Selforthogonal modules with finite injective dimension

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Abstract The category consisting of finitely generated modules which are left orthogonal with a cotilting bimodule is shown to be functorially finite. The notion of left orthogonal dimension is introduced, and then a necessary and sufficient condition of selforthogonal modules having finite injective dimension and a characterization of cotilting modules are given.

Keywords: injective dimension, selforthogonal modules, cotilting (bi) modules, homologically finite subcategories, left orthogonal dimension.

Assume that Λ is a ring. We use mod Λ (resp. modΛ^{op}) to denote the category of finitely generated left (resp. right) Λ-modules.

Cotilting (bi) modules and homologically finite subcategories are very important research objects in representation theory of algebras, which played very important roles in studying the dual properties of modules and in determining the existence of almost split sequences in subcategories of modΛ, respectively[1—5]. Let Λ be an artin algebra. It was shown in ref. [1] that the subcategory of modΛ consisting of the modules left orthogonal with a cotilting module is contravariantly finite. In this paper we introduce the notion of cotilting (bi)modules over noether rings and obtain two exact sequences which are similar to that in Proposition 1.4 of ref. [2] (also cf. Theorem A of ref. [6]). From this fact we know that the subcategory of modΛ consisting of the modules left orthogonal with a cotilting bimodule is contravariantly finite. We further show that this subcategory is functorially finite (Theorem 1). We also classify the modules in modΛ (Proposition 4), Auslander and Reiten characterized cotilting modules by using the properties of generalized Gorenstein dimension[2]. In this paper we introduce the notion of left orthogonal dimension which is "simpler" than that of generalized Gorenstein dimension. By using the properties of left orthogonal dimension, we give a necessary and sufficient condition of selforthogonal modules having finite injective dimension (Theorem 2), we then characterize cotilting modules (Theorem 3).

1 Definitions and notations

In the following, we assume that Λ is a left noether ring and Γ is a right noether ring.

Definition 1[7]. Assume that $\mathscr{C} \supset \mathscr{D}$ are subcategories of modΛ and $C \in \mathscr{C}$, $D \in \mathrm{add}\mathscr{D}$, where add$\mathscr{D}$ is the subcategory of modΛ consisting of all Λ-modules isomorphic to summands of finite sums of modules in $\mathscr{D}$. The morphism $D \to C$ is said to be a right $\mathscr{D}$-approximation of C if $\mathrm{Hom}_\Lambda(X, D) \to \mathrm{Hom}_\Lambda(X, C) \to 0$ is exact for all $X \in \mathrm{add}\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be contravariantly finite in $\mathscr{C}$ if every C in $\mathscr{C}$ has a right $\mathscr{D}$-approximation. Dually, the morphism $C \to D$ is said to be a left $\mathscr{D}$-approximation of C if $\mathrm{Hom}_\Lambda(D, X) \to \mathrm{Hom}_\Lambda(C, X) \to 0$ is exact for all $X \in \text{add}\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be covariantly finite in $\mathscr{C}$ if every C in $\mathscr{C}$ has a left $\mathscr{D}$-approximation. The subcategory $\mathscr{D}$ is said to be functorially finite in $\mathscr{C}$ if it is both contravariantly finite and covariantly finite in $\mathscr{C}$. The notions of contravariantly finite subcategories, covariantly finite subcategories and functorially finite subcategories are referred to as homologically finite subcategories.

For any left Λ-module A, we use $\text{l.id}_\Lambda(A)$ to denote left injective dimension of A.

Definition 2. Let $\omega \in \text{mod}\Lambda$. We call ω a selforthogonal module if $\text{Ext}^i_\Lambda(\omega, \omega) = 0$ for any $i \geqslant 1$. A selforthogonal module ω is called a cotilting module if $\text{l.id}_\Lambda(\omega) < \infty$ and the natural map $\Lambda \to \text{End}(\omega_{\text{End}(_\Lambda\omega)})$ is an isomorphism. Similarly, we define the notion of cotilting module in $\text{mod}\Gamma^{\text{op}}$. A (Λ, Γ)-bimodule $_\Lambda\omega_\Gamma$ is called a cotilting bimodule if $_\Lambda\omega$ and ω_Γ are cotilting modules and the natural maps $\Gamma^{\text{op}} \to \text{End}(_\Lambda\omega)$ and $\Lambda \to \text{End}(\omega_\Gamma)$ are isomorphisms.

Remark. In case Λ is an artin algebra, the definition of cotilting (bi)modules coincides with that given in refs. [1, 2]. This can be easily seen by using the dual result of ref. [4] (Proposition 1.6).

Let $\omega \in \text{mod}\Lambda$ be a selforthogonal module and $X \in \text{mod}\Lambda$. X is said to be left orthogonal with ω if $\text{Ext}^i_\Lambda(X, \omega) = 0$ for any $i \geqslant 1$. We use ${}^\perp\omega$ to denote the subcategory of $\text{mod}\Lambda$ consisting of the modules which are left orthogonal with ω. An exact sequence $\cdots \to X_n \to \cdots \to X_0 \to X \to 0$ is called a left orthogonal resolution of X if all $X_i \in {}^\perp\omega$.

Definition 3. Let $M \in \text{mod}\Lambda$ and let n be a nonnegative integer. If M has a left orthogonal resolution (of finite length):

$$0 \to X_n \to \cdots \to X_0 \to M \to 0, \qquad (1.1)$$

then set ${}^\perp\omega\text{-}\dim_\Lambda(M) = \inf\{n \mid 0 \to X_n \to \cdots \to X_0 \to M \to 0$ is a left orthogonal resolution of $M\}$. If no such a resolution exists set ${}^\perp\omega\text{-}\dim_\Lambda(M) = \infty$. We call ${}^\perp\omega\text{-}\dim_\Lambda(M)$ left orthogonal dimension of M.

It is clear that ${}^\perp\omega\text{-}\dim_\Lambda(M) = 0$ if and only if $M \in {}^\perp\omega$.

For any $A \in \text{mod}\Lambda$ (resp. $\text{mod}\Gamma^{\text{op}}$), we use $\text{add}_\Lambda A$ (resp. $\text{add}A_\Gamma$) to denote the full subcategory of $\text{mod}\Lambda$ (resp. $\text{mod}\Gamma^{\text{op}}$) consisting of all modules isomorphic to summands of finite sums of copies of $_\Lambda A$ (resp. A_Γ). Set $\widehat{\text{add}_\Lambda A}$ (resp. $\widehat{\text{add}A_\Gamma}$) $= \{X \in \text{mod}\,\Lambda$ (resp. $\text{mod}\Gamma^{\text{op}}$) $\mid$ there is an exact sequence $0 \to A_t \to \cdots \to A_1 \to A_0 \to X \to 0$ with all A_i in $\text{add}_\Lambda A$ (resp. $\text{add}A_\Gamma$) and t a non-negative integer$\}$. Suppose $_\Lambda\omega_\Gamma$ is a (Λ, Γ)-bimodule, we put $(-)^\omega = \text{Hom}(-, \omega)$. Let $\sigma_A: A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^\omega$ be the canonical evaluation homomorphism. If σ_A is a monomorphism, then A is called an ω-torsionless module. If σ_A is an isomorphism, then A is called a ω-reflexive module.

2 Homologically finite subcategories

In this section, $_\Lambda\omega_\Gamma$ is a cotilting bimodule. The following is the main result of this section.

Theorem 1. ${}^\perp\omega$ is functorially finite in $\text{mod}\Lambda$.

In the following we prove that some lemmas in $\text{mod}\Lambda$, symmetric statements in $\text{mod}\Gamma^{\text{op}}$ hold clearly.

Lemma 1. $\text{Ext}^i_\Lambda({}^\perp\omega, \widehat{\text{add}_\Lambda\omega}) = 0$ for any $i \geqslant 1$. That is, $\text{Ext}^i_\Lambda(X, Y) = 0$ for any $X \in$

${}^{\perp}\omega$ and $Y \in \widehat{\mathrm{add}_\Lambda \omega}$ and $i \geqslant 1$.

Proof. Let $Y \in \widehat{\mathrm{add}_\Lambda \omega}$. Then there is an exact sequence $0 \to X_t \xrightarrow{d_t} X_{t-1} \xrightarrow{d_{t-1}} \cdots \xrightarrow{d_1} X_0 \to Y \to 0$ with all $X_i \in \mathrm{add}_\Lambda \omega$ and $t \geqslant 0$. The conclusion is trivial for the case $t = 0$. Suppose $t \geqslant 1$. From the exactness of the sequence $0 \to X_t \xrightarrow{d_t} X_{t-1} \to \mathrm{Coker}\, d_t \to 0$ we know that $\mathrm{Ext}^i_\Lambda(X, \mathrm{Coker}\, d_t) = 0$ for any $X \in {}^{\perp}\omega$ and $i \geqslant 1$. By using induction on t, it is easy to see that $\mathrm{Ext}^i_\Lambda(X, Y) = 0$ for any $i \geqslant 1$. This finishes the proof.

Lemma 2. ${}^{\perp}\omega = \{ C \in \mathrm{mod}\, \Lambda \mid$ there is an exact sequence $0 \to C \to U_0 \to U_1 \to \cdots \to U_t \to \cdots$ with all $U_i \in \mathrm{add}_\Lambda \omega \}$.

Proof. Suppose $\mathrm{l.id}_\Lambda(\omega) = n (< \infty)$ and $C \in \mathrm{mod}\Lambda$. If there is an exact sequence $0 \to C \xrightarrow{f_0} U_0 \xrightarrow{f_1} U_1 \to \cdots \xrightarrow{f_t} U_t \to \cdots$ with all $U_i \in \mathrm{add}_\Lambda \omega$, it is easy to see that $\mathrm{Ext}^i_\Lambda(C, \omega) \cong \mathrm{Ext}^{i+n}_\Lambda(\mathrm{Im} f_n, \omega) = 0$ for any $i \geqslant 1$. So $C \in {}^{\perp}\omega$.

Conversely, suppose $C \in {}^{\perp}\omega$ and $\cdots \to P_t \to \cdots \to P_1 \to P_0 \to C^{\omega} \to 0$ is a projective resolution of C^{ω} in $\mathrm{mod}\Gamma^{\mathrm{op}}$. By Theorem 6.1 of ref. [4], $\mathrm{Ext}^i_\Gamma(C^{\omega}, \omega) = 0$ (for any $i \geqslant 1$) and $C \cong C^{\omega\omega}$. So we have an exact sequence $0 \to C \to P_0^{\omega} \to P_1^{\omega} \to \cdots \to P_t^{\omega} \to \cdots$ with all $P_i^{\omega} \in \mathrm{add}_\Lambda \omega$. This finishes the proof.

Lemma 3. Let $C \in \mathrm{mod}\Lambda$.

(1) There is an exact sequence

$$0 \to Y_C \to X_C \xrightarrow{f} C \to 0, \tag{2.1}$$

where $f: X_C \to C$ is a right ${}^{\perp}\omega$-approximation of C and $Y_C \in \widehat{\mathrm{add}_\Lambda \omega}$.

(2) There is an exact sequence

$$0 \to C \xrightarrow{g} Y^C \to X^C \to 0, \tag{2.2}$$

where $g: C \to Y^C$ is a left $\widehat{\mathrm{add}_\Lambda \omega}$-approximation of C and $X^C \in {}^{\perp}\omega$.

Proof. Let $t(C) = \sup\{ i \mid \mathrm{Ext}^i_\Lambda(C, \omega) \neq 0\}$. Clearly $t(C) \leqslant \mathrm{l.id}_\Lambda(\omega) (< \infty)$. We proceed by induction on $t(C)$.

It $t(C) = 0$, then $C \in {}^{\perp}\omega$. So $X_C = C$ and $Y_C = 0$ give the first desired exact sequence (2.1). Since $C \in {}^{\perp}\omega$, by Lemma 2 there is an exact sequence $0 \to C \xrightarrow{g} V \to C_1 \to 0$ with $V \in \mathrm{add}_\Lambda \omega$ and $C_1 \in {}^{\perp}\omega$. So by Lemma 1 it is easy to see that $g: C \to V$ is a left $\widehat{\mathrm{add}_\Lambda \omega}$-approximation C.

Now suppose $t(C) \geqslant 1$ and suppose $0 \to C_0 \to P \to C \to 0$ is an exact sequence in $\mathrm{mod}\Lambda$ with P projective. Clearly $t(C_0) = t(C) - 1$. By induction assumption, there are $X \in {}^{\perp}\omega$ and $Y \in \widehat{\mathrm{add}_\Lambda \omega}$ such that $0 \to C_0 \xrightarrow{g_0} Y \to X \to 0$ is exact with $g_0: C_0 \to Y$ a left $\widehat{\mathrm{add}_\Lambda \omega}$-approximation C_0. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C_0 & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow g_0 & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & X & = & X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

From the middle column of the above diagram we know that $X' \in {}^{\perp}\omega$. By Lemma 1, the middle row of the above diagram $0 \to Y \to X' \to C \to 0$ is our desired (2.1).

Because $X' \in {}^{\perp}\omega$, by Lemma 2 there is an exact sequence $0 \to X' \to U \to X'_1 \to 0$ with $U \in \text{add}_\Lambda \omega$ and $X'_1 \in {}^{\perp}\omega$. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & U & \longrightarrow & Y' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X'_1 & = & X'_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From the middle row of the above diagram we know that $Y' \in \widehat{\text{add}_\Lambda \omega}$. By Lemma 1, the third colomn of the above diagram $0 \to C \to Y' \to X'_1 \to 0$ is our desired (2.2). The proof is finished.

Proposition 1. ${}^{\perp}\omega$ is contravariantly finite in $\text{mod}\Lambda$ and $\widehat{\text{add}_\Lambda \omega}$ is covariantly finite in $\text{mod}\Lambda$.

Proof. It is trivial from Lemma 3.

Proposition 2. ${}^{\perp}\omega$ is covariantly finite in $\text{mod}\Lambda$.

Proof. Suppose $C \in \text{mod}\Lambda$. Then $C^{\omega} \in \text{mod}\Gamma^{\text{op}}$. By symmetric result of Lemma 3(1), there is an exact sequence $0 \to Y \to X \xrightarrow{f} C^{\omega} \to 0$, where $\text{Ext}^i_\Gamma(X, \omega) = 0$ (for any $i \geqslant 1$) and $Y \in \widehat{\text{add}\omega_\Gamma}$. Let h be the composition homomorphism: $C \xrightarrow{\sigma_C} C^{\omega\omega} \xrightarrow{f^{\omega}} X^{\omega}$, that is, $h = f^{\omega} \cdot \sigma_C$. It follows from Theorem 6.1 of ref. [4] that $X^{\omega} \in {}^{\perp}\omega$.

Suppose $g: C \to Q$ is any homomorphism of Λ-modules with $Q \in {}^{\perp}\omega$. By Theorem 6.1 of ref. [4], σ_Q is an isomorphism and $\text{Ext}^i_\Gamma(Q^{\omega}, \omega) = 0$ for any $i \geqslant 1$. By symmetric result of Lemma 1, $\text{Ext}^1_\Gamma(Q^{\omega}, Y) = 0$. So there is a homomorphism of Γ^{op}-modules $s: Q^{\omega} \to X$ such that

$g^{\omega} = f \cdot s$ and hence $g^{\omega\omega} = s^{\omega} \cdot f^{\omega}$. On the other hand, $\sigma_Q \cdot g = g^{\omega\omega} \cdot \sigma_C$ and σ_Q is an isomorphism, so $g = \sigma_Q^{-1} \cdot g^{\omega\omega} \cdot \sigma_C = \sigma_Q^{-1} \cdot s^{\omega} \cdot f^{\omega} \cdot \sigma_C = (\sigma_Q^{-1} \cdot s^{\omega}) \cdot h$. Consequently, $h: C \to X^{\omega}$ is a left ${}^{\perp}\omega$-approximation of C and ${}^{\perp}\omega$ is covariantly finite in modΛ.

Now Theorem 1 follows from Propositions 1 and 2.

Corollary 1. Let Λ be an artin algebra. Then ${}^{\perp}\omega$ has almost split sequences.

Proof. It is an immediate consequence of Theorem 1 and Theorem 2.4 of ref. [8].

We give in the following some useful properties of ${}^{\perp}\omega$ and $\widehat{\mathrm{add}_{\Lambda}\omega}$.

Proposition 3. Let $0 \to A \to B \to C \to 0$ be an exact sequence in modΛ.

(1) If two of A, B and C are in ${}^{\perp}\omega$, then the rest is also in ${}^{\perp}\omega$.

(2) If A, $B \in \widehat{\mathrm{add}_{\Lambda}\omega}$, then $C \in \widehat{\mathrm{add}_{\Lambda}\omega}$; if A, $C \in \widehat{\mathrm{add}_{\Lambda}\omega}$, then $B \in \widehat{\mathrm{add}_{\Lambda}\omega}$.

Proof. By Lemma 2 and Theorem 6.1 ref. [4], ${}_{\Lambda}\omega$ is a (relative) injective cogenerator in ${}^{\perp}\omega$. Then our conclusion follows from Propositions 3.5 and 3.8 of ref. [6].

Suppose that $\mathscr{D}$ is a subcategory of modΛ. It is straightforward from Definition 1 to verify that if one of the right $\mathscr{D}$-approximations of a module in modΛ is epimorphic, then all of the right $\mathscr{D}$-approximations of this module are epimorphic. Dually, if one of a left $\mathscr{D}$-approximations of the module in modΛ is monomorphic then all of the left $\mathscr{D}$-approximations of this module are monomorphic. Now suppose $C \in$ modΛ. C has an epimorphic right ${}^{\perp}\omega$-approximation by Lemma 3. By Proposition 2, C has a left ${}^{\perp}\omega$-approximation. However, a left ${}^{\perp}\omega$-approximation of C is in general not monomorphic. In fact, we may classify the modules in modΛ by using the properties of monomorphic left ${}^{\perp}\omega$-approximations as follows.

Proposition 4. Let $C \in$ modΛ.

(1) C is ω-torsionless if and only if C has a monomorphic left ${}^{\perp}\omega$-approximation.

(2) C is ω-[reflexive if and only if there is an exact sequence $0 \to C \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$ such that f_1: $C \to X_1$ and Im$f_2 \to X_2$ are left ${}^{\perp}\omega$-approximations of C and Imf_2, respectively.

Proof. (1) By Theorem 6.1 of ref. [4], any module in ${}^{\perp}\omega$ is ω-reflexive. So the sufficiency is trivial and the necessity follows from the proof of Proposition 2.

(2) The sufficiency. Suppose there is an exact sequence

$$0 \to C \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \tag{2.3}$$

such that f_1: $C \to X_1$ and Im$f_2 \to X_2$ are left ${}^{\perp}\omega$-approximations of C and Imf_2, respectively. Because $\omega \in {}^{\perp}\omega$, from (2.3) we get an induced exact sequence

$$X_2^{\omega} \xrightarrow{f_2^{\omega}} X_1^{\omega} \xrightarrow{f_1^{\omega}} C^{\omega} \to 0.$$

So we obtain the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & C & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 \\ & \downarrow \sigma_C & & \downarrow \sigma_{X_1} & & \downarrow \sigma_{X_2} \\ 0 \longrightarrow & C^{\omega\omega} & \xrightarrow{f_1^{\omega\omega}} & X_1^{\omega\omega} & \xrightarrow{f_2^{\omega\omega}} & X_2^{\omega\omega} \end{array}$$

Since X_1, $X_2 \in {}^{\perp}\omega$, σ_{X_1} and σ_{X_2} are isomorphisms. By diagram chasing we know that σ_C is also an isomorphism and C is an ω-reflexive module.

Necessity. Suppose C is an ω-reflexive module. By (1) there is an exact sequence

$$0 \to C \xrightarrow{f_1} X_1 \to K \to 0$$

such that $f_1: C \to X_1$ is a left ${}^{\perp}\omega$-approximation of C. Since $\omega \in {}^{\perp}\omega$, we get an exact sequence

$$0 \to K^{\omega} \to X_1^{\omega} \xrightarrow{f_1^{\omega}} C^{\omega} \to 0.$$

So we obtain the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & C & \xrightarrow{f_1} & X_1 & \longrightarrow & K & \longrightarrow 0 \\ & \downarrow \sigma_C & & \downarrow \sigma_{X_1} & & \downarrow \sigma_K & \\ 0 \longrightarrow & C^{\omega\omega} & \xrightarrow{f_1^{\omega\omega}} & X_1^{\omega\omega} & \longrightarrow & K^{\omega\omega} & \end{array}$$

Since σ_C and σ_{X_1} are isomorphisms, it follows from the snake lemma that σ_K is a monomorphism and K is an ω-torsionless module. Hence our conclusion follows from (1). This finishes the proof.

3 Left orthogonal dimension

In this section, Λ is an artin algebra, $\omega \in \operatorname{mod}\Lambda$ is a selforthogonal module and n is a non-negative integer.

The following is the main result of this section.

Theorem 2. The following statements are equivalent.

(1) $\operatorname{l.id}_{\Lambda}(\omega) < \infty$.

(2) Every module in $\operatorname{mod}\Lambda$ has finite left orthogonal dimension.

In order to prove the above result, we first prove some lemmas.

Lemma 4. (1) Suppose A, $B \in \operatorname{mod}\Lambda$. Then ${}^{\perp}\omega\text{-}\dim_{\Lambda}(A) = 0 = {}^{\perp}\omega\text{-}\dim_{\Lambda}(B)$ if and only if ${}^{\perp}\omega\text{-}\dim_{\Lambda}(A \oplus B) = 0$.

(2) If P is a finitely generated projective left Λ-module, then ${}^{\perp}\omega\text{-}\dim_{\Lambda}(P) = 0$.

Proof. It is trivial from the definition of left orthogonal dimension.

Lemma 5. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\operatorname{mod}\Lambda$. If ${}^{\perp}\omega\text{-}\dim_{\Lambda}(C) = 0$, then ${}^{\perp}\omega\text{-}\dim_{\Lambda}(A) = 0$, if and only if ${}^{\perp}\omega\text{-}\dim_{\Lambda}(B) = 0$.

Proof. From the exact sequence $0 \to A \to B \to C \to 0$ we get a long exact sequence

$$\cdots \to \operatorname{Ext}_{\Lambda}^{i}(C, \omega) \to \operatorname{Ext}_{\Lambda}^{i}(B, \omega) \to \operatorname{Ext}_{\Lambda}^{i}(A, \omega) \to \operatorname{Ext}_{\Lambda}^{i+1}(C, \omega) \to \cdots.$$

Then our conclusion follows easily.

The following lemma gives a criterion for computing left orthogonal dimension.

Lemma 6. Let $M \in \operatorname{mod}\Lambda$. Then ${}^{\perp}\omega\text{-}\dim_{\Lambda}(M) \leqslant n$ if and only if ${}^{\perp}\omega\text{-}\dim_{\Lambda}(\Omega^n(M)) = 0$, where $\Omega^n(M)$ denotes the nth syzygy module of M (note $\Omega^0(M) = M$).

Proof. The sufficiency is trivial. We next prove the necessity. Suppose that there is an exact sequence

$$0 \to \Omega^n(M) \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with all P_i finitely generated projective left Λ-modules. By Lemmas 4 and 5, ${}^{\perp}\omega$ satisfies the assumptions (3.11) of ref. [9]. Since all $P_i \in {}^{\perp}\omega$, it follows from Lemma 3.12 of ref. [6] and the definition of left orthogonal dimension that $\Omega^n(M) \in {}^{\perp}\omega$ and ${}^{\perp}\omega\text{-}\dim_{\Lambda}(\Omega^n(M)) = 0$. This finishes the proof.

The following corollary is a generalization of Lemma 5.

Corollary 2. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{mod}\Lambda$. If ${}^{\perp}\omega\text{-}\dim_{\Lambda}(C) = n < \infty$, then for any integer $t \geqslant n$, ${}^{\perp}\omega\text{-}\dim_{\Lambda}(A) \leqslant t$ if and only if ${}^{\perp}\omega\text{-}\dim_{\Lambda}(B) \leqslant t$.

Proof. Suppose $P_{t-1} \to \cdots \to P_0 \to A \to 0$ and $Q_{t-1} \to \cdots \to Q_0 \to C \to 0$ are minimal projective resolutions of A and C respectively. We have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^t(A) & \to & K & \to & \Omega^t(C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{t-1} & \to & P_{t-1} \oplus Q_{t-1} & \to & Q_{t-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_0 & \to & P_0 \oplus Q_0 & \to & Q_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It is clear that $K \cong \Omega^t(B) \oplus P$, where P is some projective module. So ${}^{\perp}\omega\text{-}\dim_{\Lambda}(K) = {}^{\perp}\omega\text{-}\dim_{\Lambda}(\Omega^t(B))$. Since ${}^{\perp}\omega\text{-}\dim_{\Lambda}(C) = n \leqslant t$, our conclusion follows easily from Lemma 6.

The following lemma gives another criterion for computing left orthogonal dimension.

Lemma 7. Let $M \in \mathrm{mod}\Lambda$ and ${}^{\perp}\omega\text{-}\dim_{\Lambda}(M) < \infty$. Then ${}^{\perp}\omega\text{-}\dim_{\Lambda}(M) = \sup\{t \mid \mathrm{Ext}_{\Lambda}^{t}(M, \omega) \neq 0\}$.

Proof. Suppose ${}^{\perp}\omega\text{-}\dim_{\Lambda}(M) = n < \infty$. By Lemma 6, $\mathrm{Ext}_{\Lambda}^{k}(M, \omega) \cong \mathrm{Ext}_{\Lambda}^{k-n}(\Omega^n(M), \omega) = 0$ for any $k > n$. So $\sup\{t \mid \mathrm{Ext}_{\Lambda}^{t}(M, \omega) \neq 0\} \leqslant n$.

Suppose $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is a minimal projective resolution of M. From the exact sequence $0 \to \Omega^n(M) \to P_{n-1} \to \Omega^{n-1}(M) \to 0$ we get a long exact sequence

$$\cdots \to \mathrm{Ext}_{\Lambda}^{i}(\Omega^n(M), \omega) \to \mathrm{Ext}_{\Lambda}^{i+1}(\Omega^{n-1}(M), \omega) \to \mathrm{Ext}_{\Lambda}^{i+1}(P_{n-1}, \omega) \to \cdots.$$

So $\mathrm{Ext}_{\Lambda}^{i}(\Omega^{n-1}(M), \omega) = 0$ for any $i \geqslant 2$. We claim $\mathrm{Ext}_{\Lambda}^{1}(\Omega^{n-1}(M), \omega) \neq 0$. Otherwise, if $\mathrm{Ext}_{\Lambda}^{1}(\Omega^{n-1}(M), \omega) = 0$, then $\mathrm{Ext}_{\Lambda}^{i}(\Omega^{n-1}(M), \omega) = 0$ for any $i \geqslant 1$ and $\Omega^{n-1}(M) \in {}^{\perp}\omega$. It follows from Lemma 6 that ${}^{\perp}\omega\text{-}\dim_{\Lambda}(M) \leqslant n-1$, which is a contradiction. In addition $\mathrm{Ext}_{\Lambda}^{n}(M, \omega) \cong \mathrm{Ext}_{\Lambda}^{1}(\Omega^{n-1}(M), \omega)$, so $\mathrm{Ext}_{\Lambda}^{n}(M, \omega) \neq 0$ which implies $\sup\{t \mid \mathrm{Ext}_{\Lambda}^{t}(M, \omega) \neq 0\} \geqslant n$. This finishes the proof.

Proof of Theorem 2. (1)$\Rightarrow$(2) Suppose $\mathrm{l.id}_{\Lambda}(\omega) = n < \infty$. Then $\mathrm{Ext}_{\Lambda}^{i}(M, \omega) = 0$ for any $M \in \mathrm{mod}\Lambda$ and $i \geqslant n+1$. It follows from Lemma 7 that ${}^{\perp}\omega\text{-}\dim_{\Lambda}(M) \leqslant n$.

(2)$\Rightarrow$(1) Suppose $\{S_1, \cdots, S_t\}$ is the set of all non-isomorphic simple modules in $\mathrm{mod}\Lambda$. By (2), each of $S_1, \cdots, S_t$ has a finite left orthogonal dimension. Put $n = \max\{{}^{\perp}\omega\text{-}\dim_{\Lambda}(S_j) \mid 1 \leqslant j \leqslant t\}$. Then $\mathrm{Ext}_{\Lambda}^{i}(S_j, \omega) = 0$ for any $i \geqslant n+1$ and $1 \leqslant j \leqslant t$.

Suppose $0 \to \omega \to E_0 \to \cdots \xrightarrow{f} E_{n-1} \to \cdots$ is a minimal injective resolution of M. Put $K = \mathrm{Coker} f$. Then $\mathrm{Ext}^i_\Lambda(S_j, K) = 0$ for any $i \geqslant 1$ and $1 \leqslant j \leqslant t$, which implies that K is injective and $\mathrm{l.id}_\Lambda(\omega) \leqslant n$. This finishes the proof.

Assume that the natural map $\Lambda \to \mathrm{End}(\omega_{\mathrm{End}(_\Lambda\omega)})$ is an isomorphism. A module M in $\mathrm{mod}\Lambda$ is said to have generalized Gorenstein dimension zero if it satisfies the conditions: (i) M is ω-reflexive; (ii) $\mathrm{Ext}^i_\Lambda(M, \omega) = 0 = \mathrm{Ext}^i_{\mathrm{End}(_\Lambda\omega)}(M^\omega, \omega)$ for any $i \geqslant 1$. M is said to have finite generalized Gorenstein dimension if there is an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$, where each X_i has generalized Gorenstein dimension zero[2].

Putting Theorem 2 together with Theorem 4.4 in ref. [2], we give the following characterizations of cotilting modules.

Theorem 3. If the natural map $\Lambda \to \mathrm{End}(\omega_{\mathrm{End}(_\Lambda\omega)})$ is an isomorphism, then the following statements are equivalent.

(1) ω is a cotilting module.

(2) Every module in $\mathrm{mod}\Lambda$ has finite left orthogonal dimension.

(3) Every module in $\mathrm{mod}\Lambda$ has finite generalized Gorenstein dimension.

Let $_\Lambda\omega = {}_\Lambda\Lambda$. The following corollary is an immediate result of Theorem 3 which generalizes Theorem in ref. [10].

Corollary 3. The following statements are equivalent.

(1) $\mathrm{l.id}_\Lambda(\Lambda) < \infty$.

(2) Every module in $\mathrm{mod}\Lambda$ has finite left orthogonal dimension.

(3) Every module in $\mathrm{mod}\Lambda$ has finite Gorenstein dimension.

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