On U-codominant dimension

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Abstract Let R and S be semiregular rings and U a semidualizing (R,S)-bimodule. We show that the U-codominant dimensions of ${}_RU$ and U_S are identical. As an application, we get that the U-codominant dimension of U is at least two if and only if the functor $U\otimes_S \operatorname{Hom}_R(U,-)$ is right exact and if and only if the functor $\operatorname{Hom}_R(U, U\otimes_S -)$ is left exact. We also get some new equivalent characterizations of (n-)Auslander algebras.

1. Introduction

The classical theory of dominant dimension was introduced by Tachikawa [21] to study QF-3 algebras. Later on, it attracted the interest of many authors (see [5]–[10], [13], [15], [16], [18], and references therein). One reason is that the notion of dominant dimension is closely related to the famous Nakayama conjecture, which says that if an artin algebra has infinite dominant dimension, then it should be self-injective (cf. [3]). In applied aspects, dominant dimension is used to study double centralizer properties, which play a central role in many parts of algebraic Lie theory (see [9], [10], [16]). Also, it has its values in classifying certain algebras [8].

Following [21], a left R-module M is said to have *dominant dimension* at least n if each of the first n terms in the minimal injective resolution of M is projective. It was shown in [21] that if R is a left and right artinian ring, then the dominant dimensions of ${}_RR$ and R_R are identical. Colby and Fuller [5] gave some equivalent characterizations for the dominant dimension of R being at least one or two in terms of the exactness of the double dual functors with respect to ${}_RR_R$. Replacing "projective" in the above definition with "cogenerated by U," Kato [15] generalized dominant dimension to U-dominant dimension, where U is a fixed left R-module, and characterized the modules with U-dominant dimension at least one. Furthermore, given two artin algebras R and S and a faithfully balanced self-orthogonal bimodule (equivalently, a semidualizing bimodule) ${}_RU_S$, Huang [13] carried over an extensive study of U-dominant dimensions and proved that the U-dominant dimensions of ${}_RU$ and U_S are identical.

On the other hand, Eerkes [6] introduced a categorically dual notion-codominant dimension as follows. A left R-module M is said to have *codominant*

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dimension at least n if each of the first n terms in the minimal projective resolution of M (if existing) is injective and proved that if R is a left and right artinian ring, then the codominant dimensions of minimal injective cogenerators for left and right R-modules are identical. Now it is natural to ask, how can one give a dual notion of U-dominant dimension? The aim of this paper is to introduce the so-called U-codominant dimension and investigate its homological behavior, especially in the case for U being a semidualizing bimodule.

Let us briefly outline the structure of the paper. In Section 2, we give some terminology and some preliminary results.

In Section 3, for a ring R and a given left (or right) R-module U, as a dual of the notion of U-dominant dimension [15], we introduce the notion of the U-codominant dimension $U\text{-codom.dim}\, M$ of a left (or right) R-module M. Let R and S be semiregular rings and U a semidualizing (R,S)-bimodule. We first prove that the U-codominant dimension of ${}_RU$ (resp. U_S) is at least one if and only if the functor $U \otimes_S \mathrm{Hom}_R(U,-)$ preserves epimorphisms, and if and only if the functor $\mathrm{Hom}_R(U, U \otimes_S -)$ preserves monomorphisms (Theorem 3.5). Then, by means of the (strong) cograde conditions of modules and the properties of the functors $U \otimes_S \mathrm{Hom}_R(-,U)$ and $\mathrm{Hom}_R(U, U \otimes_S -)$, we get that the U-codominant dimensions of ${}_RU$ and U_S are identical (Theorem 3.9 and Corollary 3.10). As an application, we have that the U-codominant dimension of U is at least two if and only if the double functor $U \otimes_S \mathrm{Hom}_R(-,U)$ is right exact, and if and only if the double functor $\mathrm{Hom}_R(U, U \otimes_S -)$ is left exact (Theorem 3.12).

In Section 4, we give some new equivalent characterizations of (n-)Auslander algebras.

2. Preliminaries

Throughout this paper, all rings are associative rings with units. For a ring R, $\mathrm{Mod}\, R$ (resp. $\mathrm{mod}\, R$) is the category of left (resp. finitely presented left) R-modules. Let M be a module in $\mathrm{Mod}\, R$. We use $\mathrm{Add}_R M$ (resp. $\mathrm{add}_R M$) to denote the full subcategory of $\mathrm{Mod}\, R$ consisting of all direct summands of direct sums of (finite) copies of M. We say that M admits a *degreewise finite R-projective resolution* if there exists an exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\mathrm{mod}\, R$ with all P_i projective.

DEFINITION 2.1 ([2], [12])

Let R and S be rings. An (R,S)-bimodule ${}_RU_S$ is called *semidualizing* if the following conditions are satisfied:

(a1) ${}_RU$ admits a degreewise finite R-projective resolution.
(a2) U_S admits a degreewise finite S^{op}-projective resolution.
(b1) The homothety map ${}_RR_R \to \mathrm{Hom}_{S^{\mathrm{op}}}(U,U)$ is an isomorphism.
(b2) The homothety map ${}_SS_S \to \mathrm{Hom}_R(U,U)$ is an isomorphism.

(c1) $\mathrm{Ext}_R^{\geqslant 1}(U,U)=0$.
(c2) $\mathrm{Ext}_{S^{\mathrm{op}}}^{\geqslant 1}(U,U)=0$.

Wakamatsu in [27] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules* (see [4], [17]). Note that a bimodule ${}_RU_S$ is semidualizing if and only if it is Wakamatsu tilting [29, Corollary 3.2]. Examples of semidualizing bimodules can be found in [12], [23], [24], and [28].

Let R and S be arbitrary rings and ${}_RU_S$ a semidualizing (R,S)-bimodule. For convenience, we write $(-)_* := \mathrm{Hom}(U,-)$. Let $M \in \mathrm{Mod}\, R$ and $N \in \mathrm{Mod}\, S$. Suppose that

$$0 \to M \to I^0(M) \xrightarrow{g^0} I^1(M)$$

is the minimal injective presentation of M, and

$$F_1(N) \xrightarrow{f_0} F_0(N) \to N \to 0$$

is the minimal flat presentation of N.

DEFINITION 2.2
Let $n \geqslant 1$.

(1) Let $M \in \mathrm{Mod}\, R$. We call $\mathrm{cTr}_U\, M := \mathrm{Coker}\, {g^0}_*$ the *cotranspose* of M with respect to ${}_RU_S$ [22].

(2) Let $N \in \mathrm{Mod}\, S$. We call $\mathrm{acTr}_U\, N := \mathrm{Ker}(1_U \otimes f_0)$ the *adjoint cotranspose* of N with respect to ${}_RU_S$ [24].

Following [25, Definition 6.2], we recall the following notions.

DEFINITION 2.3
Let $M \in \mathrm{Mod}\, R$, $N \in \mathrm{Mod}\, S$, and $n \geqslant 0$.

(1) The Ext-*cograde* of M with respect to U is defined as $\text{E-cograde}_U\, M := \inf\{i \geqslant 0 \mid \mathrm{Ext}_R^i(U,M) \neq 0\}$, and *the strong* Ext-*cograde* of M with respect to U, denoted by $\text{s.E-cograde}_U\, M$, is said to be at least n if $\text{E-cograde}_U\, X \geqslant n$ for any quotient module X of M.

(2) The Tor-*cograde* of N with respect to U is defined as $\text{T-cograde}_U\, N := \inf\{i \geqslant 0 \mid \mathrm{Tor}_i^S(U,N) \neq 0\}$, and *the strong* Tor-*cograde* of N with respect to U, denoted by $\text{s.T-cograde}_U\, N$, is said to be at least n if $\text{T-cograde}_U\, Y \geqslant n$ for any submodule Y of N.

Let $M \in \mathrm{Mod}\, R$. Then we have the following canonical evaluation homomorphism:

$$\theta_M : U \otimes_S M_* \to M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in U$ and $f \in M_*$. If θ_M is epic, then M is called *U-cotorsionless*, and if θ_M is isomorphic, then M is called *U-coreflexive* [22].

Let $N \in \operatorname{Mod} S$. Then we have the following canonical evaluation homomorphism:

$$\mu_N : N \to (U \otimes_S N)_*$$

defined by $\mu_N(y)(c) = c \otimes y$ for any $y \in N$ and $c \in U$.

3. U-codominant dimension

We introduce the notion of the relative codominant dimension of modules as follows.

DEFINITION 3.1

Let R and S be rings, and let $U, M \in \operatorname{Mod} R$ (resp. $\operatorname{Mod} S^{\mathrm{op}}$) and $n \geqslant 0$. We say that the *U-codominant dimension* of M is at least n, written U-codom.dim $M \geqslant n$, if there exists a projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M in $\operatorname{Mod} R$ (resp. $\operatorname{Mod} S^{\mathrm{op}}$) such that P_i is generated by U (equivalently, $P_i \in \operatorname{Add}_R U$ (resp. $\operatorname{Add} U_S$)) for any $0 \leqslant i \leqslant n-1$.

REMARK 3.2

Let $U, M \in \operatorname{Mod} R$.

(1) Let R be a left artinian ring. The *dominant dimension* of a finitely generated left R-module M is at least n if each of the first n terms in the minimal injective resolution of M is projective [21]. Let R be a left perfect ring. The *codominant dimension* of a left R-module M is at least n if each of the first n terms in the minimal projective resolution of M is injective (see [6], [7]). The notion of the codominant dimension of modules is the dual of that of dominant dimension of modules. When R is an artinian ring and U is injective, the U-codominant dimension of M is exactly its codominant dimension.

(2) The *U-dominant dimension* of a left R-module M is at least n if each of the first n terms in the minimal injective resolution of M is cogenerated by U [15]. The notion of the U-codominant dimension of modules is the dual of that of U-dominant dimension of modules.

(3) When M admits a minimal projective resolution, it is easy to see that U-codom.dim $M \geqslant n$ if and only if each of the first n terms in the minimal projective resolution of M is in $\operatorname{Add}_R U$.

Recall from [19] that a ring R is called *semiregular* if $R/J(R)$ is von Neumann regular and idempotents can be lifted modulo $J(R)$, where $J(R)$ is the Jacobson radical of R. The class of semiregular rings includes (1) von Neumann regular rings, (2) semiperfect rings, (3) left cotorsion rings, and (4) right cotorsion rings (see [11] for the definitions of left cotorsion rings and right cotorsion rings).

If R is a semiregular ring, then any finitely presented left or right R-module has a projective cover by [19, Theorem 2.9]. In this case, since ${}_RU$ admits a degreewise finite R-projective resolution by Definition 2.1, we may assume that

$$\cdots \xrightarrow{f_{i+1}(U)} P_i(U) \xrightarrow{f_i(U)} \cdots \xrightarrow{f_2(U)} P_1(U) \xrightarrow{f_1(U)} P_0(U) \xrightarrow{f_0(U)} {}_RU \to 0 \tag{1}$$

is the minimal projective resolution of ${}_RU$ in $\operatorname{mod} R$. Analogously, if S is a semiregular ring, then we assume that

$$\cdots \xrightarrow{g_{i+1}(U)} Q_i(U) \xrightarrow{g_i(U)} \cdots \xrightarrow{g_2(U)} Q_1(U) \xrightarrow{g_1(U)} Q_0(U) \xrightarrow{g_0(U)} U_S \to 0 \tag{2}$$

is the minimal projective resolution of U_S in $\operatorname{mod} S^{\mathrm{op}}$.

REMARK 3.3

By Remark 3.2(3), we have that if R is a semiregular ring, then U-codom.dim ${}_RU \geqslant n$ if and only if $P_i(U) \in \operatorname{add} {}_RU$ for any $0 \leqslant i \leqslant n-1$; analogously, if S is a semiregular ring, then U-codom.dim $U_S \geqslant n$ if and only if $Q_i(U) \in \operatorname{add} U_S$ for any $0 \leqslant i \leqslant n-1$.

In the rest of this paper, R and S are semiregular rings, and ${}_RU_S$ is a given semidualizing (R,S)-bimodule. We will show that the U-codominant dimensions of ${}_RU$ and U_S are identical. Some applications of this result will be given.

According to [20], the full subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{Mod} S$) consisting of modules M (resp. N) satisfying $\operatorname{Hom}_R(P_0(U), M) = 0$ (resp. $Q_0(U) \otimes_S N = 0$) forms a torsion-free (resp. torsion) class. Indeed, $P_0(U)$ (resp. $Q_0(U)$) defines a torsion theory [20, Chapter VI, Section 2]. For a module $M \in \operatorname{Mod} R$ (resp. $N \in \operatorname{Mod} S$), we use $t(M)$ (resp. $s(N)$) to denote the torsion submodule of M (resp. N).

LEMMA 3.4

(1) *For any $M \in \operatorname{Mod} R$, we have that $M/t(M) \cong \operatorname{Coker} \theta_M$ if and only if* $\operatorname{Hom}_R(P_0(U), \operatorname{Coker} \theta_M) = 0$.

(2) *For any $N \in \operatorname{Mod} S$, we have that $s(N) = \operatorname{Ker} \mu_N$ if and only if $Q_0(U) \otimes_S$* $\operatorname{Ker} \mu_N = 0$.

Proof

(1) We first prove the necessity. Let $M/t(M) \cong \operatorname{Coker} \theta_M$. Since $M/t(M)$ belongs to the class of torsion-free modules, we have $\operatorname{Hom}_R(P_0(U), \operatorname{Coker} \theta_M) = 0$.

Now we prove the sufficiency. We claim that $\operatorname{Im} \theta_M \subseteq t(M)$. Let $x \in \operatorname{Im} \theta_M$. Then by the definition of θ_M, there exist $f_1, \ldots, f_n \in M_*$ and $c_1, \ldots, c_n \in {}_RU$ such that $x = \sum_{i=1}^n f_i(c_i)$. Since we have an epimorphism $f_0(U) : P_0(U) \twoheadrightarrow {}_RU$, there exists $p_i \in P_0(U)$ such that $c_i = f_0(U)(p_i)$ for any $1 \leqslant i \leqslant n$. Note that $t(M)$ is the sum of the images of all homomorphisms from $P_0(U)$ to M. So $x = \sum_{i=1}^n f_i f_0(U)(p_i) \in t(M)$. The claim is proved. Thus, we have the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}\theta_M & \longrightarrow & M & \longrightarrow & \operatorname{Coker}\theta_M & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow f & & \\ 0 & \longrightarrow & t(M) & \longrightarrow & M & \longrightarrow & M/t(M) & \longrightarrow & 0 \end{array}$$

By the snake lemma, we have $t(M)/\operatorname{Im}\theta_M \cong \operatorname{Ker} f$. Since the class of torsion modules is closed under quotient objects, $\operatorname{Ker} f(\cong t(M)/\operatorname{Im}\theta_M)$ is in the torsion class. By assumption, $\operatorname{Hom}_R(P_0(U), \operatorname{Coker}\theta_M) = 0$; that is, $\operatorname{Coker}\theta_M$ is torsion-free. Since the class of torsion-free modules is closed under subobjects, $\operatorname{Ker} f$ is in the torsion-free class. Thus, $\operatorname{Ker} f = 0$; therefore, $M/t(M) \cong \operatorname{Coker}\theta_M$.

(2) The necessity is trivial. We will prove the sufficiency. Since there exists an epimorphism $g_0(U) : Q_0(U) \twoheadrightarrow U_S$, we get an epimorphism $g_0(U) \otimes s(N) : Q_0(U) \otimes_S s(N) \twoheadrightarrow U \otimes_S s(N)$. Notice that $Q_0(U) \otimes_S s(N) = 0$, so $U \otimes_S s(N) = 0$, which implies $c \otimes y = 0$ for any $c \in U$ and $y \in s(N)$. Thus, $s(N) \subseteq \operatorname{Ker}\mu_N$ by the definition of μ_N. Since $s(N)$ is the largest submodule of N satisfying $Q_0(U) \otimes_S s(N) = 0$, it follows from the assumption that $s(N) = \operatorname{Ker}\mu_N$. □

By using the above lemma, we get the following result.

THEOREM 3.5

The following statements are equivalent:

(1) U-codom.dim$\,{}_RU \geqslant 1$.
(2) $U \otimes_S (-)_*$ *preserves epimorphisms in* $\operatorname{Mod} R$.
(3) $(U \otimes_S -)_*$ *preserves monomorphisms in* $\operatorname{Mod} S$.
(4) $M/t(M) \cong \operatorname{Coker}\theta_M$ *for every* $M \in \operatorname{Mod} R$.
(5) $s(N) = \operatorname{Ker}\mu_N$ *for any* $N \in \operatorname{Mod} S$.
(1)′ U-codom.dim$\,U_S \geqslant 1$.
(2)′ $(-)_* \otimes_R U$ *preserves epimorphisms in* $\operatorname{Mod} S^{\mathrm{op}}$.
(3)′ $(- \otimes_R U)_*$ *preserves monomorphisms in* $\operatorname{Mod} R^{\mathrm{op}}$.

Proof

By [26, Theorem 4.8 and Corollary 4.9], we have $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (1)' \Leftrightarrow (2)' \Leftrightarrow (3)'$.

$(1) + (1)' \Rightarrow (5)$: By (1), we have $P_0(U) \in \operatorname{add}_R U$. Let $N \in \operatorname{Mod} S$. By [24, Proposition 3.2], we have $\operatorname{Ker}\mu_N \cong \operatorname{Ext}^1_R(U, \operatorname{acTr}_U N)$. It follows from [26, Theorem 4.8] that $U \otimes_S \operatorname{Ker}\mu_N \cong U \otimes_S \operatorname{Ext}^1_R(U, \operatorname{acTr}_U N) = 0$. By (1)′, we have $Q_0(U) \in \operatorname{add} U_S$. It follows that $Q_0(U) \otimes_S \operatorname{Ker}\mu_N = 0$, and then the assertion follows from Lemma 3.4(2).

$(5) \Rightarrow (2)$: Let $f : M_1 \twoheadrightarrow M_2$ be an epimorphism in $\operatorname{Mod} R$. Set $M := \operatorname{Ker} f$. It follows from [25, Corollary 6.8] that $\operatorname{Ext}^1_R(U, M) \cong \operatorname{Ker}\mu_{\operatorname{cTr}_U M}$. From the assumption and Lemma 3.4(2), we have $Q_0(U) \otimes_S \operatorname{Ext}^1_R(U, M) = 0$. Since

$\operatorname{Coker} f_*$ is isomorphic to a submodule of $\operatorname{Ext}_R^1(U,M)$ and $Q_0(U)$ is projective, $Q_0(U)\otimes_S \operatorname{Coker} f_* = 0$. So $U\otimes_S \operatorname{Coker} f_* = 0$; hence, $U\otimes f_*$ is epic.

(2) $\Rightarrow$ (4): Let $M\in \operatorname{Mod} R$, and let M' be a quotient module of $\operatorname{Coker}\theta_M$. Assume that f is the composition $M \twoheadrightarrow \operatorname{Coker}\theta_M \twoheadrightarrow M'$. Then $f\theta_M = 0$ and $f_*(\theta_M)_* = 0$. But since $(\theta_M)_*$ is a split epimorphism by [25, Lemma 6.1], we have $f_* = 0$, and so $U\otimes f_* = 0$. The assumption of (2) implies $U\otimes_S M'_* = 0$, so $M'_* = 0$ by [25, Corollary 6.6(1)]. Thanks to Lemma 3.4(1), we need only to show that $\operatorname{Hom}_R(P_0(U),\operatorname{Coker}\theta_M) = 0$. If it is not the case, then there exists $0\neq \alpha \in \operatorname{Hom}_R(P_0(U),\operatorname{Coker}\theta_M)$. Pick some modules L and L_1 such that $U\cong P_0(U)/L$ and $\operatorname{Im}\alpha \cong P_0(U)/L_1$. Because $P_0(U)$ is the projective cover of U and $\alpha\neq 0$, we get $L+L_1\neq P_0(U)$. Hence, there exists a nonzero natural epimorphism $\beta : P_0(U)/L \to P_0(U)/(L+L_1)$. Note that there are inclusions $(L+L_1)/L_1 \subseteq P_0(U)/L_1 \subseteq \operatorname{Coker}\theta_M$. Denote the natural embedding homomorphism by $i : P_0(U)/(L+L_1) \cong \frac{P_0(U)/L_1}{(L+L_1)/L_1} \to \frac{\operatorname{Coker}\theta_M}{(L+L_1)/L_1}$. Thus, we get a nonzero homomorphism $i\beta \in (\frac{\operatorname{Coker}\theta_M}{(L+L_1)/L_1})_*$, which is a contradiction to $M'_* = 0$.

(4) $\Rightarrow$ (3): Let $g : N_1 \rightarrowtail N_2$ be a monomorphism in $\operatorname{Mod} S$. Set $N := \operatorname{Coker} g$. Then $\operatorname{Ker}(U\otimes g)$ is a quotient module of $\operatorname{Tor}_1^S(U,N)$. By [23, Corollary 5.3(1)], we have $\operatorname{Tor}_1^S(U,N)\cong \operatorname{Coker}\theta_{\operatorname{acTr}_U N}$. Lemma 3.4(1) implies that $\operatorname{Hom}_R(P_0(U),\operatorname{Tor}_1^S(U,N)) = 0$; hence, $\operatorname{Hom}_R(P_0(U),\operatorname{Ker}(U\otimes g)) = 0$. It follows that $(\operatorname{Ker}(U\otimes g))_* = 0$ and $(U\otimes g)_*$ is monic. $\square$

The following proposition is useful in proving the main result.

PROPOSITION 3.6

If U-codom.dim $_RU \geqslant 1$, then the following statements are equivalent for any $n\geqslant 2$:

(1) *U-codom.dim $_RU \geqslant n$.*
(2) *For any $M\in \operatorname{Mod} R$, if $M_* = 0$, then* E-cograde$_U$ *$M\geqslant n$.*

Proof

For any $M\in\operatorname{Mod} R$ and $i\geqslant 1$, we have an exact sequence,

$$\operatorname{Hom}_R\big(P_{i-1}(U),M\big)\to \operatorname{Hom}_R\big(\operatorname{Im} f_i(U),M\big)\to \operatorname{Ext}_R^i(U,M)\to 0. \tag{3}$$

(1) $\Rightarrow$ (2): Let $M\in\operatorname{Mod} R$ with $M_*=0$. For any $0\leqslant i\leqslant n-1$, since $P_i(U)\in \operatorname{add}{}_RU$ by (1), we get $\operatorname{Hom}_R(P_i(U),M)=0$; hence, $\operatorname{Hom}_R(\operatorname{Im} f_i(U),M)=0$. Thus, $\operatorname{Ext}_R^i(U,M)=0$ for any $0\leqslant i\leqslant n-1$ by the exactness of the sequence (3).

(2) $\Rightarrow$ (1): Since U-codom.dim$_RU\geqslant 1$, $P_0(U)$ is generated by $_RU$. When $n=2$, we have to prove that $P_1(U)$ is also generated by $_RU$. For this purpose, we establish the following two claims.

Claim 1. $\operatorname{Hom}_R(U,\operatorname{Im} f_1(U)/M)\neq 0$ for any nonzero proper submodule M of $\operatorname{Im} f_1(U)$.

If $\operatorname{Hom}_R(U,\operatorname{Im} f_1(U)/M) = 0$ for some nonzero proper submodule M of $\operatorname{Im} f_1(U)$, we obtain by assumption that $\operatorname{Ext}_R^i(U,\operatorname{Im} f_1(U)/M) = 0$ for any

$i = 0, 1$. Because $P_0(U) \in \text{add}\,_RU$, $\text{Hom}_R(P_0(U), \text{Im}\, f_1(U)/M) = 0$. So from the exactness of the sequence (3), we get that $\text{Hom}_R(\text{Im}\, f_1(U), \text{Im}\, f_1(U)/M) = 0$, which is impossible. Thus, Claim 1 is proved.

Claim 2. If $\text{Im}\, f_1(U)$ is generated by $_RU$, then $P_1(U)$ is generated by $_RU$.

Suppose that $\text{Im}\, f_1(U)$ is generated by $_RU$. Then there exists an epimorphism $g : V \twoheadrightarrow \text{Im}\, f_1(U)$ with $V \in \text{Add}\,_RU$. Note that $_RU$ is a quotient module of $P_0(U)$. Hence, there exists an epimorphism $h : P_0(U)^{(I)} \twoheadrightarrow \text{Im}\, f_1(U)$ for some index set I. Since $P_1(U)$ is the projective cover of $\text{Im}\, f_1(U)$, $P_1(U)$ is isomorphic to a direct summand of $P_0(U)^{(I)}$. The fact that $P_0(U)$ is generated by $_RU$ implies that $P_1(U)$ is generated by $_RU$. Claim 2 is proved.

Since $P_0(U)$ is generated by $_RU$, by Claim 2 the proof can be finished if $\text{Im}\, f_1(U)$ is generated by $P_0(U)$. Let $L = \sum_h \text{Im}\, h$, where h runs through $\text{Hom}_R(P_0(U), \text{Im}\, f_1(U))$. If $L = \text{Im}\, f_1(U)$, then there is nothing to show. Otherwise, by Claim 1 there exists a nonzero homomorphism $\alpha \in \text{Hom}_R(U, \text{Im}\, f_1(U)/L)$. Let $\pi : \text{Im}\, f_1(U) \to \text{Im}\, f_1(U)/L$ be the natural map. Since $P_0(U)$ is projective, there exists a homomorphism $\beta \in \text{Hom}_R(P_0(U), \text{Im}\, f_1(U))$ such that $\pi\beta = \alpha f_0$. Obviously the equality produces a contradiction since $\text{Im}\, \beta \subseteq L$ and $\alpha \neq 0$. Finally, the assertion follows easily by induction on n. □

By putting $m = 1$ in [26, Proposition 4.7], we get the following lemma.

LEMMA 3.7

The following statements are equivalent for any $n \geqslant 1$:

(1) U-codom.dim $_RU \geqslant n$.
(2) s.T-cograde$_U$ $\text{Ext}^1_{S^{\text{op}}}(U, N') \geqslant n$ *for any* $N' \in \text{Mod}\, S^{\text{op}}$.
(3) s.E-cograde$_U$ $\text{Tor}^S_1(U, N) \geqslant n$ *for any* $N \in \text{Mod}\, S$.

Symmetrically, we have the following result.

LEMMA 3.8

The following statements are equivalent for any $n \geqslant 1$:

(1) U-codom.dim $U_S \geqslant n$.
(2) s.T-cograde$_U$ $\text{Ext}^1_R(U, M) \geqslant n$ *for any* $M \in \text{Mod}\, R$.
(3) s.E-cograde$_U$ $\text{Tor}^R_1(M', U) \geqslant n$ *for any* $M' \in \text{Mod}\, R^{\text{op}}$.

Now we state our main result as follows.

THEOREM 3.9

The following statements are equivalent for any $n \geqslant 1$:

(1) U-codom.dim $_RU \geqslant n$.

(2) *Applying the functor $U \otimes_S (-)_*$ to the minimal projective resolution (1) of $_RU$, the induced sequence*

$$U \otimes_S P_{n-1}(U)_* \overset{U\otimes f_{n-1}(U)_*}{\longrightarrow} \cdots \overset{U\otimes f_2(U)_*}{\longrightarrow} U \otimes_S P_1(U)_* \overset{U\otimes f_1(U)_*}{\longrightarrow}$$

$$U \otimes_S P_0(U)_* \overset{U\otimes f_0(U)_*}{\longrightarrow} U \otimes_S U_* \to 0$$

is exact.

$(1)'$ U-codom.dim$\,U_S \geqslant n$.

$(2)'$ *Applying the functor* $(-)_* \otimes_R U$ *to the minimal projective resolution (2) of* U_S*, the induced sequence*

$$Q_{n-1}(U)_* \otimes_R U \overset{g_{n-1}(U)_*\otimes U}{\longrightarrow} \cdots \overset{g_2(U)_*\otimes U}{\longrightarrow} Q_1(U)_* \otimes_R U \overset{g_1(U)_*\otimes U}{\longrightarrow}$$

$$Q_0(U)_* \otimes_R U \overset{g_0(U)_*\otimes U}{\longrightarrow} U_* \otimes_R U \to 0$$

is exact.

Proof

(1) $\Leftrightarrow$ (2): Set $F := (-)_*$ and $G =: U \otimes_S -$. We have the following commutative diagram with the bottom row exact:

$$\begin{array}{ccccccccccc}
\mathrm{GF}(P_{n-1}(U)) & \xrightarrow{\mathrm{GF}(f_{n-1}(U))} & \cdots & \longrightarrow & \mathrm{GF}(P_1(U)) & \xrightarrow{\mathrm{GF}(f_1(U))} & \mathrm{GF}(P_0(U)) & \xrightarrow{\mathrm{GF}(f_0(U))} & \mathrm{GF}(U) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_{P_{n-1}(U)}} & & & & \downarrow{\scriptstyle \theta_{P_1}} & & \downarrow{\scriptstyle \theta_{P_0}} & & \downarrow{\scriptstyle \theta_U} & & \\
P_{n-1}(U) & \xrightarrow{f_{n-1}(U)} & \cdots & \longrightarrow & P_1(U) & \xrightarrow{f_1(U)} & P_0(U) & \xrightarrow{f_0(U)} & U & \longrightarrow & 0
\end{array}$$

If the assertion (1) holds true—that is, U-codom.dim$\,_RU \geqslant n$—then $P_i(U) \in \mathrm{add}\,_RU$ for any $0 \leqslant i \leqslant n-1$. Note that $_RU$ is U-coreflexive by [22, Lemma 2.5(1)]. So $P_i(U)$ is also U-coreflexive for any $0 \leqslant i \leqslant n-1$. It follows that the upper row in the above diagram is exact, and assertion (2) follows.

Conversely, suppose that assertion (2) holds true. Then the above diagram is an exact commutative diagram. We will proceed by induction on n. Since θ_U is an isomorphism, the rightmost square in the above commutative diagram implies that $f_0(U)\theta_{P_0(U)} = \theta_U\,\mathrm{GF}(f_0(U))$ is epic. But $f_0(U)$ is superfluous, so it follows from [1, Corollary 5.15] that $\theta_{P_0(U)}$ is epic and $P_0(U)$ is U-cotorsionless. Then [22, Corollary 3.8] implies that $P_0(U)$ is generated by $_RU$; that is, $P_0(U) \in \mathrm{add}\,_RU$; hence, $\theta_{P_0(U)}$ is an isomorphism.

Now suppose $n \geqslant 2$. Then $P_i(U) \in \mathrm{add}\,_RU$ for any $0 \leqslant i \leqslant n-2$ by the induction hypothesis. Put $K_i' := \mathrm{Im}\,\mathrm{GF}(f_i(U))$ and $K_i := \mathrm{Im}\,f_i(U)$ for any $0 \leqslant i \leqslant n-1$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{GF}(P_i(U)) & \xrightarrow{\mathrm{GF}(f_i(U))} & K_i' \\
\downarrow{\scriptstyle \theta_{P_i(U)}} & & \downarrow{\scriptstyle t_i} \\
P_i(U) & \xrightarrow{f_i(U)} & K_i
\end{array}$$

where t_i is an induced isomorphism by induction. Because $f_{n-1}(U)\theta_{P_{n-1}(U)} = t_{n-1}\,\mathrm{GF}(f_{n-1}(U))$ is epic and $f_{n-1}(U)$ is superfluous, $\theta_{P_{n-1}(U)}$ is epic. It implies that $P_{n-1}(U)$ is U-cotorsionless and $P_{n-1}(U) \in \mathrm{add}\,_RU$. Thus, U-codom.dim$\,_RU \geqslant n$.

$(1)' \Rightarrow (1)$: We proceed by induction on n. The case for $n=1$ follows from Theorem 3.5.

Now suppose $n \geqslant 2$. By the induction hypothesis, we have U-codom.dim$\,_RU \geqslant n-1$. Let $N \in \mathrm{Mod}\,S$. Then s.E-cograde$_U\,\mathrm{Tor}_1^S(U,N) \geqslant n-1$ by Lemma 3.7. Let M be a quotient module of $\mathrm{Tor}_1^S(U,N)$. Then E-cograde$_U\,M \geqslant n-1$. By the dimension shifting, we have $\mathrm{Ext}_R^{n-1}(U,M) \cong \mathrm{Ext}_R^1(U, \mathrm{co}\Omega^{n-2}(M))$, where $\mathrm{co}\Omega^{n-2}(M)$ is the $(n-2)$th cosyzygy. Then by $(1)'$ and Lemma 3.8, we have T-cograde$_U\,\mathrm{Ext}_R^{n-1}(U,M) =$ T-cograde$_U\,\mathrm{Ext}_R^1(U,\mathrm{co}\Omega^{n-2}(M)) \geqslant n$. It follows from [26, Lemma 4.11(1)] that E-cograde$_U\,M \geqslant n$. Thus, we conclude that s.E-cograde$_U\,\mathrm{Tor}_1^S(U,N) \geqslant n$. Now the assertion follows from Lemma 3.7.

Symmetrically, we have $(1)' \Leftrightarrow (2)'$ and $(1) \Rightarrow (1)'$. □

As an immediate consequence of Theorem 3.9, we get the following corollary.

COROLLARY 3.10

U-codom.dim$\,_RU = U$-codom.dim$\,U_S$.

The following corollary is a supplement to Theorem 3.5.

COROLLARY 3.11

The following statements are equivalent:

(1) U-codom.dim$\,_RU \geqslant 1$.

(2) *The sequence*

$$U \otimes_S P_0(U)_* \stackrel{U\otimes f_0(U)_*}{\longrightarrow} U \otimes_S U_* \to 0$$

is exact.

(1)′ U-codom.dim$\,U_S \geqslant 1$.

(2)′ *The sequence*

$$Q_0(U)_* \otimes_R U \stackrel{g_0(U)_*\otimes U}{\longrightarrow} U_* \otimes_R U \to 0$$

is exact.

In the following result, we characterize when the U-codominant dimension of U is at least two in terms of the exactness of certain functors.

THEOREM 3.12

The following statements are equivalent:

(1) U-codom.dim$\,_RU \geqslant 2$.

(2) *The sequence*

$$U \otimes_S P_1(U)_* \stackrel{U\otimes f_1(U)_*}{\longrightarrow} U \otimes_S P_0(U)_* \stackrel{U\otimes f_0(U)_*}{\longrightarrow} U \otimes_S U_* \to 0$$

is exact.

(3) $U \otimes_S (-)_* : \operatorname{Mod} R \to \operatorname{Mod} R$ *is right exact.*

(4) $(U \otimes_S -)_* : \operatorname{Mod} S \to \operatorname{Mod} S$ *is left exact.*

(1)′ U-codom.dim $U_S \geqslant 2$.

(2)′ *The sequence*

$$Q_1(U)_* \otimes_R U \overset{g_1(U)_* \otimes U}{\longrightarrow} Q_0(U)_* \otimes_R U \overset{g_0(U)_* \otimes U}{\longrightarrow} U_* \otimes_R U \to 0$$

is exact.

(3)′ $(-)_* \otimes_R U : \operatorname{Mod} S^{\mathrm{op}} \to \operatorname{Mod} S^{\mathrm{op}}$ *is right exact.*

(4)′ $(- \otimes_R U)_* : \operatorname{Mod} R^{\mathrm{op}} \to \operatorname{Mod} R^{\mathrm{op}}$ *is left exact.*

Proof

By Theorem 3.9, we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1)′ $\Leftrightarrow$ (2)′. The implications (3) $\Rightarrow$ (2) and (3)′ $\Rightarrow$ (2)′ are trivial.

(1)′ $\Rightarrow$ (3): Let

$$0 \to M_1 \overset{\alpha}{\longrightarrow} M_2 \overset{\beta}{\longrightarrow} M_3 \to 0$$

be an exact sequence in $\operatorname{Mod} R$. Applying the functor $(-)_*$ to it induces an exact sequence

$$0 \to M_{1*} \overset{\alpha_*}{\longrightarrow} M_{2*} \overset{\beta_*}{\longrightarrow} M_{3*} \to \operatorname{Ext}^1_R(U, M_1) \tag{4}$$

in $\operatorname{Mod} S$. By (1)′, we have $Q_0(U), Q_1(U) \in \operatorname{add} U_S$. Then

$$Q_0(U) \otimes_S \operatorname{Ext}^1_R(U, M_1) = 0 = Q_1(U) \otimes_S \operatorname{Ext}^1_R(U, M_1)$$

by [26, Lemma 4.6]. Because $\operatorname{Coker} \beta_*$ is isomorphic to a submodule of $\operatorname{Ext}^1_R(U, M_1)$, we have

$$Q_0(U) \otimes_S \operatorname{Coker} \beta_* = 0 = Q_1(U) \otimes_S \operatorname{Coker} \beta_*;$$

hence,

$$U \otimes_S \operatorname{Coker} \beta_* = 0. \tag{5}$$

Moreover, applying the functor $- \otimes_S \operatorname{Coker} \beta_*$ to the minimal projective resolution (2) of U_S yields the following two exact sequences:

$$Q_1(U) \otimes_S \operatorname{Coker} \beta_* \to \operatorname{Im} g_1(U) \otimes_S \operatorname{Coker} \beta_* \to 0,$$

$$(0 =) \operatorname{Tor}^S_1\big(Q_0(U), \operatorname{Coker} \beta_*\big) \to \operatorname{Tor}^S_1(U, \operatorname{Coker} \beta_*) \to \operatorname{Im} g_1(U) \otimes_S \operatorname{Coker} \beta_*.$$

Since $Q_1(U) \otimes_S \operatorname{Coker} \beta_* = 0$, we have $\operatorname{Im} g_1(U) \otimes_S \operatorname{Coker} \beta_* = 0$; hence,

$$\operatorname{Tor}^S_1(U, \operatorname{Coker} \beta_*) = 0. \tag{6}$$

By the equalities (5) and (6), applying the functor $U \otimes_S -$ to the exact sequence (4) yields the following exact sequence:

$$U \otimes_S M_{1*} \overset{U \otimes \alpha_*}{\longrightarrow} U \otimes_S M_{2*} \overset{U \otimes \beta_*}{\longrightarrow} U \otimes_S M_{3*} \to 0.$$

(1) $\Rightarrow$ (4): Let

$$0 \to N_1 \xrightarrow{\phi} N_2 \xrightarrow{\psi} N_3 \to 0$$

be an exact sequence in $\operatorname{Mod} S$. Applying the functor $U \otimes_S -$ to it induces an exact sequence

$$\operatorname{Tor}_1^S(U, N_3) \to U \otimes_S N_1 \xrightarrow{U\otimes\phi} U \otimes_S N_2 \xrightarrow{U\otimes_S\psi} U \otimes_S N_3 \to 0 \tag{7}$$

in $\operatorname{Mod} R$. By (1), we have $P_0(U), P_1(U) \in \operatorname{add}{}_R U$. Then

$$\operatorname{Hom}_R\big(P_0(U), \operatorname{Tor}_1^S(U, N_3)\big) = 0 = \operatorname{Hom}_R\big(P_1(U), \operatorname{Tor}_1^S(U, N_3)\big)$$

by [26, Lemma 4.6]. Because $\operatorname{Ker}(U \otimes \phi)$ is isomorphic to a factor module of $\operatorname{Tor}_1^S(U, N_3)$, we have

$$\operatorname{Hom}_R\big(P_0(U), \operatorname{Ker}(U \otimes \phi)\big) = 0 = \operatorname{Hom}_R\big(P_1(U), \operatorname{Ker}(U \otimes \phi)\big);$$

hence,

$$\big(\operatorname{Ker}(U \otimes \phi)\big)_* = 0. \tag{8}$$

Moreover, applying the functor $\operatorname{Hom}_R(-, \operatorname{Ker}(U \otimes \phi))$ to the minimal projective resolution (1) of ${}_R U$ yields the following two exact sequences:

$$0 \to \operatorname{Hom}_R\big(\operatorname{Im} f_1(U), \operatorname{Ker}(U \otimes \phi)\big) \to \operatorname{Hom}_R\big(P_1(U), \operatorname{Ker}(U \otimes \phi)\big),$$

$$\operatorname{Hom}_R\big(\operatorname{Im} f_1(U), \operatorname{Ker}(U \otimes \phi)\big) \to \operatorname{Ext}_R^1\big(U, \operatorname{Ker}(U \otimes \phi)\big) \to \operatorname{Ext}_R^1\big(P_0(U), \operatorname{Ker}(U \otimes \phi)\big)(= 0).$$

Since $\operatorname{Hom}_R(P_1(U), \operatorname{Ker}(U \otimes \phi)) = 0$, we have $\operatorname{Hom}_R(\operatorname{Im} f_1(U), \operatorname{Ker}(U \otimes \phi)) = 0$; hence,

$$\operatorname{Ext}_R^1\big(U, \operatorname{Ker}(U \otimes \phi)\big) = 0. \tag{9}$$

By the equalities (8) and (9), applying the functor $(-)_*$ to the exact sequence (7) yields the following exact sequence

$$0 \to (U \otimes_S N_1)_* \xrightarrow{(U\otimes\phi)_*} (U \otimes_S N_2)_* \xrightarrow{(U\otimes_S\psi)_*} (U \otimes_S N_3)_*.$$

(4) $\Rightarrow$ (1): By (4) and Theorem 3.5, we have U-codom.dim${}_R U \geqslant 1$ and U-codom.dim$\, U_S \geqslant 1$. Let $M \in \operatorname{Mod} R$ with $M_* = 0$. By Proposition 3.6, it suffices to prove E-cograde$_U\, M \geqslant 2$.

Let

$$0 \to K \xrightarrow{f} Q \xrightarrow{g} \operatorname{Ext}_R^1(U, M) \to 0 \tag{10}$$

be an exact sequence in $\operatorname{Mod} S$ with Q projective. By Lemma 3.8, we have

$$U \otimes_S \operatorname{Ext}_R^1(U, M) = 0. \tag{11}$$

By (4), the exact sequence (10) induces the following exact sequence:

$$0 \to (U \otimes_S K)_* \xrightarrow{(U\otimes f)_*} (U \otimes_S Q)_* \xrightarrow{(U\otimes g)_*} \big(U \otimes_S \operatorname{Ext}_R^1(U, M)\big)_*(= 0),$$

which implies that $(U \otimes f)_*$ is an isomorphism; hence, $U \otimes (U \otimes f)_*$ is also an isomorphism. On the other hand, we have the following commutative diagram with the bottom row exact:

$$\begin{array}{ccccccccc} & & & & U\otimes_S (U\otimes_S K)_* & \xrightarrow{U\otimes(U\otimes f)_*} & U\otimes_S (U\otimes_S Q)_*. & & \\ & & & & \downarrow{\scriptstyle \theta_{U\otimes_S K}} & & \downarrow{\scriptstyle \theta_{U\otimes_S Q}} & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^S(U,\mathrm{Ext}_R^1(U,M)) & \longrightarrow & U\otimes_S K & \xrightarrow{U\otimes f} & U\otimes_S Q & \longrightarrow & 0 \end{array}$$

The equality (11) means that the functor $U\otimes_S \mathrm{Ext}_R^1(U,-)$ vanishes on $\mathrm{Mod}\,R$; hence, the functor $U\otimes_S \mathrm{Ext}_R^2(U,-)$ also vanishes on $\mathrm{Mod}\,R$. Then by [26, Lemma 4.18], we have that both $U\otimes_S K$ and $U\otimes_S Q$ are U-coreflexive—that is, both $\theta_{U\otimes_S K}$ and $\theta_{U\otimes_S Q}$ are isomorphisms. Then by the above commutative diagram, we have $\mathrm{Tor}_1^S(U,\mathrm{Ext}_R^1(U,M))=0$. Combining it with the equality (11) yields $\text{T-cograde}_U\,\mathrm{Ext}_R^1(U,M)\geqslant 2$. It follows from [26, Lemma 4.11(1)] that $\text{E-cograde}_U\,M\geqslant 2$.

Symmetrically, we get $(1)\Rightarrow(3)'$ and $(1)'\Leftrightarrow(4)'$. □

4. n-Auslander algebras

For any $n\geqslant 1$, recall from [14] that an artin algebra R is called an *n-Auslander algebra* if

$$\mathrm{gl.dim}\,R\leqslant n+1\leqslant \mathrm{dom.dim}\,R,$$

where $\mathrm{gl.dim}\,R$ and $\mathrm{dom.dim}\,R$ are the global and dominant dimensions of R, respectively. Note that 1-Auslander algebras are exactly classical Auslander algebras.

Let R be an artin algebra and D the usual duality between $\mathrm{mod}\,R$ and $\mathrm{mod}\,R^{\mathrm{op}}$. It is easy to verify the following observations:

(1) $D(R)$ is a semidualizing (R,R)-bimodule.

(2) $\mathrm{dom.dim}\,R_R = D(R)\text{-}\mathrm{codom.dim}\,{}_RD(R)$, and $\mathrm{dom.dim}\,{}_RR = D(R)\text{-}\mathrm{codom.dim}\,D(R)_R$.

Thus, putting ${}_RU_S={}_RD(R)_R$ in Theorem 3.9, we get some equivalent characterizations of n-Auslander algebras as follows.

COROLLARY 4.1

Let R be an artin algebra with $\mathrm{gl.dim}\,R\leqslant n+1$. Then the following statements are equivalent:

(1) *R is an n-Auslander algebra.*

(2) $D(R)\text{-}\mathrm{codom.dim}\,{}_RD(R)\geqslant n+1$.

(3) *Applying the functor $D(R)\otimes_R(-)_*$ to the minimal projective resolution of ${}_RD(R)$, the induced sequence*

$$D(R)\otimes_R P_n\big(D(R)\big)_* \xrightarrow{D(R)\otimes f_n(D(R))_*} \cdots \xrightarrow{D(R)\otimes f_2(D(R))_*} D(R)\otimes_R P_1\big(D(R)\big)_*$$

$$\xrightarrow{D(R)\otimes f_1(D(R))_*} D(R)\otimes_R P_0\big(D(R)\big)_* \xrightarrow{D(R)\otimes f_0(D(R))_*} D(R)\otimes_R D(R)_*\to 0$$

is exact.

(2)′ $D(R)$-codom.dim $D(R)_R \geqslant n+1$.

(3)′ *Applying the functor* $(-)_* \otimes_R D(R)$ *to the minimal projective resolution of* $D(R)_R$*, the induced sequence*

$$Q_n\big(D(R)\big)_* \otimes_R D(R) \overset{g_n(D(R))_* \otimes D(R)}{\longrightarrow} \cdots \overset{g_2(D(R))_* \otimes D(R)}{\longrightarrow} Q_1\big(D(R)\big)_* \otimes_R D(R)$$
$$\overset{g_1(D(R))_* \otimes D(R)}{\longrightarrow} Q_0\big(D(R)\big)_* \otimes_R D(R) \overset{g_0(D(R))_* \otimes D(R)}{\longrightarrow} D(R)_* \otimes_R D(R) \to 0$$

is exact.

Putting ${}_RU_S = {}_RD(R)_R$ in Theorem 3.12, we get some equivalent characterizations of Auslander algebras as follows.

COROLLARY 4.2

Let R be an artin algebra with gl.dim $R \leqslant 2$. *Then the following statements are equivalent:*

(1) *R is an Auslander algebra.*
(2) $D(R)$-codom.dim ${}_RD(R) \geqslant 2$.
(3) *The sequence*

$$D(R) \otimes_R P_1\big(D(R)\big)_* \overset{D(R)\otimes f_1(D(R))_*}{\longrightarrow} D(R) \otimes_R P_0\big(D(R)\big)_*$$
$$\overset{D(R)\otimes f_0(D(R))_*}{\longrightarrow} D(R) \otimes_R D(R)_* \to 0$$

is exact.

(4) $D(R) \otimes_R (-)_* : \operatorname{Mod} R \to \operatorname{Mod} R$ *is right exact.*
(5) $(D(R) \otimes_R -)_* : \operatorname{Mod} R \to \operatorname{Mod} R$ *is left exact.*
(2)′ $D(R)$-codom.dim $D(R)_R \geqslant 2$.
(3)′ *The sequence*

$$Q_1\big(D(R)\big)_* \otimes_R D(R) \overset{g_1(D(R))_* \otimes D(R)}{\longrightarrow} Q_0\big(D(R)\big)_* \otimes_R D(R)$$
$$\overset{g_0(D(R))_* \otimes D(R)}{\longrightarrow} D(R)_* \otimes_R D(R) \to 0$$

is exact.

(4)′ $(-)_* \otimes_R D(R) : \operatorname{Mod} R^{\mathrm{op}} \to \operatorname{Mod} R^{\mathrm{op}}$ *is right exact.*
(5)′ $(- \otimes_R D(R))_* : \operatorname{Mod} R^{\mathrm{op}} \to \operatorname{Mod} R^{\mathrm{op}}$ *is left exact.*

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References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, 2nd ed., Grad. Texts Math. **13**, Springer-Verlag, New York, 1974. MR 1245487. DOI 10.1007/978-1-4612-4418-9.

[2] T. Araya, R. Takahashi, and Y. Yoshino, *Homological invariants associated to semi-dualizing bimodules*, J. Math. Kyoto Univ. **45** (2005), no. 2, 287–306. MR 2161693. DOI 10.1215/kjm/1250281991.

[3] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. **36**, Cambridge Univ. Press, Cambridge, 1997. MR 1476671.

[4] A. Beligiannis and I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207. MR 2327478. DOI 10.1090/memo/0883.

[5] R. R. Colby and K. R. Fuller, *Exactness of the double dual*, Proc. Amer. Math. Soc. **82** (1981), no. 4, 521–526. MR 0614871. DOI 10.2307/2043764.

[6] G. Eerkes, *Codominant dimension of rings and modules*, Trans. Amer. Math. Soc. **176** (1973), 125–139. MR 0314906. DOI 10.2307/1996200.

[7] G. Eerkes, *Rings of equivalent dominant and codominant dimensions*, Proc. Amer. Math. Soc. **48** (1975), 297–306. MR 0360710. DOI 10.2307/2040258.

[8] M. Fang and S. König, *Schur functors and dominant dimension*, Trans. Amer. Math. Soc. **363** (2011), 1555–1576. MR 2737277. DOI 10.1090/S0002-9947-2010-05177-3.

[9] N. Gao and S. König, *Grade, dominant dimension and Gorenstein algebras*, J. Algebra **427** (2015), 118–141. MR 3312298. DOI 10.1016/j.jalgebra.2014.11.028.

[10] N. Gao and S. König, *Double centraliser property and morphism categories*, Proc. Amer. Math. Soc. **144** (2016), 971–981. MR 3447651. DOI 10.1090/proc/12807.

[11] P. A. Guil Asensio and I. Herzog, *Left cotorsion rings*, Bull. London Math. Soc. **36** (2004), no. 3, 303–309. MR 2038718. DOI 10.1112/S0024609303002844.

[12] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808. MR 2413065. DOI 10.1215/kjm/1250692289.

[13] Z. Y. Huang, *On U-dominant dimension*, J. Algebra **285** (2005), no. 2, 669–681. MR 2125458. DOI 10.1016/j.jalgebra.2004.11.008.

[14] O. Iyama, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61. MR 2735750. DOI 10.1016/j.aim.2010.03.004.

[15] T. Kato, *Rings of U-dominant dimension $\geqslant 1$*, Tohoku Math. J. (2) **21** (1969), 321–327. MR 0248169. DOI 10.2748/tmj/1178243000.

[16] S. König, I. H. Slungard, and C. C. Xi, *Double centralizer properties, dominant dimension, and tilting modules*, J. Algebra **240** (2001), no. 1, 393–412. MR 1830559. DOI 10.1006/jabr.2000.8726.

[17] F. Mantese and I. Reiten, *Wakamatsu tilting modules*, J. Algebra **278** (2004), no. 2, 532–552. MR 2071651. DOI 10.1016/j.jalgebra.2004.03.023.

[18] B. J. Müller, *The classification of algebras by dominant dimension*, Canad. J. Math. **20** (1968), 398–409. MR 0224656. DOI 10.4153/CJM-1968-037-9.

[19] W. K. Nicholson, *Semiregular modules and rings*, Canad. J. Math. **28** (1976), no. 5, 1105–1120. MR 0422343. DOI 10.4153/CJM-1976-109-2.

[20] B. Stenström, *Rings of Quotients*, Die Grundlehren der mathematischen Wissenschaften **217**, Spring-Verlag, New York–Heidelberg, 1975. MR 0389953.

[21] H. Tachikawa, *On dominant dimension of QF-3 algebras*, Trans. Amer. Math. Soc. **112** (1964), 249–266. MR 0161888. DOI 10.2307/1994293.

[22] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose*, Forum Math. **27** (2015), no. 6, 3717–3743. MR 3420357. DOI 10.1515/forum-2013-0196.

[23] X. Tang and Z. Y. Huang, *Coreflexive modules and semidualizing modules with finite projective dimension*, Taiwanese J. Math. **21** (2017), no. 6, 1283–1324. MR 3732907. DOI 10.11650/tjm/8009.

[24] X. Tang and Z. Y. Huang, *Homological aspects of the adjoint cotranspose*, Colloq. Math. **150** (2017), no. 2, 293–311. MR 3719463. DOI 10.4064/cm7121-12-2016.

[25] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose, II*, Kyoto J. Math. **57** (2017), no. 1, 17–53. MR 3621778. DOI 10.1215/21562261-3759504.

[26] X. Tang and Z. Y. Huang, *Cograde conditions and cotorsion pairs*, Publ. Res. Inst. Math. Sci. **56** (2020), no. 3, 445–502. MR 4116689. DOI 10.4171/PRIMS/56-3-2.

[27] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra **114** (1988), no. 1, 106–114. MR 0931903. DOI 10.1016/0021-8693(88)90215-3.

[28] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), no. 2, 298–325. MR 1074331. DOI 10.1016/0021-8693(90)90055-S.

[29] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra **275** (2004), no. 1, 3–39. MR 2047438. DOI 10.1016/j.jalgebra.2003.12.008.

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