

Cograde Conditions and Cotorsion Pairs

by

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Abstract

Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule. We investigate when the double functor $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms and the double functor $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, -))$ preserves monomorphisms in terms of the (strong) cograde conditions of modules. Under certain cograde condition of modules, we construct two complete cotorsion pairs. In addition, we establish the relation between some relative finitistic dimensions of rings and the right and left projective dimensions of ω .

2010 Mathematics Subject Classification: Primary 18G25; Secondary 16E10, 16E30.

Keywords: Semidualizing bimodules, (strong) Ext-cograde, (strong) Tor-cograde, double functors, n - \mathcal{X} -(co)syzygy, (adjoint) n -cotorsionfree modules, cotorsion pairs, finitistic dimensions.

§1. Introduction

Let R be a left and right Noetherian ring and $n \geq 1$. It was proved by Auslander that the flat dimension of the i th term in the minimal injective resolution of R_R is at most i for any $0 \leq i < n$ if and only if the strong grade of $\text{Ext}_R^i(M, R)$ is at least i for any finitely generated left R -module M and $1 \leq i \leq n$; this result is left–right symmetric ([FGR, Thm. 3.7]). In this case, R is called *Auslander n -Gorenstein*. If R is Auslander n -Gorenstein for all n , then it is said to satisfy the *Auslander condition*. This condition is a noncommutative version of commutative Gorenstein rings. It is known that Auslander n -Gorenstein rings and the Auslander condition play a crucial role in homological algebra, representation theory of artin algebras and noncommutative algebraic geometry; see [AR1, AR2, B, CSS, EHIS, FGR, H1, H3, HI, HQ, IS, IY1, IY2] and references therein. In particular, Auslander

Communicated by S. Mochizuki. Received February 16, 2019. Revised May 15, 2019.

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n -Gorenstein rings and some generalized versions were characterized in terms of the properties of the double functor $\text{Ext}_{R^{\text{op}}}^i(\text{Ext}_R^i(-, R), R)$ and certain (strong) grade conditions of Ext-modules, and a series of cotorsion pairs were constructed under the Auslander condition ([HI]).

It is well known that the (Auslander) transpose is one of the most powerful tools in representation theory of artin algebras and Gorenstein homological algebra; see [AB, ARS, EJ]. To dualize this important and useful notion, we introduced in [TH1] the notion of the cotranspose of modules and then obtained many dual counterparts of interesting results ([TH1, TH2, TH3, TH4]). As a dual of the notion of the (strong) grade of modules, we introduced in [TH1, TH4] the notion of the (strong) cograde of modules, and obtained the dual versions of some results about the (strong) grade of modules. Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule. In this paper, we will study when the double functor $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms and the double functor $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, -))$ preserves monomorphisms in terms of the (strong) cograde conditions of modules and some related properties of the cotranspose of modules, and also investigate the relationship between certain cograde conditions of modules and complete cotorsion pairs. This paper is organized as follows.

In Section 2 we give some terminology and some preliminary results.

Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule. In Section 3 we investigate when $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms and $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, -))$ preserves monomorphisms in terms of the (strong) cograde conditions of modules. Let $n, k \geq 0$. We prove that the Tor-cograde of $\text{Ext}_R^{i+k}(\omega, M)$ with respect to ω is at least i for any left R -module M and $1 \leq i \leq n$ if and only if $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any epimorphism of left R -modules $f: B \rightarrow C$ with B, C being a $(k+1)$ -cosyzygy and $0 \leq i \leq n-1$ (Theorem 3.5), and that the Ext-cograde of $\text{Tor}_{i+k}^S(\omega, N)$ with respect to ω is at least i for any left S -module N and $1 \leq i \leq n$ if and only if $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any monomorphism of left S -modules $g: B' \rightarrow C'$ with B', C' being a $(k+1)$ -yoke and $0 \leq i \leq n-1$ (Theorem 3.7).

Moreover, we prove that the strong Tor-cograde of $\text{Ext}_R^{i+k}(\omega, M)$ with respect to ω is at least i for any left R -module M and $1 \leq i \leq n$ if and only if for any exact sequence of left R -modules $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$ with A an $(i-1)$ - $\mathcal{P}_\omega(R)$ -syzygy of an $(i+k-1)$ -cosyzygy, $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any $0 \leq i \leq n-1$ (Theorem 3.8), and that the strong Ext-cograde of $\text{Tor}_{i+k}^S(\omega, N)$ with respect to ω is at least i for any left S -module N and $1 \leq i \leq n$ if and only if for any exact sequence of left S -modules $0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$ with C an $(i-1)$ -

$\mathcal{I}_\omega(S)$ -cosyzygy of an $(i+k-1)$ -yoke, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n-1$ (Theorem 3.9).

In Section 4 we introduce the notion of ω satisfying the (quasi) n -cograde condition in terms of the properties of the strong cograde of modules. By using the results obtained in Section 3 we give some equivalent characterizations for ω satisfying such conditions (Theorems 4.8 and 4.14). In particular, the n -cograde condition is left-right symmetric, but the quasi n -cograde condition is not. In addition, we prove that the Tor-cograde of $\text{Ext}_R^i(\omega, M)$ with respect to ω is at least $i-1$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$ if and only if the Ext-cograde of $\text{Tor}_i^S(\omega, N)$ with respect to ω is at least $i-1$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$ (Theorem 4.19).

In Section 5 we prove that if one of the equivalent conditions in Theorem 4.19 mentioned above is satisfied, then the right S -projective dimension $\text{pd}_{S^{\text{op}}} \omega$ of ω is at most $n-1$ if and only if $(\mathcal{P}_\omega\text{-id}^{\leq n-1}(R), {}_R\omega^{\perp n})$ forms a complete cotorsion pair, and that the left R -projective dimension $\text{pd}_R \omega$ of ω is at most $n-1$ if and only if $(\omega_S^{\top n}, \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S))$ forms a complete cotorsion pair (Theorem 5.6); see Sections 2 and 5 for the details of this notation.

In Section 6 we introduce the finitistic $\mathcal{P}_\omega(R)$ -injective dimension $\text{FP}_\omega\text{-id } R$ of R and the $\mathcal{I}_\omega(S)$ -projective dimension $\text{F}\mathcal{I}_\omega\text{-pd } S$ of S . We prove that if the Tor-cograde of $\text{Ext}_R^{i+k}(\omega, M)$ with respect to ω is at least i for any $M \in \text{Mod } R$ and $i \geq 1$, then $\text{FP}_\omega\text{-id } R \leq \text{pd}_R \omega \leq \text{FP}_\omega\text{-id } R + k$. Further, we prove that if the Ext-cograde of $\text{Tor}_{i+k}^S(\omega, N)$ with respect to ω is at least i for any $N \in \text{Mod } S$ and $i \geq 1$, then $\text{F}\mathcal{I}_\omega\text{-pd } S \leq \text{pd}_{S^{\text{op}}} \omega \leq \text{F}\mathcal{I}_\omega\text{-pd } S + k$ (Theorem 6.3). As an application, we get that for an artin algebra R , if R satisfies the Auslander condition then $\text{FPD } R^{\text{op}} = \text{FID } R^{\text{op}} = \text{id}_{R^{\text{op}}} R = \text{id}_R R = \text{FPD } R = \text{FID } R$, and if R satisfies the right quasi Auslander condition then $\text{FPD } R \leq \text{FID } R = \text{id}_{R^{\text{op}}} R = \text{id}_R R \leq \text{FPD } R + 1$, where $\text{FID } R$, $\text{FPD } R$, $\text{id}_{R^{\text{op}}} R$ and $\text{id}_R R$ are the finitistic injective dimension, the finitistic projective dimension, the right and left self-injective dimensions of R respectively (Corollary 6.9).

§2. Preliminaries

Throughout this paper, all rings are associative rings with units. For a ring R , $\text{Mod } R$ (resp. $\text{mod } R$) are the class of left (resp. finitely generated left) R -modules. Let $M \in \text{Mod } R$; we use $\text{Add}_R M$ to denote the subclass of $\text{Mod } R$ consisting of modules consisting of direct summands of direct sums of copies of M , and use $\text{pd}_R M$, $\text{fd}_R M$ and $\text{id}_R M$ to denote the projective, flat and injective dimensions of M respectively.

Definition 2.1 ([[ATY](#), [HW](#)]). Let R and S be rings. An $(R\text{-}S)$ -bimodule ${}_R\omega_S$ is called *semidualizing* if the following conditions are satisfied:

- (a1) ${}_R\omega$ admits a degreewise finite R -projective resolution.
- (a2) ω_S admits a degreewise finite S -projective resolution.
- (b1) The homothety map ${}_R R_R \xrightarrow{R\gamma} \text{Hom}_{S^{\text{op}}}(\omega, \omega)$ is an isomorphism.
- (b2) The homothety map ${}_S S_S \xrightarrow{S\delta} \text{Hom}_R(\omega, \omega)$ is an isomorphism.
- (c1) $\text{Ext}_R^{\geq 1}(\omega, \omega) = 0$.
- (c2) $\text{Ext}_{S^{\text{op}}}^{\geq 1}(\omega, \omega) = 0$.

Wakamatsu in [[W1](#)] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*; see [[BR](#), [MR](#)]. Note that a bimodule ${}_R\omega_S$ is semidualizing if and only if it is Wakamatsu tilting ([[W3](#), Cor. 3.2]). Examples of semidualizing bimodules are given in [[HW](#), [S](#), [TH2](#), [TH4](#), [W2](#)].

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_R\omega_S$. For convenience, we write

$$\begin{aligned} (-)_* &:= \text{Hom}(\omega, -) \quad \text{and} \quad (-)^* := \text{Hom}(-, \omega), \\ {}_R\omega^\perp &:= \{M \in \text{Mod } R \mid \text{Ext}_R^{\geq 1}(\omega, M) = 0\}, \\ \omega_S^\top &:= \{N \in \text{Mod } S \mid \text{Tor}_{\geq 1}^S(\omega, N) = 0\}. \end{aligned}$$

For any $n \geq 1$ we write

$$\begin{aligned} {}_R\omega^{\perp n} &:= \{M \in \text{Mod } R \mid \text{Ext}_R^{1 \leq i \leq n}(\omega, M) = 0\}, \\ \omega_S^{\top n} &:= \{N \in \text{Mod } S \mid \text{Tor}_{1 \leq i \leq n}^S(\omega, N) = 0\}; \end{aligned}$$

in particular, ${}_R\omega^{\perp 0} = \text{Mod } R$ and $\omega_S^{\top 0} = \text{Mod } S$. We define $\omega_S^{\perp n}$ and ${}_R\omega^{\top n}$ symmetrically. Following [[HW](#)] set

$$\begin{aligned} \mathcal{F}_\omega(R) &:= \{\omega \otimes_S F \mid F \text{ is flat in } \text{Mod } S\}, \\ \mathcal{P}_\omega(R) &:= \{\omega \otimes_S P \mid P \text{ is projective in } \text{Mod } S\}, \\ \mathcal{I}_\omega(S) &:= \{I_* \mid I \text{ is injective in } \text{Mod } R\}. \end{aligned}$$

The modules in $\mathcal{F}_\omega(R)$, $\mathcal{P}_\omega(R)$ and $\mathcal{I}_\omega(S)$ are called ω -flat, ω -projective and ω -injective respectively. Note that $\mathcal{P}_\omega(R) = \text{Add}_R \omega$ ([[TH4](#), Prop. 3.4(2)]). The classes of $\mathcal{F}_\omega(S^{\text{op}})$, $\mathcal{P}_\omega(S^{\text{op}})$ and $\mathcal{I}_\omega(R^{\text{op}})$ are defined symmetrically.

Let $M \in \text{Mod } R$ and $N \in \text{Mod } S$. Then we have the following two canonical valuation homomorphisms:

$$\theta_M : \omega \otimes_S M_* \rightarrow M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_*$, and

$$\mu_N : N \rightarrow (\omega \otimes_S N)_*$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$. Recall that a module $M \in \text{Mod } R$ is called ω -cotorsionless (resp. ω -coreflexive) if θ_M is an epimorphism (resp. an isomorphism) ([TH1]), and a module $N \in \text{Mod } S$ is called *adjoint* ω -cotorsionless (resp. *adjoint* ω -coreflexive) if μ_N is a monomorphism (resp. an isomorphism) ([TH3]).

Definition 2.2 ([HW]). (1) The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S -modules N satisfying the following conditions:

- (A1) $N \in \omega_S^\top$.
- (A2) $\omega \otimes_S N \in {}_R\omega^\perp$.
- (A3) μ_N is an isomorphism in $\text{Mod } S$.

(2) The *Bass class* $\mathcal{B}_\omega(R)$ with respect to ω consists of all left R -modules M satisfying the following conditions:

- (B1) $M \in {}_R\omega^\perp$.
- (B2) $M_* \in \omega_S^\top$.
- (B3) θ_M is an isomorphism in $\text{Mod } R$.

For a module $M \in \text{Mod } R$ we use

$$(2.1) \quad 0 \rightarrow M \rightarrow I^0(M) \xrightarrow{g^0} I^1(M) \xrightarrow{g^1} \dots \xrightarrow{g^{i-1}} I^i(M) \xrightarrow{g^i} \dots$$

to denote the minimal injective resolution of M . For any $n \geq 1$, $\text{co}\Omega^n(M) := \text{Im } g^{n-1}$ is called the n -cosyzygy of M ; in particular, $\text{co}\Omega^0(M) = M$. We use $\text{co}\Omega^n(R)$ to denote the subclass of $\text{Mod } R$ consisting of n -cosyzygy modules. We defined $\text{co}\Omega^n(S^{\text{op}})$ symmetrically.

Definition 2.3 ([TH1]). Let $M \in \text{Mod } R$ and $n \geq 1$.

- (1) $c\text{Tr}_\omega M := \text{Coker}(g^0_*)$ is called the *cotranspose* of M with respect to ${}_R\omega_S$, where g^0 is as in (2.1).
- (2) M is called n - ω -cotorsionfree if $c\text{Tr}_\omega M \in \omega_S^{\top n}$ and is called ∞ - ω -cotorsionfree if it is n - ω -cotorsionfree for all n .

We use $c\mathcal{T}_\omega^n(R)$ (resp. $c\mathcal{T}_\omega(R)$) to denote the subclass of $\text{Mod } R$ consisting of n - ω -cotorsionfree modules (resp. ∞ - ω -cotorsionfree modules). We define $c\mathcal{T}_\omega^n(S^{\text{op}})$ symmetrically. By [TH1, Prop. 3.2] we have that a module in $\text{Mod } R$ is ω -cotorsionless (resp. ω -coreflexive) if and only if it is in $c\mathcal{T}_\omega^1(R)$ (resp. $c\mathcal{T}_\omega^2(R)$).

Recall from [E] that a homomorphism $f: F \rightarrow N$ in $\text{Mod } S$ with F flat is called a *flat cover* of N if $\text{Hom}_S(F', f)$ is epic for any flat module F' in $\text{Mod } S$, and an endomorphism $h: F \rightarrow F$ is an automorphism whenever $f = fh$. Let $N \in \text{Mod } S$. Bican, El Bashir and Enochs proved in [BBE] that N has a flat cover. We use

$$(2.2) \quad \cdots \xrightarrow{f_n} F_n(N) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_1(N) \xrightarrow{f_0} F_0(N) \rightarrow N \rightarrow 0$$

to denote the minimal flat resolution of N in $\text{Mod } S$, where each $F_i(N) \rightarrow \text{Coker } f_i$ is a flat cover of $\text{Coker } f_i$. For any $n \geq 1$, $\Omega_{\mathcal{F}}^n(N) := \text{Im } f_{n-1}$ is called the *n-yoke* of N ; in particular, $\Omega_{\mathcal{F}}^0(N) = N$. We use $\Omega_{\mathcal{F}}^n(S)$ to denote the subclass of $\text{Mod } S$ consisting of *n-yoke* modules. We define $\Omega_{\mathcal{F}}^n(R^{\text{op}})$ symmetrically.

Definition 2.4 ([TH3]). Let $N \in \text{Mod } S$ and $n \geq 1$.

- (1) $\text{acTr}_{\omega} N := \text{Ker}(1_{\omega} \otimes f_0)$ is called the *adjoint cotranspose* of N with respect to ${}_{R}\omega_S$, where f_0 is as in (2.2).
- (2) N is called *adjoint n- ω -cotorsionfree* if $\text{acTr}_{\omega} N \in {}_{R}\omega^{\perp n}$ and is called *adjoint ∞ - ω -cotorsionfree* if it is adjoint *n- ω -cotorsionfree* for all n .

We use $\text{ac}\mathcal{T}_{\omega}^n(S)$ (resp. $\text{ac}\mathcal{T}_{\omega}(S)$) to denote the subclass of $\text{Mod } S$ consisting of adjoint *n- ω -cotorsionfree* modules (resp. adjoint *∞ - ω -cotorsionfree* modules). We define $\text{ac}\mathcal{T}_{\omega}^n(R^{\text{op}})$ symmetrically. By [TH3, Prop. 3.2] we have that a module in $\text{Mod } S$ is adjoint ω -cotorsionless (resp. adjoint ω -coreflexive) if and only if it is in $\text{ac}\mathcal{T}_{\omega}^1(S)$ (resp. $\text{ac}\mathcal{T}_{\omega}^2(S)$).

Definition 2.5 ([TH4]). (1) Let $M \in \text{Mod } R$ and $n \geq 0$. The *Ext-cograde* of M with respect to ω is defined as $\text{E-cograde}_{\omega} M := \inf\{i \geq 0 \mid \text{Ext}_R^i(\omega, M) \neq 0\}$ and the *strong Ext-cograde* of M with respect to ω , denoted by $\text{s.E-cograde}_{\omega} M$, is said to be at least n if $\text{E-cograde}_{\omega} X \geq n$ for any quotient module X of M . The (*strong*) *Ext-cograde* of a module in $\text{Mod } S^{\text{op}}$ is defined symmetrically.

- (2) Let $N \in \text{Mod } S$ and $n \geq 0$. The *Tor-cograde* of N with respect to ω is defined as $\text{T-cograde}_{\omega} N := \inf\{i \geq 0 \mid \text{Tor}_i^S(\omega, N) \neq 0\}$ and the *strong Tor-cograde* of N with respect to ω , denoted by $\text{s.T-cograde}_{\omega} N$, is said to be at least n if $\text{T-cograde}_{\omega} Y \geq n$ for any submodule Y of N . The (*strong*) *Tor-cograde* of a module in $\text{Mod } R^{\text{op}}$ is defined symmetrically.

Let \mathcal{X} be a subclass of $\text{Mod } R$ and $M \in \text{Mod } R$. An exact sequence (of finite or infinite length)

$$\cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

in $\text{Mod } R$ is called an \mathcal{X} -resolution of M if all X_i are in \mathcal{X} . The \mathcal{X} -projective dimension $\mathcal{X}\text{-pd}_R M$ of M is defined as $\inf\{n \mid \text{there exists an } \mathcal{X}\text{-resolution}$

$$0 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

of M in $\text{Mod } R\}$. The notions of an \mathcal{X} -coresolution and the \mathcal{X} -injective dimension $\mathcal{X}\text{-id}_R M$ of M are defined dually.

Let \mathcal{F} be a subclass of $\text{Mod } R$. A module $M \in \text{Mod } R$ is said to have *special \mathcal{F} -precover* if there exists an exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

in $\text{Mod } R$ with $F \in \mathcal{F}$ and $\text{Ext}_R^1(F', K) = 0$ for any $F' \in \mathcal{F}$. The class \mathcal{F} is called a *special precovering* if any module in $\text{Mod } R$ has a special \mathcal{F} -precover. The notions of *special \mathcal{F} -preenvelopes* and *special preenveloping classes* are defined dually (see [EJ]).

Definition 2.6 ([GT]). Let \mathcal{U}, \mathcal{V} be subclasses of $\text{Mod } R$. The pair $(\mathcal{U}, \mathcal{V})$ is called a *cotorsion pair* if $\mathcal{U} = {}^{\perp 1}\mathcal{V} := \{U \in \text{Mod } R \mid \text{Ext}_R^1(U, V) = 0 \text{ for any } V \in \mathcal{V}\}$ and $\mathcal{V} = \mathcal{U}^{\perp 1} := \{V \in \text{Mod } R \mid \text{Ext}_R^1(U, V) = 0 \text{ for any } U \in \mathcal{U}\}$.

The following is Salce’s lemma.

Lemma 2.7 (Cf. [GT, Lem. 2.2.6]). *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\text{Mod } R$. Then the following statements are equivalent:*

- (1) *Any module in $\text{Mod } R$ has a special \mathcal{U} -precover.*
- (2) *Any module in $\text{Mod } R$ has a special \mathcal{V} -preenvelope.*

In this case, the cotorsion pair $(\mathcal{U}, \mathcal{V})$ is called complete.

Definition 2.8. Let \mathcal{X} be a subcategory of an abelian category \mathcal{E} and $n \geq 1$. If there exists an exact sequence

$$0 \rightarrow N \rightarrow X_0 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow M \rightarrow 0$$

in \mathcal{E} with all X_i in \mathcal{X} , then N is called an *n - \mathcal{X} -syzygy* of M and M is called an *n - \mathcal{X} -cosyzygy* of N .

For subcategories \mathcal{X}, \mathcal{Y} of an abelian category \mathcal{E} and $n \geq 1$, we write

$$\begin{aligned} \Omega_{\mathcal{X}}^n(\mathcal{Y}) &:= \{N \in \mathcal{A} \mid N \text{ is an } n\text{-}\mathcal{X}\text{-syzygy of some object in } \mathcal{Y}\}, \\ \text{co}\Omega_{\mathcal{X}}^n(\mathcal{Y}) &:= \{M \in \mathcal{A} \mid M \text{ is an } n\text{-}\mathcal{X}\text{-cosyzygy of some object in } \mathcal{Y}\}. \end{aligned}$$

In particular, $\Omega_{\mathcal{X}}^0(\mathcal{Y}) = \mathcal{Y} = \text{co}\Omega_{\mathcal{X}}^0(\mathcal{Y})$ and $\Omega_{\mathcal{X}}^{-1}(\mathcal{Y}) = 0 = \text{co}\Omega_{\mathcal{X}}^{-1}(\mathcal{Y})$. For convenience, we write

$$\begin{aligned} \Omega_{\mathcal{A}}^n(S) &:= \Omega_{\mathcal{A}_\omega(S)}^n(\text{Mod } S), & \Omega_{\mathcal{I}_\omega}^n(S) &:= \Omega_{\mathcal{I}_\omega(S)}^n(\text{Mod } S), \\ \Omega_{\mathcal{I}_\omega}^n(R^{\text{op}}) &:= \Omega_{\mathcal{I}_\omega(R^{\text{op}})}^n(\text{Mod } R^{\text{op}}), \\ \text{co}\Omega_{\mathcal{B}}^n(R) &:= \text{co}\Omega_{\mathcal{B}_\omega(R)}^n(\text{Mod } R), & \text{co}\Omega_{\mathcal{F}_\omega}^n(R) &:= \text{co}\Omega_{\mathcal{F}_\omega(R)}^n(\text{Mod } R), \\ \text{co}\Omega_{\mathcal{P}_\omega}^n(R) &:= \text{co}\Omega_{\mathcal{P}_\omega(R)}^n(\text{Mod } R), & \text{co}\Omega_{\mathcal{P}_\omega}^n(S^{\text{op}}) &:= \text{co}\Omega_{\mathcal{P}_\omega(S^{\text{op}})}^n(\text{Mod } S^{\text{op}}). \end{aligned}$$

Lemma 2.9. *We have*

- (1) $\Omega_{\mathcal{I}_\omega}^1(S) = \text{ac}\mathcal{T}_\omega^1(S)$;
- (2) $\text{co}\Omega_{\mathcal{P}_\omega}^1(R) = c\mathcal{T}_\omega^1(R)$.

Proof. (1) By [TH3, Prop. 3.8] we have $\text{ac}\mathcal{T}_\omega^1(S) \subseteq \Omega_{\mathcal{I}_\omega}^1(S)$. Now let $N \in \Omega_{\mathcal{I}_\omega}^1(S)$ and let $f^0: N \rightarrow I^0$ be a monomorphism in $\text{Mod } S$ with $I^0 \in \mathcal{I}_\omega(S)$. Then from the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{f^0} & I^0 \\ \downarrow \mu_N & & \downarrow \mu_{I^0} \\ (\omega \otimes_S N)_* & \xrightarrow{(1_\omega \otimes f^0)_*} & (\omega \otimes_S I^0)_*, \end{array}$$

with μ_{I^0} an isomorphism, we get that μ_N is a monomorphism and $N \in \text{ac}\mathcal{T}_\omega^1(S)$. It implies $\Omega_{\mathcal{I}_\omega}^1(S) \subseteq \text{ac}\mathcal{T}_\omega^1(S)$.

(2) By [TH1, Prop. 3.7] we have $c\mathcal{T}_\omega^1(R) \subseteq \text{co}\Omega_{\mathcal{P}_\omega}^1(R)$. Now let $M \in \text{co}\Omega_{\mathcal{P}_\omega}^1(R)$ and let $f_0: W_0 \twoheadrightarrow M$ be an epimorphism in $\text{Mod } R$ with $W_0 \in \mathcal{P}_\omega(R)$. Then from the commutative diagram

$$\begin{array}{ccc} \omega \otimes_S W_{0*} & \xrightarrow{1_\omega \otimes f_{0*}} & \omega \otimes_S M_* \\ \downarrow \theta_{W_0} & & \downarrow \theta_M \\ W_0 & \xrightarrow{f_0} & M, \end{array}$$

with θ_{W_0} an isomorphism, we get that θ_M is an epimorphism and $M \in c\mathcal{T}_\omega^1(R)$. It implies $\text{co}\Omega_{\mathcal{P}_\omega}^1(R) \subseteq c\mathcal{T}_\omega^1(R)$. \square

Let \mathcal{C}, \mathcal{E} be abelian categories and $\Delta: \mathcal{C} \rightarrow \mathcal{E}$ a functor. Recall that a sequence \mathbb{T} in \mathcal{C} is called Δ -exact if $\Delta(\mathbb{T})$ is exact in \mathcal{E} .

§3. (Strong) cograde conditions and double homological functors

In this section we investigate when $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms and $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(-, \omega))$ preserves monomorphisms in terms of the (strong) cograde conditions of modules.

§3.1. Cograde conditions

We begin with the following lemma.

Lemma 3.1. (1) *Let $M \in \text{Mod } R$ with the minimal injective resolution as (2.1).*

Then there exists an exact sequence

$$(3.1) \quad 0 \rightarrow \text{Ext}_R^1(\omega, M) \xrightarrow{\lambda} \text{cTr}_\omega M \xrightarrow{\pi} I^1(M)_*/\text{co}\Omega^1(M)_* \rightarrow 0$$

in $\text{Mod } S$ such that $1_\omega \otimes \pi$ is an isomorphism.

(2) *Let $N \in \text{Mod } S$ with the minimal flat resolution as (2.2). Then there exists an exact sequence*

$$(3.2) \quad 0 \rightarrow \text{Im}(1_\omega \otimes f_1) \xrightarrow{\sigma} \text{acTr}_\omega N \xrightarrow{\tau} \text{Tor}_1^S(\omega, N) \rightarrow 0$$

in $\text{Mod } R$ such that σ_ is an isomorphism.*

Proof. (1) Let $g^0 = \alpha \cdot \beta$ (where $\beta: I^0(M) \rightarrow \text{co}\Omega^1(M) (= \text{Im } g^0)$ and $\alpha: \text{co}\Omega^1(M) \rightarrow I^1(M)$) be the natural epic-monic decomposition of g^0 . Then we have the commutative diagram with exact columns and rows

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & M_* & \longrightarrow & I^0(M)_* & \xrightarrow{\beta_*} & \text{co}\Omega^1(M)_* & \longrightarrow & \text{Ext}_R^1(\omega, M) & \longrightarrow & 0 \\
 & & \parallel & & \parallel & & \downarrow \alpha_* & & \downarrow \lambda & & \\
 0 & \longrightarrow & M_* & \longrightarrow & I^0(M)_* & \xrightarrow{g^0_*} & I^1(M)_* & \xrightarrow{\gamma} & \text{cTr}_\omega M & \longrightarrow & 0 \\
 & & & & \downarrow \pi_1 & & \downarrow \pi & & \downarrow \pi & & \\
 & & & & C & \xlongequal{\quad} & C & & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & & &
 \end{array}$$

in $\text{Mod } S$, where $C = I^1(M)_*/\text{co}\Omega^1(M)_*$, π_1 is the natural epimorphism, λ and π are induced homomorphisms. The rightmost column in the above diagram is

exactly the exact sequence (3.1). Notice that

$$0 \rightarrow \text{co}\Omega^1(M)_* \xrightarrow{\alpha_*} I^1(M)_* \xrightarrow{g^1_*} I^2(M)_*$$

is exact, so there exists a homomorphism $\delta: C \rightarrow I^2(M)_*$ in $\text{Mod } S$ such that $g^1_* = \delta \cdot \pi_1$, and hence $g^1_* = \delta \cdot \pi_1 = \delta \cdot \pi \cdot \gamma$.

By [HW, Lem. 4.1], for any injective module $I \in \text{Mod } R$ we have $\omega \otimes_S I_* \cong I$ canonically. So the upper row in the commutative diagram

$$\begin{array}{ccccc} \omega \otimes_S I^0(M)_* & \xrightarrow{1_\omega \otimes g^0_*} & \omega \otimes_S I^1(M)_* & \xrightarrow{1_\omega \otimes g^1_*} & \omega \otimes_S I^2(M)_* \\ & & \downarrow 1_\omega \otimes \gamma & & \uparrow 1_\omega \otimes \delta \\ & & \omega \otimes_S \text{cTr}_\omega M & \xrightarrow{1_\omega \otimes \pi} & \omega \otimes_S C \end{array}$$

is exact. Let $x \in \text{Ker}(1_\omega \otimes \pi)$. Then there exists $y \in \omega \otimes_S I^1(M)_*$ such that $x = (1_\omega \otimes \gamma)(y)$. It follows that

$$(1_\omega \otimes g^1_*)(y) = (1_\omega \otimes \delta) \cdot (1_\omega \otimes \pi) \cdot (1_\omega \otimes \gamma)(y) = (1_\omega \otimes \delta) \cdot (1_\omega \otimes \pi)(x) = 0.$$

So $y \in \text{Ker}(1_\omega \otimes g^1_*) = \text{Im}(1_\omega \otimes g^0_*)$, and hence there exists $z \in \omega \otimes_S I^0(M)_*$ such that $y = (1_\omega \otimes g^0_*)(z)$. Thus

$$x = (1_\omega \otimes \gamma)(y) = (1_\omega \otimes \gamma) \cdot (1_\omega \otimes g^0_*)(z) = (1_\omega \otimes (\gamma \cdot g^0_*))(z) = 0,$$

which implies that $1_\omega \otimes \pi$ is a monomorphism, and hence an isomorphism.

(2) Let $f_0 = \alpha' \cdot \beta'$ (where $\beta': F_1(N) \rightarrow \Omega^1_{\mathcal{F}}(N) (= \text{Im } f_0)$ and $\alpha': \Omega^1_{\mathcal{F}}(N) \rightarrow F_0(N)$) be the natural epic-monic decomposition of f_0 . Then we have the commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \text{Im}(1_\omega \otimes f_1) & \xlongequal{\quad} & \text{Im}(1_\omega \otimes f_1) & & & & \\ & & \downarrow \sigma & & \downarrow \sigma_1 & & & & \\ 0 & \longrightarrow & \text{acTr}_\omega N & \xrightarrow{\eta} & \omega \otimes_S F_1(N) & \xrightarrow{1_\omega \otimes f_0} & \omega \otimes_S F_0(N) & \longrightarrow & \omega \otimes_S N & \longrightarrow & 0 \\ & & \downarrow \tau & & \downarrow 1_\omega \otimes \beta' & & \parallel & & \parallel & & \\ 0 & \longrightarrow & \text{Tor}_1^S(\omega, N) & \longrightarrow & \omega \otimes_S \Omega^1_{\mathcal{F}}(N) & \xrightarrow{1_\omega \otimes \alpha'} & \omega \otimes_S F_0(N) & \xrightarrow{\gamma} & \omega \otimes_S N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & & & \\ & & 0 & & 0 & & & & & & \end{array}$$

in $\text{Mod } R$, where σ and τ are induced homomorphisms. The leftmost column in the above diagram is exactly the exact sequence (3.2). Notice that

$$\omega \otimes_S F_2(N) \xrightarrow{1_\omega \otimes f_1} \omega \otimes_S F_1(N) \xrightarrow{1_\omega \otimes \beta'} \omega \otimes_S \Omega_{\mathcal{F}}^1(N) \rightarrow 0$$

is exact, so there exists a homomorphism $\phi: \omega \otimes_S F_2(N) \rightarrow \text{Im}(1_\omega \otimes f_1)$ in $\text{Mod } R$ such that $1_\omega \otimes f_1 = \sigma_1 \cdot \phi$, and hence $1_\omega \otimes f_1 = \sigma_1 \cdot \phi = \eta \cdot \sigma \cdot \phi$.

By [HW, Lem. 4.1], for any flat module $F \in \text{Mod } S$ we have $F \cong (\omega \otimes_S F)_*$ canonically. So the upper row in the following commutative diagram is exact:

$$\begin{array}{ccccc} (\omega \otimes_S F_2(N))_* & \xrightarrow{(1_\omega \otimes f_1)_*} & (\omega \otimes_S F_1(N))_* & \xrightarrow{(1_\omega \otimes f_0)_*} & (\omega \otimes_S F_0(N))_* \\ \downarrow \phi_* & & \uparrow \eta_* & & \\ (\text{Im}(1_\omega \otimes f_1))_* & \xrightarrow{\sigma_*} & (\text{acTr}_\omega N)_* & & \end{array}$$

Let $x \in (\text{acTr}_\omega N)_*$. Since $((1_\omega \otimes f_0)_* \cdot \eta_*)(x) = (((1_\omega \otimes f_0) \cdot \eta)_*)(x) = 0$, we have that $\eta_*(x) \in \text{Ker}(1_\omega \otimes f_0)_* = \text{Im}(1_\omega \otimes f_1)_*$ and there exists $y \in (\omega \otimes_S F_2(N))_*$ such that $\eta_*(x) = (1_\omega \otimes f_1)_*(y)$. Thus

$$\eta_*(x) = (1_\omega \otimes f_1)_*(y) = (\eta_* \cdot \sigma_* \cdot \phi_*)(y).$$

As η_* is monic we have $x = \sigma_*(\phi_*(y))$. It means that σ_* is an epimorphism, and hence an isomorphism. \square

The following two lemmas are useful in this section.

Lemma 3.2. *Assume that $\text{co}\Omega^n(R) \subseteq c\mathcal{T}_\omega^m(R)$ with $m, n \geq 0$. Then the following statements are equivalent:*

- (1) $\text{T-cograde}_\omega \text{Ext}_R^{n+1}(\omega, M) \geq m$ for any $M \in \text{Mod } R$.
- (2) $\text{co}\Omega^{n+1}(R) \subseteq c\mathcal{T}_\omega^{m+1}(R)$.

Proof. Because any injective module in $\text{Mod } R$ is in $c\mathcal{T}_\omega^1(R)$ by [TH1, Lem. 2.5(2)], we have $\text{co}\Omega^{n+1}(R) \subseteq c\mathcal{T}_\omega^1(R)$ for any $n \geq 0$, and the case for $m = 0$ follows. Now suppose that $m \geq 1$ and $M \in \text{Mod } R$. By Lemma 3.1(1) there exists an exact sequence

$$0 \rightarrow \text{Ext}_R^1(\omega, \text{co}\Omega^n(M)) \xrightarrow{\lambda} c\text{Tr}_\omega \text{co}\Omega^n(M) \xrightarrow{\pi} C \rightarrow 0$$

in $\text{Mod } S$ such that $1_\omega \otimes \pi$ is an isomorphism, where $C = I^{n+1}(M)_*/\text{co}\Omega^{n+1}(M)_*$. Because $\text{co}\Omega^n(R) \subseteq c\mathcal{T}_\omega^m(R)$ by assumption, we have that both $c\text{Tr}_\omega \text{co}\Omega^n(M)$ and $c\text{Tr}_\omega \text{co}\Omega^{n+1}(M)$ are in $\omega_S^\top{}^m$. It yields that

$$\text{Tor}_i^S(\omega, \text{Ext}_R^{n+1}(\omega, M)) \cong \text{Tor}_i^S(\omega, \text{Ext}_R^1(\omega, \text{co}\Omega^n(M))) \cong \text{Tor}_{i+1}^S(\omega, C)$$

for any $0 \leq i \leq m - 1$. In addition, we also have an exact sequence

$$0 \rightarrow C \rightarrow I^{n+2}(M)_* \rightarrow \text{cTr}_\omega \text{co}\Omega^{n+1}(M) \rightarrow 0$$

in $\text{Mod } S$. By [HW, Cor. 6.1] we have $I^{n+2}(M)_* \in \omega_S^\top$. So

$$\text{Tor}_i^S(\omega, \text{Ext}_R^{n+1}(\omega, M)) \cong \text{Tor}_{i+1}^S(\omega, C) \cong \text{Tor}_{i+2}^S(\omega, \text{cTr}_\omega \text{co}\Omega^{n+1}(M))$$

for any $0 \leq i \leq m - 1$. Thus we conclude that $\text{Tor}_{0 \leq i \leq m-1}^S(\omega, \text{Ext}_R^{n+1}(\omega, M)) = 0$ if and only if $\text{cTr}_\omega \text{co}\Omega^{n+1}(M) \in \omega_S^{\top m+1}$, and if and only if $\text{co}\Omega^{n+1}(M) \in \text{cT}_\omega^{m+1}(R)$. The proof is finished. \square

Lemma 3.3. *Assume that $\Omega_{\mathcal{F}}^n(S) \subseteq \text{acT}_\omega^m(S)$ with $m, n \geq 0$. Then the following statements are equivalent:*

- (1) E-cograde $_\omega \text{Tor}_{n+1}^S(\omega, N) \geq m$ for any $N \in \text{Mod } S$.
- (2) $\Omega_{\mathcal{F}}^{n+1}(S) \subseteq \text{acT}_\omega^{m+1}(S)$.

Proof. Because any flat module in $\text{Mod } S$ is in $\text{acT}_\omega^1(S)$ by [TH3, Cor. 3.5(1)], we have $\Omega_{\mathcal{F}}^{n+1}(S) \subseteq \text{acT}_\omega^1(S)$ for any $n \geq 0$, and the case for $m = 0$ follows. Now suppose that $m \geq 1$ and $N \in \text{Mod } S$. By Lemma 3.1(2), there exists an exact sequence

$$0 \rightarrow \text{Im}(1_\omega \otimes f_{n+1}) \xrightarrow{\sigma} \text{acTr}_\omega \Omega_{\mathcal{F}}^n(N) \xrightarrow{\tau} \text{Tor}_1^S(\omega, \Omega_{\mathcal{F}}^n(N)) \rightarrow 0$$

in $\text{Mod } R$ such that σ_* is an isomorphism. Because $\Omega_{\mathcal{F}}^n(S) \subseteq \text{acT}_\omega^m(S)$ by assumption, we have that both $\text{acTr}_\omega \Omega_{\mathcal{F}}^n(N)$ and $\text{acTr}_\omega \Omega_{\mathcal{F}}^{n+1}(N)$ are in ${}_R\omega^{\perp m}$. It yields that

$$\text{Ext}_R^i(\omega, \text{Tor}_{n+1}^S(\omega, N)) \cong \text{Ext}_R^i(\omega, \text{Tor}_1^S(\omega, \Omega_{\mathcal{F}}^n(N))) \cong \text{Ext}_R^{i+1}(\omega, \text{Im}(1_\omega \otimes f_{n+1}))$$

for any $0 \leq i \leq m - 1$. In addition, we also have an exact sequence

$$0 \rightarrow \text{acTr}_\omega \Omega_{\mathcal{F}}^{n+1}(N) \rightarrow \omega \otimes_S F_{n+2}(N) \rightarrow \text{Im}(1_\omega \otimes f_{n+1}) \rightarrow 0$$

in $\text{Mod } R$. By [HW, Cor. 6.1] we have $\omega \otimes_S F_{n+2}(N) \in {}_R\omega^\perp$. So

$$\text{Ext}_R^i(\omega, \text{Tor}_{n+1}^S(\omega, N)) \cong \text{Ext}_R^{i+1}(\omega, \text{Im}(1_\omega \otimes f_{n+1})) \cong \text{Ext}_R^{i+2}(\omega, \text{acTr}_\omega \Omega_{\mathcal{F}}^{n+1}(N))$$

for any $0 \leq i \leq m - 1$. Thus we conclude that $\text{Ext}_R^{0 \leq i \leq m-1}(\omega, \text{Tor}_{n+1}^S(\omega, N)) = 0$ if and only if $\text{acTr}_\omega \Omega_{\mathcal{F}}^{n+1}(N) \in {}_R\omega^{\perp m+1}$, and if and only if $\Omega_{\mathcal{F}}^{n+1}(N) \in \text{acT}_\omega^{m+1}(S)$. The proof is finished. \square

Let $\mathcal{T} \subseteq \mathcal{W}$ be subcategories of an abelian category \mathcal{E} . Recall that \mathcal{T} is called a *generator* (resp. *cogenerator*) for \mathcal{W} if for any $W \in \mathcal{W}$, there exists an exact

sequence

$$0 \rightarrow W' \rightarrow T \rightarrow W \rightarrow 0 \quad (\text{resp. } 0 \rightarrow W \rightarrow T \rightarrow W' \rightarrow 0)$$

in \mathcal{E} with $T \in \mathcal{T}$ and $W' \in \mathcal{W}$.

Lemma 3.4. (1) $\mathcal{P}_\omega(R)$ is a generator for $\mathcal{B}_\omega(R)$.

(2) $\text{co}\Omega^n(R) \subseteq \text{co}\Omega_{\mathcal{B}}^n(R) = \text{co}\Omega_{\mathcal{F}_\omega}^n(R) = \text{co}\Omega_{\mathcal{P}_\omega}^n(R)$ for any $n \geq 1$.

Proof. (1) Let $M \in \mathcal{B}_\omega(R)$. Then by [TH1, Thm. 3.9 & Prop. 3.7] there exists an exact sequence

$$\cdots \rightarrow W_2 \rightarrow W_1 \rightarrow W_0 \rightarrow M \rightarrow 0$$

in $\text{Mod } R$ with all $W_i \in \mathcal{P}_\omega(R)$ such that it remains exact after applying the functor $\text{Hom}_R(\omega, -)$. Put $M_1 := \text{Im}(W_1 \rightarrow W_0)$. Then $M_1 \in \text{c}\mathcal{T}_\omega(R)$ by [TH1, Prop. 3.7]. Because both M and W_0 are in ${}_R\omega^\perp$, we have $M_1 \in {}_R\omega^\perp$. So $M_1 \in \mathcal{B}_\omega(R)$ by [TH1, Thm. 3.9].

(2) Let $n \geq 1$. By [HW, Lem. 4.1] we have that $\mathcal{B}_\omega(R)$ contains all injective left R -modules, which yields $\text{co}\Omega^n(R) \subseteq \text{co}\Omega_{\mathcal{B}}^n(R)$. Because $\mathcal{B}_\omega(R) \supseteq \mathcal{F}_\omega(R) \supseteq \mathcal{P}_\omega(R)$ by [HW, Cor. 6.1], we have $\text{co}\Omega_{\mathcal{B}}^n(R) \supseteq \text{co}\Omega_{\mathcal{F}_\omega}^n(R) \supseteq \text{co}\Omega_{\mathcal{P}_\omega}^n(R)$. Because $\mathcal{B}_\omega(R)$ is closed under extensions by [HW, Thm. 6.2], we have $\text{co}\Omega_{\mathcal{B}}^n(R) = \text{co}\Omega_{\mathcal{P}_\omega}^n(R)$ by (1) and [H4, Cor. 5.4(2)]. \square

In the following result, we characterize when the double functor $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms in terms of the Tor-cograde conditions of Ext-modules.

Theorem 3.5. Conditions (1)–(3) below are equivalent for any $n, k \geq 0$. If $k \geq 1$ then (1)–(4) are equivalent.

- (1) T-cograde $_\omega \text{Ext}_R^{i+k}(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (2) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any epimorphism $f: B \rightarrow C$ in $\text{Mod } R$ with $B, C \in \text{co}\Omega_{\mathcal{P}_\omega}^{k+1}(R)$ and $0 \leq i \leq n - 1$.
- (3) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any epimorphism $f: B \rightarrow C$ in $\text{Mod } R$ with $B, C \in \text{co}\Omega^{k+1}(R)$ and $0 \leq i \leq n - 1$.
- (4) $\text{co}\Omega^{i+k}(R) \subseteq \text{c}\mathcal{T}_\omega^{i+1}(R)$ for any $1 \leq i \leq n$.

Proof. By using induction on i , (1) \Leftrightarrow (4) follows from Lemma 3.2.

(1) \Rightarrow (2) Let $f: B \rightarrow C$ be an epimorphism in $\text{Mod } R$ with $B, C \in \text{co}\Omega_{\mathcal{P}_\omega}^{k+1}(R)$. Then $C = \text{co}\Omega_{\mathcal{P}_\omega}^{k+1}(C')$ for some $C' \in \text{Mod } R$. By (1) we have

$$\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, C)) \cong \text{Tor}_i^S(\omega, \text{Ext}_R^{i+k+1}(\omega, C')) = 0$$

for any $1 \leq i \leq n - 1$. Thus $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is epic. In the following we will show that $1_\omega \otimes f_*$ is epic.

If $k \geq 1$ then $\text{co}\Omega_{\mathcal{P}_\omega}^k(R) \subseteq c\mathcal{T}_\omega^1(R)$ by Lemma 2.9(2). So $\text{co}\Omega_{\mathcal{P}_\omega}^{k+1}(R) \subseteq c\mathcal{T}_\omega^2(R)$ by Lemma 3.2, and hence $B, C \in c\mathcal{T}_\omega^2(R)$. It follows that $1_\omega \otimes f_* \cong f$ and $1_\omega \otimes f_*$ is epic.

Now suppose $k = 0$. We have an epimorphism $p: W \rightarrow B$ in $\text{Mod } R$ with $W \in \text{Add}_R \omega$. From the exact sequence

$$0 \rightarrow M_1 \rightarrow W \xrightarrow{f \cdot p} C \rightarrow 0$$

in $\text{Mod } R$ with $M_1 = \text{Ker}(f \cdot p)$, we get the exact sequence

$$W_* \xrightarrow{(f \cdot p)_*} C_* \rightarrow \text{Ext}_R^1(\omega, M_1) \rightarrow 0$$

in $\text{Mod } S$. By (1), $\omega \otimes_S \text{Ext}_R^1(\omega, M_1) = 0$. So $(1_\omega \otimes f_*) \cdot (1_\omega \otimes p_*) = 1_\omega \otimes (f \cdot p)_*$ is epic, which implies that $1_\omega \otimes f_*$ is also epic.

By Lemma 3.4(2), we have (2) \Rightarrow (3).

(3) \Rightarrow (1) Let $M \in \text{Mod } R$. From the exact sequence

$$0 \rightarrow \text{co}\Omega^k(M) \rightarrow I^k(M) \xrightarrow{f} \text{co}\Omega^{k+1}(M) \rightarrow 0$$

in $\text{Mod } R$, we get the exact sequence

$$I^k(M)_* \xrightarrow{f_*} \text{co}\Omega^{k+1}(M)_* \rightarrow \text{Ext}_R^{k+1}(\omega, M) \rightarrow 0$$

in $\text{Mod } S$. Since $1_\omega \otimes f_*$ is an epimorphism by (2), we have $\omega \otimes_S \text{Ext}_R^{k+1}(\omega, M) = 0$ and $\text{T-cograde}_\omega \text{Ext}_R^{k+1}(\omega, M) \geq 1$. In addition, for any $1 \leq i \leq n - 1$,

$$0 = \text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, I^k(M))) \xrightarrow{\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))} \text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, \text{co}\Omega^{k+1}(M)))$$

is epic by (3), so we have

$$\text{Tor}_i^S(\omega, \text{Ext}_R^{i+k+1}(\omega, M)) \cong \text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, \text{co}\Omega^{k+1}(M))) = 0.$$

Thus $\text{T-cograde}_\omega \text{Ext}_R^{i+k+1}(\omega, M) \geq i + 1$ for any $0 \leq i \leq n - 1$. □

Lemma 3.6. (1) $\mathcal{I}_\omega(S)$ is a cogenerator for $\mathcal{A}_\omega(S)$.

(2) $\Omega_{\mathcal{F}}^n(S) \subseteq \Omega_{\mathcal{A}}^n(S) = \Omega_{\mathcal{L}_\omega}^n(S)$ for any $n \geq 1$.

Proof. (1) Let $N \in \mathcal{A}_\omega(S)$. Then by [TH3, Thm. 3.11(1)] there exists an $(\omega \otimes_S -)$ -exact exact sequence

$$0 \rightarrow N \rightarrow U^0 \rightarrow U^1 \rightarrow U^2 \rightarrow \dots$$

in $\text{Mod } S$ with all $U^i \in \mathcal{I}_\omega(S)$. Put $N^1 := \text{Im}(U^0 \rightarrow U^1)$. Then $N^1 \in \text{ac}\mathcal{T}_\omega(S)$ by [TH3, Cor. 3.9]. Because both N and U^0 are in ω_S^\top we have $N^1 \in \omega_S^\top$. So $N^1 \in \mathcal{A}_\omega(S)$ by [TH3, Thm. 3.11(1)] again.

(2) Let $n \geq 1$. By [HW, Lem. 4.1] we have that $\mathcal{A}_\omega(S)$ contains all flat left S -modules, which yields $\Omega_{\mathcal{F}}^n(S) \subseteq \Omega_{\mathcal{A}}^n(S)$. Because $\mathcal{A}_\omega(S)$ is closed under extensions by [HW, Thm. 6.2], we have $\Omega_{\mathcal{A}}^n(S) = \Omega_{\mathcal{I}_\omega}^n(S)$ by (1) and [H4, Cor. 5.4(1)]. \square

In the following result we characterize when the double functor $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, -))$ preserves monomorphisms in terms of the Ext-cograde conditions of Tor-modules.

Theorem 3.7. *Conditions (1)–(3) below are equivalent for any $n, k \geq 0$. If $k \geq 1$ then (1)–(4) are equivalent.*

- (1) $\text{E-cograde}_\omega \text{Tor}_{i+k}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.
- (2) $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any monomorphism $g: B' \rightarrow C'$ in $\text{Mod } S$ with $B', C' \in \Omega_{\mathcal{I}_\omega}^{k+1}(S)$ and $0 \leq i \leq n - 1$.
- (3) $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any monomorphism $g: B' \rightarrow C'$ in $\text{Mod } S$ with $B', C' \in \Omega_{\mathcal{F}}^{k+1}(S)$ and $0 \leq i \leq n - 1$.
- (4) $\Omega_{\mathcal{F}}^{i+k}(S) \subseteq \text{ac}\mathcal{T}_\omega^{i+1}(S)$ for any $1 \leq i \leq n$.

Proof. By using induction on i , (1) \Leftrightarrow (4) follows from Lemma 3.3.

(1) \Rightarrow (2) Let $g: B' \rightarrow C'$ be a monomorphism in $\text{Mod } S$ with $B', C' \in \Omega_{\mathcal{I}_\omega}^{k+1}(S)$. Then $B' = \Omega_{\mathcal{I}_\omega}^{k+1}(B'')$ for some $B'' \in \text{Mod } S$. By (1) we have

$$\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, B')) \cong \text{Ext}_R^i(\omega, \text{Tor}_{i+k+1}^S(\omega, B'')) = 0$$

for any $1 \leq i \leq n - 1$. Thus $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is monic. In the following we will show that $(1_\omega \otimes g)_*$ is monic.

If $k \geq 1$ then $\Omega_{\mathcal{I}_\omega}^k(S) \subseteq \text{ac}\mathcal{T}_\omega^1(S)$ by Lemma 2.9(1). So $\Omega_{\mathcal{I}_\omega}^{k+1}(S) \subseteq \text{ac}\mathcal{T}_\omega^2(S)$ by Lemma 3.3, and hence $B', C' \in \text{ac}\mathcal{T}_\omega^2(S)$. It follows that $(1_\omega \otimes g)_* \cong g$ and $(1_\omega \otimes g)_*$ is monic.

Now suppose $k = 0$. We have a monomorphism $i: C' \rightarrow U$ in $\text{Mod } S$ with $U \in \mathcal{I}_\omega(S)$. From the exact sequence

$$0 \rightarrow B' \xrightarrow{i \cdot g} U \rightarrow L_1 \rightarrow 0$$

in $\text{Mod } S$ with $L_1 = \text{Coker}(i \cdot g)$, we get the exact sequence

$$0 \rightarrow \text{Tor}_1^S(\omega, L_1) \rightarrow \omega \otimes_S B' \xrightarrow{1_\omega \otimes (i \cdot g)} \omega \otimes_S U$$

in $\text{Mod } R$. By (1), $(\text{Tor}_1^S(\omega, L_1))_* = 0$. So $(1_\omega \otimes i)_* \cdot (1_\omega \otimes g)_* = (1_\omega \otimes (i \cdot g))_*$ is monic, which implies that $(1_\omega \otimes g)_*$ is also monic.

By Lemma 3.6(2) we have (2) \Rightarrow (3).

(3) \Rightarrow (1) Let $N \in \text{Mod } S$. From the exact sequence

$$0 \rightarrow \Omega_{\mathcal{F}}^{k+1}(N) \xrightarrow{g} F_k(N) \rightarrow \Omega_{\mathcal{F}}^k(N) \rightarrow 0$$

in $\text{Mod } S$, we get the exact sequence

$$0 \rightarrow \text{Tor}_{k+1}^S(\omega, N) \rightarrow \omega \otimes_S \Omega_{\mathcal{F}}^{k+1}(N) \xrightarrow{1_\omega \otimes g} \omega \otimes_S F_k(N)$$

in $\text{Mod } R$. Since $(1_\omega \otimes g)_*$ is a monomorphism by (2), we have $(\text{Tor}_{k+1}^S(\omega, N))_* = 0$ and $\text{E-cograde}_\omega \text{Tor}_{k+1}^S(\omega, N) \geq 1$. In addition, for any $1 \leq i \leq n - 1$,

$$\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, \Omega_{\mathcal{F}}^{k+1}(N))) \xrightarrow{\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))} \text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, F_k(N))) = 0$$

is monic by (3), so we have

$$\text{Ext}_R^i(\omega, \text{Tor}_{i+k+1}^S(\omega, N)) \cong \text{Ext}_R^i(\omega, \text{Tor}_i^S(\Omega_{\mathcal{F}}^{k+1}(\omega, N))) = 0.$$

Thus we conclude that $\text{E-cograde}_\omega \text{Tor}_{i+k+1}^S(\omega, N) \geq i+1$ for any $0 \leq i \leq n-1$. \square

§3.2. Strong cograde conditions

Compare the following result with Theorem 3.5.

Theorem 3.8. *For any $n \geq 1$ and $k \geq 0$, the following three statements are equivalent:*

- (1) $\text{s.T-cograde}_\omega \text{Ext}_R^{i+k}(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (2) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } R$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i+k-1}(R))$, $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any $0 \leq i \leq n - 1$.

- (3) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } R$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i+k-1}(R))$, $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any $0 \leq i \leq n - 1$.

Moreover, if $k = 0$ then any of the above statements is equivalent to the following one:

(4) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } R$, $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any $0 \leq i \leq n - 1$.

Proof. (1) \Rightarrow (2) Let $A = \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i+k-1}(A'))$ with $A' \in \text{Mod } R$. For any $i \geq 0$, by dimension shifting we have an exact sequence

$$\text{Ext}_R^{i+k}(\omega, A') \xrightarrow{g} \text{Ext}_R^i(\omega, B) \xrightarrow{\text{Ext}_R^i(\omega, f)} \text{Ext}_R^i(\omega, C) \rightarrow \text{Ext}_R^{i+k+1}(\omega, A')$$

in $\text{Mod } S$, which induces exact sequences

$$\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, B)) \xrightarrow{a} \text{Tor}_i^S(\omega, \text{Im}(\text{Ext}_R^i(\omega, f))) \rightarrow \text{Tor}_{i-1}^S(\omega, \text{Ker}(\text{Ext}_R^i(\omega, f)))$$

and

$$\text{Tor}_i^S(\omega, \text{Im}(\text{Ext}_R^i(\omega, f))) \xrightarrow{b} \text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, C)) \rightarrow \text{Tor}_i^S(\omega, \text{Coker}(\text{Ext}_R^i(\omega, f)))$$

in $\text{Mod } R$. Since $\text{Coker}(\text{Ext}_R^i(\omega, f)) \subseteq \text{Ext}_R^{i+k+1}(\omega, A')$, by (1) we have

$$\text{Tor}_i^S(\omega, \text{Coker}(\text{Ext}_R^i(\omega, f))) = 0$$

for any $0 \leq i \leq n - 1$. Moreover, it follows from (1) and the exact sequence

$$0 \rightarrow \text{Ker } g \rightarrow \text{Ext}_R^{i+k}(\omega, A') \rightarrow \text{Ker}(\text{Ext}_R^i(\omega, f)) \rightarrow 0$$

in $\text{Mod } S$ that $\text{Tor}_{i-1}^S(\omega, \text{Ker}(\text{Ext}_R^i(\omega, f))) = 0$ for any $0 \leq i \leq n - 1$. Thus $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f)) = b \cdot a$ is an epimorphism for any $0 \leq i \leq n - 1$.

By Lemma 3.4(2) we have (2) \Rightarrow (3).

(3) \Rightarrow (1) Let $M \in \text{Mod } R$. Fix i ($1 \leq i \leq n$) and an S -submodule L of $\text{Ext}_R^{i+k}(\omega, M)$. Take an epimorphism $a: P \rightarrow L$ in $\text{Mod } S$ with P projective and a' the composition

$$P \xrightarrow{a} L \hookrightarrow \text{Ext}_R^{i+k}(\omega, M).$$

Then a' can be lifted to $b: P \rightarrow \text{co}\Omega^{i+k}(M)_*$. Take the pull-back diagram

$$(3.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{co}\Omega^{i+k-1}(M) & \xrightarrow{d} & X & \xrightarrow{c} & \omega \otimes_S P \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow b' \\ 0 & \longrightarrow & \text{co}\Omega^{i+k-1}(M) & \longrightarrow & I^{i+k-1}(M) & \longrightarrow & \text{co}\Omega^{i+k}(M) \longrightarrow 0, \end{array}$$

where b' is the composition

$$\omega \otimes_S P \xrightarrow{1_\omega \otimes b} \omega \otimes_S \text{co}\Omega^{i+k}(M)_* \xrightarrow{\theta_{\text{co}\Omega^{i+k}(M)}} \text{co}\Omega^{i+k}(M).$$

It induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{co}\Omega^{i+k-1}(M)_* & \xrightarrow{d_*} & X_* & \xrightarrow{c_*} & (\omega \otimes_S P)_* & \longrightarrow & L & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow b'_* & & \downarrow & & \\
 0 & \longrightarrow & \text{co}\Omega^{i+k-1}(M)_* & \longrightarrow & I^{i+k-1}(M)_* & \longrightarrow & \text{co}\Omega^{i+k}(M)_* & \longrightarrow & \text{Ext}_R^{i+k}(\omega, M) & \longrightarrow & 0.
 \end{array}$$

In the following we will proceed by induction on i . Let $i = 1$. Since $1_\omega \otimes c_*$ is epic by (3), we have $\omega \otimes_S L = 0$ and $\text{s.T-cograde}_\omega \text{Ext}_R^{1+k}(\omega, M) \geq 1$.

Assume that statement (1) holds for any $1 \leq i \leq n-1$. Now consider the case for $i = n$. By the induction hypothesis we have $\text{s.T-cograde}_\omega \text{Ext}_R^{i+k}(\omega, M) \geq i$ for any $1 \leq i \leq n-1$ and $\text{s.T-cograde}_\omega \text{Ext}_R^{n+k}(\omega, M) \geq n-1$. Then $\text{co}\Omega^{n+k-1}(M) \in \mathcal{CT}_\omega^{n-1}(R)$ by Lemma 3.2. Because $\omega \otimes_S P \in \mathcal{CT}_\omega^{n-1}(R)$ by [TH1, Prop. 3.7], it follows from [TH5, Lem. 4.3] that X in the (3.3) is in $\mathcal{CT}_\omega^{n-1}(R)$. By [TH1, Prop. 3.7] again, there exist $\text{Hom}_R(\text{Add}_R \omega, -)$ -exact exact sequences

$$0 \rightarrow Y' \rightarrow W'_{n-2} \rightarrow \cdots \rightarrow W'_0 \rightarrow \text{co}\Omega^{n+k-1}(M) \rightarrow 0$$

and

$$0 \rightarrow Y \rightarrow W_{n-2} \rightarrow \cdots \rightarrow W_0 \rightarrow X \rightarrow 0$$

in $\text{Mod } R$ with all W'_j, W_j in $\text{Add}_R \omega$. Then both Y and Y' are in ${}_R\omega^{\perp n-1}$ and we get the commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Y' & \longrightarrow & W'_{n-2} & \longrightarrow & \cdots & \longrightarrow & W'_0 & \longrightarrow & \text{co}\Omega^{n+k-1}(M) & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow & & & & \downarrow & & \downarrow d & & \\
 0 & \longrightarrow & Y & \longrightarrow & W_{n-2} & \longrightarrow & \cdots & \longrightarrow & W_0 & \longrightarrow & X & \longrightarrow & 0.
 \end{array}$$

We can guarantee that g is a monomorphism by adding a direct summand in $\text{Add}_R \omega$ (for example W'_{n-2}) to Y and W_{n-2} . Thus we get an exact sequence

$$0 \rightarrow Y' \xrightarrow{g} Y \xrightarrow{h} Z \rightarrow 0$$

in $\text{Mod } R$ with $Z = \text{Coker } g$. Since

$$\text{Coker}(\text{Ext}_R^{n-1}(\omega, h)) \cong \text{Ker}(\text{Ext}_R^n(\omega, g)) \cong \text{Ker}(\text{Ext}_R^1(\omega, d)) \cong \text{Coker } c_* \cong L$$

we obtain $L \cong \text{Ext}_R^{n-1}(\omega, Z)$. Since $Y' \in \Omega_{\mathcal{P}_\omega}^{n-1}(\text{co}\Omega^{n+k-1}(R))$, by (3) we get that $\text{Tor}_{n-1}^S(\omega, \text{Ext}_R^{n-1}(\omega, h))$ is epic. So $\text{Tor}_{n-1}^S(\omega, L) = 0$ and $\text{s.T-cograde}_\omega \text{Ext}_R^{n+k}(\omega, M) \geq n$.

When $k = 0$ the proof of (3) \Rightarrow (1) \Rightarrow (2) is in fact that of (4) \Leftrightarrow (1) by just removing the first sentence and putting $A' = A$ at the beginning of the proof of (1) \Rightarrow (2). □

Compare the following result with Theorem 3.7.

Theorem 3.9. *For any $n \geq 1$ and $k \geq 0$, the following three statements are equivalent:*

- (1) $\text{s.E-cograde}_\omega \text{Tor}_{i+k}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.
- (2) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{T}_\omega}^{i-1}(\Omega_{\mathcal{T}_\omega}^{i+k-1}(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n-1$.

- (3) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{T}_\omega}^{i-1}(\Omega_{\mathcal{F}}^{i+k-1}(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n-1$.

Moreover, if $k = 0$, then any of the above statements is equivalent to the following one:

- (4) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n-1$.

Proof. (1) \Rightarrow (2) Let $C = \text{co}\Omega_{\mathcal{T}_\omega}^{i-1}(\Omega_{\mathcal{T}_\omega}^{i+k-1}(C'))$ with $C' \in \text{Mod } S$. For any $i \geq 0$, by dimension shifting we have an exact sequence

$$\text{Tor}_{i+k+1}^S(\omega, C') \rightarrow \text{Tor}_i^S(\omega, A) \xrightarrow{\text{Tor}_i^S(\omega, g)} \text{Tor}_i^S(\omega, B) \xrightarrow{f} \text{Tor}_{i+k}^S(\omega, C')$$

in $\text{Mod } R$, which induces exact sequences

$$\text{Ext}_R^i(\omega, \text{Ker}(\text{Tor}_i^S(\omega, g))) \rightarrow \text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, A)) \xrightarrow{a} \text{Ext}_R^i(\omega, \text{Im}(\text{Tor}_i^S(\omega, g)))$$

and

$$\text{Ext}_R^{i-1}(\omega, \text{Coker}(\text{Tor}_i^S(\omega, g))) \rightarrow \text{Ext}_R^i(\omega, \text{Im}(\text{Tor}_i^S(\omega, g))) \xrightarrow{b} \text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, B))$$

in $\text{Mod } S$. Since $\text{Ker}(\text{Tor}_i^S(\omega, g))$ is an R -quotient module of $\text{Tor}_{i+k+1}^S(\omega, C')$, by (1) we have

$$\text{Ext}_R^i(\omega, \text{Ker}(\text{Tor}_i^S(\omega, g))) = 0.$$

Moreover, it follows from (1) and the exact sequence

$$0 \rightarrow \text{Coker}(\text{Tor}_i^S(\omega, g)) \rightarrow \text{Tor}_{i+k}^S(\omega, C') \rightarrow \text{Coker } f \rightarrow 0$$

in $\text{Mod } R$ that $\text{Ext}_R^{i-1}(\omega, \text{Coker}(\text{Tor}_i^S(\omega, g))) = 0$ for any $0 \leq i \leq n - 1$. Thus $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g)) = b \cdot a$ is a monomorphism for any $0 \leq i \leq n - 1$.

By Lemma 3.6(2) we have (2) \Rightarrow (3).

(3) \Rightarrow (1) Let $N \in \text{Mod } S$. Fix i ($1 \leq i \leq n$) and an R -quotient module H of $\text{Tor}_{i+k}^S(\omega, N)$. Take a monomorphism $a: H \hookrightarrow I$ in $\text{Mod } R$ with I injective and a' the composition

$$\text{Tor}_{i+k}^S(\omega, N) \rightarrow H \xrightarrow{a} I.$$

Then a' can be extended to $b: \omega \otimes_S \Omega_{\mathcal{F}}^{i+k}(N) \rightarrow I$. Take the push-out diagram

$$(3.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\mathcal{F}}^{i+k}(N) & \longrightarrow & F_{i+k-1}(N) & \longrightarrow & \Omega_{\mathcal{F}}^{i+k-1}(N) \longrightarrow 0 \\ & & \downarrow b' & & \downarrow & & \parallel \\ 0 & \longrightarrow & I_* & \xrightarrow{c} & Y & \xrightarrow{d} & \Omega_{\mathcal{F}}^{i+k-1}(N) \longrightarrow 0, \end{array}$$

where b' is the composition

$$\Omega_{\mathcal{F}}^{i+k}(N) \xrightarrow{\mu_{\Omega_{\mathcal{F}}^{i+k}(N)}} (\omega \otimes_S \Omega_{\mathcal{F}}^{i+k}(N))_* \xrightarrow{b_*} I_*.$$

It induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Tor}_{i+k}^S(\omega, N) & \longrightarrow & \omega \otimes_S \Omega_{\mathcal{F}}^{i+k}(N) & \longrightarrow & \omega \otimes_S F_{i+k-1}(N) \longrightarrow \omega \otimes_S \Omega_{\mathcal{F}}^{i+k-1}(N) \longrightarrow 0 \\ & & \downarrow & & \downarrow 1_\omega \otimes b' & & \downarrow \\ 0 & \longrightarrow & H & \longrightarrow & \omega \otimes_S I_* & \xrightarrow{1_\omega \otimes c} & \omega \otimes_S Y \xrightarrow{1_\omega \otimes d} \omega \otimes_S \Omega_{\mathcal{F}}^{i+k-1}(N) \longrightarrow 0. \end{array}$$

In the following we will proceed by induction on i . Let $i = 1$. Since $(1_\omega \otimes c)_*$ is monic by (2), we have $H_* = 0$ and $\text{s.E-cograde}_\omega \text{Tor}_{1+k}^S(\omega, N) \geq 1$.

Assume that statement (1) holds for any $1 \leq i \leq n - 1$. Now consider the case for $i = n$. By the induction hypothesis we have $\text{s.E-cograde}_\omega \text{Tor}_{i+k}^S(\omega, N) \geq i$ for any $1 \leq i \leq n - 1$ and $\text{s.E-cograde}_\omega \text{Tor}_{n+k}^S(\omega, N) \geq n - 1$. Then $\Omega_{\mathcal{F}}^{n+k-1}(N) \in \text{ac}\mathcal{T}_\omega^{n-1}(S)$ by Lemma 3.3. Because $I_* \in \text{ac}\mathcal{T}_\omega^{n-1}(S)$ by [TH3, Prop. 3.8], it follows from the dual result of [TH5, Lem. 4.3] that Y in diagram (3.4) is in $\text{ac}\mathcal{T}_\omega^{n-1}(S)$. By [TH3, Prop. 3.8] again, there exist $(\omega \otimes_S -)$ -exact exact sequences

$$0 \rightarrow Y \rightarrow U^0 \rightarrow \dots \rightarrow U^{n-2} \rightarrow X \rightarrow 0$$

and

$$0 \rightarrow \Omega_{\mathcal{F}}^{n+k-1}(L) \rightarrow V^0 \rightarrow \dots \rightarrow V^{n-2} \rightarrow X' \rightarrow 0$$

in $\text{Mod } S$ with all U^i, V^i in $\mathcal{I}_\omega(S)$. Then both X and X' are in $\omega_S^\top{}^{n-1}$ and we get the following commutative diagram:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & Y & \longrightarrow & U^0 & \longrightarrow & \dots & \longrightarrow & U^{n-2} & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow & & & & \downarrow & & \downarrow f & & \\
 0 & \longrightarrow & \Omega_{\mathcal{F}}^{n+k-1}(L) & \longrightarrow & V^0 & \longrightarrow & \dots & \longrightarrow & V^{n-2} & \longrightarrow & X' & \longrightarrow & 0.
 \end{array}$$

We can guarantee that f is an epimorphism by adding a direct summand in $\mathcal{I}_\omega(S)$ (for example V^{n-2}) to X and U^{n-2} . Thus we get an exact sequence

$$0 \rightarrow Z \xrightarrow{h} X \xrightarrow{f} X' \rightarrow 0$$

in $\text{Mod } S$ with $Z = \text{Ker } f$. Since

$$\text{Ker}(\text{Tor}_{n-1}^S(\omega, h)) \cong \text{Coker}(\text{Tor}_n^S(\omega, f)) \cong \text{Coker}(\text{Tor}_1^S(\omega, d)) \cong \text{Ker}(1_\omega \otimes c),$$

we obtain $H \cong \text{Tor}_{n-1}^S(\omega, Z)$. Since $X' \in \text{co}\Omega_{\mathcal{I}_\omega}^{n-1}(\Omega_{\mathcal{F}}^{n+k-1}(S))$, by (3) we get that $\text{Ext}_R^{n-1}(\omega, \text{Tor}_{n-1}^S(\omega, h))$ is a monomorphism. So $\text{Ext}_R^{n-1}(\omega, H) = 0$ and $\text{s.E-cograde}_\omega \text{Tor}_{n+k}^S(\omega, N) \geq n$.

When $k = 0$, the proof of (3) \Rightarrow (1) \Rightarrow (2) is in fact that of (4) \Leftrightarrow (1) by just removing the first sentence and putting $C' = C$ at the beginning of the proof of (1) \Rightarrow (2). □

§4. (Quasi) n -cograde condition

In this section, we introduce and study the (quasi) n -cograde condition of semi-dualizing bimodules.

§4.1. The n -cograde condition

Definition 4.1. For any $n \geq 1$, ω is said to satisfy the *right n -cograde condition* if $\text{s.E-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$ and ω is said to satisfy the *left n -cograde condition* if $\text{s.E-cograde}_\omega \text{Tor}_i^R(M', \omega) \geq i$ for any $M' \in \text{Mod } R^{\text{op}}$ and $1 \leq i \leq n$.

As a consequence of Theorems 3.8 and 3.9, we get the following equivalent characterizations for ω satisfying the right n -cograde condition.

Corollary 4.2. For any $n \geq 1$ the following statements are equivalent:

- (1) $\text{s.T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (2) $\text{s.E-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.

- (3) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms in $\text{Mod } R$ for $0 \leq i \leq n - 1$.
- (4) $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, -))$ preserves monomorphisms in $\text{Mod } S$ for $0 \leq i \leq n - 1$.
- (5) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } R$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i-1}(R))$, $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any $0 \leq i \leq n - 1$.

- (6) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } R$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i-1}(R))$, $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any $0 \leq i \leq n - 1$.

- (7) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{L}_\omega}^{i-1}(\Omega_{\mathcal{L}_\omega}^{i-1}(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n - 1$.

- (8) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{L}_\omega}^{i-1}(\Omega_{\mathcal{F}}^{i-1}(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n - 1$.

Proof. By [TH4, Thm. 6.9] we have (1) \Leftrightarrow (2). By Theorems 3.8 and 3.9 we have (1) \Leftrightarrow (3) \Leftrightarrow (5) \Leftrightarrow (6) and (2) \Leftrightarrow (4) \Leftrightarrow (7) \Leftrightarrow (8) respectively. \square

Symmetrically, we have the following equivalent characterizations for ω satisfying the left n -cograde condition.

Corollary 4.3. *For any $n \geq 1$ the following statements are equivalent:*

- (1) s.T-cograde $_\omega \text{Ext}_{S^{\text{op}}}^i(\omega, N') \geq i$ for any $N' \in \text{Mod } S^{\text{op}}$ and $1 \leq i \leq n$.
- (2) s.E-cograde $_\omega \text{Tor}_i^R(M', \omega) \geq i$ for any $M' \in \text{Mod } R^{\text{op}}$ and $1 \leq i \leq n$.
- (3) $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, -), \omega)$ preserves epimorphisms in $\text{Mod } S^{\text{op}}$ for $0 \leq i \leq n - 1$.
- (4) $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(-, \omega))$ preserves monomorphisms in $\text{Mod } R^{\text{op}}$ for $0 \leq i \leq n - 1$.
- (5) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } S^{\text{op}}$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i-1}(S^{\text{op}}))$, $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, f), \omega)$ is an epimorphism for any $0 \leq i \leq n-1$.

(6) For any exact sequence

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

in $\text{Mod } S^{\text{op}}$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^{i-1}(S^{\text{op}}))$, $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, f), \omega)$ is an epimorphism for any $0 \leq i \leq n-1$.

(7) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } R^{\text{op}}$ with $C \in \text{co}\Omega_{\mathcal{T}_\omega}^{i-1}(\Omega_{\mathcal{T}_\omega}^{i-1}(R^{\text{op}}))$, $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(g, \omega))$ is a monomorphism for any $0 \leq i \leq n-1$.

(8) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } R^{\text{op}}$ with $C \in \text{co}\Omega_{\mathcal{T}_\omega}^{i-1}(\Omega_{\mathcal{T}_\omega}^{i-1}(R^{\text{op}}))$, $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(g, \omega))$ is a monomorphism for any $0 \leq i \leq n-1$.

In the following we will establish the left-right symmetry of the n -cograde condition.

Lemma 4.4. *Let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence in $\text{Mod } R$ such that A is superfluous in B . Then the following assertions hold:

- (1) *Let $L \in \text{Mod } R^{\text{op}}$. If $L' \otimes_R C = 0$ for any submodule L' of L , then $L \otimes_R B = 0$.*
- (2) *Let $M \in \text{Mod } R$. If $\text{Hom}_R(C, M') = 0$ for any quotient module M' of M , then $\text{Hom}_R(B, M) = 0$.*

Proof. (1) If $L \otimes_R B \neq 0$ then there exists $x \in L$ such that $xR \otimes_R B \neq 0$. Since $xR \cong R/I$ for some right ideal I of R , we have

$$B/IB \cong R/I \otimes_R B \cong xR \otimes_R B \neq 0$$

and $IB \not\subseteq B$. In view of the assumption that A is superfluous in B , it follows that $IB + A \not\subseteq B$ and

$$xR \otimes_R C \cong R/I \otimes_R C \cong R/I \otimes_R B/A \cong \frac{B/A}{(IB + A)/A} \cong B/(IB + A) \neq 0,$$

which contradicts the assumption.

(2) If $\text{Hom}_R(B, M) \neq 0$ then there exists a nonzero homomorphism $f \in \text{Hom}_R(B, M)$. Pick the kernel L of f such that $\text{Im } f \cong B/L$. Because A is superfluous in B and $f \neq 0$, we have $A + L \not\subseteq B$. Then there exists a nonzero natural epimorphism $\pi: B/A (\cong C) \twoheadrightarrow B/(A + L)$. Note that the inclusions $(A + L)/L \subseteq B/L \subseteq M$ induce an embedding homomorphism

$$i: \frac{B/L}{(A + L)/L} (\cong B/(A + L)) \hookrightarrow \frac{M}{(A + L)/L}.$$

Then $0 \neq i \cdot \pi \in \text{Hom}_R(C, \frac{M}{(A+L)/L})$, which contradicts the assumption. □

It is straightforward to verify the following observation.

Lemma 4.5. (1) *If $P \in \text{Mod } R$ is finitely generated projective then $\text{pd}_{S^{\text{op}}} P^* = \mathcal{P}_\omega(R)\text{-id}_R P$.*

(2) *If $Q \in \text{Mod } S^{\text{op}}$ is finitely generated projective then $\text{pd}_R Q^* = \mathcal{P}_\omega(S^{\text{op}})\text{-id}_{S^{\text{op}}} Q$.*

Lemma 4.6. *Let $P \in \text{Mod } R$ be finitely generated projective and $t \geq 0$. Then the following statements are equivalent:*

- (1) $\text{pd}_{S^{\text{op}}} P^* \leq t$.
- (2) $\mathcal{P}_\omega(R)\text{-id}_R P \leq t$.
- (3) $\text{Ext}_{S^{\text{op}}}^{t+1}(\omega, H) \otimes_R P = 0$ for any $H \in \text{Mod } S^{\text{op}}$.
- (4) $\text{Hom}_R(P, \text{Tor}_{t+1}^S(\omega, N)) = 0$ for any $N \in \text{Mod } S$.

Proof. By Lemma 4.5(1), we have (1) \Leftrightarrow (2).

(1) \Leftrightarrow (3) Let $H \in \text{Mod } S^{\text{op}}$ and

$$\mathbf{I} := 0 \rightarrow H \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^i \rightarrow \dots$$

be an injective resolution of H in $\text{Mod } S^{\text{op}}$. Because $P \in \text{Mod } R$ is finitely generated projective by assumption, the functor $- \otimes_R P$ is exact. Then we have

$$\begin{aligned} \text{Ext}_{S^{\text{op}}}^{t+1}(P^*, H) &\cong H^{t+1}(\text{Hom}_{S^{\text{op}}}(P^*, \mathbf{I})) \\ &\cong H^{t+1}(\text{Hom}_{S^{\text{op}}}(\omega, \mathbf{I}) \otimes_R P) \\ &\cong H^{t+1}(\text{Hom}_{S^{\text{op}}}(\omega, \mathbf{I})) \otimes_R P \quad (\text{by [EJ, p. 30, Exer. 2]}) \\ &\cong \text{Ext}_{S^{\text{op}}}^{t+1}(\omega, H) \otimes_R P. \end{aligned}$$

Now the assertion follows easily.

(1) \Leftrightarrow (4) Since $\text{pd}_{S^{\text{op}}} P^* = \text{fd}_{S^{\text{op}}} P^*$, the assertion follows from [TH2, Lem. 7.6]. □

Recall from [N] that a ring R is called *semiregular* if $R/J(R)$ is von Neumann regular and idempotents can be lifted modulo $J(R)$, where $J(R)$ is the Jacobson radical of R . The class of semiregular rings includes (1) von Neumann regular rings, (2) semiperfect rings, (3) left cotorsion rings and (4) right cotorsion rings. See [GH] for the definitions of left cotorsion rings and right cotorsion rings.

If R is a semiregular ring then any finitely presented left or right R -module has a projective cover by [N, Thm. 2.9]. In this case, since ${}_R\omega$ admits a degreewise finite R -projective resolution by Definition 2.1, we may assume that

$$\cdots \rightarrow P_i(\omega) \rightarrow \cdots \rightarrow P_1(\omega) \rightarrow P_0(\omega) \rightarrow {}_R\omega \rightarrow 0$$

is the minimal projective resolution of ${}_R\omega$ in $\text{mod } R$. Put $\omega_i := \text{Im}(P_i(\omega) \rightarrow P_{i-1}(\omega))$ for any $i \geq 1$ and $\omega_0 := \omega$. Analogously, if S is a semiregular ring then we assume that

$$\cdots \rightarrow Q_i(\omega) \rightarrow \cdots \rightarrow Q_1(\omega) \rightarrow Q_0(\omega) \rightarrow \omega_S \rightarrow 0$$

is the minimal projective resolution of ω_S in $\text{mod } S^{\text{op}}$. By Lemma 4.6 we have the following proposition.

Proposition 4.7. *Let R be a semiregular ring and $m, n \geq 1$. Then the following statements are equivalent:*

- (1) $\text{pd}_{S^{\text{op}}} P_i(\omega)^* \leq m - 1$ for any $0 \leq i \leq n - 1$.
- (2) $\mathcal{P}_\omega(R)\text{-id}_R P_i(\omega) \leq m - 1$ for any $0 \leq i \leq n - 1$.
- (3) $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^m(\omega, N') \geq n$ for any $N' \in \text{Mod } S^{\text{op}}$.
- (4) $\text{s.E-cograde}_\omega \text{Tor}_m^S(\omega, N) \geq n$ for any $N \in \text{Mod } S$.

Proof. By [TH2, Prop. 7.7] and Lemma 4.6, we have (4) \Leftrightarrow (1) \Leftrightarrow (2).

(3) \Rightarrow (1) We proceed by induction on n . Let $N' \in \text{Mod } S^{\text{op}}$. Suppose $n = 1$. Because $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^m(\omega, N') \geq 1$ by (3), we have $L' \otimes_R \omega = 0$ for any submodule L' of $\text{Ext}_{S^{\text{op}}}^m(\omega, N')$ in $\text{Mod } R^{\text{op}}$. It follows from Lemma 4.4(1) that $\text{Ext}_{S^{\text{op}}}^m(\omega, N') \otimes_R P_0(\omega) = 0$. Therefore by Lemma 4.6 we get $\text{pd}_{S^{\text{op}}} P_0(\omega)^* \leq m - 1$ and the case for $n = 1$ is proved.

Now suppose $n \geq 2$. Let X be a submodule of $\text{Ext}_{S^{\text{op}}}^m(\omega, N')$ in $\text{Mod } R^{\text{op}}$. By (3) we have $\text{Tor}_{0 \leq i \leq n-1}^R(X, \omega) = 0$. Then for any $0 \leq i \leq n - 2$ we have

$$\text{Tor}_1^R(X, \omega_i) \cong \text{Tor}_{i+1}^R(X, \omega) = 0.$$

For any $i \geq 0$, from the exact sequence

$$0 \rightarrow \omega_{i+1} \rightarrow P_i(\omega) \rightarrow \omega_i \rightarrow 0$$

we get the exact sequence

$$(4.1) \quad 0 \rightarrow \text{Tor}_1^R(X, \omega_i) \rightarrow X \otimes_R \omega_{i+1} \rightarrow X \otimes_R P_i(\omega).$$

By the induction hypothesis we have $\text{pd}_{S^{\text{op}}} P_i(\omega)^* \leq m - 1$ for any $0 \leq i \leq n - 2$. Then it follows from Lemma 4.6 that $\text{Ext}_{S^{\text{op}}}^m(\omega, N') \otimes_R P_{n-2}(\omega) = 0$ and hence $X \otimes_R P_{n-2}(\omega) = 0$. So it is derived from (4.1) that $X \otimes_R \omega_{n-1} = 0$. Notice that $P_{n-1}(\omega)$ is the projective cover of ω_{n-1} , so $\text{Ext}_{S^{\text{op}}}^m(\omega, N') \otimes_R P_{n-1}(\omega) = 0$ by Lemma 4.4(1). It follows from Lemma 4.6 that $\text{pd}_{S^{\text{op}}} P_{n-1}(\omega)^* \leq m - 1$.

(1) \Rightarrow (3) Let X be a submodule of $\text{Ext}_{S^{\text{op}}}^m(\omega, N')$ in $\text{Mod } R^{\text{op}}$. Then by (1) and Lemma 4.6, we have $\text{Ext}_{S^{\text{op}}}^m(\omega, N') \otimes_R (\bigoplus_{i=0}^{n-1} P_i(\omega)) = 0$, and hence $X \otimes_R (\bigoplus_{i=0}^{n-1} P_i(\omega)) = 0$. Since ω_i is a quotient module of $P_i(\omega)$ for any $i \geq 0$, we then have $X \otimes_R (\bigoplus_{i=0}^{n-1} \omega_i) = 0$.

If $n = 1$ then $X \otimes_R \omega = 0$ and $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^m(\omega, N') \geq 1$. If $n \geq 2$ then from (4.1) we get $\text{Tor}_1^R(X, \bigoplus_{i=0}^{n-2} \omega_i) = 0$. Since $\text{Tor}_{i+1}^R(X, \omega) \cong \text{Tor}_1^R(X, \omega_i)$ for any $i \geq 0$, we have $\text{Tor}_{0 \leq i \leq n-1}^R(X, \omega) = 0$ and $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^m(\omega, N') \geq n$. \square

The following result means that the n -cograde condition is left–right symmetric.

Theorem 4.8. *Let R be semiregular and $n \geq 1$. Then the following statements are equivalent:*

- (1) $\text{pd}_{S^{\text{op}}} P_i(\omega)^* \leq i$ for any $0 \leq i \leq n - 1$.
- (2) $\mathcal{P}_\omega(R)\text{-id}_R P_i(\omega) \leq i$ for any $0 \leq i \leq n - 1$.
- (3) $\text{s.T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (4) $\text{s.E-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.
- (5) $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^i(\omega, N') \geq i$ for any $N' \in \text{Mod } S^{\text{op}}$ and $1 \leq i \leq n$.
- (6) $\text{s.E-cograde}_\omega \text{Tor}_i^R(M', \omega) \geq i$ for any $M' \in \text{Mod } R^{\text{op}}$ and $1 \leq i \leq n$.
- (7) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, -))$ preserves epimorphisms in $\text{Mod } R$ for $0 \leq i \leq n - 1$.
- (8) $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, -))$ preserves monomorphisms in $\text{Mod } S$ for $0 \leq i \leq n - 1$.
- (9) $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, -), \omega)$ preserves epimorphisms in $\text{Mod } S^{\text{op}}$ for $0 \leq i \leq n - 1$.
- (10) $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(-, \omega))$ preserves monomorphisms in $\text{Mod } R^{\text{op}}$ for $0 \leq i \leq n - 1$.

Proof. By Proposition 4.7 we have (1) \Leftrightarrow (2) \Leftrightarrow (4) \Leftrightarrow (5). By Corollaries 4.2 and 4.3 we have (3) \Leftrightarrow (4) \Leftrightarrow (7) \Leftrightarrow (8) and (5) \Leftrightarrow (6) \Leftrightarrow (9) \Leftrightarrow (10). \square

As a consequence, we get the following corollary.

Corollary 4.9. *Let R and S be semiregular and $n \geq 1$. Then the following statements are equivalent:*

- (1) $\text{pd}_{S^{\text{op}}} P_i(\omega)^* \leq i$ for any $0 \leq i \leq n - 1$.
- (2) $\text{pd}_R Q_i(\omega)^* \leq i$ for any $0 \leq i \leq n - 1$.
- (3) $\mathcal{P}_\omega(R)\text{-id}_R P_i(\omega) \leq i$ for any $0 \leq i \leq n - 1$.
- (4) $\mathcal{P}_\omega(S^{\text{op}})\text{-id}_{S^{\text{op}}} Q_i(\omega) \leq i$ for any $0 \leq i \leq n - 1$.

Proof. By the symmetric version of Proposition 4.7 we have

$$(2) \Leftrightarrow (4) \Leftrightarrow \text{s.T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i \quad \text{for any } M \in \text{Mod } R \text{ and } 1 \leq i \leq n.$$

Now the assertion follows from Theorem 4.8. □

§4.2. The quasi n -cograde condition

Definition 4.10. For any $n \geq 1$, ω is said to satisfy the *right quasi n -cograde condition* if $\text{s.E-cograde}_\omega \text{Tor}_{i+1}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$ and ω is said to satisfy the *left quasi n -cograde condition* if $\text{s.E-cograde}_\omega \text{Tor}_{i+1}^R(M', \omega) \geq i$ for any $M' \in \text{Mod } R^{\text{op}}$ and $1 \leq i \leq n$.

It is trivial that ω satisfies the right (resp. left) quasi n -cograde conditions if it satisfies the right (resp. left) n -cograde condition. But the converse does not hold true in general, see Section 4.4 below.

The following lemma is useful in the sequel.

Lemma 4.11. *For any $n \geq 0$, the following assertions hold:*

- (1) *Let $M \in \text{Mod } R$. If $\text{E-cograde}_\omega M \geq n$ and $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n + 1$, then $\text{E-cograde}_\omega M \geq n + 1$.*
- (2) *Let $N \in \text{Mod } S$. If $\text{T-cograde}_\omega N \geq n$ and $\text{E-cograde}_\omega \text{Tor}_n^S(\omega, N) \geq n + 1$, then $\text{T-cograde}_\omega N \geq n + 1$.*

Proof. We proceed by induction on n .

- (1) If $n = 0$ then $\omega \otimes_S M_* = 0$ by assumption. It follows from [TH4, Lem. 6.1(1)] that $M_* = 0$ and $\text{E-cograde}_\omega M \geq 1$.

Let $n \geq 1$. Consider an injective resolution

$$0 \rightarrow M \rightarrow I^0 \rightarrow \dots \rightarrow I^n \rightarrow \dots$$

of M in $\text{Mod } R$. Put $M' = \text{Im}(I^{n-1} \rightarrow I^n)$. Since $\text{E-cograde}_\omega M \geq n$ by the induction hypothesis, applying the functor $(-)_*$ to the above exact sequence yields the exact sequence

$$0 \rightarrow I^0_* \rightarrow \dots \rightarrow I^{n-1}_* \xrightarrow{g} M'_* \rightarrow \text{Ext}_R^n(\omega, M) \rightarrow 0$$

in $\text{Mod } S$. Because $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n + 1$ by assumption, we have $\text{Tor}_{0 \leq i \leq n}^S(\omega, \text{Ext}_R^n(\omega, M)) = 0$. Then by [CE, Prop. VI.5.1] we have

$$\text{Ext}_S^i(\text{Ext}_R^n(\omega, M), I^j) \cong \text{Hom}_R(\text{Tor}_i^S(\omega, \text{Ext}_R^n(\omega, M)), I^j) = 0$$

for any $0 \leq i \leq n$ and $j \geq 0$, and hence

$$\text{Ext}_S^1(\text{Ext}_R^n(\omega, M), \text{Im } g) \cong \text{Ext}_S^n(\text{Ext}_R^n(\omega, M), I^0) = 0.$$

It implies that the exact sequence

$$0 \rightarrow \text{Im } g \rightarrow M'_* \rightarrow \text{Ext}_R^n(\omega, M) \rightarrow 0$$

splits and hence $\text{Ext}_R^n(\omega, M)$ is a direct summand of M'_* . Since M'_* is adjoint 1 - ω -cotorsionfree, so is $\text{Ext}_R^n(\omega, M)$. Thus, applying [TH3, Prop. 3.2], the T-cograde condition on $\text{Ext}_R^n(\omega, M)$ proves $\text{Ext}_R^n(\omega, M) = 0$. Consequently we have $\text{E-cograde}_\omega M \geq n + 1$ and the assertion follows.

(2) If $n = 0$ then $(\omega \otimes_S N)_* = 0$ by assumption. It follows from [TH4, Lem. 6.1(2)] that $\omega \otimes_S N = 0$ and $\text{T-cograde}_\omega N \geq 1$.

Let $n \geq 1$. Consider a projective resolution

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow N \rightarrow 0$$

of N in $\text{Mod } S$. Put $N' = \text{Im}(P_n \rightarrow P_{n-1})$. Since $\text{T-cograde}_\omega N \geq n$ by the induction hypothesis, applying the functor $\omega \otimes_S -$ to the above exact sequence yields the exact sequence

$$0 \rightarrow \text{Tor}_n^S(\omega, N) \rightarrow \omega \otimes_S N' \xrightarrow{f} \omega \otimes_S P_{n-1} \rightarrow \cdots \rightarrow \omega \otimes_S P_0 \rightarrow 0$$

in $\text{Mod } R$. Because $\text{E-cograde}_\omega \text{Tor}_n^S(\omega, N) \geq n + 1$ by assumption, we have $\text{Ext}_R^{0 \leq i \leq n}(\omega, \text{Tor}_n^S(\omega, N)) = 0$. Notice that $\omega \otimes_S P \in \text{Add}_R \omega$ for any projective module P in $\text{Mod } S$, so $\text{Ext}_R^{0 \leq i \leq n}(\omega \otimes_S P_j, \text{Tor}_n^S(\omega, N)) = 0$ for any $j \geq 0$, and hence

$$\text{Ext}_R^1(\text{Im } f, \text{Tor}_n^S(\omega, N)) \cong \text{Ext}_R^n(\omega \otimes_S P_0, \text{Tor}_n^S(\omega, N)) = 0.$$

It induces an exact sequence

$$\text{Hom}_R(\omega \otimes_S N', \text{Tor}_n^S(\omega, N)) \rightarrow \text{Hom}_R(\text{Tor}_n^S(\omega, N), \text{Tor}_n^S(\omega, N)) \rightarrow 0.$$

Because $\omega \otimes_S N' \in \text{cT}_\omega^1(R)$ by [TH4, Lem. 6.1(2)], there exists an epimorphism $U \rightarrow \omega \otimes_S N'$ in $\text{Mod } R$ with $U \in \text{Add}_R \omega$ by [TH1, Lem. 3.6(1)]. Because $(\text{Tor}_n^S(\omega, N))_* = 0$ we have $\text{Hom}_R(U, \text{Tor}_n^S(\omega, N)) = 0$. It follows that $\text{Hom}_R(\omega \otimes_S N', \text{Tor}_n^S(\omega, N)) = 0$ and $\text{Hom}_R(\text{Tor}_n^S(\omega, N), \text{Tor}_n^S(\omega, N)) = 0$, which implies $\text{Tor}_n^S(\omega, N) = 0$. So $\text{T-cograde}_\omega N \geq n + 1$ and the assertion follows. \square

We have the following equivalent characterizations for ω satisfying the right quasi n -cograde condition.

Proposition 4.12. *For any $n \geq 1$, the following statements are equivalent:*

- (1) $\text{s.E-cograde}_\omega \text{Tor}_{i+1}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.
- (2) $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (3) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{T}_\omega}^{i-1}(\Omega_{\mathcal{T}_\omega}^i(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n - 1$.

- (4) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{F}}^{i-1}(\Omega_{\mathcal{F}}^i(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n - 1$.

- (5) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any epimorphism $f: B \rightarrow C$ in $\text{Mod } R$ with $B, C \in \text{co}\Omega_{\mathcal{P}_\omega}^1(R)$ and $0 \leq i \leq n - 1$.
- (6) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any epimorphism $f: B \rightarrow C$ in $\text{Mod } R$ with $B, C \in \text{co}\Omega^1(R)$ and $0 \leq i \leq n - 1$.
- (7) $\text{co}\Omega^i(R) \subseteq \text{c}\mathcal{T}_\omega^{i+1}(R)$ for any $1 \leq i \leq n$.

Proof. By Theorems 3.9 and 3.5 we have (1) \Leftrightarrow (3) \Leftrightarrow (4) and (2) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) respectively. In the following we will prove (1) \Leftrightarrow (2) by induction on n .

(1) \Rightarrow (2) Let $M \in \text{Mod } R$. By Lemma 3.1(1), for any $n \geq 1$ there exist exact sequences

$$(4.2) \quad 0 \rightarrow \text{Ext}_R^n(\omega, M) \xrightarrow{\lambda} \text{cTr}_\omega \text{co}\Omega^{n-1}(M) \xrightarrow{\pi} C \rightarrow 0,$$

$$(4.3) \quad 0 \rightarrow C \rightarrow I^{n+1}(M)_* \rightarrow \text{cTr}_\omega \text{co}\Omega^n(M) \rightarrow 0$$

in $\text{Mod } S$ such that $1_\omega \otimes \pi$ is an isomorphism, where $C = I^n(M)_*/\text{co}\Omega^n(M)_*$. Because $I^{n+1}(M)_* \in \omega_S^\top$ by [HW, Cor. 6.1], it follows from the exact sequence (4.3) that $\text{Tor}_i^S(\omega, C) \cong \text{Tor}_{i+1}^S(\omega, \text{cTr}_\omega \text{co}\Omega^n(M))$ for any $i \geq 1$.

If $n = 1$ then from the exact sequence (4.2) we get an exact sequence

$$\text{Tor}_2^S(\omega, \text{cTr}_\omega \text{co}\Omega^1(M)) (\cong \text{Tor}_1^S(\omega, C)) \rightarrow \omega \otimes_S \text{Ext}_R^1(\omega, M) \rightarrow 0$$

in $\text{Mod } R$. Because $\text{s.E-cograde}_\omega \text{Tor}_2^S(\omega, \text{cTr}_\omega \text{co}\Omega^1(M)) \geq 1$ by assumption, we have $\text{E-cograde}_\omega \omega \otimes_S \text{Ext}_R^1(\omega, M) \geq 1$. It is derived from Lemma 4.11(2) that $\text{T-cograde}_\omega \text{Ext}_R^1(\omega, M) \geq 1$.

Now suppose $n \geq 2$. Then $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i$ for any $1 \leq i \leq n - 1$ and $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n - 1$ by the induction hypothesis. It follows from Theorem 3.5 that $\text{co}\Omega^i(R) \subseteq \text{c}\mathcal{T}_\omega^i(R)$ for any $1 \leq i \leq n$. So $\text{co}\Omega^{n-1}(M) \in \text{c}\mathcal{T}_\omega^{n-1}(R)$, and hence $\text{cTr}_\omega \text{co}\Omega^{n-1}(M) \in \omega_S^{\top n-1}$. Thus from the exact sequences (4.2) and (4.3) we get the exact sequence

$$\text{Tor}_{n+1}^S(\omega, \text{cTr}_\omega \text{co}\Omega^n(M)) \rightarrow \text{Tor}_{n-1}^S(\omega, \text{Ext}_R^n(\omega, M)) \rightarrow 0.$$

By (1) we have $\text{E-cograde}_\omega \text{Tor}_{n-1}^S(\omega, \text{Ext}_R^n(\omega, M)) \geq n$. From Lemma 4.11(2) it follows that $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n$.

(2) \Rightarrow (1) Let $N \in \text{Mod } S$ and X be a quotient module of $\text{Tor}_{n+1}^S(\omega, N)$ in $\text{Mod } R$, and let $\beta: \text{Tor}_1^S(\omega, \Omega_{\mathcal{F}}^n(N)) (\cong \text{Tor}_{n+1}^S(\omega, N)) \rightarrow X$ be an epimorphism in $\text{Mod } R$. By Lemma 3.1(2) we have an exact sequence

$$0 \rightarrow \text{Im}(1_\omega \otimes f_{n+1}) \xrightarrow{\sigma} \text{acTr}_\omega \Omega_{\mathcal{F}}^n(N) \xrightarrow{\tau} \text{Tor}_{n+1}^S(\omega, N) \rightarrow 0$$

in $\text{Mod } R$ such that σ_* is an isomorphism. Then we get an exact sequence

$$(4.4) \quad 0 \rightarrow \text{Ker } f \xrightarrow{\eta} \text{acTr}_\omega \Omega_{\mathcal{F}}^n(N) \xrightarrow{f} X \rightarrow 0$$

in $\text{Mod } R$, where $f = \beta \cdot \tau$. It is easy to see that η_* is an isomorphism.

Let $n = 1$. Because $\Omega_{\mathcal{F}}^1(N) \in \text{ac}\mathcal{T}_\omega^1(S)$ by [TH3, Cor. 3.5(1)], we have $\text{acTr}_\omega \Omega_{\mathcal{F}}^1(N) \in {}_R\omega^{\perp 1}$. Then the exact sequence (4.4) gives $X_* \cong \text{Ext}_R^1(\omega, \text{Ker } f)$. So $\text{T-cograde}_\omega X_* \geq 1$ by (2), and hence $\text{E-cograde}_\omega X \geq 1$ by Lemma 4.11(1). The case for $n = 1$ is proved.

Now suppose $n \geq 2$. Then $\text{s.E-cograde}_\omega \text{Tor}_{i+1}^S(\omega, N) \geq i$ for any $1 \leq i \leq n - 1$ and $\text{s.E-cograde}_\omega \text{Tor}_{n+1}^S(\omega, N) \geq n - 1$ by the induction hypothesis. So $\text{E-cograde}_\omega X \geq n - 1$.

By Theorem 3.7 we have $\Omega_{\mathcal{F}}^i(S) \subseteq \text{ac}\mathcal{T}_\omega^i(S)$ for any $1 \leq i \leq n$. So $\Omega_{\mathcal{F}}^n(N) \in \text{ac}\mathcal{T}_\omega^n(S)$ and $\text{acTr}_\omega \Omega_{\mathcal{F}}^n(N) \in {}_R\omega^{\perp n}$. It follows from the exact sequence (4.4) that $\text{Ext}_R^{n-1}(\omega, X) \cong \text{Ext}_R^n(\omega, \text{Ker } f)$. Then by (2) we have $\text{T-cograde}_\omega \text{Ext}_R^{n-1}(\omega, X) = \text{T-cograde}_\omega \text{Ext}_R^n(\omega, \text{Ker } f) \geq n$. Thus $\text{E-cograde}_\omega X \geq n$ by Lemma 4.11(1). \square

We also have the following proposition.

Proposition 4.13. *For any $n \geq 1$ the following statements are equivalent:*

- (1) $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^{i+1}(\omega, N') \geq i$ for any $N' \in \text{Mod } S^{\text{op}}$ and $1 \leq i \leq n$.
- (2) $\text{E-cograde}_\omega \text{Tor}_i^R(M', \omega) \geq i$ for any $M' \in \text{Mod } R^{\text{op}}$ and $1 \leq i \leq n$.
- (3) For any exact sequence

$$0 \rightarrow A' \rightarrow B' \xrightarrow{f'} C' \rightarrow 0$$

in $\text{Mod } S^{\text{op}}$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega_{\mathcal{P}_\omega}^i(S^{\text{op}}))$, $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, f'), \omega)$ is an epimorphism for any $0 \leq i \leq n - 1$.

(4) For any exact sequence

$$0 \rightarrow A' \rightarrow B' \xrightarrow{f'} C' \rightarrow 0$$

in $\text{Mod } S^{\text{op}}$ with $A \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega^i(S^{\text{op}}))$, $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, f'), \omega)$ is an epimorphism for any $0 \leq i \leq n - 1$.

(5) $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(g', \omega))$ is a monomorphism for any monomorphism $g': B' \rightarrow C'$ in $\text{Mod } R^{\text{op}}$ with $B', C' \in \Omega_{\mathcal{T}_\omega}^1(R^{\text{op}})$ and $0 \leq i \leq n - 1$.

(6) $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(g', \omega))$ is a monomorphism for any monomorphism $g': B' \rightarrow C'$ in $\text{Mod } R^{\text{op}}$ with $B', C' \in \Omega_{\mathcal{F}}^1(R^{\text{op}})$ and $0 \leq i \leq n - 1$.

(7) $\Omega_{\mathcal{F}}^i(R^{\text{op}}) \subseteq \text{ac}\mathcal{T}_\omega^{i+1}(R^{\text{op}})$ for any $1 \leq i \leq n$.

Proof. By the symmetric versions of Theorems 3.8 and 3.7 we have (1) \Leftrightarrow (3) \Leftrightarrow (4) and (2) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) respectively. In the following, we will prove (1) \Leftrightarrow (2) by induction on n .

(1) \Rightarrow (2) Let $M' \in \text{Mod } R^{\text{op}}$ and let

$$\cdots \rightarrow F_{i+1}(M') \xrightarrow{f_i} F_i(M') \rightarrow \cdots \xrightarrow{f_0} F_0(M') \rightarrow M' \rightarrow 0$$

be the minimal flat resolution of M' in $\text{Mod } R^{\text{op}}$. By Lemma 3.1(2), for any $n \geq 1$, there exist exact sequences

$$(4.5) \quad 0 \rightarrow \text{Im}(1_\omega \otimes f_n) \xrightarrow{\sigma} \text{acTr}_\omega \Omega_{\mathcal{F}}^{n-1}(M') \xrightarrow{\tau} \text{Tor}_n^R(M', \omega) \rightarrow 0,$$

$$(4.6) \quad 0 \rightarrow \text{acTr}_\omega \Omega_{\mathcal{F}}^n(M') \rightarrow F_{n+1}(M') \otimes_R \omega \rightarrow \text{Im}(1_\omega \otimes f_n) \rightarrow 0$$

in $\text{Mod } S^{\text{op}}$ such that σ_* is an isomorphism. Because $F_{n+1}(M') \otimes_R \omega \in \omega_S^\perp$ by [HW, Cor. 6.1], it follows from the exact sequence (4.6) that $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Im}(1_\omega \otimes f_n)) \cong \text{Ext}_{S^{\text{op}}}^{i+1}(\omega, \text{acTr}_\omega \Omega_{\mathcal{F}}^n(M'))$ for any $i \geq 1$.

If $n = 1$ then from the exact sequence (4.5) we get an exact sequence

$$0 \rightarrow (\text{Tor}_1^R(M', \omega))_* \rightarrow \text{Ext}_{S^{\text{op}}}^2(\omega, \text{acTr}_\omega \Omega_{\mathcal{F}}^1(M')) (\cong \text{Ext}_{S^{\text{op}}}^1(\omega, \text{Im}(1_\omega \otimes f_n)))$$

in $\text{Mod } R^{\text{op}}$. Because $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^2(\omega, \text{acTr}_\omega \Omega_{\mathcal{F}}^1(M')) \geq 1$ by assumption, we have $\text{T-cograde}_\omega (\text{Tor}_1^R(M', \omega))_* \geq 1$. It is derived from Lemma 4.11(1) that $\text{E-cograde}_\omega \text{Tor}_1^R(M', \omega) \geq 1$.

Now suppose $n \geq 2$. Then $\text{E-cograde}_\omega \text{Tor}_i^R(M', \omega) \geq i$ for any $1 \leq i \leq n - 1$ and $\text{E-cograde}_\omega \text{Tor}_n^R(M', \omega) \geq n - 1$ by the induction hypothesis. It follows from Theorem 3.7 that $\Omega_{\mathcal{F}}^i(R^{\text{op}}) \subseteq \text{ac}\mathcal{T}_\omega^i(R^{\text{op}})$ for any $1 \leq i \leq n$. So $\Omega_{\mathcal{F}}^{n-1}(M') \in$

$\text{ac}\mathcal{T}_\omega^{n-1}(R^{\text{op}})$ and $\text{acTr}_\omega \Omega_{\mathcal{F}}^{n-1}(M') \in \omega_S^{\perp n-1}$. Thus from the exact sequences (4.5) and (4.6) we get the exact sequence

$$0 \rightarrow \text{Ext}_{S^{\text{op}}}^{n-1}(\omega, \text{Tor}_n^R(M', \omega)) \rightarrow \text{Ext}_{S^{\text{op}}}^{n+1}(\omega, \text{acTr}_\omega \Omega_{\mathcal{F}}^n(M')).$$

By (1) we have $\text{T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^{n-1}(\omega, \text{Tor}_n^R(M', \omega)) \geq n$. From Lemma 4.11(1) it follows that $\text{E-cograde}_\omega \text{Tor}_n^R(M', \omega) \geq n$.

(2) \Rightarrow (1) Let $N' \in \text{Mod } S^{\text{op}}$ and Y be a submodule of $\text{Ext}_{S^{\text{op}}}^{n+1}(\omega, N')$ in $\text{Mod } R^{\text{op}}$, and let $\alpha: Y \hookrightarrow \text{Ext}_{S^{\text{op}}}^1(\omega, \text{co}\Omega^n(N')) (\cong \text{Ext}_{S^{\text{op}}}^{n+1}(\omega, N'))$ be a monomorphism in $\text{Mod } R^{\text{op}}$. By Lemma 3.1(1) we have an exact sequence

$$0 \rightarrow \text{Ext}_{S^{\text{op}}}^{n+1}(\omega, N') \xrightarrow{\lambda} \text{cTr}_\omega \text{co}\Omega^n(N') \xrightarrow{\pi} I^{n+1}(N')_* / \text{co}\Omega^{n+1}(N')_* \rightarrow 0$$

in $\text{Mod } R^{\text{op}}$ such that $\pi \otimes 1_\omega$ is an isomorphism. Then we get an exact sequence

$$(4.7) \quad 0 \rightarrow Y \xrightarrow{g} \text{cTr}_\omega \text{co}\Omega^n(N') \xrightarrow{\rho} \text{Coker } g \rightarrow 0$$

in $\text{Mod } R^{\text{op}}$, where $g = \lambda \cdot \alpha$. It is easy to see that $\rho \otimes 1_\omega$ is an isomorphism.

Let $n = 1$. Because $\text{co}\Omega^1(N') \in \text{c}\mathcal{T}_\omega^1(S^{\text{op}})$ by [TH1, Lem. 2.5(2)], we have $\text{cTr}_\omega \text{co}\Omega^1(N') \in \omega_S^{\perp 1}$. Then the exact sequence (4.7) gives $Y \otimes_R \omega \cong \text{Tor}_1^R(\text{Coker } g, \omega)$. So $\text{E-cograde}_\omega Y \otimes_R \omega \geq 1$ by (2), and hence by Lemma 4.11(2) $\text{T-cograde}_\omega Y \geq 1$. The case for $n = 1$ is proved.

Now suppose $n \geq 2$. Then $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^{i+1}(\omega, N') \geq i$ for any $1 \leq i \leq n - 1$ and $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^{n+1}(\omega, N') \geq n - 1$ by the induction hypothesis. So $\text{T-cograde}_\omega Y \geq n - 1$.

By Theorem 3.5 we have $\text{co}\Omega^i(R^{\text{op}}) \subseteq \text{c}\mathcal{T}_\omega^i(R^{\text{op}})$ for any $1 \leq i \leq n$. So $\text{co}\Omega^n(N') \in \text{c}\mathcal{T}_\omega^n(R^{\text{op}})$ and $\text{cTr}_\omega \text{co}\Omega^n(N') \in {}_R\omega^{\perp n}$. It follows from the exact sequence (4.7) that $\text{Tor}_{n-1}^R(Y, \omega) \cong \text{Tor}_n^R(\text{Coker } g, \omega)$. Then by (2) we have $\text{E-cograde}_\omega \text{Tor}_{n-1}^R(Y, \omega) = \text{E-cograde}_\omega \text{Tor}_n^R(\text{Coker } g, \omega) \geq n$. Thus $\text{T-cograde}_\omega Y \geq n$ by Lemma 4.11(2). \square

Now we are in a position to state the following proposition.

Theorem 4.14. *Let R be semiregular and $n \geq 1$. Then the following statements are equivalent:*

- (1) $\text{pd}_{S^{\text{op}}} P_i(\omega)^* \leq i + 1$ for any $0 \leq i \leq n - 1$.
- (2) $\mathcal{P}_\omega(R)\text{-id}_R P_i(\omega) \leq i + 1$ for any $0 \leq i \leq n - 1$.
- (3) $\text{s.T-cograde}_\omega \text{Ext}_{S^{\text{op}}}^{i+1}(\omega, N') \geq i$ for any $N' \in \text{Mod } S^{\text{op}}$ and $1 \leq i \leq n$.
- (4) $\text{s.E-cograde}_\omega \text{Tor}_{i+1}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.
- (5) $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (6) $\text{E-cograde}_\omega \text{Tor}_i^R(M', \omega) \geq i$ for any $M' \in \text{Mod } R^{\text{op}}$ and $1 \leq i \leq n$.

- (7) $\text{Tor}_i^S(\omega, \text{Ext}_R^i(\omega, f))$ is an epimorphism for any epimorphism $f: B \twoheadrightarrow C$ in $\text{Mod } R$ with $B, C \in \text{co}\Omega^1(R)$ and $0 \leq i \leq n - 1$.
- (8) $\text{Ext}_{S^{\text{op}}}^i(\omega, \text{Tor}_i^R(f', \omega))$ is a monomorphism for any monomorphism $f': B' \hookrightarrow C'$ in $\text{Mod } R^{\text{op}}$ with $B', C' \in \Omega_{\mathcal{F}}^1(R^{\text{op}})$ and $0 \leq i \leq n - 1$.
- (9) For any exact sequence

$$0 \rightarrow A' \rightarrow B' \xrightarrow{g'} C' \rightarrow 0$$

in $\text{Mod } S^{\text{op}}$ with $A' \in \Omega_{\mathcal{P}_\omega}^{i-1}(\text{co}\Omega^i(S^{\text{op}}))$, $\text{Tor}_i^R(\text{Ext}_{S^{\text{op}}}^i(\omega, g'), \omega)$ is an epimorphism for any $0 \leq i \leq n - 1$.

- (10) For any exact sequence

$$0 \rightarrow A \xrightarrow{g} B \rightarrow C \rightarrow 0$$

in $\text{Mod } S$ with $C \in \text{co}\Omega_{\mathcal{L}_\omega}^{i-1}(\Omega_{\mathcal{F}}^i(S))$, $\text{Ext}_R^i(\omega, \text{Tor}_i^S(\omega, g))$ is a monomorphism for any $0 \leq i \leq n - 1$.

- (11) $\text{co}\Omega^i(R) \subseteq c\mathcal{T}_\omega^{i+1}(R)$ for any $1 \leq i \leq n$.
- (12) $\Omega_{\mathcal{F}}^i(R^{\text{op}}) \subseteq \text{ac}\mathcal{T}_\omega^{i+1}(R^{\text{op}})$ for any $1 \leq i \leq n$.

Proof. By Proposition 4.7 we have (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4). By Propositions 4.12 and 4.13 we have (4) \Leftrightarrow (5) \Leftrightarrow (7) \Leftrightarrow (10) \Leftrightarrow (11) and (3) \Leftrightarrow (6) \Leftrightarrow (8) \Leftrightarrow (9) \Leftrightarrow (12) respectively. \square

For the right quasi 1-cograde condition, we have some additional interesting equivalent characterizations.

Proposition 4.15. *Let R be a semiregular ring. Then the following statements are equivalent:*

- (1) $\text{pd}_{S^{\text{op}}} P_0(\omega)^* \leq 1$.
- (2) $\text{s.E-cograde}_\omega \text{Tor}_2^S(\omega, N) \geq 1$ for any $N \in \text{Mod } S$.
- (3) θ_M is a superfluous epimorphism for any $M \in \text{co}\Omega^1(R)$.
- (4) $\mu_{M'}$ is an essential monomorphism for any $M' \in \Omega_{\mathcal{F}}^1(R^{\text{op}})$.

Proof. By Theorem 4.14 we have (1) \Leftrightarrow (2).

(1) \Rightarrow (3) Let $M \in \text{co}\Omega^1(R)$. By [TH1, Lem. 2.5(2)] we have $\text{co}\Omega^1(R) \subseteq c\mathcal{T}_\omega^1(R)$. So $M \in c\mathcal{T}_\omega^1(R)$ and θ_M is an epimorphism. Because $\text{Ker } \theta_M \cong \text{Tor}_2^S(\omega, c\text{Tr}_\omega M)$ by [TH1, Prop. 3.2], we have

$$\text{Hom}_R(P_0(\omega), \text{Ker } \theta_M) \cong \text{Hom}_R(P_0(\omega), \text{Tor}_2^S(\omega, c\text{Tr}_\omega M)) = 0$$

by (1) and Lemma 4.6. It follows easily that $X_* = 0$ for any quotient module X of $\text{Ker } \theta_M$. Let A be a submodule of $\omega \otimes_S M_*$ in $\text{Mod } R$ such that $\text{Ker } \theta_M + A =$

$\omega \otimes_S M_*$. Then $(\text{Ker } \theta_M + A)/A (\cong \text{Ker } \theta_M / (A \cap \text{Ker } \theta_M))$ is isomorphic to a quotient module of $\text{Ker } \theta_M$, and so $((\text{Ker } \theta_M + A)/A)_* = 0$. Since $\omega \otimes_S M_* \in c\mathcal{T}_\omega^1(R)$ by [TH4, Lem. 6.1(2)], $(\text{Ker } \theta_M + A)/A \in c\mathcal{T}_\omega^1(R)$ by [TH1, Cor. 3.8]. It follows that $\theta_{(\text{Ker } \theta_M + A)/A}: \omega \otimes_S ((\text{Ker } \theta_M + A)/A)_* \rightarrow (\text{Ker } \theta_M + A)/A$ is epic and $(\text{Ker } \theta_M + A)/A = 0$. It induces that $A = \text{Ker } \theta_M + A = \omega \otimes_S M_*$ and θ_M is a superfluous epimorphism.

(3) \Rightarrow (2) Let $f: B \rightarrow C$ be an epimorphism in $\text{Mod } R$ with $B, C \in \text{co}\Omega^1(R) (\subseteq c\mathcal{T}_\omega^1(R))$. Then $\theta_C \cdot (1_\omega \otimes f_*) = f \cdot \theta_B$ is epic. Because θ_C is a superfluous epimorphism by (3), it follows from [AF, Cor. 5.15] that $1_\omega \otimes f_*$ is epic. Now the assertion follows from Theorem 4.14.

(1) \Rightarrow (4) Let $M' \in \Omega_{\mathcal{F}}^1(R^{\text{op}})$. By [TH3, Cor. 3.5(1)] we have $\Omega_{\mathcal{F}}^1(R^{\text{op}}) \subseteq \text{ac}\mathcal{T}_\omega^1(R^{\text{op}})$. So $M' \in \text{ac}\mathcal{T}_\omega^1(R^{\text{op}})$ and $\mu_{M'}$ is a monomorphism. Because $\text{Coker } \mu_{M'} \cong \text{Ext}_{S^{\text{op}}}^2(\omega, \text{acTr}_\omega M')$ by [TH3, Prop. 3.2], we have

$$\text{Coker } \mu_{M'} \otimes_R P_0(\omega) \cong \text{Ext}_{S^{\text{op}}}^2(\omega, \text{acTr}_\omega M') \otimes_R P_0(\omega) = 0$$

by (1) and Lemma 4.6. It follows easily that $Y \otimes_R \omega = 0$ for any submodule Y of $\text{Coker } \mu_{M'}$. Let A' be a submodule of $(M' \otimes_R \omega)_*$ in $\text{Mod } R^{\text{op}}$ with $A' \cap M' = 0$. Then $A' \cong A'/A' \cap M' \cong (A' + M')/M'$ is isomorphic to a submodule of $\text{Coker } \mu_{M'}$, and so $A' \otimes_R \omega = 0$. Since $(M' \otimes_R \omega)_* \in \text{ac}\mathcal{T}_\omega^1(R^{\text{op}})$ by [TH4, Lem. 6.1(1)], $A' \in \text{ac}\mathcal{T}_\omega^1(R^{\text{op}})$ by [TH3, Cor. 3.3(1)]. It follows that $\mu_{A'}: A' \rightarrow (A' \otimes_R \omega)_*$ is monic. It induces that $A' = 0$ and $\mu_{M'}$ is an essential monomorphism.

(4) \Rightarrow (2) Let $g: B' \rightarrow C'$ be a monomorphism in $\text{Mod } R^{\text{op}}$ with $B', C' \in \Omega_{\mathcal{F}}^1(R^{\text{op}}) (\subseteq \text{ac}\mathcal{T}_\omega^1(R^{\text{op}}))$. Then $(g \otimes 1_\omega)_* \cdot \mu_{B'} = \mu_{C'} \cdot g$ is monic. Because $\mu_{B'}$ is an essential monomorphism by (4), it follows from [AF, Cor. 5.13] that $(g \otimes 1_\omega)_*$ is monic. Now the assertion follows from Theorem 4.14. \square

§4.3. The equivalence of certain cograde condition of modules

We have the following facts: for the strong Tor-cograde condition of modules in Theorem 3.8(1) and the strong Ext-cograde condition of modules in Theorem 3.9(1), they are equivalent when $k = 0$ by Theorem 4.8, but they are not equivalent when $k = 1$ by Theorem 4.14 and Section 4.4 below. Also from Theorem 4.14 and Section 4.4 below, we know that the Tor-cograde condition of modules in Theorem 3.5(1) and the Ext-cograde condition of modules in Theorem 3.7(1) are not equivalent when $k = 0$. In this subsection we will show that these two cograde conditions of modules are equivalent when $k = 1$.

For any $i \geq 1$, by [TH3, Prop. 3.8] we have $\text{ac}\mathcal{T}_\omega^i(S) \subseteq \Omega_{\mathcal{F}}^i(S)$. The following result characterizes when they are identical.

Proposition 4.16. *For any $n \geq 1$, the following statements are equivalent:*

- (1) $E\text{-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i - 1$ for any $N \in \text{co}\Omega_{\mathcal{A}}^i(S)$ and $1 \leq i \leq n$.
- (2) $E\text{-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i - 1$ for any $N \in \text{co}\Omega_{\mathcal{I}_\omega}^i(S)$ and $1 \leq i \leq n$.
- (3) $\text{ac}\mathcal{T}_\omega^i(S) = \Omega_{\mathcal{A}}^i(S)$ for any $1 \leq i \leq n$.
- (4) $\text{ac}\mathcal{T}_\omega^i(S) = \Omega_{\mathcal{I}_\omega}^i(S)$ for any $1 \leq i \leq n$.

Proof. Because $\mathcal{I}_\omega(S) \subseteq \mathcal{A}_\omega(S)$ we have (1) \Rightarrow (2). By Lemma 3.6(2) we have (3) \Leftrightarrow (4).

(2) \Rightarrow (4) By [TH3, Prop. 3.8] it suffices to prove $\Omega_{\mathcal{I}_\omega}^i(S) \subseteq \text{ac}\mathcal{T}_\omega^i(S)$ for any $1 \leq i \leq n$. We proceed by induction on n . The case for $n = 1$ follows from Lemma 2.9(1).

Now let $N \in \Omega_{\mathcal{I}_\omega}^n(S)$ with $n \geq 2$ and let

$$(4.8) \quad 0 \longrightarrow N \xrightarrow{f^0} I^0 \xrightarrow{f^1} \dots \xrightarrow{f^{n-1}} I^{n-1}$$

be an exact sequence in $\text{Mod } S$ with all I^i in $\mathcal{I}_\omega(S)$. By the induction hypothesis we have $\text{Im } f^1 \in \text{ac}\mathcal{T}_\omega^{n-1}(S)$. Applying the functor $\omega \otimes_S -$ to (4.8) gives an exact sequence

$$(4.9) \quad 0 \rightarrow \text{Tor}_n^S(\omega, \text{Coker } f^{n-1}) \rightarrow \omega \otimes_S N \xrightarrow{1_\omega \otimes f^0} \omega \otimes_S I^0 \rightarrow \omega \otimes_S \text{Im } f^1 \rightarrow 0$$

in $\text{Mod } R$. Set $M := \text{Im}(1_\omega \otimes f^0)$ and let $1_\omega \otimes f^0 := \alpha \cdot \pi$ (where $\pi: \omega \otimes_S N \rightarrow M$ and $\alpha: M \hookrightarrow \omega \otimes_S I^0$) be the natural epic-monic decomposition of $1_\omega \otimes f^0$. Then we have the following commutative diagram with exact rows:

$$(4.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{f^0} & I^0 & \longrightarrow & \text{Im } f^1 \longrightarrow 0 \\ & & \downarrow g & & \downarrow \mu_{I^0} & & \downarrow \mu_{\text{Im } f^1} \\ 0 & \longrightarrow & (M)_* & \xrightarrow{\alpha_*} & (\omega \otimes_S I^0)_* & \longrightarrow & (\omega \otimes_S \text{Im } f^1)_* \longrightarrow \text{Ext}_R^1(\omega, M) \longrightarrow 0. \end{array}$$

Since $\mu_{\text{Im } f^1}$ is a monomorphism by the above argument, it follows from the snake lemma that g is an epimorphism. On the other hand, we have

$$\alpha_* \cdot \pi_* \cdot \mu_N = (\alpha \cdot \pi)_* \cdot \mu_N = (1_\omega \otimes f^0)_* \cdot \mu_N = \mu_{I^0} \cdot f^0 = \alpha_* \cdot g.$$

As α_* is monic we get that $\pi_* \cdot \mu_N = g$ and π_* is epic. Consider the following commutative diagram with exact rows

$$(4.11) \quad \begin{array}{ccccccc} & & N & \xlongequal{\quad} & N & & \\ & & \downarrow \mu_N & & \downarrow g & & \\ 0 & \longrightarrow & (\text{Tor}_n^S(\omega, \text{Coker } f^{n-1}))_* & \longrightarrow & (\omega \otimes_S N)_* & \xrightarrow{\pi_*} & M_* \longrightarrow 0. \end{array}$$

Because $(\text{Tor}_n^S(\omega, \text{Coker } f^{n-1}))_* = 0$ by assumption, we have that π_* is an isomorphism. So μ_N is epic by diagram (4.11), and hence an isomorphism. Thus $N \in \text{ac}\mathcal{T}_\omega^2(S)$ and the case for $n = 2$ follows.

Now suppose $n \geq 3$. By the induction hypothesis we have that $\text{Im } f^1 \in \text{ac}\mathcal{T}_\omega^{n-1}(S)$ and $\mu_{\text{Im } f^1}$ is an isomorphism. So $\text{Ext}_R^1(\omega, M) = 0$ by diagram (4.10). In addition, we have $\omega \otimes_S \text{Im } f^1 \in {}_R\omega^{\perp n-3}$ by [TH3, Cor. 3.3(3)]. Because $\text{E-cograde}_\omega \text{Tor}_n^S(\omega, \text{Coker } f^{n-1}) \geq n - 1$ (by assumption) and $\omega \otimes_S I^0 \in {}_R\omega^\perp$, applying the dimension shifting to (4.9) we obtain $\omega \otimes_S N \in {}_R\omega^{\perp n-2}$. Therefore we conclude that $N \in \text{ac}\mathcal{T}_\omega^n(S)$ by [TH3, Cor. 3.3(3)] again.

(3) \Rightarrow (1) We proceed by induction on n . The case for $n = 1$ is trivial. Let $N \in \text{co}\Omega_{\mathcal{A}}^n(S)$ with $n \geq 2$. Then there exists an exact sequence

$$0 \rightarrow H \rightarrow A_{n-1} \xrightarrow{f} A_{n-2} \rightarrow \cdots \rightarrow A_0 \rightarrow N \rightarrow 0$$

in $\text{Mod } S$ with all A_i in $\mathcal{A}_\omega(S)$. By (3) we have $H \in \text{ac}\mathcal{T}_\omega^n(S)$. By the induction hypothesis we have $\text{E-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i - 1$ for any $1 \leq i \leq n - 1$ and $\text{E-cograde}_\omega \text{Tor}_n^S(\omega, N) \geq n - 2$.

Put $M := \text{Ker}(1_\omega \otimes f)$. Because $A_i \in \text{ac}\mathcal{T}_\omega(S)$ by [TH3, Thm. 3.11(1)], we obtain $M_* \cong H$ ($\in \text{ac}\mathcal{T}_\omega^n(S)$) and $M \in {}_R\omega^{\perp n-2}$. By [TH2, Prop. 5.1] we have the exact sequences

$$(4.12) \quad 0 \rightarrow \text{Tor}_n^S(\omega, N) (\cong \text{Tor}_2^S(\omega, \text{Coker } f)) \rightarrow \omega \otimes_S M_* \xrightarrow{\pi} \text{Im } \theta_M \rightarrow 0,$$

$$(4.13) \quad 0 \rightarrow \text{Im } \theta_M \xrightarrow{\lambda} M \rightarrow \text{Tor}_{n-1}^S(\omega, N) (\cong \text{Tor}_1^S(\omega, \text{Coker } f)) \rightarrow 0$$

such that $\theta_M = \lambda \cdot \pi$. Since μ_{M_*} is an isomorphism, it follows from [TH4, Lem. 6.1(1)] that $(\theta_M)_*$ is also an isomorphism. Then both λ_* and π_* are isomorphisms.

From the exact sequence (4.13) we get $\text{Im } \theta_M \in {}_R\omega^{\perp n-2}$. Because $\omega \otimes_S M_* \in {}_R\omega^{\perp n-2}$ by [TH3, Cor. 3.3], from the exact sequence (4.12) it yields that $\text{Ext}_R^{n-2}(\omega, \text{Tor}_n^S(\omega, N)) = 0$. Thus we have $\text{E-cograde}_\omega \text{Tor}_n^S(\omega, N) \geq n - 1$. \square

For any $i \geq 1$, by [TH1, Prop. 3.7] we have $c\mathcal{T}_\omega^i(R) \subseteq \text{co}\Omega_{\mathcal{P}_\omega}^i(R)$. The following result characterizes when they are identical.

Proposition 4.17. *For any $n \geq 1$ the following statements are equivalent:*

- (1) $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i - 1$ for any $M \in \Omega_{\mathcal{B}}^i(R)$ and $1 \leq i \leq n$.
- (2) $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i - 1$ for any $M \in \Omega_{\mathcal{F}_\omega}^i(R)$ and $1 \leq i \leq n$.
- (3) $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i - 1$ for any $M \in \Omega_{\mathcal{P}_\omega}^i(R)$ and $1 \leq i \leq n$.
- (4) $c\mathcal{T}_\omega^i(R) = \text{co}\Omega_{\mathcal{B}}^i(R)$ for any $1 \leq i \leq n$.

(5) $c\mathcal{T}_\omega^i(R) = \text{co}\Omega_{\mathcal{F}_\omega}^i(R)$ for any $1 \leq i \leq n$.

(6) $c\mathcal{T}_\omega^i(R) = \text{co}\Omega_{\mathcal{P}_\omega}^i(R)$ for any $1 \leq i \leq n$.

Proof. Because $\mathcal{B}_\omega(R) \supseteq \mathcal{F}_\omega(R) \supseteq \mathcal{P}_\omega(R)$, we have (1) \Rightarrow (2) \Rightarrow (3). By Lemma 3.4(2) we have (4) \Leftrightarrow (5) \Leftrightarrow (6).

(3) \Rightarrow (6) By [TH1, Prop. 3.7], it suffices to prove $\text{co}\Omega_{\mathcal{P}_\omega}^i(R) \subseteq c\mathcal{T}_\omega^i(R)$ for any $1 \leq i \leq n$. We proceed by induction on n . The case for $n = 1$ follows from Lemma 2.9(2).

Now let $M \in \text{co}\Omega_{\mathcal{P}_\omega}^n(R)$ with $n \geq 2$ and let

$$(4.14) \quad W_{n-1} \xrightarrow{f_{n-1}} \cdots \rightarrow W_1 \xrightarrow{f_1} W_0 \xrightarrow{f_0} M \rightarrow 0$$

be an exact sequence in $\text{Mod } R$ with all W_i in $\mathcal{P}_\omega(R)$. By the induction hypothesis we have $\text{Im } f_1 \in c\mathcal{T}_\omega^{n-1}(R)$. Applying the functor $(-)_*$ to (4.14) gives an exact sequence

$$(4.15) \quad 0 \rightarrow (\text{Im } f_1)_* \rightarrow W_{0*} \xrightarrow{f_{0*}} M_* \rightarrow \text{Ext}_R^n(\omega, \text{Ker } f_{n-1}) \rightarrow 0.$$

Set $N := \text{Im}(f_{0*})$ and let $f_{0*} := \alpha \cdot \pi$ (where $\pi: W_{0*} \rightarrow N$ and $\alpha: N \hookrightarrow M_*$) be the natural epic-monic decompositions of f_{0*} . Then we have the following commutative diagram with exact rows:

$$(4.16) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \text{Tor}_1^S(\omega, N) & \longrightarrow & \omega \otimes_S (\text{Im } f_1)_* & \longrightarrow & \omega \otimes_S W_{0*} & \xrightarrow{1_\omega \otimes \pi} & \omega \otimes_S N & \longrightarrow & 0 \\ & & & & \downarrow \theta_{\text{Im } f_1} & & \downarrow \theta_{W_0} & & \downarrow g & & \\ 0 & \longrightarrow & \text{Im } f_1 & \longrightarrow & W_0 & \xrightarrow{f_0} & M & \longrightarrow & 0. & & \end{array}$$

So we have

$$\theta_M \cdot (1_\omega \otimes \alpha) \cdot (1_\omega \otimes \pi) = \theta_M \cdot (1_\omega \otimes f_{0*}) = f_0 \cdot \theta_{W_0} = g \cdot (1_\omega \otimes \pi).$$

Because $1_\omega \otimes \pi$ is epic, we have $\theta_M \cdot (1_\omega \otimes \alpha) = g$ and the following commutative diagram with exact rows:

$$(4.17) \quad \begin{array}{ccccccc} \omega \otimes_S N & \xrightarrow{1_\omega \otimes \alpha} & \omega \otimes_S M_* & \longrightarrow & \omega \otimes_S \text{Ext}_R^n(\omega, \text{Ker } f_{n-1}) & \longrightarrow & 0 \\ \downarrow g & & \downarrow \theta_M & & & & \\ M & \xlongequal{\quad} & M & & & & \end{array}$$

Since $\theta_{\text{Im } f_1}$ is an epimorphism by the above argument, it follows from the snake lemma that g is an isomorphism. Thus $1_\omega \otimes \alpha$ is a monomorphism. Because $\omega \otimes_S \text{Ext}_R^n(\omega, \text{Ker } f_{n-1}) = 0$ by assumption, we have that θ_M is an isomorphism and $M \in c\mathcal{T}_\omega^2(R)$ by diagram (4.17). It means that the assertion holds true for $n = 2$.

If $n \geq 3$ then the fact that $\text{Im } f_1 \in c\mathcal{T}_\omega^{n-1}(R)$ implies $\theta_{\text{Im } f_1}$ is an isomorphism. So $\text{Tor}_1^S(\omega, N) = 0$ by diagram (4.16). In addition, we have $(\text{Im } f_1)_* \in \omega_S^{\top n-3}$ by [TH1, Cor. 3.4(3)]. Because $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, \text{Ker } f_{n-1}) \geq n-1$ by assumption, applying the dimension shifting to (4.15) we obtain $M_* \in \omega_S^{\top n-2}$. Therefore we conclude that $M \in c\mathcal{T}_\omega^n(R)$ by [TH1, Cor. 3.4(3)] again.

(4) \Rightarrow (1) We proceed by induction on n . The case for $n = 1$ is trivial. Let $M \in \Omega_{\mathcal{B}}^n(R)$ with $n \geq 2$ and let

$$0 \rightarrow M \rightarrow B_{n-1} \rightarrow \cdots \rightarrow B_1 \xrightarrow{f} B_0 \rightarrow L \rightarrow 0$$

be an exact sequence with all B_i in $\mathcal{B}_\omega(R)$. By (4) we have $L \in c\mathcal{T}_\omega^n(R)$. By the induction hypothesis we have $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i-1$ for any $1 \leq i \leq n-1$ and $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n-2$.

Put $N := c\text{Tr}_\omega \text{Ker } f$. Because $B_i \in c\mathcal{T}_\omega(R)$ by [TH1, Thm. 3.9], we obtain that $\omega \otimes_S N \cong L$ ($\in c\mathcal{T}_\omega^n(R)$) and $N \in \omega_S^{\top n-2}$. By [TH4, Prop. 6.7] we have the exact sequences

$$(4.18) \quad 0 \rightarrow \text{Ext}_R^{n-1}(\omega, M) \rightarrow N \xrightarrow{\pi} \text{Im } \mu_N \rightarrow 0,$$

$$(4.19) \quad 0 \rightarrow \text{Im } \mu_N \xrightarrow{\lambda} (\omega \otimes_S N)_* \rightarrow \text{Ext}_R^n(\omega, M) \rightarrow 0$$

such that $\mu_N = \lambda \cdot \pi$. Since $\theta_{\omega \otimes_S N}$ is an isomorphism, it follows from [TH4, Lem. 6.1(2)] that $1_\omega \otimes \mu_N$ is also an isomorphism. Then both $1_\omega \otimes \lambda$ and $1_\omega \otimes \pi$ are isomorphisms.

From the exact sequence (4.18) we get $\text{Im } \mu_N \in \omega_S^{\top n-2}$. Because $(\omega \otimes_S N)_* \in \omega_S^{\top n-2}$ by [TH1, Cor. 3.4], from the exact sequence (4.19) it yields that $\text{Tor}_{n-2}^S(\omega, \text{Ext}_R^n(\omega, M)) = 0$. Thus we have $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n-1$. \square

Lemma 4.18. *For any $n \geq 0$, the following statements are equivalent:*

- (1) $\omega \otimes \text{Ext}_R^2(\omega, -)$ vanishes on $\text{Mod } R$.
- (2) $(\text{Tor}_2^S(\omega, -))_*$ vanishes on $\text{Mod } S$.
- (3) $M_* \in \text{ac}\mathcal{T}_\omega^2(S)$ for any $M \in \text{Mod } R$.
- (4) $\omega \otimes_S N \in c\mathcal{T}_\omega^2(R)$ for any $N \in \text{Mod } S$.

Proof. By [TH2, Cor. 6.6] we have (3) \Leftrightarrow (4).

(1) \Leftrightarrow (4) Assume that (1) holds true. Let $N \in \text{Mod } S$. By [TH4, Lem. 6.1(2)] we have

$$\theta_{\omega \otimes_S N} \cdot (1_\omega \otimes \mu_N) = 1_{\omega \otimes_S N}.$$

It follows that $\theta_{\omega \otimes_S N}$ is a split epimorphism and

$$\begin{aligned} \text{Ker } \theta_{\omega \otimes_S N} &\cong \text{Coker}(1_\omega \otimes \mu_N) \cong \omega \otimes_S \text{Coker } \mu_N \\ &\cong \omega \otimes_S \text{Ext}_R^2(\omega, \text{acTr}_\omega N) && \text{(by [TH2, Cor. 5.2(2)])} \\ &= 0 && \text{(by (1)).} \end{aligned}$$

So $\theta_{\omega \otimes_S N}$ is a monomorphism, and hence an isomorphism.

Conversely, assume that (4) holds true. Let $M \in \text{Mod } R$. By [TH4, Lem. 6.1(2)] again, we have

$$\theta_{\omega \otimes_S \text{cTr}_\omega M} \cdot (1_\omega \otimes \mu_{\text{cTr}_\omega M}) = 1_{\omega \otimes_S \text{cTr}_\omega M}.$$

It follows that

$$\begin{aligned} \omega \otimes_S \text{Ext}_R^2(\omega, M) &\cong \omega \otimes_S \text{Coker } \mu_{\text{cTr}_\omega M} && \text{(by [TH2, Cor. 5.3(2)])} \\ &\cong \text{Coker}(1_\omega \otimes \mu_{\text{cTr}_\omega M}) \cong \text{Ker } \theta_{\omega \otimes_S \text{cTr}_\omega M} \\ &= 0 && \text{(by (4)).} \end{aligned}$$

(2) \Leftrightarrow (3) Assume that (2) holds true. Let $M \in \text{Mod } R$. By [TH4, Lem. 6.1(1)] we have

$$(\theta_M)_* \cdot \mu_{M_*} = 1_{M_*}.$$

It follows that μ_{M_*} is a split monomorphism and

$$\begin{aligned} \text{Coker } \mu_{M_*} &\cong \text{Ker}(\theta_M)_* \cong (\text{Ker } \theta_M)_* \\ &\cong (\text{Tor}_2^S(\omega, \text{cTr}_\omega M))_* && \text{(by [TH1, Prop. 3.2])} \\ &= 0 && \text{(by (2)).} \end{aligned}$$

So μ_{M_*} is an epimorphism, and hence an isomorphism.

Conversely, assume that (3) holds true. Let $N \in \text{Mod } S$. By [TH4, Lem. 6.1(1)] again, we have

$$(\theta_{\text{acTr}_\omega N})_* \cdot \mu_{(\text{acTr}_\omega N)_*} = 1_{(\text{acTr}_\omega N)_*}.$$

It follows that

$$\begin{aligned} (\text{Tor}_2^S(\omega, N))_* &\cong (\text{Ker } \theta_{\text{acTr}_\omega N})_* && \text{(by [TH2, Cor. 5.3(1)])} \\ &\cong \text{Ker}(\theta_{\text{acTr}_\omega N})_* \cong \text{Coker } \mu_{(\text{acTr}_\omega N)_*} \\ &= 0 && \text{(by (3)).} \quad \square \end{aligned}$$

The following result establishes the left–right symmetry of certain cograde conditions of modules.

Theorem 4.19. *For any $n \geq 1$, the following statements are equivalent:*

- (1) T-cograde $_{\omega}$ Ext $_R^i(\omega, M) \geq i - 1$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n$.
- (2) E-cograde $_{\omega}$ Tor $_i^S(\omega, N) \geq i - 1$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n$.
- (3) $\text{co}\Omega^i(R) \subseteq \text{c}\mathcal{T}_{\omega}^i(R) = \text{co}\Omega_{\mathcal{B}}^i(R)$ for any $1 \leq i \leq n$.
- (4) $\text{co}\Omega^i(R) \subseteq \text{c}\mathcal{T}_{\omega}^i(R) = \text{co}\Omega_{\mathcal{F}_{\omega}}^i(R)$ for any $1 \leq i \leq n$.
- (5) $\text{co}\Omega^i(R) \subseteq \text{c}\mathcal{T}_{\omega}^i(R) = \text{co}\Omega_{\mathcal{P}_{\omega}}^i(R)$ for any $1 \leq i \leq n$.
- (6) $\Omega_{\mathcal{F}}^i(S) \subseteq \text{ac}\mathcal{T}_{\omega}^i(S) = \Omega_{\mathcal{A}}^i(S)$ for any $1 \leq i \leq n$.
- (7) $\Omega_{\mathcal{F}}^i(S) \subseteq \text{ac}\mathcal{T}_{\omega}^i(S) = \Omega_{\mathcal{L}_{\omega}}^i(S)$ for any $1 \leq i \leq n$.

Proof. By Theorem 3.5 and Proposition 4.17 we have (1) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5). By Theorem 3.7 and Proposition 4.16, (2) \Leftrightarrow (6) \Leftrightarrow (7).

In the following, we will prove (1) \Leftrightarrow (2) by induction on n . The case for $n = 1$ is trivial and the case for $n = 2$ follows from Lemma 4.18. Now suppose $n \geq 3$.

(1) \Rightarrow (2) Let $N \in \text{Mod } S$. By the induction hypothesis we have E-cograde $_{\omega}$ Tor $_i^S(\omega, N) \geq i - 1$ for any $1 \leq i \leq n - 1$ and E-cograde $_{\omega}$ Tor $_n^S(\omega, N) \geq n - 2$. By Lemma 3.1(2) there exists an exact sequence

$$0 \rightarrow \text{Im}(f_n \otimes 1_{\omega}) \xrightarrow{\sigma} \text{acTr}_{\omega} \Omega_{\mathcal{F}}^{n-1}(N) \xrightarrow{\tau} \text{Tor}_n^S(\omega, N) \rightarrow 0$$

in Mod R such that σ_* is an isomorphism. By Theorem 3.7 we have $\Omega_{\mathcal{F}}^{n-1}(N) \in \text{ac}\mathcal{T}_{\omega}^{n-1}(S)$ and $\text{acTr}_{\omega} \Omega_{\mathcal{F}}^{n-1}(N) \in {}_R\omega^{\perp n-1}$. So

$$\begin{aligned} \text{Ext}_R^{n-2}(\omega, \text{Tor}_n^S(\omega, N)) &\cong \text{Ext}_R^{n-1}(\omega, \text{Im}(f_n \otimes 1_{\omega})) \\ &\cong \text{Ext}_R^n(\omega, \text{acTr}_{\omega} \Omega_{\mathcal{F}}^n(N)). \end{aligned}$$

Then T-cograde $_{\omega}$ Ext $_R^{n-2}(\omega, \text{Tor}_n^S(\omega, N)) \geq n - 1$ by (1). It follows from Lemma 4.11(1) that E-cograde $_{\omega}$ Tor $_n^S(\omega, N) \geq n - 1$.

(2) \Rightarrow (1) Let $M \in \text{Mod } R$. By the induction hypothesis we have T-cograde $_{\omega}$ Ext $_R^i(\omega, M) \geq i - 1$ for any $1 \leq i \leq n - 1$ and T-cograde $_{\omega}$ Ext $_R^n(\omega, M) \geq n - 2$. By Lemma 3.1(1) there exists an exact sequence

$$0 \rightarrow \text{Ext}_R^{n+1}(\omega, M) \xrightarrow{\lambda} \text{cTr}_{\omega} \text{co}\Omega^n(M) \xrightarrow{\pi} I^{n+1}(M)_*/\text{co}\Omega^{n+1}(M)_* \rightarrow 0$$

in Mod S such that $1_{\omega} \otimes \pi$ is an isomorphism. By Theorem 3.5 we have $\text{co}\Omega^{n-1}(M) \in \text{c}\mathcal{T}_{\omega}^{n-1}(R)$ and $\text{cTr}_{\omega} \text{co}\Omega^{n-1}(M) \in \omega_S^{\perp n-1}$. So

$$\begin{aligned} \text{Tor}_{n-2}^S(\omega, \text{Ext}_R^n(\omega, M)) &\cong \text{Tor}_{n-1}^S(\omega, I^n(M)_*/\text{co}\Omega^n(M)_*) \\ &\cong \text{Tor}_n^S(\omega, \text{cTr}_{\omega} \text{co}\Omega^n(M)). \end{aligned}$$

Then E-cograde $_{\omega}$ Tor $_{n-2}^S(\omega, \text{Ext}_R^n(\omega, M)) \geq n - 1$ by (2). It follows from Lemma 4.11(2) that T-cograde $_{\omega}$ Ext $_R^n(\omega, M) \geq n - 1$. □

§4.4. Examples

In this subsection we give some examples for ω satisfying the (quasi) n -cograde condition.

Let R be an artin algebra. Recall that R is called *Auslander n -Gorenstein* if $\text{pd}_{R^{\text{op}}} I^i(R_R) \leq i$ for any $0 \leq i \leq n - 1$, or equivalently $\text{pd}_R I^i({}_R R) \leq i$ for any $0 \leq i \leq n - 1$ ([FGR, IS]), and R is called *left (resp. right) quasi n -Gorenstein* if $\text{pd}_R I^i({}_R R)$ (resp. $\text{pd}_{R^{\text{op}}} I^i(R_R) \leq i + 1$ for any $0 \leq i \leq n - 1$; [H3]).

Let D be the ordinary duality between $\text{mod } R$ and $\text{mod } R^{\text{op}}$. Then $D(R)$ is a semidualizing (R, R) -bimodule. Because

$$\text{pd}_R I^i({}_R R) = \text{id}_{R^{\text{op}}} P_i(D({}_R R)) = \text{pd}_R \text{Hom}_{R^{\text{op}}}(P_i(D({}_R R)), D(R))$$

and

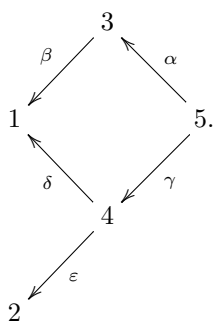
$$\text{pd}_{R^{\text{op}}} I^i(R_R) = \text{id}_R P_i(D(R_R)) = \text{pd}_{R^{\text{op}}} \text{Hom}_R(P_i(D(R_R)), D(R)),$$

we have the following example.

- Example 4.20.** (1) R is Auslander n -Gorenstein if and only if $D(R)$ satisfies the n -cograde condition.
 (2) R is left (resp. right) quasi n -Gorenstein if and only if $D(R)$ satisfies the left (resp. right) quasi n -cograde condition.

So, putting ${}_R \omega_S = {}_R D(R)_R$ in Theorem 4.8 (resp. Theorem 4.14), then all the conditions there are equivalent to R being Auslander n -Gorenstein (resp. right quasi n -Gorenstein). Note that the notion of quasi n -Gorenstein algebras is not left–right symmetric ([AR1, p.11]). So contrary to the n -cograde condition, the quasi n -cograde condition is not left–right symmetric.

Example 4.21. Let Q be the quiver



and $R = KQ/\langle \beta\alpha - \delta\gamma, \varepsilon\gamma \rangle$ with K a field. Take

$$\omega := \begin{matrix} & 0 & 0 & 0 & 1 & 0 \\ \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{matrix} & \oplus & \begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{matrix} & \oplus & \begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{matrix} & \oplus & \begin{matrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix} \end{matrix}.$$

By [ASS, Exa. VI.2.8(a)] we have that ω_R is a noninjective tilting module with $\text{pd}_R \omega = 1$. Thus it is a semidualizing $(R, \text{End}_R(\omega))$ -bimodule. It is straightforward to verify that the projective cover $P_0(\omega)$ of ω is $P(1) \oplus P(4)^2 \oplus P(5)^2$. So $\mathcal{P}_\omega(R)\text{-id}_R P_0(\omega) = 0$, and hence ω satisfies the left and right 1-cograde conditions by Theorem 4.8. Since $\text{pd}_R \omega = 1$ we have $\text{Ext}_R^{\geq 2}(\omega, M) = 0$ for any $M \in \text{Mod } R$. By Theorem 4.8 again, we have that ω satisfies the left and right n -cograde conditions for any $n \geq 1$.

§5. Two cotorsion pairs

In this section we will construct two complete cotorsion pairs under any of the equivalent conditions in Theorem 4.19.

For any $n \geq 0$, set $\mathcal{P}_\omega\text{-id}^{\leq n}(R) := \{M \in \text{Mod } R \mid \mathcal{P}_\omega(R)\text{-id}_R M \leq n\}$.

Lemma 5.1. *Let $M \in {}_R\omega^{\perp_{n-1}}$ with $n \geq 1$. If $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n - 1$, then there exists an exact sequence*

$$0 \rightarrow M \rightarrow X \rightarrow Y \rightarrow 0$$

in $\text{Mod } R$ with $X \in {}_R\omega^{\perp_n}$ and $Y \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$.

Proof. Let $M \in {}_R\omega^{\perp_{n-1}}$. From the exact sequence

$$0 \rightarrow M \rightarrow I^0(M) \rightarrow \cdots \rightarrow I^{n-1}(M) \rightarrow \text{co}\Omega^n(M) \rightarrow 0$$

in $\text{Mod } R$, we get the following commutative diagram with exact rows,

$$(5.1) \quad \begin{array}{ccccccc} P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & \text{Ext}_R^n(\omega, M) \longrightarrow 0 \\ & & & & \downarrow & & \parallel \\ I^0(M)_* & \longrightarrow & I^1(M)_* & \longrightarrow & \cdots & \longrightarrow & \text{co}\Omega^n(M)_* \longrightarrow \text{Ext}_R^n(\omega, M) \longrightarrow 0, \end{array}$$

where the upper sequence is a projective resolution of $\text{Ext}_R^n(\omega, M)$ in $\text{Mod } S$. Taking the mapping cone of diagram (5.1) we get an exact sequence

$$(5.2) \quad I^0(M)_* \oplus P_{n-1} \rightarrow \cdots \rightarrow I^{n-1}(M)_* \oplus P_0 \rightarrow \text{co}\Omega^n(M)_* \rightarrow 0.$$

Since $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, M) \geq n - 1$ we get an exact sequence

$$\omega \otimes_S P_{n-1} \rightarrow \cdots \rightarrow \omega \otimes_S P_1 \rightarrow \omega \otimes_S P_0 \rightarrow 0$$

in $\text{Mod } R$. Then we get the following commutative diagram with exact columns and rows

$$(5.3) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \dashrightarrow & M & \dashrightarrow & X & \dashrightarrow & Y & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0(M) & \longrightarrow & I^0(M) \oplus (\omega \otimes_S P_{n-1}) & \longrightarrow & \omega \otimes_S P_{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{n-1}(M) & \longrightarrow & I^{n-1}(M) \oplus (\omega \otimes_S P_0) & \longrightarrow & \omega \otimes_S P_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \text{co}\Omega^n(M) & = & \text{co}\Omega^n(M) & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & & & \end{array}$$

where

$$X = \text{Ker}(I^0(M) \oplus (\omega \otimes_S P_{n-1}) \rightarrow I^1(M) \oplus (\omega \otimes_S P_{n-2}))$$

and

$$Y = \text{Ker}(\omega \otimes_S P_{n-1} \rightarrow \omega \otimes_S P_{n-2}).$$

Then $Y \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. From the exactness of (5.2) and the middle column in diagram (5.3) we know that $X \in {}_R\omega^{\perp n}$. So the top row in diagram (5.3) is the desired exact sequence. \square

For any $n \geq 0$, set $\mathcal{I}_\omega\text{-pd}^{\leq n}(S) := \{N \in \text{Mod } S \mid \mathcal{I}_\omega(S)\text{-pd}_S N \leq n\}$.

Lemma 5.2. *Let $N \in \omega_S^\top{}^{n-1}$ with $n \geq 1$. If $\text{E-cograde}_\omega \text{Tor}_n^S(\omega, N) \geq n - 1$ then there exists an exact sequence*

$$0 \rightarrow Y' \rightarrow X' \rightarrow N \rightarrow 0$$

in $\text{Mod } S$ with $X' \in \omega_S^\top{}^n$ and $Y' \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$.

Proof. Let $N \in \omega_S^{\top n-1}$. From the exact sequence

$$0 \rightarrow \Omega_{\mathcal{F}}^n(N) \rightarrow F_{n-1}(N) \rightarrow \cdots \rightarrow F_0(N) \rightarrow N \rightarrow 0$$

in $\text{Mod } S$, we get the following commutative diagram with exact rows

$$(5.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Tor}_n^S(\omega, N) & \longrightarrow & \omega \otimes_S \Omega_{\mathcal{F}}^n(N) & \longrightarrow & \cdots \longrightarrow \omega \otimes_S F_1(N) \longrightarrow \omega \otimes_S F_0(N) \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Tor}_n^S(\omega, N) & \longrightarrow & I^0 & \longrightarrow & \cdots \longrightarrow I^{n-1}, \end{array}$$

where the lower sequence is an injective resolution of $\text{Tor}_n^S(\omega, N)$ in $\text{Mod } R$. Taking the mapping cone of diagram (5.4), we get an exact sequence

$$(5.5) \quad \omega \otimes_S \Omega_{\mathcal{F}}^n(N) \rightarrow I^0 \oplus (\omega \otimes_S F_{n-1}(N)) \rightarrow \cdots \rightarrow I^{n-1} \oplus (\omega \otimes_S F_0(N)).$$

Since $\text{E-cograde}_{\omega} \text{Tor}_n^S(\omega, N) \geq n - 1$ we get an exact sequence

$$0 \rightarrow I^0_* \rightarrow I^1_* \rightarrow \cdots \rightarrow I^{n-1}_*$$

in $\text{Mod } S$. Then we get the following commutative diagram with exact columns and rows

$$(5.6) \quad \begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \Omega_{\mathcal{F}}^n(N) & = & = & = & \Omega_{\mathcal{F}}^n(N) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^0_* & \longrightarrow & I^0_* \oplus F_{n-1}(N) & \longrightarrow & F_{n-1}(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & I^{n-1}_* & \longrightarrow & I^{n-1}_* \oplus F_0(N) & \longrightarrow & F_0(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \dashrightarrow & Y' & \dashrightarrow & X' & \dashrightarrow & N & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0, \end{array}$$

where

$$X' = \text{Coker}(I^{n-2}_* \oplus F_1(N) \rightarrow I^{n-1}_* \oplus F_0(N))$$

and

$$Y' = \text{Coker}(I^{n-2}_* \rightarrow I^{n-1}_*).$$

Then $Y' \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. From the exactness of (5.5) and the middle column in diagram (5.6) we know that $X' \in \omega_S^\top^n$. So the bottom row in diagram (5.6) is the desired exact sequence. \square

Lemma 5.3. *For any $n \geq 0$, we have*

- (1) $\mathcal{P}_\omega\text{-id}^{\leq n}(R)$ is closed under direct summands and closed under extensions;
- (2) $\mathcal{I}_\omega\text{-pd}^{\leq n}(S)$ is closed under direct summands and closed under extensions.

Proof. (1) By [TH4, Lem. 4.6], $\mathcal{P}_\omega\text{-id}^{\leq n}(R)$ is closed under direct summands. Let

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence in $\text{Mod } R$ with $A, C \in \mathcal{P}_\omega\text{-id}^{\leq n}(R)$. It is easy to see that it is $\text{Hom}_R(-, \mathcal{P}_\omega(R))$ -exact. Then $B \in \mathcal{P}_\omega\text{-id}^{\leq n}(R)$ by the generalized horseshoe lemma (cf. [H4, Lem. 3.1(2)]).

(2) By [TH3, Lem. 4.7], $\mathcal{I}_\omega\text{-pd}^{\leq n}(S)$ is closed under direct summands. Let

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence in $\text{Mod } S$ with $A, C \in \mathcal{I}_\omega\text{-pd}^{\leq n}(S)$. It is easy to see that it is $(\omega \otimes_S -)$ -exact; equivalently it is $\text{Hom}_R(-, \mathcal{I}_\omega(S))$ -exact by [TH3, p. 298, Obs.]. Then $B \in \mathcal{I}_\omega\text{-pd}^{\leq n}(S)$ by the generalized horseshoe lemma (cf. [H4, Lem. 3.1(1)]). \square

Proposition 5.4. *Let $n, k \geq 1$ and $\text{T-cograde}_\omega \text{Ext}_R^{i+k}(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n - 1$. Then for any $M \in \text{Mod } R$ and $0 \leq i \leq n - 1$ there exists an exact sequence*

$$0 \rightarrow \text{co}\Omega^{k-1}(M) \rightarrow X \rightarrow Y \rightarrow 0$$

in $\text{Mod } R$ with $X \in {}_R\omega^{\perp i+1}$ and $Y \in \mathcal{P}_\omega\text{-id}^{\leq i}(R)$.

Proof. We proceed by induction on n . The case for $n = 1$ follows from Lemma 5.1. Now suppose $n \geq 2$. By the induction hypothesis for any $0 \leq i \leq n - 2$ there exists an exact sequence

$$0 \rightarrow \text{co}\Omega^{k-1}(M) \rightarrow X_i \rightarrow Y_i \rightarrow 0$$

in $\text{Mod } R$ with $X_i \in {}_R\omega^{\perp_{i+1}}$ and $Y_i \in \mathcal{P}_\omega\text{-id}^{\leq i}(R)$. Then

$$\text{Ext}_R^n(\omega, X_{n-2}) \cong \text{Ext}_R^n(\omega, \text{co}\Omega^{k-1}(M)) \cong \text{Ext}_R^{n+k-1}(\omega, M).$$

So $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, X_{n-2}) = \text{T-cograde}_\omega \text{Ext}_R^{n+k-1}(\omega, M) \geq n - 1$ by assumption. Applying Lemma 5.1 we get an exact sequence

$$0 \rightarrow X_{n-2} \rightarrow X_{n-1} \rightarrow Y_{n-1} \rightarrow 0$$

in $\text{Mod } R$ with $X_{n-1} \in {}_R\omega^{\perp_n}$ and $Y_{n-1} \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. Consider the following push-out diagram:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \text{co}\Omega^{k-1}(M) & \longrightarrow & X_{n-2} & \longrightarrow & Y_{n-2} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{co}\Omega^{k-1}(M) & \longrightarrow & X_{n-1} & \longrightarrow & Y \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & Y_{n-1} & = & Y_{n-1} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0. \end{array}$$

By Lemma 5.3(1) we have $Y \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. So the middle row in this diagram is the desired sequence. \square

Proposition 5.5. *Let $n, k \geq 1$ and $\text{E-cograde}_\omega \text{Tor}_{i+k}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n - 1$. Then for any $N \in \text{Mod } S$ and $0 \leq i \leq n - 1$, there exists an exact sequence*

$$0 \rightarrow Y' \rightarrow X' \rightarrow \Omega_{\mathcal{F}}^{k-1}(N) \rightarrow 0$$

in $\text{Mod } S$ with $X' \in \omega_S^{\top_{i+1}}$ and $Y' \in \mathcal{I}_\omega\text{-pd}^{\leq i}(S)$.

Proof. We proceed by induction on n . The case for $n = 1$ follows from Lemma 5.2. Now suppose $n \geq 2$. By the induction hypothesis, for any $0 \leq i \leq n - 2$ there exists an exact sequence

$$0 \rightarrow Y'_i \rightarrow X'_i \rightarrow \Omega_{\mathcal{F}}^{k-1}(N) \rightarrow 0$$

in $\text{Mod } S$ with $X'_i \in \omega_S^{\top_{i+1}}$ and $Y'_i \in \mathcal{I}_\omega\text{-pd}^{\leq i}(S)$. Then

$$\text{Tor}_n^S(\omega, X'_{n-2}) \cong \text{Tor}_n^S(\omega, \Omega_{\mathcal{F}}^{k-1}(N)) \cong \text{Tor}_{n+k-1}^S(\omega, N).$$

So $E\text{-cograde}_\omega \text{Tor}_n^S(\omega, X'_{n-2}) = E\text{-cograde}_\omega \text{Tor}_{n+k-1}^S(\omega, N) \geq n - 1$ by assumption. Applying Lemma 5.2 we get an exact sequence

$$0 \rightarrow Y'_{n-1} \rightarrow X'_{n-1} \rightarrow X'_{n-2} \rightarrow 0$$

in $\text{Mod } S$ with $X'_{n-1} \in \omega_S^\top$ and $Y'_{n-1} \in \mathcal{I}_\omega\text{-pd}^{n-1}(S)$. Consider the pull-back diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & Y'_{n-1} & \longrightarrow & Y' & \longrightarrow & Y'_{n-2} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Y'_{n-1} & \longrightarrow & X'_{n-1} & \longrightarrow & X'_{n-2} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Omega_{\mathcal{F}}^{k-1}(N) & \equiv & \Omega_{\mathcal{F}}^{k-1}(N) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

By Lemma 5.3(2), we have $Y' \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. So the middle column in this diagram is the desired sequence. \square

Based on the equivalence of (1) and (2) in Theorem 4.19, we have the following result.

Theorem 5.6. *For any $n \geq 1$, we have the following properties:*

- (1) *If one of the equivalent conditions in Theorem 4.19 is satisfied then the following statements are equivalent:*
 - (1-1) $\text{pd}_{S^{\text{op}}} \omega \leq n - 1$.
 - (1-2) $\mathcal{P}_\omega(R)\text{-id}_R R \leq n - 1$.
 - (1-3) $\mathcal{P}_\omega(R)\text{-id}_R P \leq n - 1$ for any projective P in $\text{Mod } R$.
 - (1-4) $(\mathcal{P}_\omega\text{-id}^{\leq n-1}(R), {}_R\omega^{\perp n})$ forms a complete cotorsion pair.
- (2) *If one of the equivalent conditions in Theorem 4.19 is satisfied then the following statements are equivalent:*
 - (2-1) $\mathcal{I}_\omega(S)\text{-pd}_S Q \leq n - 1$ for some injective cogenerator Q in $\text{Mod } S$.
 - (2-2) $\mathcal{I}_\omega(S)\text{-pd}_S I \leq n - 1$ for any injective module I in $\text{Mod } S$.
 - (2-3) $(\omega_S^\top, \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S))$ forms a complete cotorsion pair.

If R and S are artin algebras, then statements (2-1)–(2-3) are equivalent to the following one:

$$(2-4) \text{ pd}_R \omega \leq n - 1.$$

Proof. By Lemma 4.5(1) we have (1-1) \Leftrightarrow (1-2).

If $\mathcal{P}_\omega(R)\text{-id}_R R \leq n - 1$ then $\mathcal{P}_\omega(R)\text{-id}_R F \leq n - 1$ for any free module F in $\text{Mod } R$ by [HW, Prop. 5.1(b)]. It follows from Lemma 5.3(1) that $\mathcal{P}_\omega(R)\text{-id}_R P \leq n - 1$ for any projective P in $\text{Mod } R$. This proves (1-2) \Leftrightarrow (1-3).

(1-3) \Rightarrow (1-4) It is easy to verify that $\text{Ext}_R^1(A, B) = 0$ for any $A \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$ and $B \in {}_R\omega^{\perp n}$.

Let $M \in \text{Mod } R$. By Lemma 5.1 when $n = 1$, or taking $k = 1$ in Proposition 5.4 when $n \geq 2$, we get an exact sequence

$$(5.7) \quad 0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$$

in $\text{Mod } R$ with $B \in {}_R\omega^{\perp n}$ and $A \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. It implies that M has a special ${}_R\omega^{\perp n}$ -preenvelope and ${}_R\omega^{\perp n}$ is special preenvolving in $\text{Mod } R$. If $M \in (\mathcal{P}_\omega\text{-id}^{\leq n-1}(R))^{\perp 1}$ then the exact sequence (5.7) splits. It follows that M is a direct summand of B and $M \in {}_R\omega^{\perp n}$.

Let

$$0 \rightarrow M_1 \rightarrow P \rightarrow M \rightarrow 0$$

be an exact sequence in $\text{Mod } R$ with P projective. We have $P \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$ by (1-3). By Lemma 5.1 when $n = 1$ or by Proposition 5.4 when $n \geq 2$, we have an exact sequence

$$0 \rightarrow M_1 \rightarrow B' \rightarrow A' \rightarrow 0$$

in $\text{Mod } R$ with $B' \in {}_R\omega^{\perp n}$ and $A' \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. Consider the push-out diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M_1 & \longrightarrow & P & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & B' & \longrightarrow & A'' & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & A' & \xlongequal{\quad} & A' & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since $\mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$ is closed under extensions by Lemma 5.3(1), it follows from the middle column in the above diagram that $A'' \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. If $M \in {}^{\perp 1}({}_R\omega^{\perp n})$ then the middle row in the above diagram splits and M is a direct summand of A'' . By Lemma 5.3(1) we have $M \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$. It follows from Lemma 2.7 that $(\mathcal{P}_\omega\text{-id}^{\leq n-1}(R), {}_R\omega^{\perp n})$ forms a complete cotorsion pair.

(1-4) \Rightarrow (1-2) By (1-4) we immediately have ${}_R R \in \mathcal{P}_\omega\text{-id}^{\leq n-1}(R)$ and $\mathcal{P}_\omega(R)\text{-id}_R R \leq n - 1$.

If $\mathcal{I}_\omega(S)\text{-pd}_S Q \leq n - 1$ for some injective cogenerator Q in $\text{Mod } S$, then any direct product of Q is in $\mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$ by [HW, Prop. 5.1(c)]. It follows from Lemma 5.3(2) that $\mathcal{I}_\omega(S)\text{-pd}_S I \leq n - 1$ for any injective module I in $\text{Mod } S$. This proves (2-1) \Leftrightarrow (2-2).

(2-2) \Rightarrow (2-3) It is easy to verify that $\text{Ext}_S^1(C, D) = 0$ for any $C \in \omega_S^{\top n}$ and $D \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$.

Let $N \in \text{Mod } S$. By Lemma 5.2 when $n = 1$, or taking $k = 1$ in Proposition 5.5 when $n \geq 2$, we get an exact sequence

$$(5.8) \quad 0 \rightarrow D \rightarrow C \rightarrow N \rightarrow 0$$

in $\text{Mod } S$ with $C \in \omega_S^{\top n}$ and $D \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. It implies that N has a special $\omega_S^{\top n}$ -precover and $\omega_S^{\top n}$ is precovering in $\text{Mod } S$. If $N \in {}^{\perp 1}(\mathcal{I}_\omega\text{-pd}^{\leq n-1}(S))$ then the exact sequence (5.8) splits. It follows that N is a direct summand of C and $N \in \omega_S^{\top n}$.

Let

$$0 \rightarrow N \rightarrow I \rightarrow N_1 \rightarrow 0$$

be an exact sequence in $\text{Mod } S$ with I injective. By (2-2) we have $I \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. By Lemma 5.2 when $n = 1$ or by Proposition 5.5 when $n \geq 2$, we have an exact sequence

$$0 \rightarrow D' \rightarrow C' \rightarrow N_1 \rightarrow 0$$

in $\text{Mod } S$ with $C' \in \omega_S^{\top n}$ and $D' \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. Consider the pull-back

diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & N & \xlongequal{\quad} & N & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & D' & \longrightarrow & D'' & \longrightarrow & I \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & D' & \longrightarrow & C' & \longrightarrow & N_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Since $\mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$ is closed under extensions by Lemma 5.3(2), it follows from the middle row in the above diagram that $D'' \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. If $N \in (\omega_S^{\top n})^{\perp 1}$ then the middle column in the above diagram splits and N is a direct summand of D'' . By Lemma 5.3(2) we have $N \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$. It follows from Lemma 2.7 that $(\omega_S^{\top n}, \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S))$ forms a complete cotorsion pair.

(2-3) \Rightarrow (2-2) For any injective module I in $\text{Mod } S$, by (2-3) we have $I \in \mathcal{I}_\omega\text{-pd}^{\leq n-1}(S)$ and $\mathcal{I}_\omega(S)\text{-pd}_S I \leq n - 1$.

If R and S are artin algebras then $\text{pd}_R \omega = \mathcal{I}_\omega(S)\text{-pd}_S D(S_S)$ by [TH3, Lem. 4.9]. Because $D(S_S)$ is an injective cogenerator in $\text{Mod } S$, (2-1) \Leftrightarrow (2-4) follows. \square

Observation 5.7. Let R be an artin algebra and ${}_R\omega_S = {}_R D(R)_R$. Then we have the following properties:

- (1) $\text{pd}_R \omega = \text{id}_{R^{\text{op}}} R$ and $\text{pd}_{R^{\text{op}}} \omega = \text{id}_R R$.
- (2) $\mathcal{P}_\omega(R)$ is exactly the subclass of $\text{Mod } R$ consisting of injective modules. It implies that
 - (2-1) $\mathcal{P}_\omega(R)\text{-id}_R M = \text{id}_R M$ for any $M \in \text{Mod } R$;
 - (2-2) $\mathcal{P}_\omega\text{-id}^{\leq n}(R) = \mathcal{I}^{\leq n}(R) := \{M \in \text{Mod } R \mid \text{id}_R M \leq n\}$.
- (3) $\mathcal{I}_\omega(R)$ is exactly the subclass of $\text{Mod } R$ consisting of projective modules. It implies that
 - (3-1) $\mathcal{I}_\omega(R)\text{-pd}_R N = \text{pd}_R N$ for any $N \in \text{Mod } R$;
 - (3-2) $\mathcal{I}_\omega\text{-pd}^{\leq n}(R) = \mathcal{P}^{\leq n}(R) := \{N \in \text{Mod } R \mid \text{pd}_R N \leq n\}$.
- (4) By [CE, Prop. VI.5.3] it is easy to see that $\omega_R^{\top n+1} = {}^{\perp n+1} R_R$.

- (5) If R is right quasi $(n - 1)$ -Gorenstein, then all conditions in Theorem 4.19 are satisfied; see Theorem 4.14 and Example 4.20(2).

As an application of Theorem 5.6, we have the following corollary.

Corollary 5.8. *Let R be a right quasi $(n - 1)$ -Gorenstein artin algebra with $n \geq 1$. Then the following statements are equivalent:*

- (1) $\text{id}_R R \leq n - 1$.
- (2) $\text{id}_{R^{\text{op}}} R \leq n - 1$.
- (3) $(\mathcal{I}^{\leq n-1}(R), {}_R D(R)^{\perp n})$ forms a complete cotorsion pair.
- (4) $({}^{\perp n} {}_R R, \mathcal{P}^{\leq n-1}(R))$ forms a complete cotorsion pair.

Proof. By Theorem 5.6 and Observation 5.7 we have (1) \Leftrightarrow (3) and (2) \Leftrightarrow (4).

(1) \Leftrightarrow (2) Let $\text{id}_R R \leq n - 1$. By [AR2, Thm. 4.7] and the symmetric version of [H2, Thm.] we have $\text{id}_{R^{\text{op}}} R \leq (n - 1) + (n - 2) = 2n - 3$. Conversely, let $\text{id}_{R^{\text{op}}} R \leq n - 1$. By [TH2, Thm. 7.5] we have $\text{id}_R R \leq n - 1$. Now the assertion follows from [Z, Lem. A]. □

As a consequence of Corollary 5.8, we have the following result.

Corollary 5.9. *For any artin algebra R , the following conditions are equivalent:*

- (1) $\text{id}_R R \leq 1$.
- (2) $\text{id}_{R^{\text{op}}} R \leq 1$.

Furthermore, if R is right quasi 1-Gorenstein then they are equivalent to each of the following two statements:

- (3) $(\mathcal{I}^{\leq 1}(R), {}_R D(R)^{\perp 2})$ forms a complete cotorsion pair.
- (4) $({}^{\perp 2} {}_R R, \mathcal{P}^{\leq 1}(R))$ forms a complete cotorsion pair.

Proof. The first assertion follows from [H2, Cor. 2]. If R is right quasi 1-Gorenstein then we get the second assertion by putting $n = 2$ in Corollary 5.8. □

We use $\mathcal{I}(R)$ and $\mathcal{P}(R)$ to denote the subclasses of $\text{Mod } R$ consisting of injective and projective modules respectively. Putting $n = 1$ in Corollary 5.8, we have the following result.

Corollary 5.10. *For any artin algebra R , the following statements are equivalent:*

- (1) R is self-injective.
- (2) $(\mathcal{I}(R), {}_R D(R)^{\perp 1})$ forms a complete cotorsion pair (in this case, ${}_R D(R)^{\perp 1} = \mathcal{I}(R)^{\perp 1}$).
- (3) $({}^{\perp 1} {}_R R, \mathcal{P}(R))$ forms a complete cotorsion pair (in this case, ${}^{\perp 1} {}_R R = {}^{\perp 1} \mathcal{P}(R)$).

§6. Relative finitistic dimensions

In this section, we introduce and study the finitistic $\mathcal{P}_\omega(R)$ -injective dimension and the $\mathcal{I}_\omega(S)$ -projective dimension of rings.

The *finitistic $\mathcal{P}_\omega(R)$ -injective dimension* $\text{FP}_\omega\text{-id } R$ of R is defined as

$$\text{FP}_\omega\text{-id } R := \sup \{ \mathcal{P}_\omega(R)\text{-id}_R M \mid M \in \text{Mod } R \text{ and } \mathcal{P}_\omega(R)\text{-id}_R M < \infty \}$$

and the *finitistic $\mathcal{I}_\omega(S)$ -projective dimension* $\text{FL}_\omega\text{-pd } S$ of S is defined as

$$\text{FL}_\omega\text{-pd } S := \sup \{ \mathcal{I}_\omega(S)\text{-pd}_S N \mid N \in \text{Mod } S \text{ and } \mathcal{I}_\omega(S)\text{-pd}_S N < \infty \}.$$

Lemma 6.1. *For any $n \geq 0$ and $k \geq 1$ we have*

- (1) *Let $\text{T-cograde}_\omega \text{Ext}_R^{i+k}(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $1 \leq i \leq n + 1$. If $\text{FP}_\omega\text{-id } R = n$ then $\text{pd}_R \omega \leq n + k$.*
- (2) *Let $\text{E-cograde}_\omega \text{Tor}_{i+k}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $1 \leq i \leq n + 1$. If $\text{FL}_\omega\text{-pd } S = n$ then $\text{pd}_{S^{\text{op}}} \omega \leq n + k$.*

Proof. (1) Let $M \in \text{Mod } R$. By Proposition 5.4 there exists an exact sequence

$$0 \rightarrow \text{co}\Omega^{k-1}(M) \rightarrow X \rightarrow Y \rightarrow 0$$

in $\text{Mod } R$ with $X \in {}_R\omega^{\perp n+2}$ and $\mathcal{P}_\omega(R)\text{-id}_R Y \leq n + 1$. If $\text{FP}_\omega\text{-id } R = n$ then $\mathcal{P}_\omega(R)\text{-id}_R Y \leq n$. Thus we have

$$\text{Ext}_R^{n+k+1}(\omega, M) \cong \text{Ext}_R^{n+2}(\omega, \text{co}\Omega^{k-1}(M)) \cong \text{Ext}_R^{n+1}(\omega, Y) = 0$$

and $\text{pd}_R \omega \leq n + k$.

(2) Let $N \in \text{Mod } S$. By Proposition 5.5 there exists an exact sequence

$$0 \rightarrow Y' \rightarrow X' \rightarrow \Omega_{\mathcal{F}}^{k-1}(N) \rightarrow 0$$

in $\text{Mod } S$ with $X' \in \omega_S^{\top n+2}$ and $\mathcal{P}_\omega(R)\text{-id}_S Y' \leq n + 1$. If $\text{FL}_\omega\text{-pd } S = n$ then $\mathcal{I}_\omega(R)\text{-pd}_S Y' \leq n$. Thus we have

$$\text{Tor}_{n+k+1}^S(\omega, N) \cong \text{Tor}_{n+2}^S(\omega, \Omega_{\mathcal{F}}^{k-1}(N)) \cong \text{Tor}_{n+1}^S(\omega, Y') = 0$$

and $\text{pd}_{S^{\text{op}}} \omega = \text{fd}_{S^{\text{op}}} \omega \leq n + k$. □

Lemma 6.2. *For any $n \geq 0$ we have*

- (1) *Let $\text{FP}_\omega\text{-id } R \leq n$ and $N \in \text{Mod } S$. If $\text{T-cograde}_\omega N \geq n + 1$ then $N = 0$.*
- (2) *Let $\text{FL}_\omega\text{-pd } S \leq n$ and $H \in \text{Mod } R$. If $\text{E-cograde}_\omega H \geq n + 1$ then $H = 0$.*

Proof. (1) Consider a projective resolution

$$\cdots \rightarrow Q_{n+1} \rightarrow Q_n \rightarrow \cdots \rightarrow Q_0 \rightarrow N \rightarrow 0$$

of N in $\text{Mod } S$. If $\text{T-cograde}_\omega N \geq n + 1$ then we get an exact sequence

$$0 \rightarrow M \rightarrow \omega \otimes_S Q_{n+1} \rightarrow \omega \otimes_S Q_n \rightarrow \cdots \rightarrow \omega \otimes_S Q_1 \rightarrow \omega \otimes_S Q_0 \rightarrow 0$$

in $\text{Mod } R$, where $M = \text{Ker}(\omega \otimes_S Q_{n+1} \rightarrow \omega \otimes_S Q_n)$. By [TH3, Cor. 3.5], $Q \cong (\omega \otimes_S Q)_*$ canonically for any projective Q in $\text{Mod } S$, so $N \cong \text{Ext}_R^{n+1}(\omega, M)$. Because $\text{FP}_\omega\text{-id } R \leq n$ by assumption, we have $\mathcal{P}_\omega(R)\text{-id}_R M \leq n$ and $N \cong \text{Ext}_R^{n+1}(\omega, M) = 0$.

(2) Consider an injective resolution

$$0 \rightarrow H \rightarrow I^0 \rightarrow \cdots \rightarrow I^n \rightarrow I^{n+1} \rightarrow \cdots$$

of H in $\text{Mod } R$. If $\text{E-cograde}_\omega H \geq n + 1$ then we get an exact sequence

$$0 \rightarrow I^0_* \rightarrow \cdots \rightarrow I^n_* \rightarrow I^{n+1}_* \rightarrow N \rightarrow 0$$

in $\text{Mod } S$, where $N = \text{Coker}(I^n_* \rightarrow I^{n+1}_*)$. By [TH1, Lem. 2.5(2)], $\omega \otimes_S I_* \cong I$ canonically for any injective I in $\text{Mod } R$, so $H \cong \text{Tor}_{n+1}^S(\omega, N)$. Because $\text{FL}_\omega\text{-pd } S \leq n$ by assumption, we have $\mathcal{I}_\omega(R^{\text{op}})\text{-pd}_S N \leq n$ and $H \cong \text{Tor}_{n+1}^S(\omega, N) = 0$. \square

The following is the main result in this section.

Theorem 6.3. *For any $k \geq 0$ we have the following properties:*

- (1) *If $\text{T-cograde}_\omega \text{Ext}_R^{i+k}(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $i \geq 1$, then $\text{FP}_\omega\text{-id } R \leq \text{pd}_R \omega \leq \text{FP}_\omega\text{-id } R + k$.*
- (2) *If $\text{E-cograde}_\omega \text{Tor}_{i+k}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $i \geq 1$, then $\text{FL}_\omega\text{-pd } S \leq \text{pd}_{S^{\text{op}}} \omega \leq \text{FL}_\omega\text{-pd } S + k$.*

Proof. (1) Let $\text{pd}_R \omega = n (< \infty)$ and $M \in \text{Mod } R$ with $\mathcal{P}_\omega(R)\text{-id}_R M = m (< \infty)$. Then there exists an exact sequence

$$0 \rightarrow M \xrightarrow{f^0} \omega^0 \xrightarrow{f^1} \omega^1 \rightarrow \cdots \xrightarrow{f^m} \omega^m \rightarrow 0$$

in $\text{Mod } R$ with all ω^i in $\mathcal{P}_\omega(R)$. Since $\mathcal{P}_\omega(R) \subseteq \mathcal{B}_\omega(R)$ by [HW, Cor. 6.1], we have $\mathcal{B}_\omega(R)\text{-id}_R M \leq \mathcal{P}_\omega(R)\text{-id}_R M < \infty$. If $m > n$ then it follows from [TH4, Thm. 4.2] that $\mathcal{B}_\omega(R)\text{-id}_R M \leq n$ and $\text{Im } f^n \in \mathcal{B}_\omega(R)$. On the other hand, we have the exact and split sequence

$$0 \rightarrow (\text{Im } f^n)_* \rightarrow \omega^n_* \rightarrow \cdots \rightarrow \omega^m_* \rightarrow 0$$

in $\text{Mod } S$ with all ω^i_* projective. So $(\text{Im } f^n)_*$ is projective, and hence $\text{Im } f^n \in \mathcal{P}_\omega(R)$ by [HW, Lem. 5.1(2)]. It yields that $\mathcal{P}_\omega(R)\text{-id}_R M \leq n$, a contradiction. This proves $\text{FP}_\omega\text{-id } R \leq \text{pd}_R \omega$.

In the following we will prove $\text{pd}_R \omega \leq \text{FP}_\omega\text{-id } R + k$. The case for $k \geq 1$ follows from Lemma 6.1(1). Now suppose that $k = 0$ and $\text{FP}_\omega\text{-id } R = n (< \infty)$. Let $M \in \text{Mod } R$. Then $\text{T-cograde}_\omega \text{Ext}_R^{n+1}(\omega, M) \geq n+1$ by assumption. It follows from Lemma 6.2(1) that $\text{Ext}_R^{n+1}(\omega, M) = 0$ and $\text{pd}_R \omega \leq n$.

(2) Let $\text{pd}_{S^{\text{op}}} \omega = n (< \infty)$ and $N \in \text{Mod } S$ with $\mathcal{I}_\omega(S)\text{-pd}_S N = m (< \infty)$. Then there exists an exact sequence

$$0 \rightarrow U_m \xrightarrow{g_m} \dots \rightarrow U_1 \xrightarrow{g_1} U_0 \xrightarrow{g_0} N \rightarrow 0$$

in $\text{Mod } S$ with all U_i in $\mathcal{I}_\omega(S)$. Since $\mathcal{I}_\omega(S) \subseteq \mathcal{A}_\omega(S)$ by [HW, Cor. 6.1], we have $\mathcal{A}_\omega(S)\text{-pd}_S N < \infty$. If $m > n$ then it follows from the dual result of [TH4, Thm. 4.2] that $\mathcal{A}_\omega(S)\text{-pd}_S N \leq n$ and $\text{Im } g_n \in \mathcal{A}_\omega(S)$. On the other hand, we have the exact and split sequence

$$0 \rightarrow \omega \otimes_S U_m \rightarrow \dots \rightarrow \omega \otimes_S U_n \rightarrow \omega \otimes_S \text{Im } g_n \rightarrow 0$$

in $\text{Mod } R$ with all $\omega \otimes_S U_i$ injective. So $\omega \otimes_S \text{Im } g_n$ is injective, and hence $\text{Im } g_n \in \mathcal{I}_\omega(S)$ by [HW, Lem. 5.1(3)]. It yields that $\mathcal{I}_\omega(S)\text{-pd}_S N \leq n$, a contradiction. This proves $\text{F}\mathcal{I}_\omega\text{-pd } S \leq \text{pd}_{S^{\text{op}}} \omega$.

In the following we will prove $\text{pd}_{S^{\text{op}}} \omega \leq \text{F}\mathcal{I}_\omega\text{-pd } S + k$. The case for $k \geq 1$ follows from Lemma 6.1(2). Now suppose that $k = 0$ and $\text{F}\mathcal{I}_\omega\text{-pd } S = n$. Let $N \in \text{Mod } S$. Then $\text{E-cograde}_\omega \text{Tor}_{n+1}^S(\omega, N) \geq n+1$ by assumption. It follows from Lemma 6.2(2) that $\text{Tor}_{n+1}^S(\omega, N) = 0$ and $\text{pd}_{S^{\text{op}}} \omega = \text{fd}_{S^{\text{op}}} \omega \leq n$. \square

Putting $k = 0$ in Theorem 6.3, we immediately get the following corollary.

- Corollary 6.4.** (1) *If $\text{T-cograde}_\omega \text{Ext}_R^i(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $i \geq 1$, then $\text{FP}_\omega\text{-id } R = \text{pd}_R \omega$.*
 (2) *If $\text{E-cograde}_\omega \text{Tor}_i^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $i \geq 1$, then $\text{F}\mathcal{I}_\omega\text{-pd } S = \text{pd}_{S^{\text{op}}} \omega$.*

The following is an immediate consequence of Corollaries 4.2 and 6.4.

Corollary 6.5. *If ω satisfies the n -cograde condition for all n then*

$$\text{FP}_\omega\text{-id } R = \text{pd}_R \omega \quad \text{and} \quad \text{F}\mathcal{I}_\omega\text{-pd } S = \text{pd}_{S^{\text{op}}} \omega.$$

Combining Theorem 4.19 with the case for $k = 1$ in Theorem 6.3, we get the following corollary.

Corollary 6.6. *We have*

$$\text{FP}_\omega\text{-id } R \leq \text{pd}_R \omega \leq \text{FP}_\omega\text{-id } R + 1$$

and

$$\text{FT}_\omega\text{-pd } S \leq \text{pd}_{S^{\text{op}}} \omega \leq \text{FT}_\omega\text{-pd } S + 1$$

if either of the following conditions is satisfied:

- (1) T-cograde $_\omega \text{Ext}_R^{i+1}(\omega, M) \geq i$ for any $M \in \text{Mod } R$ and $i \geq 1$.
- (2) E-cograde $_\omega \text{Tor}_{i+1}^S(\omega, N) \geq i$ for any $N \in \text{Mod } S$ and $i \geq 1$.

Corollary 6.7. *If ω satisfies the right quasi n -cograde condition for all n , then*

$$\text{FP}_\omega\text{-id } R = \text{pd}_R \omega \quad \text{and} \quad \text{FT}_\omega\text{-pd } S \leq \text{pd}_{S^{\text{op}}} \omega \leq \text{FT}_\omega\text{-pd } S + 1.$$

Proof. The former equality follows from Proposition 4.12 and Corollary 6.4(1), and the later inequalities follow from the definition of the right quasi n -cograde condition and Corollary 6.6. □

Observation 6.8. Let R be an artin algebra and ${}_R\omega_S = {}_R D(R)_R$.

- (1) By Observation 5.7 we have

$$\begin{aligned} \text{FP}_\omega\text{-id } R &= \text{FID } R := \sup\{\text{id}_R M \mid M \in \text{Mod } R \text{ and } \text{id}_R M < \infty\}, \\ \text{FT}_\omega\text{-pd } S &= \text{FPD } R := \sup\{\text{pd}_R N \mid N \in \text{Mod } R \text{ and } \text{pd}_R N < \infty\}. \end{aligned}$$

- (2) If R is right (or left) quasi n -Gorenstein for all n , then $\text{id}_{R^{\text{op}}} R = \text{id}_R R$ ([H2, Cor. 4]).

As a consequence of the above results, we have the following corollary.

Corollary 6.9. *Let R be an artin algebra.*

- (1) *If R satisfies the Auslander condition (that is, R is Auslander n -Gorenstein for all n) then*

$$\text{FPD } R^{\text{op}} = \text{FID } R^{\text{op}} = \text{id}_{R^{\text{op}}} R = \text{id}_R R = \text{FPD } R = \text{FID } R.$$

- (2) *If R satisfies the right quasi Auslander condition (that is, R is right quasi n -Gorenstein for all n) then*

$$\text{FPD } R \leq \text{FID } R = \text{id}_{R^{\text{op}}} R = \text{id}_R R \leq \text{FPD } R + 1.$$

Proof. In view of Example 4.20 and Observations 5.7 and 6.8, the assertions follow from Corollaries 6.5 and 6.7 respectively. □

Acknowledgements

This research was partially supported by NSFC (grant nos. 11971225, 11571164, 11501144), a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions and the NSF of Guangxi Province of China (grant no. 2016GXNSFAA380151). The authors thank the referee for the useful suggestions.

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