Lambda$, then $N_\Gamma \mid M_\Gamma$.
(2) Γ is a *finite extension* of Λ, that is, there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\Gamma = \sum_{i=1}^n \gamma_i \Lambda$.
(3) Γ_Λ is flat and ${}_\Lambda\Gamma$ is projective.

Recall from [26,8] that a ring extension $\Gamma \geqslant \Lambda$ is called an *excellent extension* if it is weak excellent and Γ_Λ and ${}_\Lambda\Gamma$ are free with a common basis $\{\gamma_1, \ldots, \gamma_n\}$, such that $\Lambda\gamma_i = \gamma_i\Lambda$ for any $1 \leqslant i \leqslant n$. In addition, compare the definition of the weak excellent extension with that of the almost excellent extension in [29].

Example 2.2. (See [26,8].)

(1) For a ring Λ, $M_n(\Lambda)$ (the matrix ring of Λ of degree n) is an excellent extension of Λ.
(2) Let Λ be a ring and G a finite group. If $|G|^{-1} \in \Lambda$, then the skew group ring $\Lambda * G$ is an excellent extension of Λ.
(3) Let A be a finite-dimensional algebra over a field K, and let F be a finite separable field extension of K. Then $A \otimes_K F$ is an excellent extension of A.
(4) Let K be a field, and let G be a group and H a normal subgroup of G. If $[G:H]$ is finite and is not zero in K, then KG is an excellent extension of KH.
(5) Let K be a field of characteristic p, and let G be a finite group and H a normal subgroup of G. If H contains a Sylow p-subgroup of G, then KG is an excellent extension of KH.
(6) Let K be a field and G a finite group. If G acts on K (as field automorphisms) with kernel H, then the skew group ring $K * G$ is an excellent extension of the group ring KH, and the center $Z(KH)$ of KH is an excellent extension of the center $Z(K * G)$ of $K * G$.

Recall that a Hopf algebra $(H, m, \mu, \Delta, \varepsilon, S)$ is said to *measure* a finite-dimensional K-algebra A over a field K if there exists a K-linear map $H \otimes A \to A$ given by $h \otimes a \to h \cdot a$ such that $h \cdot 1 = \varepsilon(h)1$ and $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ for any $h \in H$ and $a, b \in A$. A map $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$ is said to be *convolution invertible* if there exists a map $\delta \in \mathrm{Hom}_K(H \otimes H, A)$ such that $(\sigma * \delta)(h \otimes g) = \sum \sigma(h_1 \otimes g_1)\delta(h_2 \otimes g_2) = \varepsilon(h)\varepsilon(g)1_A = \sum \delta(h_1 \otimes g_1)\sigma(h_2 \otimes g_2) = (\delta * \sigma)(h \otimes g)$ for any $h, g \in H$. Assume that H measures A and σ is a convolution invertible map in $\mathrm{Hom}_K(H \otimes H, A)$. The *crossed product* $A \#_\sigma H$ of A with H is the set $A \otimes H$ as a vector space with multiplication $(a \# h)(b \# k) = \sum (a(h_1 \cdot b))\sigma(h_2, k_1) \# h_3 k_2$ for any $a, b \in A$ and $h, k \in H$. Here write $a \# h$ for the tensor product $a \otimes h$. By [24, Lemma 7.1.2], we have that $A \#_\sigma H$ is an associative algebra with identity element $1 \# 1$ if and only if the following conditions are satisfied: (1) A is a *twisted H-module algebra* with action $\cdot$, that is, $1 \cdot a = a$ and $h \cdot (k \cdot a) = \sum \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3)$ for any $h, k \in H$ and $a \in A$, and (2) σ is a *cocycle*, that is, $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1$ and $\sum [h_1 \cdot \sigma(k_1, m_1)]\sigma(h_1, k_2 m_2) = \sum \sigma(h_1, k_1)\sigma(h_2 k_2, m)$ for any $h, k, m \in H$.

By Definition 2.1, we have that an excellent extension is a weak excellent extension, but the converse does not hold true in general. For example, if H is a finite-dimensional semisimple Hopf algebra over a field K and A is a twisted H-module algebra, then for any cocycle $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$, the crossed product algebra $A \#_\sigma H$ is a weak excellent extension of A, but not an excellent extension of A in general [11].

Lemma 2.3. *(See [29, Lemma 1.1].) Let $\Gamma \geqslant \Lambda$ be a ring extension such that Γ is right Λ-projective. Then $M_\Gamma \mid (M \otimes_\Lambda \Gamma)_\Gamma$ for any $M \in \operatorname{mod}\Gamma$.*

Let Λ be a ring and $\operatorname{mod}\Lambda$ the subcategory of finitely generated right Λ-modules. We denote by $\operatorname{gl.dim}\Lambda$ the global dimension of Λ. For a module $M \in \operatorname{mod}\Lambda$, we denote by $\operatorname{pd} M_\Lambda$ the projective dimension of M.

Lemma 2.4. *(See [29, Theorem 1.4], [23, Theorem 3].) Let $\Gamma \geqslant \Lambda$ be an excellent extension and $M \in \operatorname{mod}\Gamma$. Then we have*:

(1) $\operatorname{pd} M_\Gamma = \operatorname{pd} M_\Lambda = \operatorname{pd}(M \otimes_\Lambda \Gamma)_\Gamma$.
(2) $\operatorname{gl.dim} \Gamma = \operatorname{gl.dim} \Lambda$.

Let Λ be a left and right Noetherian ring, and let $M \in \operatorname{mod} \Lambda$ and

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

be a projective presentation of M in $\operatorname{mod} \Lambda$. Then $\operatorname{Coker} f^*$ is called the *transpose* of M [3], and is denoted by $\operatorname{Tr} M$, where $(-)^* = \operatorname{Hom}_\Lambda(-, \Lambda)$. It is well known that the transpose of M depends on the choice of the projective resolution of M, but it is unique up to projective equivalence. Recall from [3] that M is said to have *Gorenstein dimension zero* if M is reflexive and $\operatorname{Ext}^i_\Lambda(M_\Lambda, \Lambda) = 0 = \operatorname{Ext}^i_\Lambda(M^*, \Lambda)$ for any $i \geqslant 1$ (equivalently, $\operatorname{Ext}^i_\Lambda(M_\Lambda, \Lambda) = 0 = \operatorname{Ext}^i_\Lambda(\operatorname{Tr} M_\Lambda, \Lambda)$ for any $i \geqslant 1$). Following the terminology of Enochs and Jenda, a module having Gorenstein dimension zero is called *Gorenstein projective* [12]. The *Gorenstein projective dimension* (or *Gorenstein dimension*) of M, denoted by $\operatorname{Gpd} M_\Lambda$, is defined as $\inf\{n \mid$ there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in $\operatorname{mod} \Lambda$ with G_i Gorenstein projective for any $0 \leqslant i \leqslant n\}$ (see [3] and [13]). Also recall that the *finitistic dimension* of Λ, denoted by $\operatorname{fin.dim} \Lambda$, is defined as $\sup\{\operatorname{pd} M_\Lambda \mid M \in \operatorname{mod} \Lambda$ and $\operatorname{pd} M_\Lambda < \infty\}$.

The following result was proved in [32, Lemma 4.4] for the case of Artinian algebras. The proof there remains valid in our setting.

Lemma 2.5. *For a left and right Noetherian ring Λ, $\operatorname{fin.dim} \Lambda = \sup\{\operatorname{Gpd} M_\Lambda \mid M \in \operatorname{mod} \Lambda$ and $\operatorname{Gpd} M_\Lambda < \infty\}$.*

Let M, N be in $\operatorname{mod} \Lambda$. Recall that a homomorphism $f : M \to N$ in $\operatorname{mod} \Lambda$ is called *right minimal* if every $h \in \operatorname{End}(M_\Lambda)$ such that $fh = f$ is an automorphism. Let $\mathcal{C}$ be a subcategory of $\operatorname{mod} \Lambda$ and $M \in \operatorname{mod} \Lambda$. A homomorphism $f : C \to M$ in $\operatorname{mod} \Lambda$ is called a *right $\mathcal{C}$-approximation* of M if $C \in \mathcal{C}$ and the sequence $\operatorname{Hom}_\Lambda(-, C) \xrightarrow{(-,f)} \operatorname{Hom}_\Lambda(-, M) \longrightarrow 0$ is exact in $\mathcal{C}$. We say that an exact sequence:

$$0 \longrightarrow C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow C_0 \xrightarrow{f_0} X \longrightarrow 0$$

in $\operatorname{mod} \Lambda$ is an *n-$\mathcal{C}$-resolution* of M if $C_i \in \mathcal{C}$ for any $0 \leqslant i \leqslant n$, and the sequence:

$$0 \longrightarrow \operatorname{Hom}_\Lambda(-, C_n) \xrightarrow{(-,f_n)} \operatorname{Hom}_\Lambda(-, C_{n-1}) \xrightarrow{(-,f_{n-1})} \cdots$$
$$\longrightarrow \operatorname{Hom}_\Lambda(-, C_0) \xrightarrow{(-,f_0)} \operatorname{Hom}_\Lambda(-, X) \longrightarrow 0$$

is exact in $\mathcal{C}$ [4]. We denote by $\operatorname{add} M_\Lambda$ the full subcategory of $\operatorname{mod} \Lambda$ consisting of all modules isomorphic to direct summands of finite direct sums of copies of M, and denote by $\operatorname{Gen} M_\Lambda$ the full subcategory of $\operatorname{mod} \Lambda$ consisting of all modules X such that there exists an epimorphism $M_0 \to X$ with $M_0 \in \operatorname{add} M_\Lambda$.

Lemma 2.6. *(See [1, Lemma 1.4].) Let Λ be an Artinian algebra and $M \in \operatorname{mod} \Lambda$. If $X \in \operatorname{Gen} M_\Lambda$, then there exists an epimorphism $f : M_0 \to X$ in $\operatorname{mod} \Lambda$, which is a minimal right $\operatorname{add} M_\Lambda$-approximation.*

Let Λ be an Artinian algebra. Recall that a module $M \in \operatorname{mod} \Lambda$ is called a *generator–cogenerator* for $\operatorname{mod} \Lambda$ if every indecomposable projective module and also every indecomposable injective module in $\operatorname{mod} \Lambda$ is isomorphic to a direct summand of M.

Lemma 2.7. *(See [14, Lemma 1.1].) Let Λ be an Artinian algebra and M a generator–cogenerator for* $\operatorname{mod}\Lambda$*. Then for any $n \geqslant 3$, the following statements are equivalent.*

(1) *Any indecomposable module $X \in \operatorname{mod}\Lambda$ has an $(n-2)$-$\operatorname{add} M_\Lambda$-resolution.*
(2) $\operatorname{gl.dim}\operatorname{End}(M_\Lambda) \leqslant n$.

By a similar argument to that of [13, Lemma 3.2.4] (where Γ is assumed to be commutative), we get the following

Lemma 2.8. *Let R be a commutative Noetherian ring, and let Γ be a flat R-algebra and M, N be R-modules with M finitely generated. Then we have*

$$\operatorname{Hom}_R(M, N) \otimes_R \Gamma \cong \operatorname{Hom}_\Gamma(M \otimes_R \Gamma, N \otimes_R \Gamma).$$

3. CM-finite and CM-free algebras

In this section, all rings are left and right Noetherian rings unless stated otherwise. We begin with the following easy observation.

Lemma 3.1. *Let $\Gamma \geqslant \Lambda$ be a ring extension. Then we have*:

(1) *For any $M \in \operatorname{mod}\Lambda$, $\operatorname{Tr}(M \otimes_\Lambda \Gamma)_\Gamma$ and $\Gamma \otimes_\Lambda \operatorname{Tr} M_\Lambda$ are projectively equivalent, denoted by $\operatorname{Tr}(M \otimes_\Lambda \Gamma)_\Gamma \approx \Gamma \otimes_\Lambda \operatorname{Tr} M_\Lambda$.*
(2) *If Γ is a finitely generated projective right Λ-module and $M \in \operatorname{mod}\Gamma$, then $\operatorname{Tr} M_\Lambda \approx \operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \Lambda_\Lambda) \otimes_\Gamma \operatorname{Tr} M_\Gamma$.*

Proposition 3.2. *Let $\Gamma \geqslant \Lambda$ be a ring extension such that Γ_Λ and ${}_\Lambda\Gamma$ are flat. Then $\operatorname{Gpd}(M \otimes_\Lambda \Gamma)_\Gamma \leqslant \operatorname{Gpd} M_\Lambda$ for any $M \in \operatorname{mod}\Lambda$.*

Proof. Without loss of generality, assume that $\operatorname{Gpd} M_\Lambda < \infty$. If M_Λ is Gorenstein projective, then there exists an exact sequence:

$$0 \longrightarrow K \xrightarrow{f} P \xrightarrow{g} M \longrightarrow 0 \tag{1}$$

in $\operatorname{mod}\Lambda$ with P projective and K Gorenstein projective. By applying $\operatorname{Hom}_\Lambda(-, \Lambda)$ to (1), we get the following exact sequence:

$$0 \longrightarrow \operatorname{Hom}_\Lambda(M, \Lambda) \xrightarrow{(g,\Lambda)} \operatorname{Hom}_\Lambda(P, \Lambda) \xrightarrow{(f,\Lambda)} \operatorname{Hom}_\Lambda(K, \Lambda) \longrightarrow 0. \tag{2}$$

On the other hand, the flatness of ${}_\Lambda\Gamma$ induces the following sequence:

$$0 \longrightarrow K \otimes_\Lambda \Gamma \xrightarrow{f \otimes 1_\Gamma} P \otimes_\Lambda \Gamma \xrightarrow{g \otimes 1_\Gamma} M \otimes_\Lambda \Gamma \longrightarrow 0$$

in $\operatorname{mod}\Gamma$. Because Γ_Λ is flat, for any $X \in \operatorname{mod}\Lambda$ we have

$$\begin{aligned} \Gamma \otimes_\Lambda \operatorname{Hom}_\Lambda(X, \Lambda) &\cong \operatorname{Hom}_\Lambda(X, \Gamma) \quad \big(\text{by [13, Theorem 3.2.14]}\big) \\ &\cong \operatorname{Hom}_\Gamma(X \otimes_\Lambda \Gamma, \Gamma) \quad (\text{by the adjoint isomorphism theorem}). \end{aligned}$$

Then by applying $\mathrm{Hom}_\Gamma(-,\Gamma)$ to (2), we get the following commutative diagram with exact arrows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma\otimes_\Lambda M^* & \xrightarrow{1_\Gamma\otimes g^*} & \Gamma\otimes_\Lambda P^* & \xrightarrow{1_\Gamma\otimes f^*} & \Gamma\otimes_\Lambda K^* & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & (M\otimes_\Lambda\Gamma)^\dagger & \xrightarrow{(g\otimes 1_\Gamma)^\dagger} & (P\otimes_\Lambda\Gamma)^\dagger & \xrightarrow{(f\otimes 1_\Gamma)^\dagger} & (K\otimes_\Lambda\Gamma)^\dagger & \longrightarrow & \mathrm{Ext}^1_\Gamma(M\otimes_\Lambda\Gamma,\Gamma) \longrightarrow 0 \end{array}$$

where $(-)^*$ and $(-)^\dagger$ stand for $\mathrm{Hom}_\Lambda(-,\Lambda)$ and $\mathrm{Hom}_\Gamma(-,\Gamma)$ respectively. So $\mathrm{Ext}^1_\Gamma(M\otimes_\Lambda\Gamma,\Gamma)=0$. Similarly, we have $\mathrm{Ext}^1_\Gamma(K\otimes_\Lambda\Gamma,\Gamma)=0$. So $\mathrm{Ext}^2_\Gamma(M\otimes_\Lambda\Gamma,\Gamma)\cong\mathrm{Ext}^1_\Gamma(K\otimes_\Lambda\Gamma,\Gamma)=0$. Continuing this process, we get that $\mathrm{Ext}^i_\Gamma(M\otimes_\Lambda\Gamma,\Gamma)=0$ for any $i\geqslant 1$.

Similarly, we get that $\mathrm{Ext}^i_\Gamma(\Gamma\otimes_\Lambda\mathrm{Tr}\,M_\Lambda,\Gamma)=0$ for any $i\geqslant 1$. Then $\mathrm{Ext}^i_\Gamma(\mathrm{Tr}(M\otimes_\Lambda\Gamma)_\Gamma,\Gamma)\cong\mathrm{Ext}^i_\Gamma(\Gamma\otimes_\Lambda\mathrm{Tr}\,M_\Lambda,\Gamma)=0$ for any $i\geqslant 1$ by Lemma 3.1(1). So $(M\otimes_\Lambda\Gamma)_\Gamma$ is Gorenstein projective.

If $\mathrm{Gpd}\,M_\Lambda=n\geqslant 1$, then there exists an exact sequence:

$$0\to G_n\to G_{n-1}\to\cdots\to G_0\to M\to 0$$

in $\mathrm{mod}\,\Lambda$ with G_i Gorenstein projective for any $0\leqslant i\leqslant n$. Since ${}_\Lambda\Gamma$ is flat, we get the following exact sequence:

$$0\to G_n\otimes_\Lambda\Gamma\to G_{n-1}\otimes_\Lambda\Gamma\to\cdots\to M\otimes_\Lambda\Gamma\to 0$$

in $\mathrm{mod}\,\Gamma$. By the above argument, $(G_i\otimes_\Lambda\Gamma)_\Gamma$ is Gorenstein projective for any $0\leqslant i\leqslant n$. So $\mathrm{Gpd}(M\otimes_\Lambda\Gamma)_\Gamma\leqslant n$. $\square$

Let $\alpha:\Lambda\to\Gamma$ be a homomorphism of rings. We recall the following facts.

(1) A right (resp. left) Γ-module H has a right (resp. left) Λ-module structure via $x\lambda=x\alpha(\lambda)$ (resp. $\lambda x=\alpha(\lambda)x$) for any $x\in H$ and $\lambda\in\Lambda$.
(2) Given a (Γ,Λ)-bimodule ${}_\Gamma M_\Lambda$ (it can be viewed as a (Λ,Λ)-bimodule by (1)) and a right Λ-module N_Λ, $\mathrm{Hom}_\Lambda({}_\Gamma M_\Lambda,N_\Lambda)$ has a right Γ-module structure via $f\gamma(x)=f(\gamma x)$ for any $f\in\mathrm{Hom}_\Lambda({}_\Gamma M_\Lambda,N_\Lambda)$, $\gamma\in\Gamma$ and $x\in M$, which induces a right Λ-module structure via $f\lambda(x)=f\alpha(\lambda)(x)=f(\alpha(\lambda)x)$ for any $\lambda\in\Lambda$. This right Λ-module structure can be induced equivalently by $\mathrm{Hom}_\Lambda({}_\Lambda M_\Lambda,N_\Lambda)$ via $f\lambda(x)=f(\lambda x)$, because $f(\alpha(\lambda)x)=f(\lambda x)$ by (1).

Proposition 3.3. *Let $\Gamma\geqslant\Lambda$ be a weak excellent extension. Then* $\mathrm{Gpd}\,M_\Lambda=\mathrm{Gpd}\,M_\Gamma$ *for any* $M\in\mathrm{mod}\,\Gamma$.

Proof. Let $\Gamma\geqslant\Lambda$ be a weak excellent extension. By Lemma 2.3, we have $M_\Gamma\mid(M\otimes_\Lambda\Gamma)_\Gamma$. So $\mathrm{Gpd}\,M_\Gamma\leqslant\mathrm{Gpd}(M\otimes_\Lambda\Gamma)_\Gamma\leqslant\mathrm{Gpd}\,M_\Lambda$ by Proposition 3.2. It remains to prove $\mathrm{Gpd}\,M_\Lambda\leqslant\mathrm{Gpd}\,M_\Gamma$.

Without loss of generality, assume that $\mathrm{Gpd}\,M_\Gamma<\infty$. If M_Γ is Gorenstein projective, then $\mathrm{Ext}^i_\Gamma(M_\Gamma,\Gamma)=0=\mathrm{Ext}^i_\Gamma(\mathrm{Tr}\,M_\Gamma,\Gamma)$ for any $i\geqslant 1$. Because Λ is a left and right Noetherian ring, a finitely generated flat right Λ-module is projective. So both Γ_Λ and ${}_\Lambda\Gamma$ are projective by the definition of weak excellent extensions.

We claim that $\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda)_\Gamma$ is projective. Let $f:M_\Gamma\to\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda)_\Gamma$ be an epimorphism in $\mathrm{mod}\,\Gamma$. Then f is also an epimorphism in $\mathrm{mod}\,\Lambda$. Note that the right Λ-structure of $\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda)$ can be induced equivalently by $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Lambda,\Lambda_\Lambda)$ by the argument before this proposition. So $\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda)_\Lambda$ is projective and f is split in $\mathrm{mod}\,\Lambda$, and hence $(\mathrm{Ker}\,f)_\Lambda\mid M_\Lambda$. Thus $(\mathrm{Ker}\,f)_\Gamma\mid M_\Gamma$ by the definition of weak excellent extensions, which implies that $\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda)_\Gamma$ is projective. The claim is proved. Thus $\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda)\otimes_\Gamma\mathrm{Ext}^i_\Gamma(M_\Gamma,\Gamma)\cong\mathrm{Ext}^i_\Gamma(M_\Gamma,\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda,\Lambda_\Lambda))=0$ for any $i\geqslant 1$ by [13, Theorem 3.2.15]. Then for any $i\geqslant 1$ we have

$$\begin{aligned}\operatorname{Ext}^i_\Lambda(M_\Lambda, \Lambda) &\cong \operatorname{Ext}^i_\Lambda(M \otimes_\Gamma \Gamma, \Lambda)\\ &\cong \operatorname{Ext}^i_\Gamma\big(M_\Gamma, \operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \Lambda_\Lambda)\big) \quad \big(\text{by [27, Corollary 10.65]}\big)\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ext}^i_\Lambda(\operatorname{Tr} M_\Lambda, \Lambda) &\cong \operatorname{Ext}^i_\Lambda\big(\operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \Lambda_\Lambda) \otimes_\Gamma \operatorname{Tr} M_\Gamma, \Lambda\big) \quad \big(\text{by Lemma 3.1(2)}\big)\\ &\cong \operatorname{Ext}^i_\Gamma\big(\operatorname{Tr} M_\Gamma, \operatorname{Hom}_\Lambda\big(\operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \Lambda_\Lambda), \Lambda\big)\big) \quad \big(\text{by [27, Corollary 10.65]}\big)\\ &\cong \operatorname{Ext}^i_\Gamma(\operatorname{Tr} M_\Gamma, \Gamma) = 0.\end{aligned}$$

It implies that M_Λ is Gorenstein projective.

If $\operatorname{Gpd} M_\Gamma = m$ $(\geqslant 1)$, then there exists an exact sequence:

$$0 \to V_m \to V_{m-1} \to \cdots \to V_0 \to M \to 0$$

in $\operatorname{mod}\Gamma$ with $V_{i\Gamma}$ Gorenstein projective for $0 \leqslant i \leqslant m$, which is also exact in $\operatorname{mod}\Lambda$. By the above argument, $V_{i\Lambda}$ is Gorenstein projective, so we have that $\operatorname{Gpd} M_\Lambda \leqslant m$. The proof is finished. □

Corollary 3.4. *Let $\Gamma \geqslant \Lambda$ be an excellent extension. Then* $\operatorname{Gpd}(M \otimes_\Lambda \Gamma)_\Gamma = \operatorname{Gpd} M_\Lambda$ *for any* $M \in \operatorname{mod}\Lambda$.

Proof. By Proposition 3.2, it suffices to prove $\operatorname{Gpd} M_\Lambda \leqslant \operatorname{Gpd}(M \otimes_\Lambda \Gamma)_\Gamma$. Because $\Gamma \geqslant \Lambda$ is an excellent extension, $M_\Lambda \mid (M \otimes_\Lambda \Gamma)_\Lambda$. So $\operatorname{Gpd} M_\Lambda \leqslant \operatorname{Gpd}(M \otimes_\Lambda \Gamma)_\Lambda = \operatorname{Gpd}(M \otimes_\Lambda \Gamma)_\Gamma$ by Proposition 3.3. □

By the definition of weak excellent extensions, it is easy to prove the following

Lemma 3.5. *Let $\Gamma \geqslant \Lambda$ be a weak excellent extension. If Λ is an Artinian algebra, then so is Γ.*

Recall that a ring is called *Gorenstein* if its left and right self-injective dimensions are finite.

Proposition 3.6.

(1) *If $\Gamma \geqslant \Lambda$ is a weak excellent extension, then* $\operatorname{fin.dim}\Gamma \leqslant \operatorname{fin.dim}\Lambda$. *The equality holds true if $\Gamma \geqslant \Lambda$ is an excellent extension.*
(2) *If $\Gamma \geqslant \Lambda$ is a weak excellent extension, then* $\operatorname{gl.dim}\Gamma \leqslant \operatorname{gl.dim}\Lambda$.
(3) *Let Γ be a weak excellent extension of an Artinian algebra Λ. If Λ is Gorenstein, then so is Γ. Furthermore, if $\Gamma \geqslant \Lambda$ is an excellent extension, then Λ is Gorenstein if and only if so is Γ.*

Proof. (1) According to Lemma 2.5, the first assertion follows from Proposition 3.3, and the second assertion follows from the first one and Corollary 3.4.

(2) Let $M \in \operatorname{mod}\Gamma$. Then it is easy to get that $\operatorname{pd}(M \otimes_\Lambda \Gamma)_\Gamma \leqslant \operatorname{pd} M_\Lambda$. Because $M_\Gamma \mid (M \otimes_\Lambda \Gamma)$ by Lemma 2.3, $\operatorname{pd} M_\Gamma \leqslant \operatorname{pd} M_\Lambda$. Then the assertion follows.

(3) Let Γ be a weak excellent extension of an Artinian algebra Λ. Then Γ is also an Artinian algebra by Lemma 3.5. By [17, Theorem], we have that an Artinian algebra is Gorenstein if and only if each of its finitely generated right modules has finite Gorenstein projective dimension. Then the first assertion follows from Proposition 3.3, and the second assertion follows from the first one and Corollary 3.4. □

Recall from [10] that an Artinian algebra Λ is called *Cohen–Macaulay free*, or simply, *CM-free*, if any Gorenstein projective module in $\operatorname{mod}\Lambda$ is projective. It was proved in [10, Theorem 1.1] that a connected Artinian algebra with radical square zero is either self-injective or CM-free.

Theorem 3.7. *Let Γ be a weak excellent extension of an Artinian algebra Λ. If Λ is CM-free, then so is Γ; furthermore, if $\Gamma \geqslant \Lambda$ is an excellent extension, then Λ is CM-free if and only if so is Γ.*

Proof. Let Λ be CM-free and let $M \in \operatorname{mod}\Gamma$ be Gorenstein projective. Then M_Λ is Gorenstein projective by Proposition 3.3. So M_Λ is projective and hence M_Γ is also projective by Lemma 2.4(1). Thus Γ is CM-free.

Assume that $\Gamma \geqslant \Lambda$ is an excellent extension and Γ is CM-free. Let $M \in \operatorname{mod}\Lambda$ be Gorenstein projective. Then $(M \otimes_\Lambda \Gamma)_\Gamma$ is Gorenstein projective by Corollary 3.4. So $(M \otimes_\Lambda \Gamma)_\Gamma$ is projective and hence $(M \otimes_\Lambda \Gamma)_\Lambda$ is also projective by Lemma 2.4(1). Because $M_\Lambda \mid (M \otimes_\Lambda \Gamma)_\Lambda$, M_Λ is projective and so Λ is CM-free. $\square$

Recall that a module $M \in \operatorname{mod}\Lambda$ is called an *additive generator* for $\operatorname{mod}\Lambda$ if any indecomposable module in $\operatorname{mod}\Lambda$ is in $\operatorname{add} M_\Lambda$. Obviously, an Artinian algebra Λ is of finite representation type if and only if $\operatorname{mod}\Lambda$ has an additive generator. Let $\mathcal{GP}(\Lambda)$ be the full subcategory of $\operatorname{mod}\Lambda$ consisting of Gorenstein projective modules. Recall from [6] that an Artinian algebra Λ is said to be of *finite Cohen–Macaulay type*, or simply, *CM-finite*, if there exist only finitely many isomorphism classes of indecomposable Gorenstein projective modules in $\operatorname{mod}\Lambda$. Clearly, Λ is CM-finite if and only if there exists a module $E \in \operatorname{mod}\Lambda$ such that $\mathcal{GP}(\Lambda) = \operatorname{add} E_\Lambda$. It is clear that Λ is CM-finite if Λ is of finite representation type. Furthermore, if $\operatorname{gl.dim}\Lambda < \infty$, then $\mathcal{GP}(\Lambda) = \mathcal{P}(\Lambda)$ (where $\mathcal{P}(\Lambda)$ is the full subcategory of $\operatorname{mod}\Lambda$ consisting of all projective modules) and Λ is CM-finite. These are "trivial" examples of CM-finite algebras. But, in general, little examples of "non-trivial" CM-finite algebras have been known.

Theorem 3.8. *Let Γ be a weak excellent extension of an Artinian algebra Λ.*

(1) *If Λ is of finite representation type, then so is Γ; furthermore, if $\Gamma \geqslant \Lambda$ is an excellent extension, then Λ is of finite representation type if and only if so is Γ.*
(2) *If Λ is CM-finite, then so is Γ; furthermore, if $\Gamma \geqslant \Lambda$ is an excellent extension, then Λ is CM-finite if and only if so is Γ.*

Proof. (1) Let Λ be of finite representation type and $M \in \operatorname{mod}\Lambda$ an additive generator for $\operatorname{mod}\Lambda$. It suffices to prove that $(M \otimes_\Lambda \Gamma)_\Gamma$ is an additive generator for $\operatorname{mod}\Gamma$. Let $X \in \operatorname{mod}\Gamma$ be indecomposable. Then $X \in \operatorname{mod}\Lambda$ and $X_\Lambda \mid (M_\Lambda)^n$ for some positive integer n. So $(X \otimes_\Lambda \Gamma)_\Gamma \mid (M \otimes_\Lambda \Gamma)^n_\Gamma$. It follows from Lemma 2.3 that $X_\Gamma \in \operatorname{add}(M \otimes_\Lambda \Gamma)_\Gamma$ and $(M \otimes_\Lambda \Gamma)_\Gamma$ is an additive generator for $\operatorname{mod}\Gamma$.

Furthermore, if $\Gamma \geqslant \Lambda$ is an excellent extension and Γ is of finite representation type, then there exists an additive generator M_Γ for $\operatorname{mod}\Gamma$. It suffices to prove that M_Λ is an additive generator for $\operatorname{mod}\Lambda$. Let $Y \in \operatorname{mod}\Lambda$ be indecomposable. Then $Y_\Lambda \mid (Y \otimes_\Lambda \Gamma)_\Lambda$. Notice that $(Y \otimes_\Lambda \Gamma)_\Gamma \mid M^m_\Gamma$ for some positive integer m, so $(Y \otimes_\Lambda \Gamma)_\Lambda \mid M^m_\Lambda$ and M_Λ is an additive generator for $\operatorname{mod}\Lambda$.

(2) The proof is similar to that of (1), but for the sake of completeness, we also give it.

Let Λ be CM-finite. Then there exists a module $E \in \operatorname{mod}\Lambda$ such that $\mathcal{GP}(\Lambda) = \operatorname{add} E_\Lambda$. It suffices to prove that $\mathcal{GP}(\Gamma) = \operatorname{add}(E \otimes_\Lambda \Gamma)_\Gamma$. By Proposition 3.2, $\operatorname{add}(E \otimes_\Lambda \Gamma)_\Gamma \subseteq \mathcal{GP}(\Gamma)$. Let $M \in \operatorname{mod}\Gamma$ be indecomposable Gorenstein projective. By Proposition 3.3, we have that M_Λ is Gorenstein projective. So $M_\Lambda \mid E^n_\Lambda$ for some positive integer n and hence $(M \otimes_\Lambda \Gamma)_\Gamma \mid (E \otimes_\Lambda \Gamma)^n_\Gamma$. By Lemma 2.3, we have $M_\Gamma \mid (M \otimes_\Lambda \Gamma)_\Gamma$. Thus $M_\Gamma \mid (E \otimes_\Lambda \Gamma)^n_\Gamma$ and therefore $\mathcal{GP}(\Gamma) = \operatorname{add}(E \otimes_\Lambda \Gamma)_\Gamma$.

Furthermore, if $\Gamma \geqslant \Lambda$ is an excellent extension and Γ is CM-finite, then there exists a module $V \in \operatorname{mod}\Gamma$ such that $\mathcal{GP}(\Gamma) = \operatorname{add} V_\Gamma$. It suffices to prove that $\mathcal{GP}(\Lambda) = \operatorname{add} V_\Lambda$. By Proposition 3.3, $\operatorname{add} V_\Lambda \subseteq \mathcal{GP}(\Lambda)$. Let $Y \in \operatorname{mod}\Lambda$ be indecomposable Gorenstein projective. Then $Y_\Lambda \mid (Y \otimes_\Lambda \Gamma)_\Lambda$. On the other hand, $(Y \otimes_\Lambda \Gamma)_\Gamma$ is Gorenstein projective by Corollary 3.4. So $(Y \otimes_\Lambda \Gamma)_\Gamma \mid V^m_\Gamma$ for some positive integer m and hence $(Y \otimes_\Lambda \Gamma)_\Lambda \mid V^m_\Lambda$. Thus $Y_\Lambda \mid V^m_\Lambda$ and $\mathcal{GP}(\Lambda) = \operatorname{add} V_\Lambda$. $\square$

By [11, Theorem 6.1.7], we have that if H is a finite-dimensional semisimple Hopf algebra over a field K and A is a twisted H-module algebra, then for any cocycle $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$, $A \#_\sigma H$ is a weak excellent extension of A and $A \#_\sigma H \cong H \otimes_K A$ as right A-modules, but $A \#_\sigma H$ is not an excellent extension of A. By Theorem 3.8, we have the following

Corollary 3.9. *Let H be a finite-dimensional semisimple Hopf algebra over a field K and A a finitely generated twisted H-module algebra. Then for any cocycle $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$, A is of finite representation type (resp. CM-finite, CM-free) if and only if so is $A \#_\sigma H$.*

Proof. The necessity follows from Theorems 3.8 and 3.7. Conversely, if $A \#_\sigma H$ is of finite representation type (resp. CM-finite, CM-free), then so is $(A \#_\sigma H) \# \widehat{H}$ also by Theorems 3.8 and 3.7, where $\widehat{H} = \mathrm{Hom}_K(H, K)$. Notice that $\widehat{H}$ is a semisimple Hopf algebra, so by the Blattner–Montgomery duality theorem (see [24, Section 9.4]), $(A \#_\sigma H) \# \widehat{H} \cong M_n(A)$ where $n = \dim_K H$. Because A is Morita equivalent to $M_n(A)$, A is also Morita equivalent to $(A \#_\sigma H) \# \widehat{H}$. On the other hand, it is not difficult to prove that the representation type (resp. CM-finiteness, CM-freeness) of algebras is invariant under Morita equivalences. Then the assertion follows. □

Note that Li and Zhang showed in [21, Corollary–Example 1.3] that for an algebraically closed field K, $\Lambda = T_2(K[x]/\langle x^n \rangle)$ (the upper triangular algebra of $K[x]/\langle x^n \rangle$ of degree two) is a "non-trivial" CM-finite Gorenstein algebra when n =4 or 5. By Theorem 3.8, Lemma 2.4(2) and Proposition 3.6(3), any excellent extension of Λ given above is also a "non-trivial" CM-finite Gorenstein algebra. For example, let F_i be a finite separable field extension of K for any $i \geqslant 1$ and $\Lambda = T_2(K[x]/\langle x^n \rangle)$ where n =4 or 5. Then all of $\Lambda \otimes_K F_1$, $(\Lambda \otimes_K F_1) \otimes_K F_2$, $((\Lambda \otimes_K F_1) \otimes_K F_2) \otimes_K F_3$, ... are "non-trivial" CM-finite Gorenstein algebras by Example 2.2(3).

4. The representation dimension

In this section, Λ is an Artinian algebra. Auslander introduced in [2] the notion of the representation dimension of an Artinian algebra as follows.

Definition 4.1. The *representation dimension* $\mathrm{rep.dim}\,\Lambda$ of Λ is defined as $\inf\{\mathrm{gl.dim}\,\mathrm{End}(M_\Lambda) \mid M$ is a generator–cogenerator for $\mathrm{mod}\,\Lambda\}$ if Λ is non-semisimple; and $\mathrm{rep.dim}\,\Lambda = 1$ if Λ is semisimple.

Let $\Gamma \geqslant \Lambda$ be an excellent extension. Then we have that Λ is semisimple if and only if so is Γ by Lemma 2.4(2), and that $\mathrm{rep.dim}\,\Lambda = \mathrm{rep.dim}\,\Gamma$ provided either of them is at most two by Theorem 3.8(1). On the other hand, if Λ is hereditary, then Γ is also hereditary by Lemma 2.4(2) and so $\mathrm{rep.dim}\,\Lambda = \mathrm{rep.dim}\,\Gamma$ by [2] and Theorem 3.8(1). Based on these facts, it is natural to raise the following

Conjecture. *If $\Gamma \geqslant \Lambda$ is an excellent extension, then* $\mathrm{rep.dim}\,\Lambda = \mathrm{rep.dim}\,\Gamma$.

As applications of Theorem 3.8(1), in this section we will study this conjecture and prove it partially. To compute the representation dimension of an Artinian algebra, we need the following easy observation, which is maybe known.

Lemma 4.2. *Let $X, M \in \mathrm{mod}\,\Lambda$ and $X = X_1 \oplus X_2 \in \mathrm{Gen}\,M_\Lambda$. If X has an n-$\mathrm{add}\,M_\Lambda$-resolution*:

$$0 \longrightarrow M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{f_0} X \longrightarrow 0,$$

then X_1 also has an n-$\mathrm{add}\,M_\Lambda$-resolution:

$$0 \longrightarrow M'_n \xrightarrow{f'_n} M'_{n-1} \longrightarrow \cdots \longrightarrow M'_0 \xrightarrow{f'_0} X_1 \longrightarrow 0.$$

Proof. We claim that if there exists an exact sequence:

$$0 \longrightarrow K_0 \longrightarrow M_0 \xrightarrow{f_0} X \longrightarrow 0$$

in $\operatorname{mod} \Lambda$ with f_0 a right $\operatorname{add} M_\Lambda$-approximation of X, then there exists an epimorphism $f_0' : M_0' \to X_1$ in $\operatorname{mod} \Lambda$, which is a right $\operatorname{add} M_\Lambda$-approximation of X_1 and K_0' $(= \operatorname{Ker} f') \mid K_0$. Because $X_1 \in \operatorname{Gen} M_\Lambda$, there exists an epimorphism $f_0' : M_0' \to X_1$ in $\operatorname{mod} \Lambda$ which is a minimal right $\operatorname{add} M_\Lambda$-approximation of X_1 by Lemma 2.6. So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0' & \longrightarrow & M_0' & \xrightarrow{f_0'} & X_1 & \longrightarrow & 0 \\
 & & \downarrow s & & \downarrow \alpha & & \downarrow i & & \\
0 & \longrightarrow & K_0 & \longrightarrow & M_0 & \xrightarrow{f_0} & X & \longrightarrow & 0 \\
 & & \downarrow t & & \downarrow \beta & & \downarrow p & & \\
0 & \longrightarrow & K_0' & \longrightarrow & M_0' & \xrightarrow{f_0'} & X_1 & \longrightarrow & 0
\end{array}$$

where $pi = 1_{X_1}$. The minimality of f_0' implies that $\beta\alpha$ is an isomorphism and hence ts is also an isomorphism, which implies that Γ is a split monomorphism. The claim is proved. Then by using induction on n, we get the assertion easily. □

Lemma 4.3. *(See [26, p. 273, Lemma 2.3].) Let A be a finite-dimensional algebra over a field K, and let F be a finite separable field extension of K. Then $A \otimes_K F$ is an excellent extension of A.*

Remark 4.4. The condition "separable" is necessary for this lemma. For example, let K be a field of characteristic p. If F is a finite field extension of K but not separable, then there exists a finite-dimensional semisimple algebra A such that $A \otimes_K F$ is not semisimple [20]. Thus $A \otimes_K F$ is not an excellent extension of A by Lemma 2.4(2).

By Lemmas 2.4(2) and 4.3 and Theorem 3.8(1), we immediately get the following

Corollary 4.5. *(See [20, Theorem 3.3].) Let A be a finite-dimensional algebra over a field K, and let F be a finite separable field extension of K. Then* $\operatorname{rep.dim} A \otimes_K F = \operatorname{rep.dim} A$ *provided either of them is at most two.*

Now we are in a position to establish the relation between the representation dimensions of A and $A \otimes_K F$ in general case as follows.

Theorem 4.6. *Let A be a finite-dimensional algebra over a field K, and let F be a finite separable field extension of K. Then* $\operatorname{rep.dim} A \otimes_K F = \operatorname{rep.dim} A$.

Proof. The assertion holds true if $\operatorname{rep.dim} A \otimes_K F \leqslant 2$ by Corollary 4.5.

Assume that $\operatorname{rep.dim} A \otimes_K F = n$ $(\geqslant 3)$ and $V_{A \otimes_K F}$ is a generator–cogenerator for $\operatorname{mod} A \otimes_K F$ such that $\operatorname{gl.dim} \operatorname{End}(V_{A \otimes_K F}) = n$. It is easy to see that V_A is a generator–cogenerator for $\operatorname{mod} A$. Let $X \in \operatorname{mod} A$ be indecomposable. Then $X \otimes_K F \in \operatorname{mod} A \otimes_K F$. So $X \otimes_K F$ has an $(n-2)$-$\operatorname{add} V_{A \otimes_K F}$-resolution:

$$0 \longrightarrow V_{n-2} \xrightarrow{f_{n-2}} V_{n-3} \longrightarrow \cdots \longrightarrow V_0 \xrightarrow{f_0} X \otimes_K F \longrightarrow 0 \tag{3}$$

in $\operatorname{mod} A \otimes_K F$ by Lemma 2.7.

We claim that $X \otimes_K F$ as a right A-module has an $(n-2)$-add V_A-resolution. Obviously, (3) is in $\operatorname{mod} A$. Let $K_i = \operatorname{Ker} f_i$ for any $0 \leqslant i \leqslant n-2$ and $K_{-1} = X \otimes_K F$. Then we have exact sequences:

$$0 \to \operatorname{Hom}_{A\otimes_K F}(V, K_i) \to \operatorname{Hom}_{A\otimes_K F}(V, V_i) \to \operatorname{Hom}_{A\otimes_K F}(V, K_{i-1}) \to 0$$

and

$$\begin{aligned} 0 \to \operatorname{Hom}_{A\otimes_K F}(V, K_i) \otimes_K F &\to \operatorname{Hom}_{A\otimes_K F}(V, V_i) \otimes_K F \\ &\to \operatorname{Hom}_{A\otimes_K F}(V, K_{i-1}) \otimes_K F \to 0 \end{aligned}$$

for any $0 \leqslant i \leqslant n-2$.

Because

$$\begin{aligned} \operatorname{Hom}_A(A \otimes_K F, -) &\cong \operatorname{Hom}_K\big(F, \operatorname{Hom}_A(A, -)\big) \quad \text{(by the adjoint isomorphism theorem)} \\ &\cong \operatorname{Hom}_K(F, -) \\ &\cong \operatorname{Hom}_K\big(\operatorname{Hom}_K(-, K), \operatorname{Hom}_K(F, K)\big) \quad \text{(by the Yoneda lemma)} \\ &\cong \operatorname{Hom}_K\big(\operatorname{Hom}_K(-, K), F\big) \\ &\cong - \otimes_K F, \end{aligned}$$

we have

$$\begin{aligned} \operatorname{Hom}_{A\otimes_K F}(V, -) \otimes_K F &\cong \operatorname{Hom}_{A\otimes_K F}(V, - \otimes_K F) \\ &\cong \operatorname{Hom}_{A\otimes_K F}\big(V, \operatorname{Hom}_A(A \otimes_K F, -)\big) \\ &\cong \operatorname{Hom}_A\big(V \otimes_{A\otimes_K F} (A \otimes_K F), -\big) \quad \text{(by the adjoint isomorphism theorem)} \\ &\cong \operatorname{Hom}_A(V, -). \end{aligned}$$

So from the last exact sequence we get the following exact sequence:

$$0 \to \operatorname{Hom}_A(V, K_i) \to \operatorname{Hom}_A(V, V_i) \to \operatorname{Hom}_A(V, K_{i-1}) \to 0$$

for any $0 \leqslant i \leqslant n-2$, which induces an exact sequence:

$$\begin{aligned} 0 \to \operatorname{Hom}_A(V, V_{n-2}) &\to \operatorname{Hom}_A(V, V_{n-3}) \to \cdots \\ &\to \operatorname{Hom}_A(V, V_0) \to \operatorname{Hom}_A(V, X \otimes_K F) \to 0. \end{aligned}$$

The claim is proved.

Since $X_A \mid (X \otimes_K F)_A$, X_A has an $(n-2)$-add V_A-resolution by Lemma 4.2. Thus we conclude that $\operatorname{gl.dim} \operatorname{End}(V_A) \leqslant n$ by Lemma 2.7 and therefore $\operatorname{rep.dim} A \leqslant n$.

Conversely, assume that $\operatorname{rep.dim} A = m$ and M_A is a generator–cogenerator for $\operatorname{mod} A$ such that $\operatorname{gl.dim} \operatorname{End}(M_A) = m$. It is easy to see that $M \otimes_K F$ is a generator–cogenerator for $\operatorname{mod} A \otimes_K F$. Since $\operatorname{End}((M \otimes_K F)_{A\otimes_K F}) \cong \operatorname{End}(M_A) \otimes_K F$, $\operatorname{End}((M \otimes_K F)_{A\otimes_K F})$ is an excellent extension of $\operatorname{End}(M_A)$ by Lemma 4.3. So $\operatorname{gl.dim} \operatorname{End}((M \otimes_K F)_{A\otimes_K F}) = \operatorname{gl.dim} \operatorname{End}(M_A)$ by Lemma 2.4(2). It follows that $\operatorname{rep.dim} A \otimes_K F \leqslant m$. The proof is finished. □

For any $\Gamma \geqslant \Lambda$, by the adjoint isomorphism theorem we have the following adjoint pair (F, H):

$$F = - \otimes_\Lambda \Gamma : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma,$$

$$H = \operatorname{Hom}_\Gamma(\Gamma, -) : \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda.$$

Lemma 4.7. *Let $\Gamma \geqslant \Lambda$ be an excellent extension and F and H as above. Then (H, F) is an adjoint pair.*

Proof. Because $\Gamma \geqslant \Lambda$ is an excellent extension, both ${}_\Lambda\Gamma$ and Γ_Λ are finitely generated free, $\Gamma \cong \operatorname{Hom}_\Lambda(\Gamma, \Lambda)$ as left Λ-modules and as right Λ-modules, as well as right Γ-modules. Then $F \cong \operatorname{Hom}_\Lambda(\Gamma, -)$ by [5, Chapter III, Proposition 4.12]. On the other hand, $H \cong - \otimes_\Gamma \Gamma$. Note that $(- \otimes_\Gamma \Gamma, \operatorname{Hom}_\Lambda(\Gamma, -))$ is an adjoint pair by the adjoint isomorphism theorem, so (H, F) is also an adjoint pair. $\square$

For a commutative Artinian ring R and an Artinian R-algebra Γ, we denote by $\mathbb{D} = \operatorname{Hom}_R(-, E(R/J(R)))$ the Matlis duality between $\operatorname{mod} \Gamma$ and $\operatorname{mod} \Gamma^{op}$, where $J(R)$ is the radical of R and $E(R/J(R))$ is the injective envelope of $R/J(R)$. We establish the relation between the representation dimensions of a commutative Artinian ring and its excellent extension as follows.

Theorem 4.8. *Let R be a commutative Artinian ring and Γ an R-algebra. If Γ is an excellent extension of R, then* $\operatorname{rep.dim} \Gamma = \operatorname{rep.dim} R$.

Proof. By Lemma 3.5, Γ is an Artinian algebra. Then by Lemma 2.4(2) and Theorem 3.8(1), the assertion holds true provided either $\operatorname{rep.dim} R$ or $\operatorname{rep.dim} \Gamma$ is at most two.

Now assume that $\operatorname{rep.dim} R = n$ $(\geqslant 3)$ and M_R is a generator–cogenerator for $\operatorname{mod} R$ such that $\operatorname{gl.dim} \operatorname{End}(M_R) = n$. Because $R \oplus \mathbb{D}R \in \operatorname{add} M_R$ and $\Gamma \cong R \otimes_R \Gamma \in \operatorname{add}(M \otimes_R \Gamma)_\Gamma$, $(M \otimes_R \Gamma)_\Gamma$ is a generator for $\operatorname{mod} \Gamma$. Let $Y \in \operatorname{mod} \Gamma$. Then there exists a positive integer n such that $0 \to Y_R \to M^n$ is exact in $\operatorname{mod} R$ and so $0 \to Y \otimes_R \Gamma \to (M \otimes_R \Gamma)^n$ is exact in $\operatorname{mod} \Gamma$. Because $\Gamma \geqslant R$ is an excellent extension by assumption, ${}_R\Gamma$ is free. So $Y_\Gamma \mid (Y \otimes_R \Gamma)_\Gamma$, and hence $(M \otimes_R \Gamma)_\Gamma$ is a cogenerator for $\operatorname{mod} \Gamma$. Thus we get that $(M \otimes_R \Gamma)_\Gamma$ is a generator–cogenerator for $\operatorname{mod} \Gamma$.

Let $X \in \operatorname{mod} \Gamma$ be indecomposable. Then by Lemma 2.7, X as an R-module has an $(n-2)$-$\operatorname{add} M_R$-resolution:

$$0 \longrightarrow M_{n-2} \xrightarrow{f_{n-2}} M_{n-3} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{f_0} X \longrightarrow 0.$$

We claim that $(X \otimes_R \Gamma)_\Gamma$ has an $(n-2)$-$\operatorname{add}(\Gamma \otimes_R M)_\Gamma$-resolution:

$$0 \to M_{n-2} \otimes_R \Gamma \to M_{n-3} \otimes_R \Gamma \to \cdots \to M_0 \otimes_R \Gamma \to X \otimes_R \Gamma \to 0.$$

Let $K_i = \operatorname{Ker} f_i$ for any $0 \leqslant i \leqslant n-2$ and $K_{-1} = X$. Because Γ is a finitely generated free R-module, we have the following exact sequence:

$$0 \to K_i \otimes_R \Gamma \to M_i \otimes_R \Gamma \to K_{i-1} \otimes_R \Gamma \to 0,$$

which is exact as right Γ-modules and as R-modules for any $0 \leqslant i \leqslant n-2$. On the other hand, we have the following exact sequence:

$$0 \to \operatorname{Hom}_R(M, K_i) \to \operatorname{Hom}_R(M, M_i) \to \operatorname{Hom}_R(M, K_{i-1}) \to 0$$

in $\operatorname{mod} R$, which induces the following exact sequence:

$$0 \to \mathrm{Hom}_R(M, K_i) \otimes_R \Gamma \to \mathrm{Hom}_R(M, M_i) \otimes_R \Gamma \to \mathrm{Hom}_R(M, K_{i-1}) \otimes_R \Gamma \to 0$$

for any $0 \leqslant i \leqslant n-2$. By Lemma 2.8, the sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, K_i \otimes_R \Gamma) \to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, M_i \otimes_R \Gamma) \\ &\to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, K_{i-1} \otimes_R \Gamma) \to 0 \end{aligned}$$

is also exact for any $0 \leqslant i \leqslant n-2$, which implies that the following sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, M_{n-2} \otimes_R \Gamma) \to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, M_{n-3} \otimes_R \Gamma) \to \cdots \\ &\to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, M_0 \otimes_R \Gamma) \to \mathrm{Hom}_\Gamma(M \otimes_R \Gamma, X \otimes_R \Gamma) \to 0 \end{aligned}$$

is exact. The claim is proved.

Notice that $X_\Gamma \mid (X \otimes_R \Gamma)_\Gamma$, so X_Γ has an $(n-2)$-add$(M \otimes_R \Gamma)_\Gamma$-resolution by Lemma 4.2. Thus gl.dim End$((M \otimes_R \Gamma)_\Gamma) \leqslant n$ by Lemma 2.7 and therefore rep.dim $\Gamma \leqslant n$.

Conversely, assume that rep.dim $\Gamma = m$ $(\geqslant 3)$ and V_Γ is a generator–cogenerator for mod Γ such that gl.dim End$(V_\Gamma) = m$. It is easy to see that V_R is a generator–cogenerator for mod R. Let $Y \in \mathrm{mod}\, R$ be indecomposable. Then $Y \otimes_R \Gamma \in \mathrm{mod}\, \Gamma$. So by Lemma 2.7, we have the following exact sequence:

$$0 \to V_{m-2} \to V_{m-3} \to \cdots \to V_0 \to Y \otimes_R \Gamma \to 0$$

in mod Γ (and hence in mod R) such that

$$\begin{aligned} 0 &\to \mathrm{Hom}_\Gamma(V, V_{m-2}) \to \mathrm{Hom}_\Gamma(V, V_{m-3}) \to \cdots \\ &\to \mathrm{Hom}_\Gamma(V, V_0) \to \mathrm{Hom}_\Gamma(V, Y \otimes_R \Gamma) \to 0 \end{aligned}$$

is also exact in mod R. Because $\Gamma \geqslant R$ is an excellent extension by assumption, ${}_R\Gamma$ is free. Then we get the following exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_\Gamma(V, V_{m-2}) \otimes_R \Gamma \to \mathrm{Hom}_\Gamma(V, V_{m-3}) \otimes_R \Gamma \to \cdots \\ &\to \mathrm{Hom}_\Gamma(V, V_0) \otimes_R \Gamma \to \mathrm{Hom}_\Gamma(V, Y \otimes_R \Gamma) \otimes_R \Gamma \to 0. \end{aligned}$$

Since $(\mathrm{Hom}_\Gamma(\Gamma, -), - \otimes_R \Gamma)$ is an adjoint pair by Lemma 4.7, for any $U \in \mathrm{mod}\, \Gamma$ we have

$$\begin{aligned} \mathrm{Hom}_\Gamma(V, U) \otimes_R \Gamma &\cong \mathrm{Hom}_\Gamma(V, U \otimes_R \Gamma) \quad \text{(by [13, Theorem 3.2.14])} \\ &\cong \mathrm{Hom}_R\big(\mathrm{Hom}_\Gamma(\Gamma, V), U\big) \quad \text{(by the adjoint isomorphism theorem)} \\ &\cong \mathrm{Hom}_R(V, U). \end{aligned}$$

So from the last exact sequence we get the following exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(V, V_{m-2}) \to \mathrm{Hom}_R(V, V_{m-3}) \to \cdots \\ &\to \mathrm{Hom}_R(V, V_0) \to \mathrm{Hom}_R(V, Y \otimes_R \Gamma) \to 0. \end{aligned}$$

Thus $Y \otimes_R \Gamma$ as an R-module has an $(m-2)$-add V_R-resolution. Since $Y_R \mid (Y \otimes_R \Gamma)_R$, Y_R has an $(m-2)$-add V_R-resolution by Lemma 4.2. Thus gl.dim End$(V_R) \leqslant m$ by Lemma 2.7 and therefore rep.dim $R \leqslant m$. The proof is finished. □

Corollary 4.9.

(1) *Let R be a commutative Artinian ring and G a finite group with $|G|^{-1} \in R$. Then* $\operatorname{rep.dim} R * G = \operatorname{rep.dim} R$.
(2) *Let K be a field of characteristic p and H a subgroup of the center of a finite group G. If H contains a Sylow p-subgroup of G, then* $\operatorname{rep.dim} KG = \operatorname{rep.dim} KH$.

Proof. By Example 2.2, both $R * G \geqslant R$ (in (1)) and $KG \geqslant KH$ (in (2)) are excellent extensions. So both assertions follow from Theorem 4.8. $\square$

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