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Selforthogonal modules with finite injective dimension II

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Abstract

Let Λ be a left and right Artin ring and ${}_\Lambda\omega_\Lambda$ a faithfully balanced selforthogonal bimodule. We give a sufficient condition that the injective dimension of ω_Λ is finite implies that of ${}_\Lambda\omega$ is also finite.

Keywords: Selforthogonal modules; Cotilting modules; Injective dimension

1. Introduction

Unless stated otherwise, Λ is a left noetherian ring, Γ is a right noetherian ring. We use $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{\mathrm{op}}$) to denote the category of finitely generated left Λ-modules (resp. right Γ-modules). The modules considered are finitely generated. For a module ω in $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{\mathrm{op}}$) we use $\mathrm{l.id}_\Lambda(\omega)$ (resp. $\mathrm{r.id}_\Gamma(\omega)$) to denote the left (resp. right) injective dimension of ω.

Definition 1 [10]. Let ω be in $\operatorname{mod}\Lambda$. We call ω a selforthogonal module if $\operatorname{Ext}^i_\Lambda(\omega,\omega)=0$ for any $i \geqslant 1$. A selforthogonal module ω is called a cotilting module if $\mathrm{l.id}_\Lambda(\omega)<\infty$ and the natural map $\Lambda \to \operatorname{End}(\omega_{\operatorname{End}({}_\Lambda\omega)})$ is an isomorphism. Similarly, we define the notion of cotilting modules in $\operatorname{mod}\Gamma^{\mathrm{op}}$. Dually, we define the notion of tilting modules in $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{\mathrm{op}}$).

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Remark. In case Λ (resp. Γ) is an Artin algebra, the definitions of tilting modules and cotilting modules coincide with those given in [2,3]. These can be seen by using [12, Proposition 1.6] and its dual result.

A bimodule ${}_\Lambda\omega_\Gamma$ is called a faithfully balanced selforthogonal bimodule if it satisfies the following conditions:

(1) The natural maps $\Gamma \to \mathrm{End}({}_\Lambda\omega)^{\mathrm{op}}$ and $\Lambda \to \mathrm{End}(\omega_\Gamma)$ are isomorphisms.
(2) $\mathrm{Ext}^i_\Lambda({}_\Lambda\omega, {}_\Lambda\omega) = 0 = \mathrm{Ext}^i_\Gamma(\omega_\Gamma, \omega_\Gamma)$ for any $i \geqslant 1$.

Miyashita in [12] showed that for a faithfully balanced selforthogonal bimodule ${}_\Lambda\omega_\Gamma$, ${}_\Lambda\omega$ is tilting if and only if ω_Γ is tilting. Assume that Λ and Γ are Artin algebras. If ${}_\Lambda\omega$ and ω_Γ are cotilting then $\mathrm{l.id}_\Lambda(\omega) = \mathrm{r.id}_\Gamma(\omega)$ by [3, Lemma 1.7]. However, in general we do not know whether ${}_\Lambda\omega$ (resp. ω_Γ) is necessarily cotilting or not provided that ω_Γ (resp. ${}_\Lambda\omega$) is cotilting. Then it is natural to ask when ${}_\Lambda\omega$ is cotilting if ω_Γ is cotilting. This question is a general case of an important question raised by Auslander and Reiten [2, p. 150] (that is, does $\mathrm{r.id}_\Lambda(\Lambda) < \infty$ imply $\mathrm{l.id}_\Lambda(\Lambda) < \infty$ (where Λ is an Artin algebra)?). In this paper, for a faithfully balanced selforthogonal bimodule ${}_\Lambda\omega_\Lambda$ over a left and right Artin ring Λ, we give a sufficient condition that ω_Λ is cotilting implies that ${}_\Lambda\omega$ is also cotilting. As a consequence, we have that ${}_\Lambda\omega$ is classical cotilting if and only if ω_Λ is classical cotilting.

2. Main result

Let A be in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\mathrm{op}}$) and i a non-negative integer. We say that the grade of A, written grade A, is greater than or equal to i if $\mathrm{Ext}^j_\Lambda(A, \Lambda) = 0$ (resp. $\mathrm{Ext}^j_\Gamma(A, \Gamma) = 0$) for any $0 \leqslant j < i$. We denote $\mathrm{s.grade}\, A \geqslant i$ if $\mathrm{grade}\, X \geqslant i$ for each submodule X of A. Let W be in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\mathrm{op}}$). We say that the grade of A with respect to W, written $\mathrm{grade}_W A$, is greater than or equal to i if $\mathrm{Ext}^j_\Lambda(A, W) = 0$ (resp. $\mathrm{Ext}^j_\Gamma(A, W) = 0$) for any $0 \leqslant j < i$.

Assume that Λ is a left and right Artin ring and ${}_\Lambda\omega_\Lambda$ is a faithfully balanced selforthogonal bimodule. Our main result is the following

Theorem. *Let m and n be positive integers. Suppose that* $\mathrm{r.id}_\Lambda(\omega) \leqslant n$ *and* $\mathrm{grade}_\omega \mathrm{Ext}^m_\Lambda(M, \omega) \geqslant n-1$ *for any* $M \in \mathrm{mod}\,\Lambda$. *Then* $\mathrm{l.id}_\Lambda(\omega) \leqslant m+n-1$.

A cotilting module ${}_\Lambda\omega$ (resp. ω_Λ) is call classical cotilting if $\mathrm{l.id}_\Lambda(\omega)$ (resp. $\mathrm{r.id}_\Lambda(\omega)$) $\leqslant 1$. Consider the case $n = 1$ in theorem above. It is clear that the second assumption ($\mathrm{grade}_\omega \mathrm{Ext}^m_\Lambda(M, \omega) \geqslant n-1$ for any $M \in \mathrm{mod}\,\Lambda$) is always satisfied and we get

Corollary 1. *${}_\Lambda\omega$ is classical cotilting if and only if ω_Λ is classical cotilting.*

Put ${}_\Lambda\omega_\Gamma = {}_\Lambda\Lambda_\Lambda$. Then we have

Corollary 2. $\mathrm{l.id}_\Lambda(\Lambda) \leqslant 1$ *if and only if* $\mathrm{r.id}_\Lambda(\Lambda) \leqslant 1$.

Let $\mathrm{r.id}_\Lambda(\Lambda) \leqslant n(<\infty)$ and

$$0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_n \to 0$$

be a minimal injective resolution of Λ as a right Λ-module. Assume that the right flat dimension of $\bigoplus_{i=0}^{n-1} I_i$ is less than or equal to $r(<\infty)$. We may assume that $r \geqslant n$ and $r = n + s$ where s is a non-negative integer. Then by [8, Theorem 2.8] we have $\mathrm{s.grade}\,\mathrm{Ext}_\Lambda^{n+s+1}(M, \Lambda) \geqslant n$, and certainly $\mathrm{grade}\,\mathrm{Ext}_\Lambda^{n+s+1}(M, \Lambda) \geqslant n$ for any $M \in \mathrm{mod}\,\Lambda$. By theorem above, $\mathrm{l.id}_\Lambda(\Lambda) \leqslant (n + s + 1) + n - 1 = 2n + s\ (<\infty)$. It follows from [13, Lemma A] that $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda)$. Hence we have established

Corollary 3. *If* $\mathrm{r.id}_\Lambda(\Lambda) = n$ *and the first* n *terms of the minimal injective resolution of* Λ_Λ *have finite right flat dimension, then* $\mathrm{l.id}_\Lambda(\Lambda) = n$.

Suppose k is a positive integer. An Artin algebra Λ is called quasi k-Gorenstein [9] (resp. k-Gorenstein [4]) if the ith term of the minimal injective resolution of ${}_\Lambda\Lambda$ has left flat dimension at most $i + 1$ (resp. i) for any $0 \leqslant i \leqslant k - 1$. By theorem above, [8, Theorem 3.3] (or [5, Theorem 4.7]) and [13, Lemma A] we have

Corollary 4. $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda)$ *if* Λ *is a* (*quasi*) k*-Gorenstein algebra for all* k.

Auslander showed in [7, Theorem 3.7] that the notion of k-Gorenstein algebras is left-right symmetric (note: on the contrary, the notion of quasi k-Gorenstein algebras is not left-right symmetric [9]). An Artin algebra Λ is called Auslander–Gorenstein [6] if Λ is k-Gorenstein for all k and it has finite left and right self-injective dimension (that is, $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda) < \infty$). By Corollary 4 we may weaken the condition of this definition, that is, we have that an Artin algebra Λ is Auslander–Gorenstein if Λ is k-Gorenstein for all k and it has finite either sided self-injective dimension (see [4, Corollary 5.5(b)]).

3. The proof of main result

We first recall some notions. Let A be in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\mathrm{op}}$). We call $\mathrm{Hom}_\Lambda({}_\Lambda A, {}_\Lambda\omega_\Gamma)$ (resp. $\mathrm{Hom}_\Gamma(A_\Gamma, {}_\Lambda\omega_\Gamma)$) the dual module of A with respect to ω, and denote either of these modules by A^ω. For a homomorphism f between Λ-modules (resp. Γ^{op}-modules), we put $f^\omega = \mathrm{Hom}(f, {}_\Lambda\omega_\Gamma)$. Let $\sigma_A : A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^\omega$ be the canonical evaluation homomorphism. A is called ω-torsionless (resp. ω-reflexive) if σ_A is a monomorphism (resp. an isomorphism). It is easy to see that any projective module in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\mathrm{op}}$) is ω-reflexive.

Let A be in $\mathrm{mod}\,\Lambda$ and

$$\cdots \to P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to A \to 0$$

be a projective resolution of A in mod Λ. Put $A_i = \operatorname{Coker} f_{i+1}$ and $X_i = \operatorname{Coker} f_i^{\omega}$. For each f_i ($i \geqslant 1$) there is a natural epic-monic decomposition: $f_i = \alpha_i \pi_i$ with π_i epic and α_i monic.

Lemma 1. *$X_i^{\omega} \cong A_{i+1}$ and $X_i^{\omega\omega} \cong A_{i+1}^{\omega} \cong \operatorname{Ker} f_{i+2}^{\omega}$ for any $i \geqslant 1$.*

Proof. For any $i \geqslant 1$ we have exact sequences:

$$0 \to A_{i+1} \xrightarrow{\alpha_{i+1}} P_i \xrightarrow{f_i} P_{i-1} \xrightarrow{\pi_{i-1}} A_{i-1} \to 0,$$

$$0 \to A_{i-1}^{\omega} \xrightarrow{\pi_{i-1}^{\omega}} P_{i-1}^{\omega} \xrightarrow{f_i^{\omega}} P_i^{\omega} \xrightarrow{\beta_i} X_i \to 0.$$

Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{i+1} & \xrightarrow{\alpha_{i+1}} & P_i & \xrightarrow{f_i} & P_{i-1} & \xrightarrow{\pi_{i-1}} & A_{i-1} \longrightarrow 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \sigma_{P_i}} & & \downarrow{\scriptstyle \sigma_{P_{i-1}}} & & \\
0 & \longrightarrow & X_i^{\omega} & \xrightarrow{\beta_i^{\omega}} & P_i^{\omega\omega} & \xrightarrow{f_i^{\omega\omega}} & P_{i-1}^{\omega\omega} & &
\end{array} \tag{1}$$

where σ_{P_i} and $\sigma_{P_{i-1}}$ are isomorphisms. Hence f is an isomorphism and $X_i^{\omega} \cong A_{i+1}$. The other assertions follow easily. $\square$

Lemma 2. *For any $i \geqslant 1$ there is an exact sequence*:

$$\zeta_i : 0 \to \operatorname{Ext}_{\Lambda}^{i}(A, \omega) \to X_i \xrightarrow{\phi_i} P_{i+1}^{\omega} \to X_{i+1} \to 0.$$

Proof. Let ϕ_i be the composition:

$$X_i \xrightarrow{\sigma_{X_i}} X_i^{\omega\omega} \xrightarrow{f^{\omega}} A_{i+1}^{\omega} \xrightarrow{\pi_{i+1}^{\omega}} P_{i+1}^{\omega},$$

that is, $\phi_i = \pi_{i+1}^{\omega} f^{\omega} \sigma_{X_i}$.

Since π_{i+1}^{ω} is a monomorphism and f^{ω} is an isomorphism, $\operatorname{Ker} \phi_i = \operatorname{Ker}(\pi_{i+1}^{\omega} f^{\omega} \sigma_{X_i}) \cong \operatorname{Ker} \sigma_{X_i} \cong \operatorname{Ext}_{\Lambda}^{1}(A_{i-1}, \omega) \cong \operatorname{Ext}_{\Lambda}^{i}(A, \omega)$ by [11, Lemma 2.1].

Now we calculate $\operatorname{Coker} \phi_i$.

Since $f_{i+1} = \alpha_{i+1}\pi_{i+1}$, $f_{i+1}^{\omega} = \pi_{i+1}^{\omega}\alpha_{i+1}^{\omega}$. From diagram (1) we know that $\sigma_{P_i}\alpha_{i+1} = \beta_i^{\omega} f$ and $\alpha_{i+1}^{\omega}\sigma_{P_i}^{\omega} = f^{\omega}\beta_i^{\omega\omega}$, and so $f_{i+1}^{\omega}\sigma_{P_i}^{\omega} = \pi_{i+1}^{\omega}\alpha_{i+1}^{\omega}\sigma_{P_i}^{\omega} = \pi_{i+1}^{\omega} f^{\omega}\beta_i^{\omega\omega}$. Since $\sigma_{P_i}^{\omega}\sigma_{P_i^{\omega}} = 1_{P_i^{\omega}}$ (cf. [1, Proposition 20.14]) and $\beta_i^{\omega\omega}\sigma_{P_i^{\omega}} = \sigma_{X_i}\beta_i$, we have that $f_{i+1}^{\omega} = \pi_{i+1}^{\omega} f^{\omega}\beta_i^{\omega\omega}\sigma_{P_i^{\omega}} = \pi_{i+1}^{\omega} f^{\omega}\sigma_{X_i}\beta_i = \phi_i\beta_i$. Since β_i is epic, $\operatorname{Im} f_{i+1}^{\omega} = \operatorname{Im}(\phi_i\beta_i) \cong \operatorname{Im}\phi_i$ and $\operatorname{Coker}\phi_i \cong P_{i+1}^{\omega}/\operatorname{Im}\phi_i \cong P_{i+1}^{\omega}/\operatorname{Im} f_{i+1}^{\omega} \cong \operatorname{Coker} f_{i+1}^{\omega} = X_{i+1}$. We are done. $\square$

Lemma 3. $\operatorname{Ext}_{\Gamma}^{1}(X_i, \omega) = 0$ *for any $i \geqslant 2$.*

Proof. By [11, Lemma 2.1] there is an exact sequence:

$$0 \to \mathrm{Ext}^1_\Gamma(X_i, \omega) \to A_{i-1} \xrightarrow{\sigma_{A_{i-1}}} A^{\omega\omega}_{i-1} \to \mathrm{Ext}^2_\Gamma(X_i, \omega) \to 0.$$

If $i \geqslant 2$, then A_{i-1} is ω-torsionless because A_{i-1} is a submodule of P_{i-2}. It follows that $\sigma_{A_{i-1}}$ is monic and $\mathrm{Ext}^1_\Gamma(X_i, \omega) = 0$. □

Lemma 4. *Suppose m and n are positive integers and* $\mathrm{grade}_\omega \mathrm{Ext}^m_\Lambda(M, \omega) \geqslant n - 1$ *for any $M \in \mathrm{mod}\,\Lambda$. Then $\mathrm{Ext}^j_\Gamma(X_{i+j-1}, \omega) = 0$ for any $i \geqslant m + 1$ and $1 \leqslant j \leqslant n$.*

Proof. The case $n = 1$ follows from Lemma 3. Now suppose $n \geqslant 2$. Since $\mathrm{grade}_\omega \mathrm{Ext}^m_\Lambda(M, \omega) \geqslant n - 1$ for any $M \in \mathrm{mod}\,\Lambda$, it is easy to see that $\mathrm{grade}_\omega \mathrm{Ext}^i_\Lambda(M, \omega) \geqslant n - 1$ for any $M \in \mathrm{mod}\,\Lambda$ and $i \geqslant m$. Applying $\mathrm{Hom}_\Gamma(-, \omega)$ to the exact sequences $(\zeta_i), \ldots, (\zeta_{i+n-2})$ (where $i \geqslant m + 1$) in Lemma 2, we get a chain of embeddings:

$$\mathrm{Ext}^n_\Gamma(X_{i+n-1}, \omega) \hookrightarrow \mathrm{Ext}^{n-1}_\Gamma(X_{i+n-2}, \omega) \hookrightarrow \cdots \hookrightarrow \mathrm{Ext}^1_\Gamma(X_i, \omega).$$

Now our assertion follows from Lemma 3. □

From now on, assume that m and n are positive integers, $\mathrm{r.id}_\Gamma(\omega) \leqslant n$ and $\mathrm{grade}_\omega \mathrm{Ext}^m_\Lambda(M, \omega) \geqslant n - 1$ for any $M \in \mathrm{mod}\,\Lambda$.

Lemma 5. $\mathrm{Ext}^j_\Gamma(X_i, \omega) = 0$ *for any $i \geqslant m + n$ and $j \geqslant 1$.*

Proof. Since $\mathrm{r.id}_\Gamma(\omega) \leqslant n$, $\mathrm{Ext}^j_\Gamma(X_i, \omega) = 0$ for any $j \geqslant n + 1$. On the other hand, $\mathrm{Ext}^j_\Gamma(X_i, \omega) = 0$ for any $1 \leqslant j \leqslant n$ by Lemma 4. Hence we are done. □

Lemma 6. *A_i is ω-reflexive for any $i \geqslant m + n - 1$.*

Proof. By [11, Lemma 2.1] there is an exact sequence:

$$0 \to \mathrm{Ext}^1_\Gamma(X_{i+1}, \omega) \to A_i \xrightarrow{\sigma_{A_i}} A^{\omega\omega}_i \to \mathrm{Ext}^2_\Gamma(X_{i+1}, \omega) \to 0.$$

By Lemma 5, $\mathrm{Ext}^1_\Gamma(X_{i+1}, \omega) = 0 = \mathrm{Ext}^2_\Gamma(X_{i+1}, \omega)$ for any $i \geqslant m + n - 1$. So σ_{A_i} is an isomorphism and A_i is ω-reflexive. □

Lemma 7. $\mathrm{Ext}^j_\Gamma(A^\omega_i, \omega) = 0$ *for any $i \geqslant m + n - 1$ and $j \geqslant 1$.*

Proof. Because there is an exact sequence

$$0 \to A^\omega_i \to P^\omega_i \xrightarrow{f^\omega_{i+1}} P^\omega_{i+1} \to X_{i+1} \to 0,$$

our conclusion follows from Lemma 5. □

Lemma 8. $\operatorname{grade}_\omega \operatorname{Ext}^i_\Lambda(A,\omega)=\infty$ *for any* $i \geqslant m+n$.

Proof. From the definition of ϕ_i (see the proof of Lemma 2) we know that $\phi_i = \pi^\omega_{i+1} f^\omega \sigma_{X_i}$ and $\phi^\omega_i = \sigma^\omega_{X_i} f^{\omega\omega}\pi^{\omega\omega}_{i+1}$. Notice that A_{i+1} is ω-reflexive by Lemma 6, so $\sigma_{A_{i+1}}$ is an isomorphism. Since $\pi^{\omega\omega}_{i+1}\sigma_{P_{i+1}} = \sigma_{A_{i+1}}\pi_{i+1}$ and π_{i+1} is epic, $\pi^{\omega\omega}_{i+1}$ is also epic. On the other hand, $\sigma^\omega_{X_i}$ is epic by [1, Proposition 20.14], $f^{\omega\omega}$ is an isomorphism since f is an isomorphism (see the proof of Lemma 1). Hence we have that ϕ^ω_i is epic.

Put $K = \operatorname{Im}\phi_i$. Then we have an epic-monic decomposition $\phi_i = \alpha\pi$ with $\pi : X_i \to K$ epic and $\alpha : K \to P^\omega_{i+1}$ monic. Since ϕ^ω_i is epic and π^ω is monic, from $\phi^\omega_i = \pi^\omega\alpha^\omega$ we know that π^ω is an epimorphism and hence an isomorphism. Moreover, from the exact sequence $0 \to \operatorname{Ext}^i_\Lambda(A,\omega) \to X_i \xrightarrow{\pi} K \to 0$ we get a long exact sequence:

$$\begin{aligned}
0 \;\to\; & K^\omega \xrightarrow{\pi^\omega} X^\omega_i \to \big[\operatorname{Ext}^i_\Lambda(A,\omega)\big]^\omega \to \operatorname{Ext}^1_\Gamma(K,\omega) \to \operatorname{Ext}^1_\Gamma(X_i,\omega)\\
\to\; & \operatorname{Ext}^1_\Gamma\big(\operatorname{Ext}^i_\Lambda(A,\omega),\omega\big) \to \operatorname{Ext}^2_\Gamma\big(K,\omega\big) \to \operatorname{Ext}^2_\Gamma(X_i,\omega) \to \cdots \to \operatorname{Ext}^j_\Gamma\big(X_i,\omega\big)\\
\to\; & \operatorname{Ext}^j_\Gamma\big(\operatorname{Ext}^i_\Lambda(A,\omega),\omega\big) \to \operatorname{Ext}^{j+1}_\Gamma(K,\omega) \to \operatorname{Ext}^{j+1}_\Gamma(X_i,\omega) \to \cdots;
\end{aligned}$$

on the other hand, applying $\operatorname{Hom}_\Gamma(-,\omega)$ to the exact sequence $0 \to K \xrightarrow{\alpha} P^\omega_{i+1} \to X_{i+1} \to 0$ we get the following isomorphisms:

$$\operatorname{Ext}^j_\Gamma(K,\omega) \cong \operatorname{Ext}^{j+1}_\Gamma(X_{i+1},\omega)$$

for any $j \geqslant 1$.

Note that $i \geqslant m+n$, so $\operatorname{Ext}^j_\Gamma(X_i,\omega) = 0 = \operatorname{Ext}^{j+1}_\Gamma(X_{i+1},\omega)$ for any $j \geqslant 1$ by Lemma 5. It follows from the long exact sequence above that $[\operatorname{Ext}^i_\Lambda(A,\omega)]^\omega = 0 = \operatorname{Ext}^j_\Gamma(\operatorname{Ext}^i_\Lambda(A,\omega),\omega)$ for any $j \geqslant 1$ and $\operatorname{grade}_\omega \operatorname{Ext}^i_\Lambda(A,\omega) = \infty$. □

Assume that Λ is a left and right Artin ring and ${}_\Lambda\omega_\Gamma = {}_\Lambda\omega_\Lambda$. We now give the proof of the main result.

Proof of Theorem. Because $\operatorname{r.id}_\Lambda(\omega) \leqslant n (< \infty)$, there is a well defined linear map $\beta : K_0(\operatorname{mod}\Lambda^{\mathrm{op}}) \to K_0(\operatorname{mod}\Lambda)$ via $\beta([X]) = \sum_{i\geqslant 0}(-1)^i[\operatorname{Ext}^i_\Lambda(X,\omega)]$ for any X in $\operatorname{mod}\Lambda^{\mathrm{op}}$.

For $i \geqslant m+n-1$, by Lemmas 6 and 7 we have

$$\begin{aligned}
[A] &= \sum_{j=0}^{i-1}(-1)^j[P_j] + (-1)^i[A_i] = \sum_{j=0}^{i-1}(-1)^j\big[P^{\omega\omega}_j\big] + (-1)^i\big[A^{\omega\omega}_i\big]\\
&= \sum_{j=0}^{i-1}(-1)^j\beta\big(\big[P^\omega_j\big]\big) + (-1)^i\beta\big(\big[A^\omega_i\big]\big)\\
&= \beta\left(\sum_{j=0}^{i-1}(-1)^j\big[P^\omega_j\big] + (-1)^i\big[A^\omega_i\big]\right),
\end{aligned}$$

which implies that β is surjective.

Note that Λ is a left and right Artin ring, so both $K_0(\operatorname{mod}\Lambda^{\mathrm{op}})$ and $K_0(\operatorname{mod}\Lambda)$ are finitely generated free abelian groups with $\operatorname{rank} K_0(\operatorname{mod}\Lambda^{\mathrm{op}}) = \operatorname{rank} K_0(\operatorname{mod}\Lambda)$, and $[X] = 0$ if and only if $X = 0$ for any X in $\operatorname{mod}\Lambda^{\mathrm{op}}$. On the other hand, $\operatorname{grade}_\omega \operatorname{Ext}^i_\Lambda(A,\omega) = \infty$ for any $i \geqslant m+n$ by Lemma 8. So $\operatorname{Ext}^j_\Lambda(\operatorname{Ext}^i_\Lambda(A,\omega),\omega) = 0$ for any $j \geqslant 0$ and $i \geqslant m+n$ and $\beta([\operatorname{Ext}^i_\Lambda(A,\omega)]) = 0$ for any $i \geqslant m+n$. Consequently $[\operatorname{Ext}^i_\Lambda(A,\omega)] = 0$ and $\operatorname{Ext}^i_\Lambda(A,\omega) = 0$ for any $i \geqslant m+n$, which implies $\mathrm{l.id}_\Lambda(\omega) \leqslant m+n-1$. $\square$

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