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On generalized *k*-syzygy modules

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Abstract In this paper, we first introduce the notion of generalized k-syzygy modules, and then give an equivalent characterization that the class of generalized k-syzygy modules coincides with that of ω-k-torsionfree modules. We further study the extension closure of the category consisting of generalized k-syzygy modules. Some known results are obtained as corollaries.

Keywords: generalized *k*-syzygy module, *ω*-*k*-torsionfree module, extension closure

MSC(2000): 16E05, 16E30

1 Introduction and main result

Throughout this paper Λ is a left Noetherian ring and Γ is a right Noetherian ring, $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{\mathrm{op}}$) is the category of finitely generated left Λ-modules (resp. right Γ-modules). All modules considered are finitely generated.

Let ${}_\Lambda\omega_\Gamma$ be a (Λ,Γ)-bimodule with ${}_\Lambda\omega$ in $\operatorname{mod}\Lambda$ and ω_Γ in $\operatorname{mod}\Gamma^{\mathrm{op}}$. We use $\mathrm{add}_\Lambda\omega$ (resp. $\mathrm{add}\omega_\Gamma$) to denote the full subcategory of $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Gamma^{\mathrm{op}}$) consisting of all modules isomorphic to the direct summands of finite direct sums of copies of ${}_\Lambda\omega$ (resp. ω_Γ).

Definition 1.1[1]**.** *Let $A\in\operatorname{mod}\Lambda$ and i be a non-negative integer. We say that the grade of A with respect to ω, written* $\mathrm{grade}_\omega A$, *is greater than or equal to i if $\mathrm{Ext}^j_\Lambda(A,\omega)=0$ for any $0\leqslant j<i$. We say that the strong grade of A with respect to ω, written* $\mathrm{s.grade}_\omega A$, *is greater than or equal to i if $\mathrm{grade}_\omega B\geqslant i$ for all submodules B of A.*

Definition 1.2. *Let $A\in\operatorname{mod}\Lambda$ and k be a positive integer. We call A a generalized k-syzygy module if there exists an exact sequence $0\to A\to X_0\to X_1\to\cdots\to X_{k-1}$ with all X_i in $\overline{\mathrm{add}_\Lambda\omega}$, where $\overline{\mathrm{add}_\Lambda\omega}=\mathrm{add}_\Lambda\omega\cup\{$projective modules in $\operatorname{mod}\Lambda\}$. We use $\bar{\Omega}^k_\omega(\operatorname{mod}\Lambda)$ to denote the full subcategory of $\operatorname{mod}\Lambda$ consisting of generalized k-syzygy modules. Dually we may define generalized k-syzygy modules in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and $\bar{\Omega}^k_\omega(\operatorname{mod}\Gamma^{\mathrm{op}})$.*

Remark. If all X_i above are projective (resp. in $\mathrm{add}_\Lambda\omega$), then the notion of generalized k-syzygy modules is just that of k-syzygy modules[2] (resp. ω-k-syzygy modules[1]).

For any $A\in\operatorname{mod}\Lambda$, there exists a projective resolution $P_1\xrightarrow{f}P_0\to A\to 0$ in $\operatorname{mod}\Lambda$. Then we have an exact sequence $0\to A^\omega\to P_0^\omega\xrightarrow{f^\omega}P_1^\omega\to X\to 0$ in $\operatorname{mod}\Gamma^{\mathrm{op}}$, where

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$(\ )^{\omega}$ =$\mathrm{Hom}_{\Lambda}(\ ,\omega)$ and X =Cokerf^{ω}.

Definition 1.3[3]. *Suppose that the natural maps $\Lambda \to \mathrm{End}(\omega_{\Gamma})$ and $\Gamma^{\mathrm{op}} \to \mathrm{End}({}_{\Lambda}\omega)$ are isomorphisms and $\mathrm{Ext}^i_{\Gamma}(\omega,\omega) = 0$ for any $i \geqslant 1$. Let A and X be as above. A is called a ω-k-torsionfree module if $\mathrm{Ext}^i_{\Gamma}(X,\omega) = 0$ for any $1 \leqslant i \leqslant k$.*

Let $\sigma_A : A \to A^{\omega\omega}$ be the canonical evaluation homomorphism. A is called a ω-torsionless module if σ_A is a monomorphism; and A is called a ω-reflexive module if σ_A is an isomorphism.

From now on, k is a positive integer, ${}_{\Lambda}\omega_{\Gamma}$ with ${}_{\Lambda}\omega$ in $\mathrm{mod}\,\Lambda$ and ω_{Γ} in $\mathrm{mod}\,\Gamma^{\mathrm{op}}$ is a faithfully balanced and self-orthogonal bimodule, that is, the natural maps $\Lambda \to \mathrm{End}(\omega_{\Gamma})$ and $\Gamma \to \mathrm{End}({}_{\Lambda}\omega)^{\mathrm{op}}$ are isomorphisms and $\mathrm{Ext}^i_{\Lambda}(\omega,\omega) = 0 = \mathrm{Ext}^i_{\Gamma}(\omega,\omega)$ for any $i \geqslant 1$. In this case, it is easy to see that any module in $\overline{\mathrm{add}_{\Lambda}\,\omega}$ (resp. $\overline{\mathrm{add}\,\omega_{\gamma}}$) is ω-reflexive.

Note that when ${}_{\Lambda}\omega_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$, the notions of ω-k-syzygy modules and ω-k-torsionfree modules above are just those of k-syzygy modules and k-torsionfree modules in the usual sense[2,4] respectively. We use $\Omega^k_{\omega}(\mathrm{mod}\,\Lambda)$ (resp. $\mathcal{T}^k_{\omega}(\mathrm{mod}\,\Lambda)$) to denote the full subcategory of mod Λ consisting of ω-k-syzygy modules (resp. ω-k-torsionfree modules), and use $\Omega^k_{\Lambda}(\mathrm{mod}\,\Lambda)$ (resp. $\mathcal{T}^k_{\Lambda}(\mathrm{mod}\,\Lambda)$) to denote the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of k-syzygy modules (resp. k-torsionfree modules). It is not difficult to verify that $\mathcal{T}^k_{\omega}(\mathrm{mod}\,\Lambda) \subset \Omega^k_{\omega}(\mathrm{mod}\,\Lambda)$ (especially, $\mathcal{T}^k_{\Lambda}(\mathrm{mod}\,\Lambda) \subset \Omega^k_{\Lambda}(\mathrm{mod}\,\Lambda)$). Auslander and Bridger[2] gave an equivalent condition of $\mathcal{T}^k_{\Lambda}(\mathrm{mod}\,\Lambda) = \Omega^k_{\Lambda}(\mathrm{mod}\,\Lambda)$ by using the properties of grade of modules as follows.

Theorem A[2,Proposition 2.26]. *The following statements are equivalent.*

(1) $\mathrm{grade}_{\Lambda}\mathrm{Ext}^{i+1}_{\Lambda}(M,\Lambda) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$;

(2) $\Omega^i_{\Lambda}(\mathrm{mod}\ \Lambda) = \mathcal{T}^i_{\Lambda}(\mathrm{mod}\ \Lambda)$ *for any* $1 \leqslant i \leqslant k$.

Huang[5] established the left-right symmetry of the above Auslander and Bridger's result.

Theorem B[5,Theorem 2.4]. *The following statements are equivalent.*

(1) $\mathrm{grade}_{\Lambda}\mathrm{Ext}^{i+1}_{\Lambda}(M,\Lambda) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$;

(2) $\Omega^i_{\Lambda}(\mathrm{mod}\ \Lambda) = \mathcal{T}^i_{\Lambda}(\mathrm{mod}\ \Lambda)$ *for any* $1 \leqslant i \leqslant k$;

$(1)^{\mathrm{op}}$ $\mathrm{grade}_{\Lambda}\mathrm{Ext}^{i+1}_{\Lambda}(N,\Lambda) \geqslant i$ *for any* $N \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$ *and* $1 \leqslant i \leqslant k-1$;

$(2)^{\mathrm{op}}$ $\Omega^i_{\Lambda}(\mathrm{mod}\ \Lambda^{\mathrm{op}}) = \mathcal{T}^i_{\Lambda}(\mathrm{mod}\ \Lambda^{\mathrm{op}})$ *for any* $1 \leqslant i \leqslant k$.

On the other hand, Huang in [1] generalized Theorem A and gave a ω-dual version of this Auslander and Bridger's result as follows.

Theorem C[1,Theorem 2.1]. *The following statements are equivalent.*

(1) $\mathrm{grade}_{\omega}\mathrm{Ext}^{i+1}_{\Lambda}(M,\omega) \geqslant i$ *for any* $M \in \Omega^{-(i+1)}_{\omega}(\mathrm{mod}\ \Lambda)$ *and* $1 \leqslant i \leqslant k-1$ *(here* $M \in \Omega^{-(i+1)}_{\omega}(\mathrm{mod}\ \Lambda)$ *means that there exists an exact sequence* $0 \to X_i \to \cdots \to X_1 \to X_0 \to M \to 0$ *with* X_t *in* $\mathrm{add}_{\Lambda}\omega$ *for any* $0 \leqslant t \leqslant i$*)*;

(2) $\Omega^i_{\omega}(\mathrm{mod}\ \Lambda) = \mathcal{T}^i_{\omega}(\mathrm{mod}\ \Lambda)$ *for any* $1 \leqslant i \leqslant k$.

Then, it is natural to ask the following question: Is the result of Theorem C also left-right symmetric? The main aim of this paper is to study this problem. The following is the main result of this paper, which can be regarded as a ω-dual version of Theorem B.

Theorem 1.4. *The following statements are equivalent.*

(1) $\mathrm{grade}_{\omega}\mathrm{Ext}^{i+1}_{\Lambda}(M,\omega) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$;

(2) $\bar{\Omega}^i_{\omega}(\mathrm{mod}\ \Lambda) = \mathcal{T}^i_{\omega}(\mathrm{mod}\ \Lambda)$ *for any* $1 \leqslant i \leqslant k$;

(1)$^{\rm op}$ $\mathrm{grade}_\omega \mathrm{Ext}^{i+1}_\Gamma(N,\omega) \geqslant i$ *for any* $N \in \mathrm{mod}\,\Gamma^{\rm op}$ *and* $1 \leqslant i \leqslant k-1$;

(2)$^{\rm op}$ $\bar{\Omega}^i_\omega(\mathrm{mod}\ \Gamma^{\rm op}) = \mathcal{T}^i_\omega(\mathrm{mod}\ \Gamma^{\rm op})$ *for any* $1 \leqslant i \leqslant k$.

We prove this theorem in sec. 2 and then give some applications of it. In sec. 3 we study the extension closure of $\bar{\Omega}^k_\omega(\mathrm{mod}\ \Lambda)$.

2 Proof of Theorem 1.4

In this section we prove our main result (Theorem 1.4). We first give some lemmas.

Let $A \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\rm op}$) and $X_1 \xrightarrow{f} X_0 \to A \to 0$ be an exact sequence in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\rm op}$) with X_0 and X_1 in $\overline{\mathrm{add}_\Lambda\omega}$ (resp. $\overline{\mathrm{add}\omega_\Gamma}$). Then we get an exact sequence:

$$0 \to A^\omega \to X_0^\omega \xrightarrow{f^\omega} X_1^\omega \to X \to 0$$

in $\mathrm{mod}\,\Gamma^{\rm op}$ (resp. $\mathrm{mod}\,\Lambda$), where $X = \mathrm{Coker} f^\omega$.

Lemma 2.1. *Let A and X be as above. Then we have the following exact sequences:*

$$0 \to \mathrm{Ext}^1_\Gamma(X,\omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}^2_\Gamma(X,\omega) \to 0,$$

$$0 \to \mathrm{Ext}^1_\Lambda(A,\omega) \to X \xrightarrow{\sigma_X} X^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(A,\omega) \to 0.$$

Proof. The proof here is similar to that in [6, Lemma 2.1], and we omit it.

By Lemma 2.1, it is trivial that a module A in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{\rm op}$) is ω-torsionless (resp. ω-reflexive) if and only if A is ω-1-torsionfree (resp. ω-2-torsionfree).

Lemma 2.2[1,Lemma 2.4]**.** *Let* $0 \to A \to H \xrightarrow{f} B$ *be an exact sequence in* $\mathrm{mod}\,\Lambda$ (*resp.* $\mathrm{mod}\,\Gamma^{\rm op}$) *with H ω-reflexive and B ω-torsionless. Then* $A \cong (\mathrm{Coker} f^\omega)^\omega$.

Lemma 2.3. *The following statements are equivalent.*

(1) $M \in \Omega^2_\omega(\mathrm{mod}\ \Lambda)$;

(2) $M \in \bar{\Omega}^2_\omega(\mathrm{mod}\ \Lambda)$;

(3) *there is a module* $N \in \mathrm{mod}\,\Gamma^{\rm op}$ *such that* $M \cong N^\omega$.

Proof. (1) ⇒ (2) obviously. By Lemma 2.2, it is easy to get (2) ⇒ (3).

(3) ⇒ (1) Suppose $M \cong N^\omega$ with $N \in \mathrm{mod}\ \Gamma^{\rm op}$. Because there exists a projective resolution $Q_1 \to Q_0 \to N \to 0$ in $\mathrm{mod}\,\Gamma^{\rm op}$, we have an exact sequence $0 \to N^\omega \to Q_0^\omega \to Q_1^\omega$ with Q_0^ω, $Q_1^\omega \in \mathrm{add}_\Lambda\omega$ and $M(\cong N^\omega) \in \Omega^2_\omega(\mathrm{mod}\ \Lambda)$.

Lemma 2.4. *The following statements are equivalent.*

(1) A^ω *is ω-reflexive for any* $A \in \mathrm{mod}\,\Lambda$;

(1)$^{\rm op}$ B^ω *is ω-reflexive for any* $B \in \mathrm{mod}\,\Gamma^{\rm op}$;

(2) $[\mathrm{Ext}^2_\Lambda(A,\omega)]^\omega = 0$ *for any* $A \in \mathrm{mod}\,\Lambda$;

(2)$^{\rm op}$ $[\mathrm{Ext}^2_\Gamma(B,\omega)]^\omega = 0$ *for any* $B \in \mathrm{mod}\,\Gamma^{\rm op}$;

(3) *Every module in* $\bar{\Omega}^2_\omega(\mathrm{mod}\ \Lambda)$ *is ω-reflexive*;

(3)$^{\rm op}$ *Every module in* $\bar{\Omega}^2_\omega(\mathrm{mod}\ \Gamma^{\rm op})$ *is ω-reflexive.*

Proof. By Lemma 2.3, we have $\Omega^2_\omega(\mathrm{mod}\,\Lambda) = \bar{\Omega}^2_\omega(\mathrm{mod}\,\Lambda)$. So the proof of [1, Lemma 1.7] remains valid here and we omit it.

Lemma 2.5[1,Lemma 1.9]**.** *Let* $k \geqslant 3$. *Then a ω-reflexive module A in* $\mathrm{mod}\,\Lambda$ *is ω-k-torsionfree if and only if* $\mathrm{Ext}^i_\Gamma(A^\omega,\omega) = 0$ *for any* $1 \leqslant i \leqslant k-2$.

The following result is a generalization of Theorem A.

Theorem 2.6. *The following statements are equivalent.*

(1) $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$;

(2) $\bar{\Omega}_\omega^i(\mathrm{mod}\,\Lambda) = \mathcal{T}_\omega^i(\mathrm{mod}\,\Lambda)$ *for any* $1 \leqslant i \leqslant k$.

Proof. We proceed by induction on k. It is not difficult to verify that a module in $\mathrm{mod}\,\Lambda$ is ω-torsionless if and only if it is in $\bar{\Omega}_\omega^1(\mathrm{mod}\ \Lambda)$. Then we have $\bar{\Omega}_\omega^1(\mathrm{mod}\ \Lambda) = \mathcal{T}_\omega^1(\mathrm{mod}\,\Lambda)$. On the other hand, when $k=1$ the assumption of (1) is empty. So the case for $k=1$ is trivial. The case for $k=2$ follows from Lemma 2.4. Now suppose $k \geqslant 3$.

(1) $\Rightarrow$ (2) Firstly, $\mathcal{T}_\omega^k(\mathrm{mod}\ \Lambda) \subset \bar{\Omega}_\omega^k(\mathrm{mod}\,\Lambda)$. So we only need to prove $\mathcal{T}_\omega^k(\mathrm{mod}\,\Lambda) \supset \bar{\Omega}_\omega^k(\mathrm{mod}\,\Lambda)$.

Let $L \in \bar{\Omega}_\omega^k(\mathrm{mod}\,\Lambda)$. Then there exists an exact sequence $0 \to L \to X_{k-1} \xrightarrow{f} X_{k-2} \to \cdots \to X_0 \to M \to 0$ in $\mathrm{mod}\,\Lambda$ with all $X_i \in \overline{\mathrm{add}_\Lambda \omega}$. By induction hypothesis, we have that $\bar{\Omega}_\omega^i(\mathrm{mod}\,\Lambda) = \mathcal{T}_\omega^i(\mathrm{mod}\,\Lambda)$ for any $1 \leqslant i \leqslant k-1$. So $L \in \mathcal{T}_\omega^{k-1}(\mathrm{mod}\,\Lambda)$.

Let $P_1 \xrightarrow{g} P_0 \to L \to 0$ be a projective resolution of L in $\mathrm{mod}\,\Lambda$. Then we have an exact sequence $0 \to L^\omega \to P_0^\omega \xrightarrow{g^\omega} P_1^\omega \to X \to 0$ in $\mathrm{mod}\,\Gamma^{\mathrm{op}}$ with P_0^ω and P_1^ω in $\mathrm{add}\omega_\Gamma$, where $X = \mathrm{Coker} g^\omega$. We will show that L is ω-k-torsionfree.

Notice that $L \in \mathcal{T}_\omega^{k-1}(\mathrm{mod}\,\Lambda)$ and $k \geqslant 3$, so L is ω-reflexive and hence it suffices to show that $\mathrm{Ext}_\Gamma^i(L^\omega,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$ by Lemma 2.5.

Put $N = \mathrm{Coker} f^\omega$. Then, by Lemma 2.2, $L \cong N^\omega$ and $L^\omega \cong N^{\omega\omega}$. We claim that $\mathrm{Ext}_\Gamma^i(N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$. If $k=3$, then $\mathrm{Coker} f$ is a submodule of X_0. But X_0 is ω-reflexive, so $\mathrm{Coker} f$ is ω-torsionless. By Lemma 2.1, $\mathrm{Ext}_\Gamma^1(N,\omega) \cong \mathrm{Ker}\sigma_{\mathrm{Coker} f} = 0$. If $k=4$, then $\mathrm{Coker} f \in \bar{\Omega}_\omega^2(\mathrm{mod}\,\Lambda)(= \mathcal{T}_\omega^2(\mathrm{mod}\ \Lambda))$ and $\mathrm{Coker} f$ is ω-reflexive. Thus $\mathrm{Ext}_\Gamma^1(N,\omega) \cong \mathrm{Ker}\sigma_{\mathrm{Coker} f} = 0$ and $\mathrm{Ext}_\Gamma^2(N,\omega) \cong \mathrm{Coker}\sigma_{\mathrm{Coker} f} = 0$ and the case for $k=4$ follows. If $k \geqslant 5$, then $\mathrm{Coker} f \in \bar{\Omega}_\omega^{k-2}(\mathrm{mod}\,\Lambda)$ and $\mathrm{Coker} f \in \mathcal{T}_\omega^{k-2}(\mathrm{mod}\,\Lambda)$. Thus $\mathrm{Ext}_\Gamma^i((\mathrm{Coker} f)^\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k-4$ by Lemma 2.5. It follows from the exact sequence $0 \to (\mathrm{Coker} f)^\omega \to X_{k-2}^\omega \xrightarrow{f^\omega} X_{k-1}^\omega \to N \to 0$ that $\mathrm{Ext}_\Gamma^i(N,\omega) = 0$ for any $3 \leqslant i \leqslant k-2$. So $\mathrm{Ext}_\Gamma^i(N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

By Lemma 2.1, we have an exact sequence:

$$0 \to \mathrm{Ext}_\Lambda^1(\mathrm{Coker} f,\omega) \to N \xrightarrow{\sigma_N} N^{\omega\omega} \to \mathrm{Ext}_\Lambda^2(\mathrm{Coker} f,\omega) \to 0.$$

Then $\mathrm{Ker}\sigma_N \cong \mathrm{Ext}_\Lambda^1(\mathrm{Coker} f,\omega) \cong \mathrm{Ext}_\Lambda^{k-1}(M,\omega)$ and $\mathrm{Coker}\sigma_N \cong \mathrm{Ext}_\Lambda^2(\mathrm{Coker} f,\omega) \cong \mathrm{Ext}_\Lambda^k(M,\omega)$. So we get the following exact sequences:

$$0 \to \mathrm{Ext}_\Lambda^{k-1}(M,\omega) \to N \xrightarrow{\pi} \mathrm{Im}\sigma_N \to 0, \tag{a}$$

$$0 \to \mathrm{Im}\sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_\Lambda^k(M,\omega) \to 0, \tag{b}$$

where $\sigma_N = \mu\pi$. Since $\mathrm{Ext}_\Gamma^i(N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$ and $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{k-1}(M,\omega) \geqslant k-2$, from the exact sequence (a) we have $\mathrm{Ext}_\Gamma^i(\mathrm{Im}\sigma_N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$. Moreover, since $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^k(M,\omega) \geqslant k-1$, from the exact sequence (b) we get that $\mathrm{Ext}_\Gamma^i(N^{\omega\omega},\omega) = 0$ for any $1 \leqslant i \leqslant k-2$, which yields $\mathrm{Ext}_\Gamma^i(L^\omega,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

(2) $\Rightarrow$ (1) Let $M \in \mathrm{mod}\,\Lambda$. Then there exists an exact sequence $0 \to L \to P_{k-1} \xrightarrow{f} P_{k-2} \to \cdots \to P_0 \to M \to 0$ in $\mathrm{mod}\ \Lambda$ with each P_i projective. By (2), $L \in \mathcal{T}_\omega^k(\mathrm{mod}\,\Lambda)$. By

induction hypothesis, $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ for any $1 \leqslant i \leqslant k-2$. So it remains to show that $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{k}(M,\omega) \geqslant k-1$. Put $N = \mathrm{Coker} f^\omega$. From the proof of (1) ⇒ (2), we have the following facts:

(i) there exist exact sequences $0 \to \mathrm{Ext}_\Lambda^{k-1}(M,\omega) \to N \xrightarrow{\pi} \mathrm{Im}\sigma_N \to 0$ and $0 \to \mathrm{Im}\sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_\Lambda^{k}(M,\omega) \to 0$, where $\sigma_N = \mu\pi$;

(ii) $L \cong N^\omega$;

(iii) $\mathrm{Ext}_\Gamma^i(N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$;

(iv) $\mathrm{Ext}_\Gamma^i(\mathrm{Im}\sigma_N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$.

Since $L \in \mathcal{T}_\omega^k(\mathrm{mod}\,\Lambda)$ and $L \cong N^\omega$, N^ω is ω-reflexive and $\mathrm{Ext}_\Gamma^i(N^{\omega\omega},\omega) \cong \mathrm{Ext}_\Gamma^i(L^\omega,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$ by Lemma 2.5. Since $\mathrm{Ext}_\Gamma^i(\mathrm{Im}\sigma_N,\omega) = 0$ for any $1 \leqslant i \leqslant k-2$ and we have the exact sequence $0 \to \mathrm{Im}\sigma_N \xrightarrow{\mu} N^{\omega\omega} \to \mathrm{Ext}_\Lambda^k(M,\omega) \to 0$, $\mathrm{Ext}_\Gamma^i(\mathrm{Ext}_\Lambda^k(M,\omega),\omega) = 0$ for any $2 \leqslant i \leqslant k-2$. On the other hand, N^ω is ω-reflexive, so $\pi^\omega\mu^\omega = \sigma_N^\omega$ is an isomorphism by [7, Proposition 20.14], and it follows easily that π^ω and μ^ω are isomorphisms. Moreover, we have a long exact sequence:

$$0 \to [\mathrm{Ext}_\Lambda^k(M,\omega)]^\omega \to N^{\omega\omega\omega} \xrightarrow{\mu^\omega} (\mathrm{Im}\sigma_N)^\omega \to \mathrm{Ext}_\Gamma^1(\mathrm{Ext}_\Lambda^k(M,\omega),\omega) \to \mathrm{Ext}_\Gamma^1(N^{\omega\omega},\omega) = 0.$$

So $[\mathrm{Ext}_\Lambda^k(M,\omega)]^\omega \cong \mathrm{Ker}\mu^\omega = 0$ and $\mathrm{Ext}_\Gamma^1(\mathrm{Ext}_\Lambda^k(M,\omega),\omega) \cong \mathrm{Coker}\mu^\omega = 0$. Therefore we conclude that $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^k(M,\omega) \geqslant k-1$.

It is easy to see that a 1-syzygy module in $\mathrm{mod}\,\Lambda$ is in $\Omega_\omega^1(\mathrm{mod}\,\Lambda)$. But we do not know whether a k-syzygy module in $\mathrm{mod}\,\Lambda$ is in $\Omega_\omega^k(\mathrm{mod}\,\Lambda)$ in general. As a corollary of Theorem 2.6 we have the following result.

Corollary 2.7. *If* $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$, *then each* t*-syzygy module in* $\mathrm{mod}\,\Lambda$ *is in* $\Omega_\omega^t(\mathrm{mod}\,\Lambda)$ *for any* $1 \leqslant t \leqslant k$, *that is, if there exists an exact sequence* $0 \to K \to P_0 \to P_1 \to \cdots \to P_{t-1}$ *in* $\mathrm{mod}\,\Lambda$ *with each* P_i *projective, then there exists an exact sequence* $0 \to K \to X_0 \to X_1 \to \cdots \to X_{t-1}$ *with each* $X_i \in \mathrm{add}_\Lambda\omega$.

Proof. Assume that $\mathrm{grade}_\omega \mathrm{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leqslant i \leqslant k-1$. For any $1 \leqslant t \leqslant k$, since a t-syzygy module K in $\mathrm{mod}\,\Lambda$ is in $\bar{\Omega}_\omega^t(\mathrm{mod}\,\Lambda)$, K is in $\mathcal{T}_\omega^t(\mathrm{mod}\,\Lambda)$ by Theorem 2.6. On the other hand, we have that $\Omega_\omega^t(\mathrm{mod}\,\Lambda) = \mathcal{T}_\omega^t(\mathrm{mod}\,\Lambda)$ by Theorem C, so K is in $\Omega_\omega^t(\mathrm{mod}\,\Lambda)$.

Lemma 2.8. *Let* n *be a non-negative integer and* $X \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$. *If* $\mathrm{grade}_\omega X \geqslant n$ *and* $\mathrm{grade}_\omega \mathrm{Ext}_\Gamma^n(X,\omega) \geqslant n+1$, *then* $\mathrm{Ext}_\Gamma^n(X,\omega) = 0$.

Proof. The result was proved in [8, Lemma 2.6] when Λ and Γ are left and right Noetherian rings. The proof in [8] remains valid in the setting here, so we omit it.

Lemma 2.9. *Let* $\cdots \to X_{k+1} \to X_k \to X_{k-1} \to \cdots \to X_0 \to M \to 0$ *be an exact sequence in* $\mathrm{mod}\,\Lambda$ *with* X_i *in* $\overline{\mathrm{add}_\Lambda\omega}$ *for any* $i \geqslant 0$. *Putting* $Y_k = \mathrm{Coker}(X_{k-1}^\omega \to X_k^\omega)$ *and* $Y_{k+1} = \mathrm{Coker}(X_k^\omega \to X_{k+1}^\omega)$, *then there exists an exact sequence*:

$$0 \to \mathrm{Ext}_\Lambda^k(M,\omega) \to Y_k \to X_{k+1}^\omega \to Y_{k+1} \to 0.$$

Proof. Put $M_k = \mathrm{Coker}(X_k \to X_{k-1})$ and $M_{k+1} = \mathrm{Coker}(X_{k+1} \to X_k)$. Notice that each $X_i \in \overline{\mathrm{add}_\omega\Lambda}$, so $\mathrm{Ext}_\Lambda^i(X_i,\omega) = 0$ for any $j \geqslant 1$ and $i \geqslant 0$ and we then have the commutative diagram (Figure 1) with exact columns and rows:

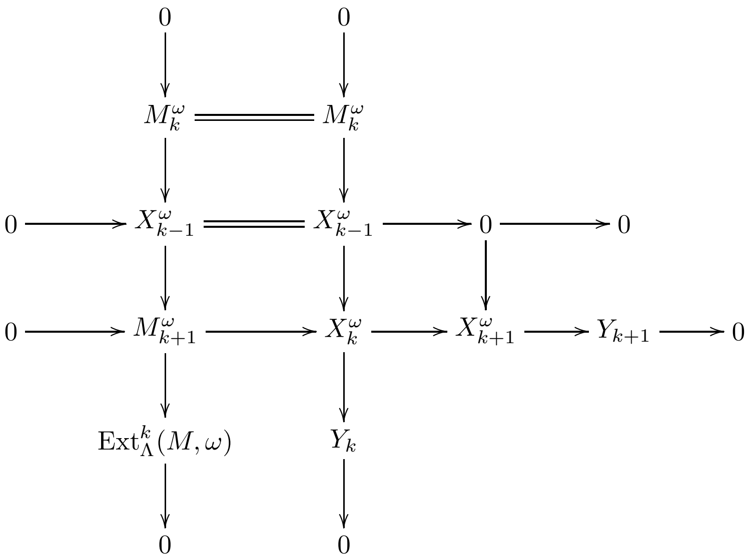

Figure 1

By the snake lemma, we get the desired exact sequence.

We are now in a position to prove our main result.

Proof of Theorem 1.4. By Theorem 2.6 and its dual statement we get the equivalence of (1) and (2) and that of $(1)^{\rm op}$ and $(2)^{\rm op}$. In the following we prove that $(1)^{\rm op}$ implies (1) by induction on k. The case for $k = 1$ is trivial. The case for $k = 2$ follows from Lemma 2.4. Now suppose $k \geqslant 3$.

Let $M \in \operatorname{mod} \Lambda$ and

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a projective resolution of M in $\operatorname{mod}\Lambda$. Put $M_i = \operatorname{Coker}(P_i \to P_{i-1})$ (where $M_1 = M$) and $X_i = \operatorname{Coker}(P_{i-1}^{\omega} \to P_i^{\omega})$ for any $i \geqslant 1$. By an induction hypothesis, we have $\operatorname{grade}_{\omega}\operatorname{Ext}_{\Lambda}^{i+1}(M,\omega) \geqslant i$ for any $1 \leqslant i \leqslant k-2$ and $\operatorname{grade}_{\omega}\operatorname{Ext}_{\Lambda}^{k}(M,\omega) \geqslant k-2$. So it suffices to prove $\operatorname{Ext}_{\Gamma}^{k-2}(\operatorname{Ext}_{\Lambda}^{k}(M,\omega),\omega) = 0$. By Theorem 2.6, $\bar{\Omega}_{\omega}^{i}(\operatorname{mod}\Lambda) = \mathcal{T}_{\omega}^{i}(\operatorname{mod}\Lambda)$ for any $1 \leqslant i \leqslant k-1$. Since $M_t \in \bar{\Omega}_{\omega}^{k-1}(\operatorname{mod}\Lambda)$ for any $t \geqslant k$, $M_t \in \mathcal{T}_{\omega}^{k-1}(\operatorname{mod}\Lambda)$ for any $t \geqslant k$. It follows that X_t satisfies $\operatorname{Ext}_{\Gamma}^{i}(X_t,\omega) = 0$ $(1 \leqslant i \leqslant k-1)$ for any $t \geqslant k$.

On the other hand, by Lemma 2.9 we have an exact sequence:

$$0 \to \operatorname{Ext}_{\Lambda}^{k}(M,\omega) \to X_k \to P_{k+1}^{\omega} \to X_{k+1} \to 0.$$

Put $K = \operatorname{Im}(X_k \to P_{k+1}^{\omega})$. From the exactness of $0 \to K \to P_{k+1}^{\omega} \to X_{k+1} \to 0$ we know that K satisfies $\operatorname{Ext}_{\Gamma}^{i}(K,\omega) = 0$ $(1 \leqslant i \leqslant k-2)$ and $\operatorname{Ext}_{\Gamma}^{k}(X_{k+1},\omega) \cong \operatorname{Ext}_{\Gamma}^{k-1}(K,\omega)$. Moreover, from the exactness of

$$0 \to \operatorname{Ext}_{\Lambda}^{k}(M,\omega) \to X_k \to K \to 0$$

we know that $\operatorname{Ext}_{\Gamma}^{k-1}(K,\omega) \cong \operatorname{Ext}_{\Gamma}^{k-2}(\operatorname{Ext}_{\Lambda}^{k}(M,\omega),\omega)$. So

$$\operatorname{Ext}_{\Gamma}^{k-2}(\operatorname{Ext}_{\Lambda}^{k}(M,\omega),\omega) \cong \operatorname{Ext}_{\Gamma}^{k}(X_{k+1},\omega).$$

By $(1)^{\rm op}$ we then have

$$\text{grade}_\omega \text{Ext}_\Gamma^{k-2}(\text{Ext}_\Lambda^k(M,\omega),\omega) = \text{grade}_\omega \text{Ext}_\Gamma^k(X_{k+1},\omega) \geqslant k-1.$$

It follows from Lemma 2.8 that $\text{Ext}_\Gamma^{k-2}(\text{Ext}_\Lambda^k(M,\omega),\omega) = 0$. Dually we have (1) implies $(1)^{\rm op}$.

It is easy to see that $\Omega_\omega^1(\text{mod }\Lambda) = \bar{\Omega}_\omega^1(\text{mod}\,\Lambda)$. On the other hand, by Lemma 2.3 we have that $\Omega_\omega^2(\text{mod}\,\Lambda) = \bar{\Omega}_\omega^2(\text{mod}\,\Lambda)$. However, in general $\Omega_\omega^k(\text{mod}\,\Lambda) \neq \bar{\Omega}_\omega^k(\text{mod}\,\Lambda)$ for any $k \geqslant 3$. So it is natural to ask when $\Omega_\omega^k(\text{mod}\,\Lambda) = \bar{\Omega}_\omega^k(\text{mod}\,\Lambda)$? The following corollary gives an answer to this question.

Corollary 2.10. *The following statements are equivalent.*

(1) $\text{grade}_\omega \text{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ *for any* $M \in \text{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$;

(2) $\bar{\Omega}_\omega^i(\text{mod}\,\Lambda) = \Omega_\omega^i(\text{mod}\,\Lambda) = \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ *for any* $1 \leqslant i \leqslant k$;

$(1)^{\rm op}$ $\text{grade}_\omega \text{Ext}_\Gamma^{i+1}(N,\omega) \geqslant i$ *for any* $N \in \text{mod}\,\Gamma^{\rm op}$ *and* $1 \leqslant i \leqslant k-1$;

$(2)^{\rm op}$ $\bar{\Omega}_\omega^i(\text{mod}\,\Gamma^{\rm op}) = \Omega_\omega^i(\text{mod}\,\Gamma^{\rm op}) = \mathcal{T}_\omega^i(\text{mod}\,\Gamma^{\rm op})$ *for any* $1 \leqslant i \leqslant k$.

Proof. By Theorem 1.4 and Theorem C.

3 Extension closure of generalized *k*-syzygy modules

A full subcategory $\mathcal{X}$ of $\text{mod}\,\Lambda$ is said to be extension closed if the middle term B of any short exact sequence $0 \to A \to B \to C \to 0$ is in $\mathcal{X}$ provided that the end terms A and C are in $\mathcal{X}$. We discuss in this section the extension closure of $\bar{\Omega}_\omega^k(\text{mod}\,\Lambda)$.

Lemma 3.1[1,Corollary 2.2]**.** *The following statements are equivalent.*

(1) $\mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ *is extension closed for any* $1 \leqslant i \leqslant k$;

(2) $\text{s.grade}_\omega \text{Ext}_\Lambda^1(N,\omega) \geqslant i$ *for any* $N \in \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ *and* $1 \leqslant i \leqslant k$.

Proposition 3.2. *If* $\bar{\Omega}_\omega^i(\text{mod}\Lambda)$ *is extension closed for any* $1 \leqslant i \leqslant k$, *then* $\bar{\Omega}_\omega^i(\text{mod}\,\Lambda) = \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ *for any* $1 \leqslant i \leqslant k$.

Proof. we proceed by induction on k. The case $k=1$ is trivial.

Now suppose $k \geqslant 2$. Then, by induction hypothesis, for any $1 \leqslant i \leqslant k-1$ we have that $\bar{\Omega}_\omega^i(\text{mod }\Lambda) = \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$, which is extension closed. For any $M \in \text{mod}\,\Lambda$ and $1 \leqslant i \leqslant k-1$, there exists a projective resolution $P_i \xrightarrow{f_i} \cdots \to P_0 \to M \to 0$ in mod Λ. Then $\text{Im}f_i \in \bar{\Omega}_\omega^i(\text{mod }\Lambda) = \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ $(1 \leqslant i \leqslant k-1)$ and $\text{Ext}_\Lambda^{i+1}(M,\omega) \cong \text{Ext}_\Lambda^1(\text{Im}f_i,\omega)$. So we have

$$\text{s.grade}_\omega \text{Ext}_\Lambda^{i+1}(M,\omega) = \text{s.grade}_\omega \text{Ext}_\Lambda^1(\text{Im}f_i,\omega) \geqslant i$$

for any $1 \leqslant i \leqslant k-1$ by Lemma 3.1. Then by Theorem 2.6 we have that $\bar{\Omega}_\omega^k(\text{mod}\,\Lambda) = \mathcal{T}_\omega^k(\text{mod}\,\Lambda)$, which finishes the proof.

The following result is a generalization of [4, Theorem 1.7].

Theorem 3.3. *The following statements are equivalent.*

(1) $\text{s.grade}_\omega \text{Ext}_\Lambda^{i+1}(M,\omega) \geqslant i$ *for any* $M \in \text{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k$,

(2) $\bar{\Omega}_\omega^i(\text{mod}\,\Lambda)$ *is extension closed for any* $1 \leqslant i \leqslant k$,

(3) $\bar{\Omega}_\omega^i(\text{mod}\,\Lambda)$ *is extension closed and* $\bar{\Omega}_\omega^i(\text{mod }\Lambda) = \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ *for any* $1 \leqslant i \leqslant k$.

Proof. (1) $\Rightarrow$ (2) By (1) and Theorem 2.6 we have $\bar{\Omega}_\omega^i(\text{mod}\,\Lambda) = \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ for any $1 \leqslant i \leqslant k$. Let $N \in \mathcal{T}_\omega^i(\text{mod}\,\Lambda)$ $(1 \leqslant i \leqslant k)$. Then $N \in \bar{\Omega}_\omega^i(\text{mod}\,\Lambda)$ and there exists an exact sequence in $\text{mod}\,\Lambda$:

$$0 \to N \to X_{i-1} \to \cdots \to X_0 \to M \to 0$$

with all $X_j \in \overline{\mathrm{add}_\Lambda \omega}$. It follows that $\mathrm{Ext}^1_\Lambda(N,\omega) \cong \mathrm{Ext}^{i+1}_\Lambda(M,\omega)$. So

$$\mathrm{s.grade}_\omega \mathrm{Ext}^1_\Lambda(N,\omega) = \mathrm{s.grade}_\omega \mathrm{Ext}^{i+1}_\Lambda(M,\omega) \geqslant i, \quad (1 \leqslant i \leqslant k)$$

by (1) and hence $\mathcal{T}^i_\omega(\mathrm{mod}\,\Lambda)$ is extension closed for any $1 \leqslant i \leqslant k$ by Lemma 3.1. Therefore we conclude that $\bar{\Omega}^i_\omega(\mathrm{mod}\,\Lambda)$ is also extension closed for any $1 \leqslant i \leqslant k$.

(2) $\Rightarrow$ (3) By Proposition 3.2.

(3) $\Rightarrow$ (1) By (3) and Lemma 3.1, $\mathrm{s.grade}_\omega \mathrm{Ext}^1_\Lambda(N,\omega) \geqslant i$ for any $N \in \mathcal{T}^i_\omega(\mathrm{mod}\,\Lambda) = \bar{\Omega}^i_\omega(\mathrm{mod}\,\Lambda)$ and $1 \leqslant i \leqslant k$. So $\mathrm{s.grade}_\omega \mathrm{Ext}^{i+1}_\Lambda(M,\omega) \geqslant i$ for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leqslant i \leqslant k$.

References

[1] Huang Z Y. Extension closure of relative syzygy modules. *Sci China Ser A*: *Math*, **46**(5): 611–620 (2003)

[2] Auslander M, Bridger M. Stable Module Theory. Memoirs Amer Math Soc, Vol 94. Providence: American Mathematical Society, 1969

[3] Huang Z Y. ω-k-torsionfree modules and ω-left approximation dimension. *Sci China Ser A*: *Math*, **44**(2): 184–192 (2001)

[4] Auslander M, Reiten I. Syzygy modules for Noetherian rings. *J Algebra*, **183**(1): 167–185 (1996)

[5] Huang Z Y. On the grade of modules over Noetherian rings. Preprint

[6] Huang Z Y, Tang G H. Self-orthogonal modules over coherent rings. *J Pure Appl Algebra*, **161**(1-2): 167–176 (2001)

[7] Anderson F W, Fuller K R. Rings and Categories of Modules. 2nd ed. Grad Texts in Math, Vol 13. Berlin: Springer-Verlag, 1992

[8] Huang Z Y. On U-dominant dimension. *J Algebra*, **285**(2): 669–681 (2005)