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Relative homology and the structure of relative approximations

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Abstract The structure of right F-${}^{\perp}T$-approximations of any finitely generated module over an artin algebra Λ is given, relative to an additive subbifunctor F of $\mathrm{Ext}^1_\Lambda(-,-)$ and an F-cotilting module T.

Keywords: **F-exact, F-cotilting modules, approximations.**

1 Introduction and main theorems

Let Λ be an artin algebra. We use modΛ to denote the category of finitely generated (left) Λ-modules. By a subcategory of modΛ we always mean a full subcategory closed under isomorphisms, finite direct sums and direct summands.

Relative homology was studied by Hochschild[1] and Butler[2], and was used systematically in the representation theory of artin algebras by Auslander and Solberg[3−6]. It should be mentioned that Auslander and Solberg's papers above had stimulated several investigations (see refs. [7—10]).

Now we introduce some notions and results of relative homology (cf. refs. [3, 4, 6]). Let F be an additive subbifunctor of $\mathrm{Ext}^1_\Lambda(-,-) : (\mathrm{mod}\Lambda)^{op} \times \mathrm{mod}\Lambda \to Ab$. An exact sequence $\eta : 0 \to A \to B \to C \to 0$ in mod Λ is F-exact if η is in $F(C, A)$. A Λ-module P (resp. I) is said to be F-projective (resp. F-injective) if for each F-exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to \mathrm{Hom}_\Lambda(P, A) \to \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(P, C) \to 0$ (resp. $0 \to \mathrm{Hom}_\Lambda(C, I) \to \mathrm{Hom}_\Lambda(B, I) \to \mathrm{Hom}_\Lambda(A, I) \to 0$) is exact. We use $\mathcal{P}(F)$(resp. $\mathcal{I}(F)$) to denote the subcategory of modΛ consisting of all F-projective (resp. F-injective) modules. F is said to have enough projectives (resp. injectives) if for any $A \in \mathrm{mod}\Lambda$ there is an F-exact sequence $0 \to B \to P \to A \to 0$ (resp. $0 \to A \to I \to C \to 0$) with P in $\mathcal{P}(F)$ (resp. I in $\mathcal{I}(F)$). An exact sequence $0 \to A_0 \to A_1 \to \cdots \to A_n \to 0$ in modΛ is said to be F-exact if $0 \to \mathrm{Im}(A_{i-1} \to A_i) \to A_i \to \mathrm{Im}(A_i \to A_{i+1}) \to 0$ is F-exact for any $1 \leqslant i \leqslant n-1$.

Let F be a subbifunctor of $\mathrm{Ext}^1_\Lambda(-,-)$. Suppose F has enough projectives. Then for any $A \in \mathrm{mod}\ \Lambda$ there is an exact sequence of Λ-modules

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0,$$

where $P_i \in \mathcal{P}(F)$ and $0 \to \mathrm{Im}d_{i+1} \to P_i \to \mathrm{Im}d_i \to 0$ is F-exact for any $i \geqslant 0$. Such a sequence is called an F-exact projective resolution of A. We say that the relative projective dimension $\mathrm{pd}_F A$ is the minimum of n including infinity such that there is an F-exact projective resolution

$0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$. If all the F-exact sequences $0 \to \mathrm{Im}d_{i+1} \to P_i \xrightarrow{d_i} \mathrm{Im}d_i \to 0$ have the property that d_i is a right minimal homomorphism, then we denote by $\Omega_F^i(A)$ the i-th syzygy $\mathrm{Ker}d_{i-1}$. Dually, for any $A \in \mathrm{mod}\ \Lambda$ we can define the notions of F-exact injective resolution of A, the relative injective dimension $\mathrm{id}_F A$ respectively if F has enough injectives.

If F has enough projectives and injectives, then for any A and C in mod Λ the right derived functors of $\mathrm{Hom}_\Lambda(C, -)$ and $\mathrm{Hom}_\Lambda(-, A)$ using F-exact injective and F-exact projective resolutions, respectively, coincide. We denote by $\mathrm{Ext}_F^i(C, -)$ the right derived functors of $\mathrm{Hom}_\Lambda(C, -)$ and by $\mathrm{Ext}_F^i(-, A)$ the right derived functor of $\mathrm{Hom}_\Lambda(-, A)$. It is not difficult to check that $\mathrm{pd}_F A = \inf\{n | \mathrm{Ext}_F^{n+1}(A, B) = 0$ for any $B \in \mathrm{mod}\ \Lambda\}$ and $\mathrm{id}_F A = \inf\{n | \mathrm{Ext}_F^{n+1}(B, A) = 0$ for any $B \in \mathrm{mod}\ \Lambda\}$. We use $\mathcal{P}^\infty(F)$ (resp. $\mathcal{I}^\infty(F)$) to denote the subcategory of mod Λ consisting of the modules with finite relative projective (resp. injective) dimension.

From now on, assume that Λ is an artin algebra and F is a subbifunctor of $\mathrm{Ext}_\Lambda^1(-, -)$, which has enough projectives and injectives. For a module T in mod Λ, we use ${}^\perp T$ (resp. $\widehat{\mathrm{add}_\Lambda T}$) to denote the subcategory of mod Λ given by $\{X \in \mathrm{mod}\ \Lambda | \mathrm{Ext}_F^i(X, T) = 0$ for any $i \geqslant 1\}$ (resp. $\{Y \in \mathrm{mod}\ \Lambda |$ there is an F-exact sequence $0 \to T_n \to T_{n-1} \to \cdots \to T_0 \to Y \to 0$ with each T_i in $\mathrm{add}_\Lambda T$, where $\mathrm{add}_\Lambda T$ denotes the subcategory of mod Λ consisting of all modules isomorphic to summands of direct sums of copies of ${}_\Lambda T\}$).

Definition A[4]**.** A module T in mod Λ is called F-cotilting if the following conditions are satisfied.

(i) $\mathrm{id}_F T < \infty$.

(ii) $\mathrm{Ext}_F^i(T, T) = 0$ for any $i \geqslant 1$.

(iii) $\mathcal{I}(F)$ is contained in $\widehat{\mathrm{add}_\Lambda T}$.

Dually, we may define the notion of F-tilting modules.

Definition B[3]**.** Let $\mathcal{D}$ be a subcategory of mod Λ. A right F-$\mathcal{D}$-approximation of a module C in mod Λ is an F-exact sequence $0 \to Y \to X \xrightarrow{f} C \to 0$ with X in $\mathcal{D}$ such that $\mathrm{Hom}_\Lambda(\mathcal{D}, X) \to \mathrm{Hom}_\Lambda(\mathcal{D}, C) \to 0$ is exact. The approximation is called minimal if f is a right minimal homomorphism. Dually, a left F-$\mathcal{D}$-approximation of a module A in mod Λ is an F-exact sequence $0 \to A \xrightarrow{g} X \to Z \to 0$ with X in $\mathcal{D}$ such that $\mathrm{Hom}_\Lambda(X, \mathcal{D}) \to \mathrm{Hom}_\Lambda(A, \mathcal{D}) \to 0$ is exact. The approximation is called minimal if g is a left minimal homomorphism.

By Proposition 2.2 and Theorem 3.2 of ref. [4] we get easily the following result, which is a relative version of Proposition 1.4 of ref. [11] and Lemma 3 of ref. [12].

Theorem A. Let T be an F-cotilting module and A a module in mod Λ. Then there are a minimal right F-${}^\perp T$-approximation of A:

$$0 \to Y_A \to X_A \to A \to 0$$

with $Y_A \in \widehat{\mathrm{add}_\Lambda T}$ and a minimal left F-$\widehat{\mathrm{add}_\Lambda T}$- approximation of A:

$$0 \to A \to Y^A \to X^A \to 0$$

with $X^A \in {}^\perp T$.

Let T be an F-cotilting module. By Proposition 3.2 of ref. [4], for any $X \in {}^\perp T$ there is an

F-exact sequence

$$0 \to X \to T_0 \xrightarrow{f_0} T_1 \to \cdots \to T_n \xrightarrow{f_n} T_{n+1} \to \cdots$$

with each $T_i \in \mathrm{add}_\Lambda T$ and $\mathrm{Im} f_i \in {}^\perp T$ for all $i \geqslant 0$. Then the F-exact sequence $0 \to \mathrm{Ker} f_i \to T_i \to \mathrm{Im} f_i \to 0$ is a left F-$\mathrm{add}_\Lambda T$-approximation of $\mathrm{Ker} f_i$ for all $i \geqslant 0$. It is not difficult to show that we may choose suitable T_i such that $\mathrm{Im} f_i$ has no nonzero summands in $\mathrm{add}_\Lambda T$, and in this case we denote $\mathrm{Im} f_i$ by $\Omega_{CM}^{-(i+1)}(X)$ for all $i \geqslant 0$. For any $A \in \mathrm{mod}\ \Lambda$ we use $\underline{A}$ to denote the part of A having no summands in $\mathrm{add}_\Lambda T$.

The main purpose of this paper is to study how one gets X_A from A, or more accurately, determines what $\underline{X_A}$ is. In fact, we will prove the following result.

Theorem 1. Let T be an F-cotilting module with $\mathrm{id}_F T = n (\geqslant 1)$ and $\mathcal{P}(F) \subset \mathrm{add}_\Lambda T$. Then $\underline{X_A} \cong \Omega_{CM}^{-n} \Omega_F^n(A)$ for any $A \in \mathrm{mod}\ \Lambda$.

A subcategory $\mathcal{D}$ of mod Λ is called an F-generator in mod Λ if $\mathcal{D}$ contains $\mathcal{P}(F)$[4].

Corollary 1. If T is an F-cotilting F-generator with $\mathrm{id}_F T = n (\geqslant 1)$, then $\underline{X_A} \cong \Omega_{CM}^{-n} \Omega_F^n(A)$ for any $A \in \mathrm{mod}\ \Lambda$.

An artin algebra Λ is called F-Gorenstein if $\mathcal{I}^\infty(F) = \mathcal{P}^\infty(F)$[6]. If Λ is F-Gorenstein then $\mathcal{P}(F)$ is an additive subcategory generated by some F-cotilting F-tilting module T (see the proof of Proposition 3.3 of ref. [6]), and thus $\mathcal{P}(F) \subset \mathrm{add}_\Lambda T$.

Corollary 2. Let Λ be an F-Gorenstein algebra and T the F-cotilting F-tilting module generating $\mathcal{P}(F)$ with $\mathrm{id}_F T = n (\geqslant 1)$. Then $\underline{X_A} \cong \Omega_{CM}^{-n} \Omega_F^n(A)$ for any $A \in \mathrm{mod}\ \Lambda$.

Remark. The above corollary is a relative version of Proposition 1.2 of ref. [11] and Lemma 3.6 of ref. [13].

We further give the relation between any right F-${}^\perp T$-approximation and the minimal right F-${}^\perp T$-approximation of any given module in mod Λ.

Theorem 2. Under the assumptions of Theorem 1, for a module $A \in \mathrm{mod}\ \Lambda$ any right F-${}^\perp T$-approximation of A has the form: $0 \to Y \to X \to A \to 0$, where $\underline{Y} \cong \underline{Y_A}$ and $\underline{X} \cong \underline{X_A} \cong \Omega_{CM}^{-n} \Omega_F^n(A)$, and furthermore, there is a module $S \in \mathrm{add}_\Lambda T$ such that $Y \cong Y_A \oplus S$ and $X \cong X_A \oplus S$.

2 Lemmas

Lemma 1. Assume that there is an exact commutative diagram in mod Λ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0
\end{array}$$

and $\pi_1 : P_1 \to A_2$ is any homomorphism. Then we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 \oplus P_1 & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix}} & B_1 \oplus P_1 & \xrightarrow{(g_1, 0)} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\alpha, \pi_1)} & & \downarrow{\scriptstyle (\beta, f_2 \pi_1)} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0
\end{array}$$

Moreover, if $\pi_2 : P_2 \to C_2$ and $\delta : P_2 \to B_2$ are homomorphisms such that $\pi_2 = g_2\delta$, then we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 \oplus P_1 & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & B_1 \oplus P_1 \oplus P_2 & \xrightarrow{\begin{pmatrix} g_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & C_1 \oplus P_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\alpha,\pi_1)} & & \downarrow{\scriptstyle (\beta, f_2\pi_1, \delta)} & & \downarrow{\scriptstyle (\gamma,\pi_2)} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \longrightarrow & 0
\end{array}$$

Proof. It is straightforward to verifty.

Lemma 2(A generalization of Schanuel's Lemma). Let $A \in \operatorname{mod} \Lambda$ and $\mathcal{D}$ a subcategory of $\operatorname{mod} \Lambda$. If

$$0 \to X_1 \to Y_1 \xrightarrow{f_1} A \to 0,$$

and

$$0 \to X_2 \to Y_2 \xrightarrow{f_2} A \to 0$$

are two right F-$\mathcal{D}$-approximations of A, then

$$X_1 \oplus Y_2 \cong X_2 \oplus Y_1.$$

Proof. The proof is essentially dual to that of Lemma 6 of ref. [14]. For the sake of completeness, we give it here.

Consider the following pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_1 & = & X_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_2 & \longrightarrow & Y & \longrightarrow & Y_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_1} & & \\
0 & \longrightarrow & X_2 & \longrightarrow & Y_2 & \xrightarrow{f_2} & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Because $0 \to X_1 \to Y_1 \xrightarrow{f_1} A \to 0$ is a right F-$\mathcal{D}$-approximation of A and $Y_2 \in \mathcal{D}$, there is a homomorphism $f : Y_2 \to Y_1$ such that $f_1 f = f_2 = f_2 1_{Y_2}$. Noting that the above diagram is a pull-back diagram, so there is a homomorphism $g' : Y_2 \to Y$ such that $gg' = 1_{Y_2}$ which implies that the middle column of above diagram splits and $Y \cong X_1 \oplus Y_2$. Similarly the middle row of the above diagram splits and $Y \cong X_2 \oplus Y_1$. Hence we are done.

Lemma 3. If we have the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \eta_1 & & \eta_2 & & \eta_3 & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e_1: 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e_2: 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e_3: 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Diagram (2.1)

with e_2 F-exact. If η_1 is F-exact, then e_1 is F-exact; if both η_2 and e_3 are F-exact, then η_3 is F-exact.

Proof. Noting that an exact sequence $0 \to A \to B \to C \to 0$ in mod Λ is F-exact if and only if $0 \to \mathrm{Hom}_\Lambda(P, A) \to \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(P, C) \to 0$ is exact for any $P \in \mathcal{P}(F)$ (see Proposition 1.5 of ref. [3]).

Since e_2 is F-exact, for any $P \in \mathcal{P}(F)$ applying $\mathrm{Hom}_\Lambda(P, -)$ to Diagram (2.1) we get the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_\Lambda(P, A_1) & \longrightarrow & \mathrm{Hom}_\Lambda(P, B_1) & \xrightarrow{g} & \mathrm{Hom}_\Lambda(P, C_1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_\Lambda(P, A_2) & \longrightarrow & \mathrm{Hom}_\Lambda(P, B_2) & \longrightarrow & \mathrm{Hom}_\Lambda(P, C_2) & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow r & & \downarrow t & & \\
0 & \longrightarrow & \mathrm{Hom}_\Lambda(P, A_3) & \longrightarrow & \mathrm{Hom}_\Lambda(P, B_3) & \xrightarrow{s} & \mathrm{Hom}_\Lambda(P, C_3) & &
\end{array}
$$

If η_1 is F-exact, then f is epic and g is also epic by Snake Lemma, which implies that e_1 is F-exact. If both η_2 and e_3 are F-exact, then r and s are epic and t is also epic again by Snake Lemma, which implies that η_3 is F-exact. The proof is finished.

Lemma 4. Let T be an F-cotilting module. Then $\mathrm{Ext}_F^i({}^\perp T, \widehat{\mathrm{add}_\Lambda T}) = 0$ for any $i \geqslant 1$, that is, $\mathrm{Ext}_F^i(X, Y) = 0$ for any $X \in {}^\perp T$ and $Y \in \widehat{\mathrm{add}_\Lambda T}$.

Proof. See Theorem 3.2(b) of ref. [4].

Lemma 5. Let T be an F-cotilting module. Then ${}^\perp T \cap \widehat{\mathrm{add}_\Lambda T} = \mathrm{add}_\Lambda T$.

Proof. Let $X \in {}^\perp T \cap \widehat{\mathrm{add}_\Lambda T}$. Then $\mathrm{Ext}_F^i(X, P) = 0$ for any $P \in \mathrm{add}_\Lambda T$ and $i \geqslant 1$, and there is an F-exact sequence $0 \to T_n \to T_{n-1} \to \cdots \to T_0 \to X \to 0$ with each $T_i \in \mathrm{add}_\Lambda T$. Applying the functor $\mathrm{Hom}_\Lambda(X, -)$ to the above F-exact sequence, it is easy to see that $X \in \mathrm{add}_\Lambda T$. The proof is finished.

Lemma 6. Let $\mathcal{D}$ be a subcategory of mod Λ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & \downarrow f & & & & & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & Y_1 & \xrightarrow{\beta_1} & B_1 & \longrightarrow & 0 \end{array}$$

an F-exact diagram with $Y_1 \in \mathcal{D}$ and $0 \to A \xrightarrow{\alpha} Y \xrightarrow{\beta} B \to 0$ a left F-$\mathcal{D}$-approximation of A. Then there are homomorphisms $g : Y \to Y_1$ and $h : B \to B_1$ such that the above diagram commutes.

Proof. The existence of g follows from the fact that $Y_1 \in \mathcal{D}$ and $0 \to A \xrightarrow{\alpha} Y \xrightarrow{\beta} B \to 0$ is a left F-$\mathcal{D}$-approximation of A. By diagram chasing we have $\mathrm{Ker}\beta \subseteq \mathrm{Ker}\beta_1 g$, then the existence of h follows from Theorem 3.6 of ref. [15]. We are done.

3 The proofs of Theorems

Proof of Theorem 1. Assume that $P_n \xrightarrow{g_n} P_{n-1} \xrightarrow{g_{n-1}} \cdots \to P_1 \xrightarrow{g_1} P_0 \xrightarrow{g} A \to 0$ is an F-exact minimal projective resolution. Then $\Omega_F^n(A) = \mathrm{Ker} g_{n-1}$.

Because $\mathrm{id}_F T = n, \mathrm{Ext}_F^i(\Omega_F^n(A), T) \cong \mathrm{Ext}_F^{n+i}(A, T) = 0$ for any $i \geqslant 1$ and $\Omega_F^n(A) \in {}^{\perp} T$. By Theorem 3.2 of ref. [4], there is an F-exact sequence

$$0 \to \Omega_F^n(A) \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \to \cdots \to T_{n-1} \xrightarrow{f_n} T_n \to \cdots$$

with $T_i \in \mathrm{add}_\Lambda T$ and $\mathrm{Im} f_i \in^{\perp} T$. It is not difficult to see that we may choose suitable $T_i (0 \leqslant i \leqslant n-1)$ such that $\mathrm{Im} f_n$ has no nonzero summands in $\mathrm{add}_\Lambda T$. Then $\Omega_{CM}^{-n}\Omega_F^n(A) = \mathrm{Im} f_n$. Since $\mathrm{Im} f_i \in^{\perp} T$ for any $i \geqslant 0$, it is easy to see from Lemma 4 that $0 \to \mathrm{Im} f_i \to T_i \to \mathrm{Im} f_{i+1} \to 0$ is a left F-${}^{\perp}T$-approximation. Since $\mathcal{P}(F) \subset \mathrm{add}_\Lambda T \subset^{\perp} T$, by Lemma 6 we have an F-exact commutative diagram:

$$\begin{array}{ccccccccccccc} 0 \to \Omega_F^n(A) & \xrightarrow{f_0} & T_0 & \xrightarrow{f_1} & T_1 & \to \cdots & \xrightarrow{f_{n-1}} & T_{n-1} & \xrightarrow{f} & \Omega_{CM}^{-n}\Omega_F^n(A) & \to & 0 \\ \| & & \downarrow \alpha_0 & & \downarrow \alpha_1 & & & \downarrow \alpha_{n-1} & & \downarrow t & & \\ 0 \to \Omega_F^n(A) & \xrightarrow{g_n} & P_{n-1} & \xrightarrow{g_{n-1}} & P_{n-2} & \to \cdots & \xrightarrow{g_1} & P_0 & \xrightarrow{g} & A & \to & 0 \end{array}$$

Diagram (3.1)

For any $0 \leqslant j \leqslant n-1$, let $f_j = i_j \pi_j$ where $\pi_j : T_{j-1} \to \mathrm{Im} f_j$ is epic and $i_j : \mathrm{Im} f_j \to T_j$ is monic, and let $g_{n-j} = i'_{n-j}\pi'_{n-j}$ where $\pi'_{n-j} : P_{n-j} \to \mathrm{Im} g_{n-j}$ is epic and $i'_{n-j} : \mathrm{Im} g_{n-j} \to P_{n-j-1}$ is monic.

From Diagram (3.1) we have an F-exact commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \Omega_F^n(A) & \xrightarrow{f_0} & T_0 & \xrightarrow{\pi_1} & \mathrm{Im} f_1 & \to 0 \\ & \| & & \downarrow \alpha_0 & & \downarrow \alpha'_1 & \\ 0 \to & \Omega_F^n(A) & \xrightarrow{g_n} & P_{n-1} & \xrightarrow{\pi'_{n-1}} & \mathrm{Im} g_{n-1} & \to 0 \end{array}$$

where α_1' is an induced homomorphism.

It is clear that the above diagram is a pull-back diagram, so there is an exact sequence

$$0 \longrightarrow T_0 \xrightarrow{\begin{pmatrix} \pi_1 \\ \alpha_0 \end{pmatrix}} \operatorname{Im} f_1 \oplus P_{n-1} \xrightarrow{(\alpha_1', \pi_{n-1}')} \operatorname{Im} g_{n-1} \longrightarrow 0$$

and the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_0 & = & T_0 & & \\
 & & & & \downarrow \begin{pmatrix} 1 \\ -\alpha_0 \end{pmatrix} & & \downarrow \begin{pmatrix} \pi_1 \\ -\alpha_0 \end{pmatrix} & & \\
0 & \longrightarrow & \Omega_F^n(\Lambda) & \xrightarrow{\begin{pmatrix} f_0 \\ 0 \end{pmatrix}} & T_0 \oplus P_{n-1} & \xrightarrow{\begin{pmatrix} \pi_1 & 0 \\ 0 & 1 \end{pmatrix}} & \operatorname{Im} f_1 \oplus P_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow (\alpha_0, 1) & & \downarrow (\alpha_1', \pi_{n-1}') & & \\
0 & \longrightarrow & \Omega_F^n(\Lambda) & \xrightarrow{g_n} & P_{n-1} & \xrightarrow{\pi_{n-1}'} & \operatorname{Im} g_{n-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle column of the above diagram splits by Proposition 1 of ref. [16], that is, it is the zero element in $\operatorname{Ext}^1_\Lambda(P_{n-1}, T_0)$. By ref. [3], $F(P_{n-1}, T_0)$ is a subgroup of $\operatorname{Ext}^1_\Lambda(P_{n-1}, T_0)$, so this middle column is also in $F(P_{n-1}, T_0)$, that is, it is F-exact. Because the above diagram is a pull-back diagram and the third row is F-exact, the middle row is F-exact by ref. [3]. Therefore, the third column is also F-exact by Lemma 3.

On the other hand, by Lemma 1 we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_0 & \xrightarrow{\begin{pmatrix} f_1 \\ -\alpha_0 \end{pmatrix}} & T_1 \oplus P_n & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow \begin{pmatrix} \pi_1 \\ -\alpha_0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\alpha_1 & -g_{n-1} \end{pmatrix} & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Im} f_1 \oplus P_{n-1} & \xrightarrow{\begin{pmatrix} i_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} & T_1 \oplus P_{n-1} \oplus P_{n-2} & \xrightarrow{\begin{pmatrix} \pi_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & \operatorname{Im} f_2 \oplus P_{n-2} & \longrightarrow & 0 \\
 & & \downarrow (\alpha_1', \pi_{n-1}') & & \downarrow (\alpha_1, g_{n-1}, 1) & & \downarrow (\alpha_2', \pi_{n-2}') & & \\
0 & \longrightarrow & \operatorname{Im} g_{n-1} & \xrightarrow{i_{n-1}'} & P_{n-2} & \xrightarrow{\pi_{n-2}'} & \operatorname{Im} g_{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The middle column of the above diagram is F-exact since it splits also by Proposition 1 of ref. [16]. On the other hand, the middle row is F-exact by ref. [3] because it is a direct sum of $0 \to \operatorname{Im} f_1 \xrightarrow{i_1} T_1 \xrightarrow{\pi_2} \operatorname{Im} f_2 \to 0, 0 \to P_{n-1} \xrightarrow{1} P_{n-1} \xrightarrow{0} 0 \to 0$ and $0 \to 0 \xrightarrow{0} P_{n-2} \xrightarrow{1} P_{n-2} \to 0$ and each direct summand is F-exact. Then, by Lemma 3, we have F-exact sequences:

$$0 \longrightarrow T_0 \xrightarrow{\begin{pmatrix} f_1 \\ -\alpha_0 \end{pmatrix}} T_1 \oplus P_{n-1} \longrightarrow K_1 \longrightarrow 0$$

$$0 \longrightarrow K_1 \longrightarrow \operatorname{Im} f_2 \oplus P_{n-2} \xrightarrow{(\alpha_2', \pi_{n-2}')} \operatorname{Im} g_{n-2} \longrightarrow 0$$

with T_0 and $T_1 \oplus P_{n-1}$ in $\mathrm{add}_\Lambda T$ (note: $P_{n-1} \in \mathcal{P}(F) \subset \mathrm{add}_\Lambda T$). Then $\mathrm{add}_\Lambda T\text{-resdim}_F(K_1) \leqslant 1$ (see ref. [4] p.3044 for the definition of $\mathrm{add}_\Lambda T\text{-resdim}_F$).

Proceed in this way, we finally get an F-exact sequence $0 \to K_n \to \Omega_{CM}^{-n}\Omega_F^n(A) \oplus P_0 \xrightarrow{(t,g)} A \to 0$ with $\mathrm{add}_\Lambda T\text{-resdim}_F(K_n) \leqslant n$ and $\Omega_{CM}^{-n}\Omega_F^n(A) \oplus P_0 \in {}^\perp T$. Then the above F-exact sequence is a right F-${}^\perp T$-approximation of A by Lemma 4. Now our conclusion follows easily by Lemmas 2 and 5. This finishes the proof.

Proof of Theorem 2. Let $0 \to Y \to X \to A \to 0$ be a right F-${}^\perp T$-approximation of A. Then by Lemma 2, $Y_A \oplus X \cong Y \oplus X_A$. Notice that $\mathrm{Ext}_F^1({}^\perp T, Y) = 0$. On the other hand, $\widehat{\mathrm{add}_\Lambda T} = ({}^\perp T)^\perp$ by Lemma 4, and ${}^\perp T$ is an F-resolving subcategory of mod Λ by ref. [4] p.3041. So $Y \in \widehat{\mathrm{add}_\Lambda T}$ by Lemma 2.1 of ref. [4].

Assume that

$$\begin{aligned} Y_A &\cong Y_1 \oplus \cdots \oplus Y_r, \\ X_A &\cong X_1 \oplus \cdots \oplus X_v, \\ Y &\cong Y_1' \oplus \cdots \oplus Y_u', \\ X &\cong X_1' \oplus \cdots \oplus X_s' \end{aligned}$$

are the finite indecomposable decompositions of Y_A, X_A, Y and X, respectively. So we have an isomorphism:

$$Y_1 \oplus \cdots \oplus Y_r \oplus X_1' \oplus \cdots \oplus X_s' \cong Y_1' \oplus \cdots \oplus Y_u' \oplus X_1 \oplus \cdots \oplus X_v. \tag{3.1}$$

Then by Krull-Schmidt Theorem (see Theorem 12.9 of ref. [15]) we have $r + s = u + v$ and the indecomposable modules from each side of the isomorphism (3.1) are pairwise isomorphic. By Lemma 5, it is easy to get that $\underline{Y} \cong \underline{Y_A}$ and $\underline{X} \cong \underline{X_A}$. Now our first claim follows from Theorem 1.

Since both $0 \to Y_A \to X_A \to A \to 0$ and $0 \to Y \to X \to A \to 0$ are right F-${}^\perp T$-approximations of A, there are homomorphisms f and g such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_A & \longrightarrow & X_A & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g & & \| & & \\ 0 & \longrightarrow & Y_A & \longrightarrow & X_A & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

From the minimality of the first row we know that gf is an isomorphism, which implies that f is a split monomorphism and X_A is a direct summand of X. Then the second claim follows from the first one and the fact that $Y_A \oplus X \cong Y \oplus X_A$. This finishes the proof.

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