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Syzygy modules for quasi k-Gorenstein rings ☆

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Abstract

Let Λ be a right quasi k-Gorenstein ring. For each dth syzygy module M in $\operatorname{mod}\Lambda$ (where $0 \leqslant d \leqslant k-1$), we obtain an exact sequence $0 \to B \to M \oplus P \to C \to 0$ in $\operatorname{mod}\Lambda$ with the properties that it is dual exact, P is projective, C is a $(d+1)$st syzygy module, B is a dth syzygy of $\operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda)$ and the right projective dimension of B^* is less than or equal to $d-1$. We then give some applications of such an exact sequence as follows. (1) We obtain a chain of epimorphisms concerning M, and by dualizing it we then get the spherical filtration of Auslander and Bridger for M^*. (2) We get Auslander and Bridger's Approximation Theorem for each reflexive module in $\operatorname{mod}\Lambda^{\mathrm{op}}$. (3) We show that for any $0 \leqslant d \leqslant k-1$ each dth syzygy module in $\operatorname{mod}\Lambda$ has an Evans–Griffith presentation. As an immediate consequence of (3), we have that, if Λ is a commutative Noetherian ring with finite self-injective dimension, then for any non-negative integer d, each dth syzygy module in $\operatorname{mod}\Lambda$ has an Evans–Griffith presentation, which generalizes an Evans and Griffith's result to much more general setting.

Keywords: Syzygy modules; Quasi k-Gorenstein rings; Spherical filtration; Evans–Griffith presentations

1. Introduction

Let Λ be a left and right Noetherian ring and $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules.

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It is well known that Λ possesses rather interesting properties when it satisfies the condition that $\operatorname{grade}\operatorname{Ext}^i_{\Lambda^{\mathrm{op}}}(N, \Lambda) \geqslant i$ for any $N \in \operatorname{mod}\Lambda^{\mathrm{op}}$ and $1 \leqslant i \leqslant k$ (where k is a positive integer). Assume that Λ satisfies this grade condition. For any T in $\operatorname{mod}\Lambda^{\mathrm{op}}$, Auslander and Bridger in [3, Spherical Filtration Theorem 2.37] produced a projective module Q and a filtration $T \oplus Q = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$ such that each T_i/T_{i+1} is "spherical" in the sense that the cohomological $\operatorname{Ext}^j_{\Lambda^{\mathrm{op}}}(T_i/T_{i+1}, \Lambda) \neq 0$ only if $j = 0$ or $j = i$. They also showed in [3] that under this grade condition the following statements are true: (1) the full subcategory of $\operatorname{mod}\Lambda^{\mathrm{op}}$ consisting of the modules with projective dimension less than or equal to k is covariantly finite (see [4] for the definition of covariantly finite); (2) a dth syzygy module in $\operatorname{mod}\Lambda^{\mathrm{op}}$ is d-torsionfree for any $1 \leqslant d \leqslant k$. We remark that the second statement does not hold in general although the converse is always true.

It follows from Auslander and Reiten [5, Theorem 0.1] and Hoshino and Nishida [9, Theorem 4.1] that the following conditions (1) and (2) are equivalent for a left and right Noetherian ring Λ:

(1) For a minimal injective resolution $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ of Λ as a right Λ-module, we have flat $\dim_\Lambda(I_i) \leqslant i + 1$ for any $0 \leqslant i \leqslant k - 1$.
(2) $\operatorname{grade}\operatorname{Ext}^i_{\Lambda^{\mathrm{op}}}(N, \Lambda) \geqslant i$ for any $N \in \operatorname{mod}\Lambda^{\mathrm{op}}$ and $1 \leqslant i \leqslant k$.

We call a ring *right quasi k-Gorenstein* provided it satisfies one of these equivalent conditions. A ring is called *right quasi Auslander* if it is right quasi k-Gorenstein for all k.

Recall that Λ is called a *k-Gorenstein ring* if the right flat dimension of the $(i+1)$st term in a minimal injective resolution of Λ as a right Λ-module is less than or equal to i for any $0 \leqslant i \leqslant k-1$. This notion was introduced by Auslander in [8]. Iwanaga and Sato in [11] called Λ an *Auslander ring* if it is k-Gorenstein for all k. In [8, Theorem 3.7] Auslander showed that the notion of k-Gorenstein rings (and hence that of Auslander rings) is left–right symmetric and that Λ is k-Gorenstein if and only if the grade of any submodule of $\operatorname{Ext}^i_{\Lambda^{\mathrm{op}}}(N, \Lambda)$ is greater than or equal to i for any $N \in \operatorname{mod}\Lambda^{\mathrm{op}}$ and $1 \leqslant i \leqslant k$. However, as already pointed out in [5], the notion of quasi k-Gorenstein rings (and hence that of quasi Auslander rings) is not left–right symmetric.

Notice that Bass showed in [6] that a commutative Noetherian ring Λ has finite self-injective dimension if and only if $\operatorname{grade}\operatorname{Ext}^i_\Lambda(N, \Lambda) \geqslant i$ for any $N \in \operatorname{mod}\Lambda$ and $i \geqslant 1$. So the notion of Auslander rings is in fact a generalization of that of commutative Noetherian rings with finite self-injective dimension.

The discussion in this paper is based on the results mentioned above. For a right quasi k-Gorenstein ring Λ and each dth syzygy module in $\operatorname{mod}\Lambda$ (where $0 \leqslant d \leqslant k-1$) we obtain here an exact sequence with "nice" properties as follows.

Theorem. *Let Λ be a right quasi k-Gorenstein ring and M a dth syzygy module in* $\operatorname{mod}\Lambda$ (*where d is an integer with* $0 \leqslant d \leqslant k-1$). *Then there is a projective module P in* $\operatorname{mod}\Lambda$ *such that the dth syzygy B of* $\operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda)$ (*see Section* 2 *for the definition of* $\operatorname{Tr} M$) *is a submodule of $M \oplus P$ and such that the exact sequence*

$$0 \to B \to M \oplus P \to C \to 0$$

has the following properties:

(1) *C is a $(d+1)$st syzygy module.*

(2) $\mathrm{r.pd}_{\Lambda}(B^*) \leqslant d - 1$.
(3) *The sequence* $0 \to B \to M \oplus P \to C \to 0$ *is dual exact, that is, the induced sequence* $0 \to C^* \to M^* \oplus P^* \to B^* \to 0$ *is exact.*

The above theorem is the main result in this paper, we will prove it in Section 3. To prove it, we collect some preliminary results in Section 2. In Section 4 we give some applications of our main theorem. For example, as an application of the theorem, we obtain a chain of epimorphisms concerning a module M in $\mathrm{mod}\,\Lambda$, by dualizing it we then get the spherical filtration of Auslander and Bridger for M^*; and furthermore we get Auslander–Bridger's Approximation Theorem for each reflexive module in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$.

Evans and Griffith in [7, Theorem 2.1] showed that if Λ is a commutative Noetherian local ring with finite global dimension and contains a field then each non-free dth syzygy of rank d has an Evans–Griffith presentation. As another application of theorem above, we show that for a right quasi k-Gorenstein ring Λ and any $0 \leqslant d \leqslant k - 1$ each dth syzygy module in $\mathrm{mod}\,\Lambda$ has an Evans–Griffith presentation; especially, we have that, if Λ is a commutative Noetherian ring with finite self-injective dimension, then for any non-negative integer d, each dth syzygy module in $\mathrm{mod}\,\Lambda$ has an Evans–Griffith presentation, which generalizes Evans and Griffith's result in [7, Theorem 2.1] to much more general setting. This generalization coincides with Maşek [12, Proposition 48] when Λ is a commutative Noetherian local ring with finite self-injective dimension.

2. Preliminaries

In this section, we give some definitions in our terminology and collect some facts which are used in this paper.

Throughout this paper, Λ is a left and right Noetherian ring, $\mathrm{mod}\,\Lambda$ is the category of finitely generated left Λ-modules and $\Omega^k\ (\mathrm{mod}\,\Lambda)$ is the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of kth syzygy modules. Let A be a module in $\mathrm{mod}\,\Lambda$ (respectively $\mathrm{mod}\,\Lambda^{\mathrm{op}}$). We use $\mathrm{l.pd}_{\Lambda}(A)$ (respectively $\mathrm{r.pd}_{\Lambda}(A)$) to denote the left (respectively right) projective dimension of A. We use $\sigma_A : A \to A^{**}$, defined by $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$, to denote the canonical evaluation homomorphism. A is called *torsionless* if σ_A is a monomorphism; and A is called *reflexive* if σ_A is an isomorphism. For a non-negative integer i, we denote $\mathrm{grade}\,A \geqslant i$ if $\mathrm{Ext}^j_{\Lambda}(A, \Lambda) = 0$ (respectively $\mathrm{Ext}^j_{\Lambda^{\mathrm{op}}}(A, \Lambda) = 0$) for any $0 \leqslant j < i$.

Let M be in $\mathrm{mod}\,\Lambda$ and

$$P_1 \to P_0 \to M \to 0$$

a projective resolution of M in $\mathrm{mod}\,\Lambda$. Then we have an exact sequence

$$0 \to M^* \to P_0^* \to P_1^* \to \mathrm{Tr}\,M \to 0$$

in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$, where $\mathrm{Tr}\,M = \mathrm{Coker}(P_0^* \to P_1^*)$ is the *transpose* of M. The following lemma is due to Auslander.

Lemma 2.1. [2, Proposition 6.3] *Let M and* $\mathrm{Tr}\,M$ *be as above. Then we have the following exact sequence*:

$$0 \to \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M, \Lambda) \to M \xrightarrow{\sigma_M} M^{**} \to \mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\,M, \Lambda) \to 0.$$

It is clear that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) \cong \mathrm{Ext}^{i-2}_{\Lambda^{\mathrm{op}}}(M^*, \Lambda)$ for any $i \geqslant 3$. On the other hand, $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) \cong \mathrm{Ker}\, \sigma_M$ and $\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) \cong \mathrm{Coker}\, \sigma_M$ by Lemma 2.1. So we get that, although $\mathrm{Tr}\, M$ depends on the choice of the projective resolution of M, each of $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda)$ (for any $i \geqslant 1$) is independent of the choice of the projective resolution of M and hence is identical up to isomorphisms.

Recall that M is called *k-torsionfree* if $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) = 0$ for any $1 \leqslant i \leqslant k$ (see [3]). By Lemma 2.1, we have that M is 1-torsionfree (respectively 2-torsionfree) if and only if it is torsionless (respectively reflexive). We use $\mathcal{T}^k\ (\mathrm{mod}\, \Lambda)$ to denote the full subcategory of $\mathrm{mod}\, \Lambda$ consisting of k-torsionfree modules. It follows from [3, Theorem 2.17] that $\mathcal{T}^k\ (\mathrm{mod}\, \Lambda) \subseteq \Omega^k\ (\mathrm{mod}\, \Lambda)$. Furthermore, we have the following useful result, which gives some equivalent conditions of $\mathcal{T}^i\ (\mathrm{mod}\, \Lambda) = \Omega^i\ (\mathrm{mod}\, \Lambda)$ for any $1 \leqslant i \leqslant k$.

Lemma 2.2. *For a positive integer k, the following statements are equivalent.*

(1) $\mathrm{grade}\, \mathrm{Ext}^{i+1}_{\Lambda}(M, \Lambda) \geqslant i$ *for any* $M \in \mathrm{mod}\, \Lambda$ *and* $1 \leqslant i \leqslant k-1$.
(2) $\Omega^i\ (\mathrm{mod}\, \Lambda) = \mathcal{T}^i\ (\mathrm{mod}\, \Lambda)$ *for any* $1 \leqslant i \leqslant k$.
(3) $\mathrm{grade}\, \mathrm{Ext}^{i+1}_{\Lambda^{\mathrm{op}}}(N, \Lambda) \geqslant i$ *for any* $N \in \mathrm{mod}\, \Lambda^{\mathrm{op}}$ *and* $1 \leqslant i \leqslant k-1$.
(4) $\Omega^i\ (\mathrm{mod}\, \Lambda^{\mathrm{op}}) = \mathcal{T}^i\ (\mathrm{mod}\, \Lambda^{\mathrm{op}})$ *for any* $1 \leqslant i \leqslant k$.

Proof. The equivalence of (1) and (2) is proved in [3, Proposition 2.26] (or see [5, Proposition 1.6]). The other implications are proved in [10, Theorem 2.4]. □

Corollary 2.3. *If* Λ *is a right quasi k-Gorenstein ring, then* $\Omega^i\ (\mathrm{mod}\, \Lambda) = \mathcal{T}^i\ (\mathrm{mod}\, \Lambda)$ *and* $\Omega^i\ (\mathrm{mod}\, \Lambda^{\mathrm{op}}) = \mathcal{T}^i\ (\mathrm{mod}\, \Lambda^{\mathrm{op}})$ *for any* $1 \leqslant i \leqslant k$.

Proof. Our conclusion follows from Lemma 2.2. □

3. The proof of the Theorem

In this section, we will prove the theorem mentioned in the introduction. We proceed in several steps.

Proof of Theorem. *The case* $d = 0$. Put $C = \mathrm{Im}\, \sigma_M$ and $B = \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda)$. Then we have an exact sequence in $\mathrm{mod}\, \Lambda$:

$$0 \to B \to M \to C \to 0.$$

Since C is a submodule of M^{**}, C is torsionless and $C \in \Omega^1\ (\mathrm{mod}\, \Lambda)$. On the other hand, $\mathrm{grade}\, \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) \geqslant 1$ by assumption, that is, $B^* = 0$, so the obtained exact sequence is desired.

The case $d = 1$. Assume that

$$0 \to B \xrightarrow{f} P \xrightarrow{g} \mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) \to 0$$

is an exact sequence in mod Λ with P projective. Consider the following pull-back diagram with the middle row splitting:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B & = & B & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & M & \longrightarrow & M \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \xrightarrow{\sigma_M} & M^{**} & \longrightarrow & \mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Because M^{**} is a dual, M^{**} is a second syzygy. Since grade $\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda) \geqslant 2$ by assumption, $P^* \xrightarrow{f^*} B^*$ is an isomorphism and B^* is projective. We know from [1, Proposition 20.14] that σ_M^* is epic, so, by applying the functor $\mathrm{Hom}_\Lambda(-, \Lambda)$ to the above diagram, we get the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & [\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda)]^*(=0) & \longrightarrow & M^{***} & \xrightarrow{\sigma_M^*} & M^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g^*} & & \downarrow & & \| & & \\
0 & \longrightarrow & P^* & \longrightarrow & P^* \oplus M^* & \longrightarrow & M^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow & & & & \\
 & & B^* & = & B^* & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

It is easy to see that $\mathrm{Coker}(M^{***} \to P^* \oplus M^*) = B^*$. Then the middle column in the former diagram:

$$0 \to B \to M \oplus P \to M^{**} \to 0$$

is desired.

The case $d \geqslant 2$. Let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M^* \to 0 \tag{1}$$

be a projective resolution of M^* in mod Λ^{op}. If $d = 2$, then M is reflexive by Lemma 2.2. So we have an exact sequence:

$$0 \to M\ (\cong M^{**}) \to P_0^* \to P_1^* \to N \to 0,$$

where $N = \operatorname{Coker}(P_0^* \to P_1^*)$. Now suppose $d \geqslant 3$. Since M is a dth syzygy module, it follows from Lemma 2.2 that M is d-torsionfree and so $\operatorname{Ext}^i_{\Lambda^{\mathrm{op}}}(M^*, \Lambda) = 0$ for any $1 \leqslant i \leqslant d-2$. From this fact and the exact sequence (1) we yield the following exact sequence:

$$0 \to M\ (\cong M^{**}) \to P_0^* \to P_1^* \to \cdots \to P_{d-2}^* \to P_{d-1}^* \to N \to 0, \tag{2}$$

where $N = \operatorname{Coker}(P_{d-2}^* \to P_{d-1}^*)$. So for any $d \geqslant 2$ we have an exact sequence of the form (2).

By Lemma 2.1 we get easily the following exact sequence:

$$0 \to \operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) \to N \xrightarrow{\sigma_N} N^{**} \to \operatorname{Ext}^{d+2}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) \to 0.$$

Write $K = \operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda)$ and $Y = \operatorname{Im} \sigma_N$. Let $\mathbb{U}^\bullet$ and $\mathbb{V}^\bullet$ be projective resolutions of K and Y, respectively. Then there is a projective module W in mod Λ such that we have the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & M' \oplus W & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_{d-1} & \longrightarrow & U_{d-1} \oplus V_{d-1} & \longrightarrow & V_{d-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_0 & \longrightarrow & U_0 \oplus V_0 & \longrightarrow & V_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where M' is the greatest direct summand of M without projective summands.

Since Y is a submodule of N^{**}, Y is torsionless and hence it is in $\Omega^1\ (\mathrm{mod}\,\Lambda)$. So D is in $\Omega^{d+1}\ (\mathrm{mod}\,\Lambda)$. On the other hand, $\operatorname{grade} K = \operatorname{grade}\operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda) \geqslant d+1$ by assumption, so we get an exact sequence $0 \to U_0^* \to \cdots \to U_{d-1}^* \to B^* \to 0$ and hence $\mathrm{r.pd}_\Lambda(B^*) \leqslant d-1$. It is trivial that every homomorphism $f : B \to \Lambda$ may extend to a homomorphism $g : U_{d-1} \to \Lambda$, so f may extend to a homomorphism $h : M' \oplus W \to \Lambda$ and hence the sequence $0 \to D^* \to M'^* \oplus W^* \to B^* \to 0$ is exact.

Let $M' \oplus W \oplus Q \cong M \oplus P$ for projective modules P, Q and put $C = D \oplus Q$. Then from the exact sequence $0 \to B \to M' \oplus W \to D \to 0$ we yield the following exact sequence:

$$0 \to B \to M \oplus P \to C \to 0,$$

which is desired. □

4. Applications

In this section we will give some applications of the main theorem. We first have the following result.

Corollary 4.1. *If Λ is a right quasi Auslander ring, then for any non-negative integer d and M in $\Omega^d\ (\mathrm{mod}\,\Lambda)$, there is a projective module P in $\mathrm{mod}\,\Lambda$ such that the dth syzygy B of $\operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda)$ is a submodule of $M \oplus P$ and such that the exact sequence $0 \to B \to M \oplus P \to C \to 0$ has the following properties*:

(1) $C \in \Omega^{d+1}\ (\mathrm{mod}\,\Lambda)$.
(2) $\mathrm{r.pd}_\Lambda(B^*) \leqslant d-1$.
(3) *The sequence $0 \to B \to M \oplus P \to C \to 0$ is dual exact.*

Proposition 4.2 *(A dual version of spherical filtration). Let Λ be a right quasi k-Gorenstein ring. Then, for each M in $\mathrm{mod}\,\Lambda$, there is a projective module P in $\mathrm{mod}\,\Lambda$ and a chain of epimorphisms*:

$$M \oplus P = M_0 \twoheadrightarrow M_1 \twoheadrightarrow \cdots \twoheadrightarrow M_{k-1} \twoheadrightarrow M_k,$$

such that

(1) $B_d = \operatorname{Ker}(M_d \to M_{d+1})$ *is a dth syzygy of* $\operatorname{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} M, \Lambda)$ (*or equivalently, B_d is a dth syzygy of* $\operatorname{Ext}^{d-1}_{\Lambda^{\mathrm{op}}}(M^*, \Lambda)$ *if $d \geqslant 2$*) *for any* $0 \leqslant d \leqslant k-1$.
(2) $M_d \in \Omega^d\ (\mathrm{mod}\,\Lambda)$ *for any* $0 \leqslant d \leqslant k$.
(3) $\mathrm{r.pd}_\Lambda(B_d^*) \leqslant d-1$ *for any* $0 \leqslant d \leqslant k-1$.
(4) *Each exact sequence $0 \to B_d \to M_d \to M_{d+1} \to 0$ is dual exact for any* $0 \leqslant d \leqslant k-1$.

Proof. We proceed by employing induction with successive applications of Theorem in the introduction.

First, by Theorem and its proof, we have an exact sequence in $\mathrm{mod}\,\Lambda$:

$$0 \to B_0 \to M \to C_1 \to 0 \tag{3}$$

with the properties that it is dual exact, $B_0 = \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda)$ and $C_1 = \mathrm{Im}\, \sigma_M$. Notice that $B_0^* = 0$ and $\mathrm{Im}\, \sigma_M$ is torsionless, then by Lemma 2.1 we have that $\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, C_1, \Lambda) = \mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(\mathrm{Im}\, \sigma_M), \Lambda) \cong (\mathrm{Im}\, \sigma_M)^{**}/\mathrm{Im}\, \sigma_M \cong M^{**}/\mathrm{Im}\, \sigma_M \cong \mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda)$.

Next, by Theorem and its proof, we have an exact sequence in $\mathrm{mod}\, \Lambda$:

$$0 \to B_1 \to C_1 \oplus P_1 \to C_2 \to 0$$

with the properties that it is dual exact, $C_2 = C_1^{**}$, P_1 is projective, B_1 is a first syzygy of $\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, C_1, \Lambda) \cong (\mathrm{Ext}^2_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda))$, B_1^* is projective and $C_2 \in \Omega^2$ $(\mathrm{mod}\, \Lambda)$. Then we have that $C_2^* \oplus B_1^* \cong C_1^* \oplus P_1^* \cong M^* \oplus P_1^*$ and $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(C_2^*, \Lambda) \cong \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(M^*, \Lambda)$ for any $i \geqslant 1$.

Now suppose that $k \geqslant 3$ and for any $0 \leqslant d \leqslant k-2$ there is an exact sequence in $\mathrm{mod}\, \Lambda$:

$$0 \to B_d \to C_d \oplus P_d \to C_{d+1} \to 0$$

with the properties that it is dual exact, P_d is projective, B_d is a dth syzygy of $\mathrm{Ext}^{d+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, C_d, \Lambda)$, $\mathrm{r.pd}_\Lambda(B_d^*) \leqslant d-1$, $C_0 = M$ and $C_{d+1} \in \Omega^{d+1}$ $(\mathrm{mod}\, \Lambda)$. Then we have that $\mathrm{Ext}^{k-2}_{\Lambda^{\mathrm{op}}}(C_{k-1}^*, \Lambda) \cong \mathrm{Ext}^{k-2}_{\Lambda^{\mathrm{op}}}(C_{k-2}^*, \Lambda) \cong \cdots \cong \mathrm{Ext}^{k-2}_{\Lambda^{\mathrm{op}}}(C_2^*, \Lambda) \cong \mathrm{Ext}^{k-2}_{\Lambda^{\mathrm{op}}}(M^*, \Lambda)$.

By Theorem, there is a projective module P_{k-1} and an exact sequence in $\mathrm{mod}\, \Lambda$:

$$0 \to B_{k-1} \to C_{k-1} \oplus P_{k-1} \to C_k \to 0,$$

such that

(1) B_{k-1} is a $(k-1)$st syzygy of $\mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, C_{k-1}, \Lambda)$; or equivalently, B_{k-1} is a $(k-1)$st syzygy of $\mathrm{Ext}^{k-2}_{\Lambda^{\mathrm{op}}}(C_{k-1}^*, \Lambda)$ $(\cong \mathrm{Ext}^{k-2}_{\Lambda^{\mathrm{op}}}(M^*, \Lambda) \cong \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\, M, \Lambda))$.
(2) $C_k \in \Omega^k$ $(\mathrm{mod}\, \Lambda)$.
(3) $\mathrm{r.pd}_\Lambda(B_{k-1}^*) \leqslant k-2$.
(4) The induced sequence $0 \to C_k^* \to C_{k-1}^* \oplus P_{k-1}^* \to B_{k-1}^* \to 0$ is exact.

Put $P = \bigoplus_{i=1}^{k-1} P_i$, $M_0 = M \oplus P$, $M_i = C_i \oplus (\bigoplus_{j=i}^{k-1} P_j)$ for any $1 \leqslant i \leqslant k-1$ and $M_k = C_k$. Then we get our conclusion. □

Let $T \in \mathrm{mod}\, \Lambda^{\mathrm{op}}$. We remark that if one takes a chain of epimorphisms from T^* as in Proposition 4.2 and dualizes it, one then obtains the spherical filtration of Auslander and Bridger for T^{**}. So we in fact obtain the spherical filtration of Auslander and Bridger for each reflexive module in $\mathrm{mod}\, \Lambda^{\mathrm{op}}$, and thus we may regard Proposition 4.2 a dual version of the spherical filtration of Auslander and Bridger.

For a module $A \in \mathrm{mod}\, \Lambda^{\mathrm{op}}$, we use A' to denote the greatest direct summand of A without projective summands. As a corollary of Proposition 4.2, we get Auslander–Bridger's Approximation Theorem (see [3, Theorem 2.41]) for T^{**} (or for T if T is reflexive) as follows.

Corollary 4.3. *Let Λ be a right quasi k-Gorenstein ring. Then, for any $T \in \mathrm{mod}\, \Lambda^{\mathrm{op}}$, there are a projective module P and an exact sequence in* $\mathrm{mod}\, \Lambda^{\mathrm{op}}$:

$$0 \to X \to T^{**} \oplus P \to Y \to 0$$

such that

(1) *It is dual exact.*
(2) $\mathrm{r.pd}_{\Lambda}(Y) \leqslant k-2$.
(3) *The homomorphism* $T^{**} \oplus P \to Y$ *induces isomorphisms* $\mathrm{Ext}^{i}_{\Lambda^{\mathrm{op}}}(Y, \Lambda) \xrightarrow{\cong} \mathrm{Ext}^{i}_{\Lambda^{\mathrm{op}}}(T^{**}, \Lambda)$ *for any* $1 \leqslant i \leqslant k-2$.
(4) *If* $(T^{**})' \to H'$ *is a homomorphism with* $\mathrm{r.pd}_{\Lambda}(H) \leqslant k-2$, *then the above exact sequence induces an isomorphism* $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y', H') \xrightarrow{\cong} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}((T^{**})', H')$.

Proof. Let T be in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$. Then T^* is in $\mathrm{mod}\,\Lambda$. By Proposition 4.2, there are a projective module P and exact sequences in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$:

$$0 \to M_2^* \to M_1^* \ (\cong T^{**} \oplus P) \to B_1^* \to 0$$

and

$$0 \to M_3^* \to M_2^* \to B_2^* \to 0$$

with $M_2 \in \Omega^2\ (\mathrm{mod}\,\Lambda)$, $M_3 \in \Omega^3\ (\mathrm{mod}\,\Lambda)$, B_1^* projective and $\mathrm{r.pd}_{\Lambda}(B_2^*) \leqslant 1$.

Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_3^* & = & M_3^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M_2^* & \to & M_1^* & \to & B_1^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & B_2^* & \to & Y_1 & \to & B_1^* & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the bottom row in the above diagram we know that $\mathrm{r.pd}_{\Lambda}(Y_1) \leqslant 1$. Notice that the first column $0 \to M_3^* \to M_2^* \to B_2^* \to 0$ is dual exact and the middle row $0 \to M_2^* \to M_1^* \to B_1^* \to 0$ splits, then it is easy to verify that the middle column $0 \to M_3^* \to M_1^* \to Y_1 \to 0$ is dual exact.

We then consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_4^* & = & M_4^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M_3^* & \longrightarrow & M_1^* & \longrightarrow & Y_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B_3^* & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the middle row is the middle column in the former diagram, the exactness of the first column follows from Proposition 4.2, $M_4 \in \Omega^4$ (mod Λ) and $\text{r.pd}_\Lambda(B_3^*) \leqslant 2$. From the bottom row in the above diagram we know that $\text{r.pd}_\Lambda(Y_2) \leqslant 2$. Notice that both the middle row and the first column in above diagram are dual exact, we then get the following exact commutative diagram:

$$\begin{array}{ccccc}
M_1^{**} & \longrightarrow & M_3^{**} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
M_4^{**} & = & M_4^{**} & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

So $M_1^{**} \to M_4^{**}$ is epic and hence the middle column in the above diagram is dual exact.

Continuing this process, we finally get an exact sequence in mod Λ^{op}:

$$0 \to X \to T^{**} \oplus P \left(\cong M_1^*\right) \to Y \to 0 \tag{4}$$

which is dual exact, where $X = M_k^*$ (where $M_k \in \Omega^k$ (mod Λ)) and $\text{r.pd}_\Lambda(Y) \leqslant k-2$.

Since $M_k \in \Omega^k$ (mod Λ), $\text{Ext}^i_{\Lambda^{\text{op}}}(X, \Lambda) = \text{Ext}^i_{\Lambda^{\text{op}}}(M_k^*, \Lambda) = 0$ for any $1 \leqslant i \leqslant k-2$. So $\text{Ext}^i_{\Lambda^{\text{op}}}(Y, \Lambda) \cong \text{Ext}^i_{\Lambda^{\text{op}}}(T^{**}, \Lambda)$ for any $2 \leqslant i \leqslant k-2$. On the other hand, from the fact that $\text{Ext}^1_{\Lambda^{\text{op}}}(M_k^*, \Lambda) = 0$ and the dual exactness of the sequence (4) we have that $\text{Ext}^1_{\Lambda^{\text{op}}}(Y, \Lambda) \cong \text{Ext}^1_{\Lambda^{\text{op}}}(T^{**}, \Lambda)$ and thus $\text{Ext}^i_{\Lambda^{\text{op}}}(Y, \Lambda) \cong \text{Ext}^i_{\Lambda^{\text{op}}}(T^{**}, \Lambda)$ for any $1 \leqslant i \leqslant k-2$. So, if $(T^{**})' \to H'$ is a homomorphism with $\text{r.pd}_\Lambda(H) \leqslant k-2$, it then follows from [3, Lemma 2.42] that the exact sequence (4) induces an isomorphism $\text{Hom}_{\Lambda^{\text{op}}}(Y', H') \xrightarrow{\cong} \text{Hom}_{\Lambda^{\text{op}}}((T^{**})', H')$. We are done. □

Let Λ be a commutative Noetherian ring and let n be a non-negative integer and M in $\Omega^n\ (\text{mod}\,\Lambda)$. An *Evans–Griffith presentation* of M is an exact sequence in $\text{mod}\,\Lambda$:

$$0 \to S \to B \to M \to 0$$

where B is an nth syzygy of $\text{Ext}^{n+1}_{\Lambda^{\text{op}}}(\text{Tr}\,M, \Lambda)$ and S is in $\Omega^{n+2}\ (\text{mod}\,\Lambda)$ (cf. [7,12]). In the case Λ is not necessarily commutative we also call such an exact sequence an *Evans–Griffith presentation* of M.

Proposition 4.4. *Let Λ be a right quasi k-Gorenstein ring. Then, for any $0 \leqslant d \leqslant k-1$, each module in $\Omega^d\ (\text{mod}\,\Lambda)$ has an Evans–Griffith presentation.*

Proof. The proof here is similar to that of [12, Proposition 48].

Let M be in $\Omega^d\ (\text{mod}\,\Lambda)$. By Theorem there is a projective module P and an exact sequence in $\text{mod}\,\Lambda$:

$$0 \to B \xrightarrow{\alpha} M \oplus P \xrightarrow{\beta} C \to 0$$

satisfying the properties that B is a dth syzygy of $\text{Ext}^{d+1}_{\Lambda^{\text{op}}}(\text{Tr}\,M, \Lambda)$ and $C \in \Omega^{d+1}\ (\text{mod}\,\Lambda)$.

Let γ be the composition: $B \xrightarrow{\alpha} M \oplus P \xrightarrow{(1,0)} M$, that is, $\gamma = (1,0)\alpha$. Suppose that $Q \xrightarrow{\delta} M \to 0$ is exact in $\text{mod}\,\Lambda$ with Q projective. Then we have the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S & \longrightarrow & P\oplus Q & \longrightarrow & C & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptsize\begin{pmatrix}0&-\delta\\1&0\\0&1\end{pmatrix}} & & \| & & \\
0 & \longrightarrow & B\oplus Q & \xrightarrow{\left(\begin{smallmatrix}\alpha&0\\0&1\end{smallmatrix}\right)} & M\oplus P\oplus Q & \xrightarrow{(\beta,0)} & C & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle(\gamma,\ \delta)} & & \downarrow{\scriptstyle(1,\ 0,\ \delta)} & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $S = \text{Ker}(\gamma, \delta)$. By the exactness of the first row we have $S \in \Omega^{d+2}\ (\text{mod}\,\Lambda)$. Thus the first column in the above diagram:

$$0 \to S \to B \oplus Q \to M \to 0$$

is an Evans–Griffith presentation of M. We are done. □

The following is an immediate consequence of Proposition 4.4.

Corollary 4.5. *If Λ is a right quasi Auslander ring, then for any non-negative integer d, each module in Ω^d $(\bmod \Lambda)$ has an Evans–Griffith presentation.*

If Λ is an Auslander ring, that is, Λ is k-Gorenstein for all k, then the grade condition in Corollary 4.5 is satisfied for Λ (see [8, Theorem 3.7]). We point out that a k-Gorenstein ring with both left and right self-injective dimensions being k is an Auslander ring (see [11, Proposition 1]).

On the other hand, by [6] we have that a commutative Noetherian ring Λ has finite self-injective dimension if and only if $\operatorname{grade} \operatorname{Ext}^i_\Lambda(N, \Lambda) \geqslant i$ for any $N \in \bmod \Lambda$ and $i \geqslant 1$. So, by Corollary 4.5, we immediately have the following

Corollary 4.6. *If Λ is a commutative Noetherian with finite self-injective dimension, then for any non-negative integer d, each module in Ω^d $(\bmod \Lambda)$ has an Evans–Griffith presentation.*

Observe that a special instance of Corollary 4.6 was already considered by Evans and Griffith in [7, Theorem 2.1]. They showed that if Λ is a commutative Noetherian local ring with finite global dimension and contains a field then each non-free dth syzygy of rank d has an Evans–Griffith presentation. Corollary 4.6 generalizes this result to much more general setting.

On the other hand, Maşek in [12, Proposition 48] also generalized the above Evans and Griffith's result and showed that any d-torsionfree module (note: such a module is called in [12] a d-torsionless module) in $\bmod \Lambda$ has an Evans–Griffith presentation over a commutative Noetherian local ring Λ. Notice that if Λ is a commutative Noetherian ring with finite self-injective dimension, then Ω^i $(\bmod \Lambda) = \mathcal{T}^i$ $(\bmod \Lambda)$ for any $i \geqslant 1$ by the above argument and Lemma 2.2. So Corollary 4.6 coincides with this Maşek's result over a commutative Noetherian local ring with finite self-injective dimension.

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