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Resolving Subcategories of Triangulated Categories and Relative Homological Dimension

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Abstract We introduce and study (pre)resolving subcategories of a triangulated category and the homological dimension relative to these subcategories. We apply the obtained properties to relative Gorenstein categories.

Keywords (Pre)resolving subcategories, triangulated categories, relative homological dimension, Gorenstein categories

MR(2010) Subject Classification 18E30, 18G20, 18G25

1 Introduction

It is well known that triangulated categories play an important role in representation theory of algebras, see [1–3, 7–11, 13–15] and references therein.

Beligiannis developed in [2] a relative version of homological algebra in triangulated categories in analogy to relative homological algebra in abelian categories, in which the notion of a proper class of exact sequences is replaced by a proper class of exact triangles. Later on, Asadollahi and Salarian extended the Beligiannis' theory in [1] by combining it with Gorenstein homological theory for abelian categories; in particular, they introduced Gorenstein objects in triangulated categories and gave estimates on certain Gorenstein projective dimensions. Some further investigations of Gorenstein homological theory and proper classes of triangles for triangulated categories were carried in [11] and [13–15]. On the other hand, Huang introduced and studied in [6] relative preresolving subcategories and precoresolving subcategories of an abelian category and homological dimensions and codimensions relative to these subcategories respectively, and unified some important properties possessed by some known homological dimensions. Motivated by these, in this paper we introduce and study (pre)resolving subcategories of a triangulated category and the homological dimension relative to these subcategories. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

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In Section 3, we first give the definition of (pre)resolving subcategories of a triangulated category. Then we give some criteria for computing homological dimensions relative to (pre)resolving subcategories. Let $\mathcal{E}$ be an additive and full subcategory of a triangulated category $\mathcal{T}$ and ξ a proper class of triangles in $\mathcal{T}$, and let $\mathcal{J}$ be an $\mathcal{E}$-preresolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$. Assume that $0 \longrightarrow M \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0$ is a ξ-exact complex in $\mathcal{T}$ with both T_0 and T_1 objects in $\mathcal{J}$. Then there exists a ξ-exact complex $0 \longrightarrow M \longrightarrow T \longrightarrow C \longrightarrow A \longrightarrow 0$ in $\mathcal{T}$ with T an object in $\mathcal{J}$ and C an object in $\mathcal{C}$; and furthermore, if the former complex is $\mathcal{T}(\mathcal{E},-)$-exact, then so is the later one. As an application of this result, we get that the $\mathcal{J}_{\mathcal{E}}$-dimension of an object A in $\mathcal{T}$ is at most n if and only if there exists a $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex $0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$ in $\mathcal{T}$ with all C_i objects in $\mathcal{C}$ and K_n an object in $\mathcal{J}$. Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and $\mathcal{J}$ an $\mathcal{E}$-resolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$, and let A be an object of $\mathcal{T}$ with $\mathcal{J}_{\mathcal{E}}$-dimension at most n. Then for any $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex $0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$ in $\mathcal{T}$ with all C_i objects in $\mathcal{C}$, we have that K_n is an object in $\mathcal{J}$; moreover, if $X \longrightarrow C \longrightarrow A \longrightarrow \Sigma X$ is a $\mathcal{T}(\mathcal{E},-)$-exact triangle in ξ with C an object in $\mathcal{C}$, then the $\mathcal{J}_{\mathcal{E}}$-dimension of X is at most $n-1$.

Let $\mathcal{C}$ and $\mathcal{E}$ be subcategories of $\mathcal{T}$. In Section 4, we introduce $(\mathcal{E},\mathcal{C})$-Gorenstein category $\mathcal{GC}_{\mathcal{E}}(\xi)$. Let $\mathcal{C} \subseteq \mathcal{E}$. Then we prove that $\mathcal{GC}_{\mathcal{E}}(\xi)$ is an $\mathcal{E}$-resolving and $\mathcal{E}$-coresolving subcategory of $\mathcal{T}$; furthermore, if $\mathcal{C}$ is closed under direct summands, then for any object A in $\mathcal{T}$ with finite $\mathcal{C}_{\mathcal{E}}$-dimension, the $\mathcal{GC}_{\mathcal{E}}(\xi)_{\mathcal{E}}$-dimension and the $\mathcal{C}_{\mathcal{E}}$-dimension of A are identical.

2 Preliminaries

Let $\mathcal{T}$ be an additive category with Σ an autoequivalence of $\mathcal{T}$. A subcategory $\mathcal{C}$ of $\mathcal{T}$ is called Σ-*stable* if $\Sigma\mathcal{C} = \mathcal{C}$. We use $\mathrm{Diag}(\mathcal{T},\Sigma)$ to denote the category whose objects are the sequences of morphisms

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X,$$

and morphisms between $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ are a triple (α,β,γ) such that the following diagram commutes.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'. \end{array}$$

A triple $(\mathcal{T},\Sigma,\Delta)$ is called a *pre-triangulated category* ([10]), where Δ is a full subcategory of $\mathrm{Diag}(\mathcal{T},\Sigma)$ which satisfies the following axioms. The elements of Δ are then called *triangles*.

(TR1) Every sequence of morphisms which is isomorphic to a triangle is a triangle. For every object X in $\mathcal{T}$, the sequence of morphisms $X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$ is a triangle. Every morphism $u : X \longrightarrow Y$ in $\mathcal{T}$ can be embedded into a triangle $X \xrightarrow{u} Y \longrightarrow Z \longrightarrow \Sigma X$.

(TR2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle, then so is $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$.

(TR3) Given triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$, then each commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

can be completed to a morphism of triangles (but not necessarily uniquely).

Moreover, a pre-triangulated category $(\mathcal{T}, \Sigma, \Delta)$ is called a *triangulated category* if Δ satisfies the equivalent conditions in the following proposition.

Proposition 2.1 ([2, Proposition 2.1] and [8]) *Let $(\mathcal{T}, \Sigma, \Delta)$ be a pre-triangulated category. Then the following statements are equivalent.*

(1) (TR4) *The octahedral axiom. For any two morphisms $u : X \longrightarrow Y$ and $v : Y \longrightarrow Z$, there exists a commutative diagram*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{u'} & Z' & \xrightarrow{u''} & \Sigma X \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \alpha} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{w} & Y' & \xrightarrow{w'} & \Sigma X \\
\downarrow{\scriptstyle u} & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma u} \\
Y & \xrightarrow{v} & Z & \xrightarrow{v'} & X' & \xrightarrow{v''} & \Sigma Y \\
\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (\Sigma u')v''} & & \downarrow \\
0 & \longrightarrow & \Sigma Z' & = & \Sigma Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

(2) *Base change. For any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in Δ and any morphism $\alpha : Z' \longrightarrow Z$, there exists the following commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & X' & = & X' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \downarrow \\
X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\
\| & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\
0 & \longrightarrow & \Sigma X' & = & \Sigma X' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

(3) *Cobase change. For any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in Δ and any morphism*

$\beta : X \longrightarrow X'$, *there exists the following commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}Z' & = & \Sigma^{-1}Z' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle -\Sigma^{-1}\gamma} & & \downarrow{\scriptstyle -\Sigma^{-1}\gamma'} & & \downarrow \\
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} & & \| \\
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow \\
0 & \longrightarrow & Z' & = & Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

From now on, $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ is a triangulated category with Σ the suspension functor and Δ the triangulation, and all subcategories of $\mathcal{T}$ are additive, full, closed under isomorphisms and Σ-stable.

Definition 2.2 ([2]) *A triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called split if it is isomorphic to the triangle

$$X \xrightarrow{\binom{1}{0}} X \oplus Z \xrightarrow{(0,1)} Z \xrightarrow{0} \Sigma X.$$

We use Δ_0 to denote the full subcategory of Δ consisting of the split triangles.

Definition 2.3 ([2]) *Let ξ be a class of triangles in $\mathcal{T}$.*

(1) *ξ is said to be closed under base change* (*resp. cobase change*) *if for any triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in ξ and any morphism $\alpha : Z' \longrightarrow Z$ (*resp.* $\beta : X \longrightarrow X'$) *as in Proposition* 2.1 (2) (*resp. Proposition* 2.1 (3)), *the triangle*

$$X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X \quad (\textit{resp.} \;\; X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X')$$

is in ξ.

(2) *ξ is said to be closed under suspension if for any triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in ξ and any $i \in \mathbb{Z}$ (*the set of all integers*), *the triangle*

$$\Sigma^i X \xrightarrow{(-1)^i \Sigma^i u} \Sigma^i Y \xrightarrow{(-1)^i \Sigma^i v} \Sigma^i Z \xrightarrow{(-1)^i \Sigma^i w} \Sigma^{i+1} X$$

is in ξ.

(3) *ξ is called saturated if in the situation of base change as in Proposition* 2.1 (2), *whenever the third vertical and the second horizontal triangles are in ξ, then the triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is in ξ.

Definition 2.4 ([2]) *A class ξ of triangles in $\mathcal{T}$ is called proper if the following conditions are satisfied.*

(1) *ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi$.*

(2) *ξ is closed under suspensions and is saturated.*

(3) *ξ is closed under base and cobase changes.*

In the following, ξ is a proper class of triangles in $\mathcal{T}$.

Definition 2.5 ([2]) *Let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle in ξ. Then the morphism u (resp. v) is called ξ-monic (resp. ξ-epic), and u (resp. v) is called the hokernel of v (resp. the hocokernel of u).

Lemma 2.6 ([13, Proposition 2.7]) *Let $u : X \longrightarrow Y$ and $v : Y \longrightarrow Z$ be morphisms in $\mathcal{T}$. If vu is ξ-monic (resp. ξ-epic), then so is u (resp. v).*

Lemma 2.7 ([13, Proposition 2.4]) *Given a commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & Z' & = & Z' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z \\
\| & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \| \\
\Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\downarrow & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\
0 & \longrightarrow & \Sigma Z' & = & \Sigma Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

(1) *If the third vertical triangle and the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ are in ξ, then so is the triangle $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X'$.*

(2) *If the second vertical triangle and the triangle $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X'$ are in ξ, then so is the third vertical triangle.*

Definition 2.8 *Let $\mathcal{E}$ be a subcategory of $\mathcal{T}$.*

(1) *A triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ is called $\mathcal{T}(\mathcal{E}, -)$-exact (resp. $\mathcal{T}(-, \mathcal{E})$-exact) if for any object E in $\mathcal{E}$, the induced complex

$$0 \longrightarrow \mathcal{T}(E, X) \longrightarrow \mathcal{T}(E, Y) \longrightarrow \mathcal{T}(E, Z) \longrightarrow 0$$
$$(\textit{resp. } 0 \longrightarrow \mathcal{T}(Z, E) \longrightarrow \mathcal{T}(Y, E) \longrightarrow \mathcal{T}(X, E) \longrightarrow 0)$$

is exact.

(2) ([1]) *A ξ-exact complex is a complex*

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \tag{2.1}$$

in $\mathcal{T}$ such that for any $n \in \mathbb{Z}$, there exists a triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1} \tag{2.2}$$

in ξ and the differential d_n is defined as $d_n = g_{n-1} f_n$. A ξ-exact complex as (2.1) *is called $\mathcal{T}(\mathcal{E}, -)$-exact* (*resp. $\mathcal{T}(-, \mathcal{E})$-exact*) *if the triangle* (2.2) *is $\mathcal{T}(\mathcal{E}, -)$-exact* (*resp. $\mathcal{T}(-, \mathcal{E})$-exact*) *for any $n \in \mathbb{Z}$.*

We need the following easy and useful observation.

Lemma 2.9 *Let C be an object in $\mathcal{T}$ and*

$$\begin{array}{ccccccc}
0 & \longrightarrow & X' & = & X' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
X & \longrightarrow & Y & \xrightarrow{u'} & Z & \longrightarrow & \Sigma X \\
\| & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \| \\
X & \longrightarrow & Y' & \xrightarrow{u} & Z' & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma X' & = & \Sigma X' & \longrightarrow & 0
\end{array}$$

be a commutative diagram in $\mathcal{T}$ in which all rows and columns are triangles in Δ.

(1) *If $\mathcal{T}(C, \beta)$ is epic, then so is $\mathcal{T}(C, \beta')$. Moreover, if the third vertical triangle is $\mathcal{T}(C, -)$-exact, then so is the second vertical triangle.*

(2) *If $\mathcal{T}(u', C)$ is epic, then so is $\mathcal{T}(u, C)$. Moreover, if the triangle $Y \longrightarrow Z \longrightarrow \Sigma X \longrightarrow \Sigma Y$ is $\mathcal{T}(-, C)$-exact, then so is the triangle $Y' \longrightarrow Z' \longrightarrow \Sigma X \longrightarrow \Sigma Y'$.*

(3) *If the third vertical and the second horizontal triangles are $\mathcal{T}(-, C)$-exact, then so are the second vertical and the third horizontal triangles.*

(4) *If the triangles $Y \longrightarrow Y' \longrightarrow \Sigma X' \longrightarrow \Sigma Y$ and $Y' \longrightarrow Z' \longrightarrow \Sigma X \longrightarrow \Sigma Y'$ are $\mathcal{T}(C, -)$-exact, then so are the triangles $Y \longrightarrow Z \longrightarrow \Sigma X \longrightarrow \Sigma Y$ and $Z \longrightarrow Z' \longrightarrow \Sigma X' \longrightarrow \Sigma Z$.*

Proof (1) The first assertion has been proved in [15, Lemma 2.1], so the second one is clear.

Similarly, we get the assertions (2), (3) and (4). □

3 (Pre)resolving Subcategories and Homological Dimension

Before giving the definition of (pre)resolving subcategories of a triangulated category, we give the following

Definition 3.1 *Let $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{J}$ be subcategories of $\mathcal{T}$ with $\mathcal{C} \subseteq \mathcal{J}$. Then $\mathcal{C}$ is called a ξ-proper generator for $\mathcal{J}$ if for any object Z in $\mathcal{J}$, there exists a $\mathcal{T}(\mathcal{E}, -)$-exact triangle*

$$X \longrightarrow C \longrightarrow Z \longrightarrow \Sigma X$$

in ξ with C an object in $\mathcal{C}$ and X an object in $\mathcal{J}$. Dually, a ξ-coproper cogenerator for $\mathcal{J}$ is defined.

Now we introduce the notion of (pre)resolving subcategories of a triangulated category $\mathcal{T}$.

Definition 3.2 *Let $\mathcal{E}$ and $\mathcal{J}$ be subcategories of $\mathcal{T}$. Then $\mathcal{J}$ is called an $\mathcal{E}$-preresolving subcategory of $\mathcal{T}$ if the following conditions are satisfied.*

(1) *$\mathcal{J}$ admits a ξ-proper generator $\mathcal{C}$.*

(2) *$\mathcal{J}$ is closed under ξ-proper extensions, that is, for any $\mathcal{T}(\mathcal{E},-)$-exact triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ, if X and Z are in $\mathcal{J}$, then so is Y.

An $\mathcal{E}$-preresolving subcategory $\mathcal{J}$ is called $\mathcal{E}$-resolving if the following condition is satisfied.

(3) *$\mathcal{J}$ is closed under hokernels of ξ-proper epimorphisms, that is, for any $\mathcal{T}(\mathcal{E},-)$-exact triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ, if Y and Z are in $\mathcal{J}$, then so is X.

Dually, an $\mathcal{E}$-(pre)coresolving subcategory *of $\mathcal{T}$ is defined.*

The following two results play an important role in the sequel.

Proposition 3.3 *Let $\mathcal{J}$ be an $\mathcal{E}$-preresolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$, and let*

$$0 \longrightarrow M \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0 \tag{3.1}$$

be a ξ-exact complex in $\mathcal{T}$ with both T_0 and T_1 objects in $\mathcal{J}$. Then

(1) *There exists a ξ-exact complex*

$$0 \longrightarrow M \longrightarrow T \longrightarrow C \longrightarrow A \longrightarrow 0 \tag{3.2}$$

in $\mathcal{T}$ with T an object in $\mathcal{J}$ and C an object in $\mathcal{C}$.

(2) *If* (3.1) *is $\mathcal{T}(\mathcal{E},-)$-exact, then so is* (3.2).

Proof (1) Since $\mathcal{C}$ is a ξ-proper generator for $\mathcal{J}$, there exists a $\mathcal{T}(\mathcal{E},-)$-exact triangle

$$T_0' \longrightarrow C \longrightarrow T_0 \longrightarrow \Sigma T_0'$$

in ξ with C an object in $\mathcal{C}$ and T_0' an object in $\mathcal{J}$. By assumption, we have the following two triangles

$$M \longrightarrow T_1 \longrightarrow K_1 \longrightarrow \Sigma M \quad \text{and} \quad K_1 \longrightarrow T_0 \longrightarrow A \longrightarrow \Sigma K_1.$$

Applying base change for the triangle $\Sigma^{-1}A \longrightarrow K_1 \longrightarrow T_0 \longrightarrow A$ along the morphism $C \longrightarrow T_0$, we get the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_0' & = & T_0' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
\Sigma^{-1}A & \longrightarrow & W & \xrightarrow{g} & C & \longrightarrow & A \\
\| & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle f'} & & \| \\
\Sigma^{-1}A & \longrightarrow & K_1 & \xrightarrow{g'} & T_0 & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma T_0' & = & \Sigma T_0' & \longrightarrow & 0.
\end{array} \tag{3.3}$$

Because $gh = f$ is ξ-monic, h is ξ-monic by Lemma 2.6. So the triangle

$$T_0' \longrightarrow W \longrightarrow K_1 \longrightarrow \Sigma T_0'$$

is in ξ. Notice that the third vertical triangle is $\mathcal{T}(\mathcal{E},-)$-exact, so the second vertical triangle is also $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9. Since the third vertical triangle and the triangle $K_1 \longrightarrow T_0 \longrightarrow A \longrightarrow \Sigma K_1$ are in ξ, the triangle

$$W \longrightarrow C \longrightarrow A \longrightarrow \Sigma W$$

is in ξ by Lemma 2.7 (1). Applying base change for the triangle $M \longrightarrow T_1 \longrightarrow K_1 \longrightarrow \Sigma M$ along the morphism $W \longrightarrow K_1$, we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_0' & = & T_0' & \longrightarrow & 0 \\ \downarrow & & \downarrow u & & \downarrow h & & \downarrow \\ M & \longrightarrow & T & \xrightarrow{v} & W & \longrightarrow & \Sigma M \\ \| & & \downarrow u' & & \downarrow h' & & \| \\ M & \longrightarrow & T_1 & \xrightarrow{v'} & K_1 & \longrightarrow & \Sigma M \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma T_0' & = & \Sigma T_0' & \longrightarrow & 0. \end{array} \tag{3.4}$$

Because $vu = h$ is ξ-monic, u is ξ-monic by Lemma 2.6. So the second vertical triangle is in ξ. Notice that the third vertical triangle is $\mathcal{T}(\mathcal{E},-)$-exact, so the second vertical triangle is also $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9. Since $\mathcal{J}$ is closed under ξ-proper extensions, we have that T is an object in $\mathcal{J}$. Since ξ is closed under base change, the triangle

$$M \longrightarrow T \longrightarrow W \longrightarrow \Sigma M$$

is in ξ. Thus

$$0 \longrightarrow M \longrightarrow T \longrightarrow C \longrightarrow A \longrightarrow 0$$

is a ξ-exact complex.

(2) By assumption, the triangles $M \longrightarrow T_1 \longrightarrow K_1 \longrightarrow \Sigma M$ and $K_1 \longrightarrow T_0 \longrightarrow A \longrightarrow \Sigma K_1$ are both $\mathcal{T}(\mathcal{E},-)$-exact. Notice that in the diagram (3.3), the third vertical triangle is $\mathcal{T}(\mathcal{E},-)$-exact, so the triangle

$$W \longrightarrow C \longrightarrow A \longrightarrow \Sigma W$$

is $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9 and the snake lemma. Because the third horizontal triangle in the diagram (3.4) is $\mathcal{T}(\mathcal{E},-)$-exact, the triangle

$$M \longrightarrow T \longrightarrow W \longrightarrow \Sigma M$$

is also $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9. Therefore, we conclude that the ξ-exact complex (3.2) is $\mathcal{T}(\mathcal{E},-)$-exact. □

Furthermore, we have the following

Proposition 3.4 *Let $\mathcal{J}$ be an $\mathcal{E}$-preresolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$ and $n \geq 1$. Let*

$$0 \longrightarrow M \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0 \tag{3.5}$$

be a ξ-exact complex in $\mathcal{T}$ with all T_i objects in $\mathcal{J}$. Then

(1) *There exist a ξ-exact complex*

$$0 \longrightarrow N \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0 \tag{3.6}$$

in $\mathcal{T}$ with all C_i objects in $\mathcal{C}$ and a $\mathcal{T}(\mathcal{E},-)$-exact triangle

$$T \longrightarrow N \longrightarrow M \longrightarrow \Sigma T$$

in ξ with T an object in $\mathcal{J}$.

(2) *If* (3.5) *is $\mathcal{T}(\mathcal{E},-)$-exact, then so is* (3.6).

Proof Since $\mathcal{C}$ is a ξ-proper generator for $\mathcal{J}$, there exists a $\mathcal{T}(\mathcal{E},-)$-exact triangle

$$T_0' \longrightarrow C_0 \longrightarrow T_0 \longrightarrow \Sigma T_0'$$

in ξ with C_0 an object in $\mathcal{C}$ and T_0' an object in $\mathcal{J}$.

We proceed by induction on n. The case for $n = 1$ has been proved in the proof of Proposition 3.3. Now suppose $n \geq 2$. By assumption, we have the following two ξ-exact complexes

$$0 \longrightarrow M \longrightarrow T_{n-1} \longrightarrow T_{n-2} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow K_2 \longrightarrow 0,$$

$$0 \longrightarrow K_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0.$$

By Proposition 3.3, we get a ξ-exact complex

$$0 \longrightarrow K_2 \longrightarrow T_1' \longrightarrow C_0 \longrightarrow A \longrightarrow 0,$$

in $\mathcal{T}$ with T_1' an object in $\mathcal{J}$ and C_0 an object in $\mathcal{C}$ with both $K_2 \longrightarrow T_1' \longrightarrow K_1' \longrightarrow \Sigma K_2$ and $K_1' \longrightarrow C_0 \longrightarrow A \longrightarrow \Sigma K_1'$ triangles in ξ. Then we get a ξ-exact complex

$$0 \longrightarrow M \longrightarrow T_{n-1} \longrightarrow T_{n-2} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow T_1' \longrightarrow K_1' \longrightarrow 0.$$

By the induction hypothesis, we get a ξ-exact complex

$$0 \longrightarrow N \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow K_1' \longrightarrow 0,$$

and

$$0 \longrightarrow N \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

is the desired ξ-exact complex.

(2) It follows inductively from Proposition 3.3 (2). □

We introduce the homological dimension and codimension of an object in $\mathcal{T}$ relative to subcategories of $\mathcal{T}$ as follows.

Definition 3.5 *Let $\mathcal{C}$ and $\mathcal{E}$ be subcategories of $\mathcal{T}$ and A an object in $\mathcal{T}$. The $\mathcal{C}_\mathcal{E}$-dimension of A, written $\mathcal{C}_\mathcal{E}$-dim A, is defined by*

$$\mathcal{C}_\mathcal{E}\text{-}\dim A = \inf\{n \geq 0 \mid \textit{there exists a } \mathcal{T}(\mathcal{E},-)\textit{-exact } \xi\textit{-exact complex } 0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_0 \longrightarrow A \longrightarrow 0 \textit{ in } \mathcal{T} \textit{ with all } C_i \textit{ objects in } \mathcal{C}\}.$$

Dually, the $\mathcal{C}_\mathcal{E}$-codimension of A, written $\mathcal{C}_\mathcal{E}$-codim A, is defined by

$$\mathcal{C}_\mathcal{E}\text{-codim}\, A = \inf\{n \geq 0 \mid \textit{there exists a } \mathcal{T}(-,\mathcal{E})\textit{-exact } \xi\textit{-exact complex } 0 \longrightarrow A \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0 \textit{ in } \mathcal{T} \textit{ with all } C^i \textit{ objects in } \mathcal{C}\}.$$

In the case for $\mathcal{C} = \mathcal{E}$, we write $\mathcal{C}$-dim $A := \mathcal{C}_{\mathcal{E}}$-dim A and $\mathcal{C}$-codim $A := \mathcal{C}_{\mathcal{E}}$-codim A.

The following result gives a criterion for computing the $\mathcal{J}_{\mathcal{E}}$-dimension of an object in $\mathcal{T}$.

Theorem 3.6 *Let $\mathcal{J}$ be an $\mathcal{E}$-preresolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$. Then for any object A in $\mathcal{T}$ and $n \geq 0$, the following statements are equivalent.*

(1) $\mathcal{J}_{\mathcal{E}}$*-dim* $A \leq n$.

(2) *There exists a $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex*

$$0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with all C_i objects in $\mathcal{C}$ and K_n an object in $\mathcal{J}$.

Proof (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) We proceed by induction on n. The case for $n = 0$ is trivial. If $n = 1$, then by Proposition 3.3 with $M = 0$, we get the desired $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$0 \longrightarrow T \longrightarrow C \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with T an object in $\mathcal{J}$ and C an object in $\mathcal{C}$.

Now suppose $n \geq 2$. By assumption, we have the following two $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complexes

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow T_{n-2} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow K_2 \longrightarrow 0,$$

$$0 \longrightarrow K_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0.$$

By Proposition 3.3, we get a $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$0 \longrightarrow K_2 \longrightarrow T_1' \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with T_1' an object in $\mathcal{J}$ and C_0 an object in $\mathcal{C}$ with both $K_2 \longrightarrow T_1' \longrightarrow K_1' \longrightarrow \Sigma K_2$ and $K_1' \longrightarrow C_0 \longrightarrow A \longrightarrow \Sigma K_1'$ $\mathcal{T}(\mathcal{E}, -)$-exact triangles in ξ. Thus

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow T_{n-2} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow T_1' \longrightarrow K_1' \longrightarrow 0$$

is a $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex. Then by the induction hypothesis, we get a $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow K_1' \longrightarrow 0.$$

Thus we get the desired $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0.$$

The following three results generalize [15, Theorems 2.3, 2.7 and 2.9]. The arguments here are similar to that in [15], so we omit them.

Theorem 3.7 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and*

$$X \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \Sigma X$$

a $\mathcal{T}(\mathcal{E}, -)$-exact triangle in ξ. Assume that $\mathcal{C}$ is closed under hokernels of ξ-proper epimorphisms, and let

$$\cdots \longrightarrow C_n^0 \longrightarrow C_{n-1}^0 \longrightarrow \cdots \longrightarrow C_1^0 \longrightarrow C_0^0 \longrightarrow X^0 \longrightarrow 0$$

and

$$\cdots \longrightarrow C_n^1 \longrightarrow C_{n-1}^1 \longrightarrow \cdots \longrightarrow C_1^1 \longrightarrow C_0^1 \longrightarrow X^1 \longrightarrow 0$$

be $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complexes with all C_i^0, C_i^1 objects in $\mathcal{C}$. Then there exist a $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$\cdots \longrightarrow C_{n+1}^1 \oplus C_n^0 \longrightarrow C_n^1 \oplus C_{n-1}^0 \longrightarrow \cdots \longrightarrow C_2^1 \oplus C_1^0 \longrightarrow C \longrightarrow X \longrightarrow 0$$

and a $\mathcal{T}(\mathcal{E}, -)$-exact triangle

$$C \longrightarrow C_1^1 \oplus C_0^0 \longrightarrow C_0^1 \longrightarrow \Sigma C$$

in ξ.

Theorem 3.8 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and*

$$X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow \Sigma X_1 \tag{3.7}$$

a triangle in ξ. Let

$$\cdots \longrightarrow C_0^n \longrightarrow C_0^{n-1} \longrightarrow \cdots \longrightarrow C_0^1 \longrightarrow C_0^0 \longrightarrow X_0 \longrightarrow 0 \tag{3.8}$$

and

$$\cdots \longrightarrow C_1^n \longrightarrow C_1^{n-1} \longrightarrow \cdots \longrightarrow C_1^1 \longrightarrow C_1^0 \longrightarrow X_1 \longrightarrow 0 \tag{3.9}$$

be $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complexes with all C_0^i, C_1^i objects in $\mathcal{C}$. Then

(1) *If* (3.7) *is $\mathcal{T}(\mathcal{E}, -)$-exact, then we have the following $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex*

$$\cdots \longrightarrow C_0^n \oplus C_1^{n-1} \longrightarrow \cdots \longrightarrow C_0^2 \oplus C_1^1 \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_0^0 \longrightarrow X \longrightarrow 0. \tag{3.10}$$

(2) *If* (3.7)–(3.9) *are $\mathcal{T}(-, \mathcal{E})$-exact, then so is* (3.10).

Theorem 3.9 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and*

$$Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \Sigma Y \tag{3.11}$$

a triangle in ξ. Let

$$0 \longrightarrow Y^0 \longrightarrow C_0^0 \longrightarrow C_1^0 \longrightarrow \cdots \longrightarrow C_n^0 \longrightarrow \cdots \tag{3.12}$$

and

$$0 \longrightarrow Y^1 \longrightarrow C_0^1 \longrightarrow C_1^1 \longrightarrow \cdots \longrightarrow C_n^1 \longrightarrow \cdots \tag{3.13}$$

be $\mathcal{T}(-, \mathcal{E})$-exact ξ-exact complexes with all C_i^0, C_i^1 objects in $\mathcal{C}$. Then

(1) *If the triangle* (3.11) *is $\mathcal{T}(-, \mathcal{E})$-exact, then we have the following $\mathcal{T}(-, \mathcal{E})$-exact ξ-exact complex*

$$0 \longrightarrow Y \longrightarrow C_0^0 \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_1^1 \oplus C_2^0 \longrightarrow \cdots \longrightarrow C_{n-1}^1 \oplus C_n^0 \longrightarrow \cdots \tag{3.14}$$

(2) *If* (3.11)–(3.13) *are $\mathcal{T}(\mathcal{E}, -)$-exact, then so is* (3.14).

The following lemma is similar to Horseshoe Lemma, which plays an important role in this paper.

Lemma 3.10 *Let $\mathcal{E}$ be a subcategory of $\mathcal{T}$ and*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{3.15}$$

a triangle in ξ.

(1) *Consider the following commutative diagram*

$$\begin{array}{ccccccc}
K_1^X & & & & K_1^Z & & \\
\downarrow & & & & \downarrow & & \\
C_0^X & \longrightarrow & C_0^X \oplus C_0^Z & \longrightarrow & C_0^Z & \xrightarrow{0} & \Sigma C_0^X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & & & \downarrow & & \\
\Sigma K_1^X & & & & \Sigma K_1^Z, & &
\end{array}$$

in which the first vertical and the third vertical triangles are in ξ. Then we have the following commutative diagram except the middle square on the top which anticommutes

$$\begin{array}{ccccccc}
K_1^X & \longrightarrow & W_1 & \longrightarrow & K_1^Z & \longrightarrow & \Sigma K_1^X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_0^X & \longrightarrow & C_0^X \oplus C_0^Z & \longrightarrow & C_0^Z & \longrightarrow & \Sigma C_0^X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma K_1^X & \longrightarrow & \Sigma W_1 & \longrightarrow & \Sigma K_1^Z & \longrightarrow & \Sigma^2 K_1^X,
\end{array}$$

in which the first horizontal and the second vertical triangles are in ξ. Moreover,

(i) *If the first vertical and the third vertical triangles and the triangle* (3.15) *are $\mathcal{T}(\mathcal{E},-)$-exact, then so are the first horizontal and the second vertical triangles.*

(ii) *If the first vertical and the third vertical triangles and the triangle* (3.15) *are $\mathcal{T}(-,\mathcal{E})$-exact, then so are the first horizontal and the second vertical triangles.*

(2) *Consider the following commutative diagram*

$$\begin{array}{ccccccc}
\Sigma^{-1}K_X^1 & & & & \Sigma^{-1}K_Z^1 & & \\
\downarrow & & & & \downarrow & & \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_X^0 & \longrightarrow & C_X^0 \oplus C_Z^0 & \longrightarrow & C_Z^0 & \xrightarrow{0} & \Sigma C_X^0 \\
\downarrow & & & & \downarrow & & \\
K_X^1 & & & & K_Z^1, & &
\end{array}$$

in which the triangles $X \longrightarrow C_X^0 \longrightarrow K_X^1 \longrightarrow \Sigma X$ *and* $Z \longrightarrow C_Z^0 \longrightarrow K_Z^1 \longrightarrow \Sigma Z$ *are in* ξ*. Then we have the following commutative diagram except the middle square on the bottom which anticommutes*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_X^0 & \longrightarrow & C_X^0 \oplus C_Z^0 & \longrightarrow & C_Z^0 & \longrightarrow & \Sigma C_X^0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_X^1 & \longrightarrow & W^1 & \longrightarrow & K_Z^1 & \longrightarrow & \Sigma K_X^1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma X & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X, \end{array}$$

in which the third horizontal and the second vertical triangles are in ξ*. Moreover,*

(i) *If the first vertical and the third vertical triangles and the triangle* (3.15) *are* $\mathcal{T}(\mathcal{E},-)$*-exact, then so are the third horizontal and the second vertical triangles.*

(ii) *If the first vertical and the third vertical triangles and the triangle* (3.15) *are* $\mathcal{T}(-,\mathcal{E})$*-exact, then so are the third horizontal and the second vertical triangles.*

Proof It is similar to the proof of [2, Proposition 4.11], so we omit it. □

The following results give some relations of $\mathcal{J}_{\mathcal{E}}$-dimension with the terms of a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ in ξ.

Proposition 3.11 *Let* $\mathcal{C} \subseteq \mathcal{E}$ *be subcategories of* $\mathcal{T}$ *and* $\mathcal{J}$ *an* $\mathcal{E}$*-resolving subcategory of* $\mathcal{T}$ *admitting a* ξ*-proper generator* $\mathcal{C}$*, and let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a $\mathcal{T}(\mathcal{E},-)$*-exact triangle in* ξ *with* Z *an object in* $\mathcal{J}$*. Then* $\mathcal{J}_{\mathcal{E}}\text{-}\dim X = \mathcal{J}_{\mathcal{E}}\text{-}\dim Y$*.*

Proof Because $\mathcal{J}$ is $\mathcal{E}$-resolving by assumption, we have that $\mathcal{J}_{\mathcal{E}}\text{-}\dim X = 0$ if and only if $\mathcal{J}_{\mathcal{E}}\text{-}\dim Y = 0$.

Now suppose that $\mathcal{J}_{\mathcal{E}}\text{-}\dim X = n \geq 1$ and

$$K_1^X \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma K_1^X$$

is a $\mathcal{T}(\mathcal{E},-)$-exact triangle with T_0 an object in $\mathcal{J}$ and $\mathcal{J}_{\mathcal{E}}\text{-}\dim K_1^X \leq n-1$. Since $\mathcal{J}$ admits a ξ-proper generator $\mathcal{C}$, there exists a $\mathcal{T}(\mathcal{E},-)$-exact triangle

$$K_1 \longrightarrow C_0 \longrightarrow Z \longrightarrow \Sigma K_1$$

in ξ with C_0 an object in $\mathcal{C}$ and K_1 an object in $\mathcal{J}$. By [15, Lemma 2.2], we get the following commutative diagram

$$\begin{array}{ccccccc} T_0 & \longrightarrow & T_0 \oplus C_0 & \longrightarrow & C_0 & \stackrel{0}{\longrightarrow} & \Sigma T_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X. \end{array}$$

By Lemma 3.10, we get the following commutative diagram except the middle square on the top which anticommutes

$$\begin{array}{ccccccc}
K_1^X & \longrightarrow & W_1 & \longrightarrow & K_1 & \longrightarrow & \Sigma K_1^X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_0 & \longrightarrow & T_0 \oplus C_0 & \longrightarrow & C_0 & \longrightarrow & \Sigma T_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma K_1^X & \longrightarrow & \Sigma W_1 & \longrightarrow & \Sigma K_1 & \longrightarrow & \Sigma^2 K_1^X,
\end{array}$$

in which the first horizontal and the second vertical triangles are in ξ, and the first horizontal and the second vertical triangles are $\mathcal{T}(\mathcal{E}, -)$-exact. Note that $T_0 \oplus C_0$ is an object in $\mathcal{J}$. By the induction hypothesis, we have $\mathcal{J}_{\mathcal{E}}\text{-dim}\, Y \leq \mathcal{J}_{\mathcal{E}}\text{-dim}\, W_1 + 1 \leq \mathcal{J}_{\mathcal{E}}\text{-dim}\, K_1^X + 1 \leq n = \mathcal{J}_{\mathcal{E}}\text{-dim}\, X$.

Conversely, suppose that $\mathcal{J}_{\mathcal{E}}\text{-dim}\, Y = n \geq 1$ and

$$K_1^Y \longrightarrow T_0 \longrightarrow Y \longrightarrow \Sigma K_1^Y$$

is a $\mathcal{T}(\mathcal{E}, -)$-exact triangle with T_0 an object in $\mathcal{J}$ and $\mathcal{J}_{\mathcal{E}}\text{-dim}\, K_1^Y \leq n - 1$. Applying base change for the triangle $\Sigma^{-1}Z \longrightarrow X \longrightarrow Y \longrightarrow Z$ along the morphism $T_0 \longrightarrow Y$, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1^Y & = & K_1^Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
\Sigma^{-1}Z & \longrightarrow & T & \stackrel{g}{\longrightarrow} & T_0 & \longrightarrow & Z \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1^Y & = & \Sigma K_1^Y & \longrightarrow & 0.
\end{array}$$

Since $gh = f$ is ξ-monic, by Lemma 2.6 we have that h is ξ-monic and the second vertical triangle is in ξ. Since the third vertical triangle and the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ are in ξ, the triangle $T \longrightarrow T_0 \longrightarrow Z \longrightarrow \Sigma T$ is also in ξ by Lemma 2.7 (1). Since the third vertical triangle and the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ are $\mathcal{T}(\mathcal{E}, -)$-exact, the second vertical triangle and the triangle $T \longrightarrow T_0 \longrightarrow Z \longrightarrow \Sigma T$ are $\mathcal{T}(\mathcal{E}, -)$-exact by Lemma 2.9 and the snake lemma. Moreover, since $\mathcal{J}$ is closed under hokernels of ξ-proper epimorphisms, T is an object in $\mathcal{J}$. Thus $\mathcal{J}_{\mathcal{E}}\text{-dim}\, X \leq \mathcal{J}_{\mathcal{E}}\text{-dim}\, K_1^Y + 1 \leq n = \mathcal{J}_{\mathcal{E}}\text{-dim}\, Y$. □

As a consequence of Proposition 3.11, we get the following

Corollary 3.12 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and $\mathcal{J}$ an $\mathcal{E}$-resolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$. If $\mathcal{J}_{\mathcal{E}}\text{-dim}\, K = n$ for an object K in $\mathcal{T}$, then $\mathcal{J}_{\mathcal{E}}\text{-dim}\, K \oplus M = n$ for any object M in $\mathcal{J}$.*

Proof Applying Proposition 3.11 to the triangle $K \longrightarrow K \oplus M \longrightarrow M \longrightarrow \Sigma K$ in ξ, we have $\mathcal{J}_{\mathcal{E}}\text{-}\dim K \oplus M = \mathcal{J}_{\mathcal{E}}\text{-}\dim K = n$. □

Proposition 3.13 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and $\mathcal{J}$ an $\mathcal{E}$-resolving subcategory admitting a ξ-proper generator $\mathcal{C}$. Let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a $\mathcal{T}(\mathcal{E}, -)$-exact triangle in ξ with X an object in $\mathcal{J}$ and neither Y nor Z in $\mathcal{J}$. Then $\mathcal{J}_{\mathcal{E}}\text{-}\dim Y = \mathcal{J}_{\mathcal{E}}\text{-}\dim Z$.

Proof Suppose that $\mathcal{J}_{\mathcal{E}}\text{-}\dim Y = n \geq 1$ and

$$K_1^Y \longrightarrow T_0^Y \longrightarrow Y \longrightarrow \Sigma K_1^Y$$

is a $\mathcal{T}(\mathcal{E}, -)$-exact triangle in ξ with T_0^Y an object in $\mathcal{J}$ and $\mathcal{J}_{\mathcal{E}}\text{-}\dim K_1^Y \leq n-1$. Applying base change for the triangle $\Sigma^{-1} Z \longrightarrow X \longrightarrow Y \longrightarrow Z$ along the morphism $T_0^Y \longrightarrow Y$, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1^Y & = & K_1^Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
\Sigma^{-1} Z & \longrightarrow & T & \xrightarrow{g} & T_0^Y & \longrightarrow & Z \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1} Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1^Y & = & \Sigma K_1^Y & \longrightarrow & 0.
\end{array}$$

Since $gh = f$ is ξ-monic, by Lemma 2.6 we have that h is ξ-monic and the second vertical triangle is in ξ. Since the third vertical triangle and the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ are in ξ, the triangle $T \longrightarrow T_0^Y \longrightarrow Z \longrightarrow \Sigma T$ is also in ξ by Lemma 2.7 (1). Since the third vertical triangle and the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ are $\mathcal{T}(\mathcal{E}, -)$-exact, the second vertical triangle and the triangle $T \longrightarrow T_0^Y \longrightarrow Z \longrightarrow \Sigma T$ are $\mathcal{T}(\mathcal{E}, -)$-exact by Lemma 2.9 and the snake lemma. It follows from Proposition 3.11 that $\mathcal{J}_{\mathcal{E}}\text{-}\dim Z \leq \mathcal{J}_{\mathcal{E}}\text{-}\dim T + 1 = \mathcal{J}_{\mathcal{E}}\text{-}\dim K_1^Y + 1 \leq n = \mathcal{J}_{\mathcal{E}}\text{-}\dim Y$.

Conversely, suppose that $\mathcal{J}_{\mathcal{E}}\text{-}\dim Z = n \geq 1$ and

$$K_1^Z \to T_0^Z \to Z \to \Sigma K_1^Z$$

is a $\mathcal{T}(\mathcal{E}, -)$-exact triangle in ξ with T_0^Z an object in $\mathcal{J}$ and $\mathcal{J}_{\mathcal{E}}\text{-}\dim K_1^Z \leq n-1$. Applying base change for the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ along the morphism $T_0^Z \longrightarrow Z$, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1^Z & = & K_1^Z & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
X & \longrightarrow & W & \xrightarrow{g} & T_0^Z & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1^Z & = & \Sigma K_1^Z & \longrightarrow & 0,
\end{array}$$

in which the second horizontal triangle is in ξ. Since the third vertical and the third horizontal triangles are $\mathcal{T}(\mathcal{E},-)$-exact, the second vertical and the second horizontal triangles are also $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9. Since $\mathcal{J}$ is closed under ξ-proper extensions, we have that W is an object in $\mathcal{J}$ and $\mathcal{J}_{\mathcal{E}}\text{-}\dim Y \leq \mathcal{J}_{\mathcal{E}}\text{-}\dim K_1^Z + 1 \leq n = \mathcal{J}_{\mathcal{E}}\text{-}\dim Z$. $\square$

The following result gives a sufficient condition such that for an object A in $\mathcal{T}$, if $\mathcal{J}_{\mathcal{E}}$-$\dim A \leq n$, then all "$n$-$\mathcal{C}$-syzygies" of A are objects in $\mathcal{J}$.

Theorem 3.14 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and $\mathcal{J}$ an $\mathcal{E}$-resolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$, and let A be an object of $\mathcal{T}$ with $\mathcal{J}_{\mathcal{E}}\text{-}\dim A \leq n$. Then we have*

(1) *For any $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex*

$$0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with all C_i objects in $\mathcal{C}$, we have that K_n is an object in $\mathcal{J}$.

(2) *If*

$$X \longrightarrow C \longrightarrow A \longrightarrow \Sigma X$$

is a $\mathcal{T}(\mathcal{E},-)$-exact triangle in ξ with C an object in $\mathcal{C}$, then $\mathcal{J}_{\mathcal{E}}\text{-}\dim X \leq n-1$.

Proof Let $\mathcal{J}_{\mathcal{E}}\text{-}\dim A \leq n$. By Theorem 3.6, there exists a $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex

$$0 \longrightarrow T_n \longrightarrow C'_{n-1} \longrightarrow C'_{n-2} \longrightarrow \cdots \longrightarrow C'_2 \longrightarrow C'_1 \longrightarrow C'_0 \longrightarrow A \longrightarrow 0 \tag{3.16}$$

with T_n an object in $\mathcal{J}$ and all C'_i objects in $\mathcal{C}$.

(1) Applying Theorem 3.8 to a $\mathcal{T}(\mathcal{E},-)$-exact triangle $A \longrightarrow A \longrightarrow 0 \longrightarrow \Sigma A$ in ξ, we get a $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow T_n \oplus C_{n-1} \longrightarrow C'_{n-1} \oplus C_{n-2} \longrightarrow \cdots \longrightarrow C'_1 \oplus C_0 \longrightarrow C'_0 \longrightarrow 0.$$

Since $\mathcal{J}$ is closed under ξ-proper epimorphisms, K_n is an object in $\mathcal{J}$.

(2) From (3.16) we get a $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact triangle

$$K_1 \longrightarrow C'_0 \longrightarrow A \longrightarrow \Sigma K_1$$

in ξ with $\mathcal{J}_{\mathcal{E}}\text{-}\dim K_1 \leq n-1$. Applying base change for the triangle $X \longrightarrow C \longrightarrow A \longrightarrow \Sigma X$

along the morphism $C_0' \longrightarrow A$, we get the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1 & = & K_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle w} & & \downarrow \\
X & \longrightarrow & W & \xrightarrow{v} & C_0' & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & C & \longrightarrow & A & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1 & = & \Sigma K_1 & \longrightarrow & 0,
\end{array}$$

in which the second horizontal triangle is in ξ. Since the third vertical and the third horizontal triangles are $\mathcal{T}(\mathcal{E},-)$-exact, the second vertical and the second horizontal triangles are also $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9. Because C is an object in $\mathcal{C}(\subseteq \mathcal{E})$, we have that the second vertical triangle splits and $W \cong K_1 \oplus C$. Since ξ is closed under isomorphisms, the triangle $X \longrightarrow K_1 \oplus C \longrightarrow C_0' \longrightarrow \Sigma X$ is in ξ with C_0' an object in $\mathcal{C}(\subseteq \mathcal{J})$. By Proposition 3.11 and Corollary 3.12, we have $\mathcal{J}_{\mathcal{E}}\text{-}\dim X = \mathcal{J}_{\mathcal{E}}\text{-}\dim K_1 \oplus C = \mathcal{J}_{\mathcal{E}}\text{-}\dim K_1 \leq n-1$. □

We use $\mathcal{J}_{\mathcal{E}}\text{-}\dim^{\leq n}$ to denote the subcategory of $\mathcal{T}$ consisting of objects with $\mathcal{J}_{\mathcal{E}}$-dimension at most n.

Corollary 3.15 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ and $\mathcal{J}$ an $\mathcal{E}$-resolving subcategory of $\mathcal{T}$ admitting a ξ-proper generator $\mathcal{C}$. If $\mathcal{J}$ is closed under direct summands, then so is $\mathcal{J}_{\mathcal{E}}\text{-}\dim^{\leq n}$ for any $n \geq 0$.*

Proof We proceed by induction on n. The case $n = 0$ is trivial. Let $n \geq 1$ and let X be an object in $\mathcal{T}$ with $\mathcal{J}_{\mathcal{E}}\text{-}\dim X \leq n$ and $X = X_1 \oplus X_2$. By Theorem 3.6, there exists a $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{T}$ with K_n an object in $\mathcal{J}$ and all C_i objects in $\mathcal{C}$. Now applying base change for the triangle $\Sigma^{-1}X_1 \longrightarrow X_2 \longrightarrow X \longrightarrow X_1$ along the morphism $C_0 \longrightarrow X$, we get the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1 & = & K_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}X_1 & \longrightarrow & W_1 & \longrightarrow & C_0 & \longrightarrow & X_1 \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}X_1 & \longrightarrow & X_2 & \longrightarrow & X & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1 & = & \Sigma K_1 & \longrightarrow & 0.
\end{array}$$

Since the third vertical triangle and the triangle $X_2 \longrightarrow X \longrightarrow X_1 \longrightarrow \Sigma X_2$ are in ξ, the triangle $W_1 \longrightarrow C_0 \longrightarrow X_1 \longrightarrow \Sigma W_1$ is also in ξ by Lemma 2.7 (1). Since the third vertical triangle and the triangle $X_2 \longrightarrow X \longrightarrow X_1 \longrightarrow \Sigma X_2$ are $\mathcal{T}(\mathcal{E},-)$-exact, the triangle

$$W_1 \longrightarrow C_0 \longrightarrow X_1 \longrightarrow \Sigma W_1$$

is $\mathcal{T}(\mathcal{E},-)$-exact by Lemma 2.9 and the snake lemma. Similarly, we get a $\mathcal{T}(\mathcal{E},-)$-exact triangle

$$W_2 \longrightarrow C_0 \longrightarrow X_2 \longrightarrow \Sigma W_2$$

in ξ. By Theorem 3.8, we get the following two $\mathcal{T}(\mathcal{E},-)$-exact triangles

$$C_0 \oplus C_1 \longrightarrow C_0 \longrightarrow X_1 \longrightarrow \Sigma(C_0 \oplus C_1),$$

$$C_0 \oplus C_1 \longrightarrow C_0 \longrightarrow X_2 \longrightarrow \Sigma(C_0 \oplus C_1)$$

in ξ. Repeating this process, we get the following two $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complexes

$$0 \longrightarrow Y_1 \longrightarrow \bigoplus_{i=1}^{n-1} C_i \longrightarrow \bigoplus_{i=1}^{n-2} C_i \longrightarrow \cdots \longrightarrow C_0 \oplus C_1 \longrightarrow C_0 \longrightarrow X_1 \longrightarrow 0,$$

$$0 \longrightarrow Y_2 \longrightarrow \bigoplus_{i=1}^{n-1} C_i \longrightarrow \bigoplus_{i=1}^{n-2} C_i \longrightarrow \cdots \longrightarrow C_0 \oplus C_1 \longrightarrow C_0 \longrightarrow X_2 \longrightarrow 0.$$

Since ξ is closed under finite coproducts, we get a $\mathcal{T}(\mathcal{E},-)$-exact ξ-exact complex

$$\begin{aligned} 0 \longrightarrow Y_1 \oplus Y_2 \longrightarrow \bigoplus_{i=1}^{n-1} C_i \oplus \bigoplus_{i=1}^{n-1} C_i \longrightarrow \bigoplus_{i=1}^{n-2} C_i \oplus \bigoplus_{i=1}^{n-2} C_i \longrightarrow \\ \cdots \longrightarrow C_0 \oplus C_1 \oplus C_0 \oplus C_1 \longrightarrow C_0 \oplus C_0 \longrightarrow X \longrightarrow 0. \end{aligned}$$

By Theorem 3.14, $Y_1 \oplus Y_2$ is an object in $\mathcal{J}$. Because $\mathcal{J}$ is closed under direct summands by assumption, both Y_1 and Y_2 are objects in $\mathcal{J}$. Thus $\mathcal{J}_{\mathcal{E}}\text{-dim}\, X_1 \leq n$ and $\mathcal{J}_{\mathcal{E}}\text{-dim}\, X_2 \leq n$. □

4 Relative Gorenstein Categories

In this section, we give some applications of the results obtained in Section 3. We first introduce the following

Definition 4.1 *Let $\mathcal{C}$ and $\mathcal{E}$ be subcategories of $\mathcal{T}$ and X an object in $\mathcal{T}$. A complete $\mathcal{C}_{\mathcal{E}}(\xi)$-resolution of X is a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact ξ-exact complex*

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

in $\mathcal{T}$ with all C_i, C^i objects in $\mathcal{C}$ such that both

$$K_1 \longrightarrow C_0 \longrightarrow X \longrightarrow \Sigma K_1 \quad \text{and} \quad X \longrightarrow C^0 \longrightarrow K^1 \longrightarrow \Sigma X$$

are corresponding triangles in ξ. The $(\mathcal{E},\mathcal{C})$-Gorenstein category *is defined as*

$$\mathcal{GC}_{\mathcal{E}}(\xi) = \{X \text{ is in } \mathcal{T} \mid X \text{ admits a complete } \mathcal{C}_{\mathcal{E}}(\xi)\text{-resolution}\}.$$

Recall from [2] that an object P (resp. I) in $\mathcal{T}$ is called *ξ-projective* (resp. *ξ-injective*) if for any triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ in ξ, the induced complex

$$0 \longrightarrow \mathcal{T}(P,X) \longrightarrow \mathcal{T}(P,Y) \longrightarrow \mathcal{T}(P,Z) \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow \mathcal{T}(Z,I) \longrightarrow \mathcal{T}(Y,I) \longrightarrow \mathcal{T}(X,I) \longrightarrow 0)$$

is exact. We use $\mathcal{P}(\xi)$ (resp. $\mathcal{I}(\xi)$) to denote the full subcategory of $\mathcal{T}$ consisting of ξ-projective (resp. ξ-injective) objects.

Remark 4.2 (1) If $\mathcal{E} = \mathcal{C}$, then $\mathcal{GC}_{\mathcal{E}}(\xi) = \mathcal{GC}(\xi)$, where $\mathcal{GC}(\xi)$ is the Gorenstein category defined in [15].

(2) If $\mathcal{E} = \mathcal{C} = \mathcal{P}(\xi)$ (resp. $\mathcal{C} = \mathcal{E} = \mathcal{I}(\xi)$), then $\mathcal{GC}_{\mathcal{E}}(\xi)$ coincides with $\mathcal{GP}(\xi)$ (resp. $\mathcal{GI}(\xi)$), where $\mathcal{GP}(\xi)$ (resp. $\mathcal{GI}(\xi)$) is the full subcategory of $\mathcal{T}$ consisting of ξ-Gorenstein projective (resp. injective) objects in [1].

We have the following result.

Theorem 4.3 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$. Then we have*

(1) *$\mathcal{GC}_{\mathcal{E}}(\xi)$ is an $\mathcal{E}$-resolving subcategory of $\mathcal{T}$.*

(2) *$\mathcal{GC}_{\mathcal{E}}(\xi)$ is an $\mathcal{E}$-coresolving subcategory of $\mathcal{T}$.*

Proof (1) By assumption, for any object X in $\mathcal{GC}_{\mathcal{E}}(\xi)$, there exists a $\mathcal{T}(\mathcal{E}, -)$-exact triangle

$$K_1^X \longrightarrow C_0^X \longrightarrow X \longrightarrow \Sigma K_1$$

in ξ with K_1^X an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$ and C_0^X an object in $\mathcal{C}$. So $\mathcal{C}$ is a ξ-proper generator for $\mathcal{GC}_{\mathcal{E}}(\xi)$.

Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a $\mathcal{T}(\mathcal{E}, -)$-exact triangle in ξ. Suppose that Z is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$. Then there exists a $\mathcal{T}(\mathcal{E}, -)$-exact and $\mathcal{T}(-, \mathcal{E})$-exact triangle

$$K_1^Z \longrightarrow C_0^Z \longrightarrow Z \longrightarrow \Sigma K_1^Z$$

in ξ with K_1^Z an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$ and C_0^Z an object in $\mathcal{C}$. Applying base change for the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ along the morphism $C_0^Z \longrightarrow Z$, we get the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1^Z & = & K_1^Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & W & \longrightarrow & C_0^Z & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1^Z & = & \Sigma K_1^Z & \longrightarrow & 0,
\end{array}$$

in which the second horizontal triangle is in ξ. Since the third horizontal triangle is $\mathcal{T}(\mathcal{E}, -)$-exact, so is the second horizontal triangle by Lemma 2.9. Because C_0^Z is an object in $\mathcal{C}(\subseteq \mathcal{E})$, the second horizontal triangle splits and is $\mathcal{T}(-, \mathcal{E})$-exact. Because the third vertical triangle is also $\mathcal{T}(-, \mathcal{E})$-exact, so is the third horizontal triangle by Lemma 2.9.

Claim 1 If X and Z are objects in $\mathcal{GC}_{\mathcal{E}}(\xi)$, then so is Y.

Since X and Z are objects in $\mathcal{GC}_{\mathcal{E}}(\xi)$, there exist $\mathcal{T}(\mathcal{E}, -)$-exact and $\mathcal{T}(-, \mathcal{E})$-exact triangles

$$K_1^X \longrightarrow C_0^X \longrightarrow X \longrightarrow \Sigma K_1^X, \quad X \longrightarrow C_X^0 \longrightarrow K_X^1 \longrightarrow \Sigma X,$$

$$K_1^Z \longrightarrow C_0^Z \longrightarrow Z \longrightarrow \Sigma K_1^Z, \quad Z \longrightarrow C_Z^0 \longrightarrow K_Z^1 \longrightarrow \Sigma Z$$

with C_0^X, C_X^0, C_0^Z, C_Z^0 objects in $\mathcal{C}$ and K_1^X, K_X^1, K_1^Z, K_Z^1 objects in $\mathcal{GC}_{\mathcal{E}}(\xi)$. By [15, Lemma 2.2], we have the following commutative diagram

$$\begin{array}{ccccccc} C_0^X & \longrightarrow & C_0^X \oplus C_0^Z & \longrightarrow & C_0^Z & \xrightarrow{0} & \Sigma C_0^X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X. \end{array}$$

By Lemma 3.10, we get the following commutative diagram except the middle square on the top which anticommutes

$$\begin{array}{ccccccc} K_1^X & \longrightarrow & W_1 & \longrightarrow & K_1^Z & \longrightarrow & \Sigma K_1^X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_0^X & \longrightarrow & C_0^X \oplus C_0^Z & \longrightarrow & C_0^Z & \longrightarrow & \Sigma C_0^X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma K_1^X & \longrightarrow & \Sigma W_1 & \longrightarrow & \Sigma K_1^Z & \longrightarrow & \Sigma^2 K_1^X, \end{array}$$

in which the first horizontal and the second vertical triangles are in ξ, and the first horizontal and the second vertical triangles are $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact.

Again by [15, Lemma 2.2], we have the following commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_X^0 & \longrightarrow & C_X^0 \oplus C_Z^0 & \longrightarrow & C_Z^0 & \xrightarrow{0} & \Sigma C_X^0. \end{array}$$

By Lemma 3.10, we get the following commutative diagram except the middle square on the bottom which anticommutes

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_X^0 & \longrightarrow & C_X^0 \oplus C_Z^0 & \longrightarrow & C_Z^0 & \longrightarrow & \Sigma C_X^0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_X^1 & \longrightarrow & W^1 & \longrightarrow & K_Z^1 & \longrightarrow & \Sigma K_X^1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma X & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X \end{array}$$

in which the third horizontal and the second vertical triangles are in ξ, and the third horizontal and the second vertical triangles are $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact. Continuing this process, we get a complete $\mathcal{C}_{\mathcal{E}}(\xi)$-resolution of Y. Thus Y is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$.

Claim 2 If Y and Z are in $\mathcal{GC}_{\mathcal{E}}(\xi)$, then so is X.

By Theorem 3.9, we have a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact ξ-exact complex

$$0 \longrightarrow X \longrightarrow C_X^0 \longrightarrow C_X^1 \longrightarrow \cdots \longrightarrow C_X^n \longrightarrow \cdots \tag{4.1}$$

with all C_X^i objects in $\mathcal{C}$.

Since Y is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$, there exists a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact triangle $K_1^Y \longrightarrow C_0^Y \longrightarrow Y \longrightarrow \Sigma K_1^Y$ in ξ with K_1^Y an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$ and C_0^Y an object in $\mathcal{C}$. Applying base change for the triangle $\Sigma^{-1}Z \longrightarrow X \longrightarrow Y \longrightarrow Z$ along the morphism $C_0^Y \longrightarrow Y$, we get the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1^Y & = & K_1^Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}Z & \longrightarrow & W_1 & \longrightarrow & C_0^Y & \longrightarrow & Z \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K_1^Y & = & \Sigma K_1^Y & \longrightarrow & 0.
\end{array}$$

Since the third vertical triangle and the triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ are in ξ, by Lemma 2.7 (1) we have that the triangle $W_1 \longrightarrow C_0^Y \longrightarrow Z \longrightarrow \Sigma W_1$ is also in ξ and it is $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact. Also we have that the second vertical triangle is $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact. Since Z is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$, there exists a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact triangle $K_1^Z \longrightarrow C_0^Z \longrightarrow Z \longrightarrow \Sigma K_1^Z$ in ξ with K_1^Z an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$ and C_0^Z an object in $\mathcal{C}$. By [7, Axioms B' and E], we get the following commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}Z & \longrightarrow & K_1^Z & \longrightarrow & C_0^Z & \longrightarrow & Z \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}Z & \longrightarrow & W_1 & \longrightarrow & C_0^Y & \longrightarrow & Z
\end{array}$$

with $K_1^Z \longrightarrow C_0^Z \oplus W_1 \longrightarrow C_0^Y \longrightarrow \Sigma K_1^Z$ a triangle in Δ. By [13, Proposition 2.1], we have that this triangle is in ξ. So it is $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact. Because K_1^Z and C_0^Y are objects in $\mathcal{GC}_{\mathcal{E}}(\xi)$, we have a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact ξ-exact complex of $C_0^Z \oplus W_1$ by Claim 1. Applying Theorem 3.8 to the triangle $C_0^Z \longrightarrow C_0^Z \oplus W_1 \longrightarrow W_1 \longrightarrow \Sigma C_0^Z$, we get a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact ξ-exact complex

$$\cdots \longrightarrow C_n^{W_1} \longrightarrow C_{n-1}^{W_1} \longrightarrow \cdots \longrightarrow C_0^{W_1} \longrightarrow W_1 \longrightarrow 0$$

with all $C_i^{W_1}$ objects in $\mathcal{C}$.

Applying Theorem 3.8 to the triangle $K_1^Y \longrightarrow W_1 \longrightarrow X \longrightarrow \Sigma K_1^Y$, we get a $\mathcal{T}(\mathcal{E},-)$-exact and $\mathcal{T}(-,\mathcal{E})$-exact ξ-exact complex

$$\cdots \longrightarrow C_n^X \longrightarrow C_{n-1}^X \longrightarrow \cdots \longrightarrow C_0^X \longrightarrow X \longrightarrow 0 \tag{4.2}$$

with all C_i^X objects in $\mathcal{C}$.

Combining (4.1) and (4.2), we get a complete $\mathcal{C}_{\mathcal{E}}(\xi)$-resolution of X. Thus X is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$.

(2) It is dual to (1). $\square$

As an immediate consequence of Theorem 4.3, we get the following

Corollary 4.4 *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$. Then $\mathcal{GC}(\xi)$ is a $\mathcal{C}$-resolving and $\mathcal{C}$-coresolving subcategory of $\mathcal{T}$. In particular, $\mathcal{GP}(\xi)$ is a $\mathcal{P}(\xi)$-resolving and $\mathcal{P}(\xi)$-coresolving subcategory of $\mathcal{T}$, and $\mathcal{GI}(\xi)$ is an $\mathcal{I}(\xi)$-resolving and $\mathcal{I}(\xi)$-coresolving subcategory of $\mathcal{T}$.*

Now we are in a position to prove the following

Theorem 4.5 *Let $\mathcal{C} \subseteq \mathcal{E}$ be subcategories of $\mathcal{T}$ with $\mathcal{C}$ closed under direct summands, and let A be an object in $\mathcal{T}$ with $\mathcal{C}_{\mathcal{E}}$-$\dim A < \infty$. Then $\mathcal{GC}_{\mathcal{E}}(\xi)_{\mathcal{E}}$-$\dim A = \mathcal{C}_{\mathcal{E}}$-$\dim A$.*

Proof Clearly, $\mathcal{GC}_{\mathcal{E}}(\xi)_{\mathcal{E}}$-$\dim A \le \mathcal{C}_{\mathcal{E}}$-$\dim A$. In the following we prove $\mathcal{C}_{\mathcal{E}}$-$\dim A \le \mathcal{GC}_{\mathcal{E}}(\xi)_{\mathcal{E}}$-$\dim A$. Let $\mathcal{GC}_{\mathcal{E}}(\xi)_{\mathcal{E}}$-$\dim A = n < \infty$ and $\mathcal{C}_{\mathcal{E}}$-$\dim A = m < \infty$. If $m > n$, then consider the following $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$0 \longrightarrow C_m \longrightarrow C_{m-1} \longrightarrow C_{m-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with all C_i objects in $\mathcal{C}$. By Theorem 3.14, we get a $\mathcal{T}(\mathcal{E}, -)$-exact ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with K_n an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$. Since $\mathcal{GC}_{\mathcal{E}}(\xi)$ is closed under hokernels of ξ-proper epimorphisms, K_i is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$ for any $n \le i \le m$; in particular, K_{m-1} is an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$. Thus there exists a $\mathcal{T}(\mathcal{E}, -)$-exact and $\mathcal{T}(-, \mathcal{E})$-exact triangle

$$T \longrightarrow C \longrightarrow K_{m-1} \longrightarrow \Sigma T$$

in ξ with C an object in $\mathcal{C}$ and T an object in $\mathcal{GC}_{\mathcal{E}}(\xi)$. Now applying base change for the triangle $C_m \longrightarrow C_{m-1} \longrightarrow K_{m-1} \longrightarrow \Sigma C_m$ along the morphism $C \longrightarrow K_{m-1}$, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T & = & T & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
C_m & \xrightarrow{u} & W & \xrightarrow{g} & C & \longrightarrow & \Sigma C_m \\
\| & & \downarrow{\scriptstyle v} & & \downarrow & & \| \\
C_m & \xrightarrow{w} & C_{m-1} & \longrightarrow & K_{m-1} & \longrightarrow & \Sigma C_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma T & = & \Sigma T & \longrightarrow & 0.
\end{array}$$

Since $gh = f$ and $vu = w$ are ξ-monic, we have that h and u are ξ-monic by Lemma 2.6. So the second vertical and the second horizontal triangles are in ξ. Since the third vertical and the third horizontal triangles are $\mathcal{T}(\mathcal{E}, -)$-exact, the second vertical and the second horizontal triangles are also $\mathcal{T}(\mathcal{E}, -)$-exact by Lemma 2.9. So the second horizontal triangle splits, and hence is $\mathcal{T}(-, \mathcal{E})$-exact. Because the third vertical triangle is $\mathcal{T}(-, \mathcal{E})$-exact, by Lemma 2.9 we have that the third horizontal triangle is $\mathcal{T}(-, \mathcal{E})$-exact, and hence splits. Thus $C_{m-1} \cong C_m \oplus K_{m-1}$.

Because $\mathcal{C}$ is closed under direct summands by assumption, K_{m-1} is an object in $\mathcal{C}$. Repeating this process, we get that K_n is in $\mathcal{C}$ and $\mathcal{C}_{\mathcal{E}}$-dim $A \leq n$, which is a contradiction. Thus $m \leq n$. □

Putting $\mathcal{E} = \mathcal{C}$ in Theorem 4.5, we get the following

Corollary 4.6 *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$ closed under direct summands, and let A be an object in $\mathcal{T}$ with $\mathcal{C}$-dim $A < \infty$. Then $\mathcal{GC}(\xi)_{\mathcal{C}}$-dim $A = \mathcal{C}$-dim A.*

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