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The Auslander-type condition of triangular matrix rings

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Abstract Let R be a left and right Noetherian ring and n, k be any non-negative integers. R is said to satisfy the Auslander-type condition $G_n(k)$ if the right flat dimension of the $(i+1)$-th term in a minimal injective resolution of R_R is at most $i+k$ for any $0 \leqslant i \leqslant n-1$. In this paper, we prove that R is $G_n(k)$ if and only if so is a lower triangular matrix ring of any degree t over R.

Keywords Auslander-type condition, triangular matrix rings, flat dimension, minimal injective resolutions, minimal flat resolutions

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1 Introduction

Let R be a ring and M be a right R-module. We use

$$0 \to M \to I^0(M) \to I^1(M) \to \cdots \to I^i(M) \to \cdots$$

to denote a minimal injective resolution of M_R. For a positive integer n, recall from [8] that a left and right Noetherian ring R is called an *n-Gorenstein ring* if the right flat dimension of $I^i(R)$ is at most i for any $0 \leqslant i \leqslant n-1$, and R is said to satisfy the *Auslander condition* if R is n-Gorenstein for all n. The notion of the Auslander condition may be regarded as a non-commutative version of commutative Gorenstein rings. A remarkable property of n-Gorenstein rings (and hence rings satisfying the Auslander condition) is the left-right symmetry, which was proved by Auslander (see [8, Theorem 3.7]). Motivated by the philosophy of Auslander, Huang and Iyama introduced in [15] the notion of the Auslander-type condition as follows. For any $n, k \geqslant 0$, a left and right Noetherian ring R is said to be $G_n(k)$ if the right flat dimension of $I^i(R)$ is at most $i+k$ for any $0 \leqslant i \leqslant n-1$. It is trivial that R is an n-Gorenstein ring if and only if R is $G_n(0)$. In general, the Auslander-type condition $G_n(k)$ does not possess the left-right symmetry [15]. Note that the Auslander-type condition plays a crucial role in the representation theory of algebras and homological algebra (e.g., [2, 3, 5, 7, 9, 12–18, 20–25]).

It was proved by Iwanaga and Wakamatsu in [19, Theorem 8] that a left and right Artinian ring R is an n-Gorenstein ring if and only if so is a lower triangular matrix ring of any degree t over R. Observe

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that this is a generalization of [8, Theorem 3.10] where the case $k = 2$ was established. In this paper, we will generalize the Iwanaga and Wakamatsu's result mentioned above, and prove the following result.

Theorem. *Let R be a left and right Noetherian ring and $n, k \geqslant 0$. Then R is $G_n(k)$ if and only if so is a lower triangular matrix ring $T_t(R)$ of any degree t over R.*

In Section 2, we recall some notions and notations and give some preliminary results about triangular matrix rings. Then in Section 3, we give the proof of the above theorem by establishing the relation between the flat dimensions of the corresponding terms in the minimal injective resolutions of R_R and $T_t(R)_{T_t(R)}$. In [20], Iyama introduced the notions of the $(l, n)^{\mathrm{op}}$-condition (which has a close relation with the Auslander-type condtion) and the dominant number. In Section 3, we also prove the following results. Let R be a left and right Noetherian ring and $l, n \geqslant 0$, $t \geqslant 1$. If R satisfies the $(l, n)^{\mathrm{op}}$-condition, then $T_t(R)$ satisfies the $(l+1, n)^{\mathrm{op}}$-condition. Conversely, if $T_t(R)$ satisfies the $(l, n)^{\mathrm{op}}$-condition, then so does R. In addition, if n is a dominant number of R, then $n + 1$ is a dominant number of $T_t(R)$.

2 Preliminaries

In this section, we give some notions and notations and collect some elementary facts which are useful for the rest of this paper.

Throughout this paper, R and S are rings and ${}_SM_R$ is a left S right R-bimodule. We denote by $\Lambda = \left(\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix}\right)$ the triangular matrix ring, and denote by ${}^*(-)$ the functor $\mathrm{Hom}_R(M, -)$. For the ring R, we use $\mathrm{Mod}\, R$ to denote the category of right R-modules.

By [11], $\mathrm{Mod}\,\Lambda$ is equivalent to a category $\mathscr{T}$ of triples $(X, Y)_f$, where $X \in \mathrm{Mod}\, R$ and $Y \in \mathrm{Mod}\, S$ and $f:\ Y \otimes_S M_R \to X_R$ is a homomorphism in $\mathrm{Mod}\, R$ (which is called the *associated homomorphism*). The right Λ-module corresponding to the triple $(X, Y)_f$ is the additive group $X \oplus Y$ with the right Λ-action given by

$$(x, y)\begin{pmatrix} r & 0 \\ m & s \end{pmatrix} = (xr + f(y \otimes m), ys)$$

for any $x \in X, y \in Y, r \in R, s \in S$ and $m \in M$.

Another description of a right Λ-module $X \oplus Y$ is a triple ${}_\varphi(X, Y)$, where $\varphi:\ Y_S \to \mathrm{Hom}_R({}_SM_R, X_R)_S$ is a homomorphism in $\mathrm{Mod}\, S$ (which is also called the *associated homomorphism*). The right Λ-module corresponding to the triple ${}_\varphi(X, Y)$ is the additive group $X \oplus Y$ with the right Λ-action given by

$$(x, y)\begin{pmatrix} r & 0 \\ m & a \end{pmatrix} = (xr + \varphi(y)(m), ys)$$

for any $x \in X, y \in Y, r \in R, s \in S$ and $m \in M$.

In particular, we have the following isomorphism:

$$\mathrm{Hom}_S(Y_S, \mathrm{Hom}_R({}_SM_R, X_R)_S) \cong \mathrm{Hom}_R(Y \otimes_S M_R, X_R).$$

So it is convenient for us to adopt either of these two descriptions of $X \oplus Y$ in the following argument.

If $(U, V)_g$ and $(X, Y)_f$ are in $\mathscr{T}$, then the homomorphisms from $(U, V)_g$ to $(X, Y)_f$ are pairs (h_1, h_2), where $h_1:\ U \to X$ is a homomorphism in $\mathrm{Mod}\, R$ and $h_2:\ V \to Y$ is a homomorphism in $\mathrm{Mod}\, S$ satisfying the condition $h_1 g = f(h_2 \otimes 1_M)$. It is not difficult to verify that (h_1, h_2) is monic (resp. epic) if and only if so are both of h_1 and h_2.

Lemma 2.1 (See [8, Proposition 1.14]). *Let X, Y and f be as above. Then $(X, Y)_f \in \mathrm{Mod}\,\Lambda$ is flat if and only if the following conditions are satisfied*:

(1) $Y \in \mathrm{Mod}\, S$ *is flat*;

(2) $\mathrm{Coker}\, f \in \mathrm{Mod}\, R$ *is flat*;

(3) f *is a monomorphism.*

Lemma 2.2 (See [26, Corollary 6])**.** *Λ is a left (resp. right) Noetherian ring if and only if both R and S are left (resp. right) Noetherian rings and ${}_SM$ (resp. M_R) is finitely generated.*

For the ring R and any positive integer t, we use $T_t(R)$ to denote the triangular matrix ring

$$\begin{pmatrix} R & & & \\ R & R & & \\ \vdots & \vdots & \ddots & \\ R & R & \cdots & R \end{pmatrix}$$

of degree t.

Lemma 2.3. *For any $t \geqslant 2$, $T_t(R)$ is a triangular matrix ring of the form*

$$T_t(R) = \begin{pmatrix} T_{t-1}(R) & 0 \\ {}_RR^{(t-1)}_{T_{t-1}(R)} & R \end{pmatrix}.$$

In particular, $R^{(t-1)}_{T_{t-1}(R)}$ is faithful and finitely generated projective and $\mathrm{End}_{T_{t-1}(R)}(R^{(t-1)}) \cong R$.

Proof. We can regard $R^{(t-1)}$ as a right $T_{t-1}(R)$-module in a natural way. Let

$$e = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

be the matrix in $T_{t-1}(R)$ such that the $(t-1, t-1)$-component is 1 and 0 elsewhere. Then e is an idempotent and $R^{(t-1)}_{T_{t-1}(R)} \cong eT_{t-1}(R)_{T_{t-1}(R)}$, which implies that $R^{(t-1)}_{T_{t-1}(R)}$ is faithful and finitely generated projective and $\mathrm{End}_{T_{t-1}(R)}(R^{(t-1)}) \cong eT_{t-1}(R)e \cong R$. □

Proposition 2.4. *If R is a left (resp. right) Noetherian ring, then so is $T_t(R)$ for any $t \geqslant 1$.*

Proof. We proceed by induction on t. The case for $t = 1$ is trivial, and the case for $t = 2$ follows from Lemma 2.2. Now assume $t \geqslant 3$. By Lemma 2.3,

$$T_t(R) = \begin{pmatrix} T_{t-1}(R) & 0 \\ {}_RR^{(t-1)}_{T_{t-1}(R)} & R \end{pmatrix}$$

with both ${}_RR^{(t-1)}$ and $R^{(t-1)}_{T_{t-1}(R)}$ finitely generated. Then by the induction hypothesis and Lemma 2.2, we get the assertion. □

Definition 2.5 (See [6])**.** Assume that $\mathscr{F}$ is a subclass of $\mathrm{Mod}\, R$, $X \in \mathscr{F}$ and $Y \in \mathrm{Mod}\, R$. The homomorphism $f : X \to Y$ is said to be an *$\mathscr{F}$-precover* of Y if $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_\Lambda(X', Y) \to 0$ is exact for any $X' \in \mathscr{F}$. An $\mathscr{F}$-precover $f : X \to Y$ is said to be an *$\mathscr{F}$-cover* of Y if an endomorphism $g : X \to X$ is an automorphism whenever $f = fg$. If $\mathscr{F}$ is the subclass of $\mathrm{Mod}\, R$ consisting of all flat right R-modules, then an $\mathscr{F}$-cover is called a *flat cover*.

Bican, Bashir and Enochs proved in [4, Theorem 3] that every module in $\mathrm{Mod}\, R$ has a flat cover. For a module $N \in \mathrm{Mod}\, R$, we call the following exact sequence

$$\cdots \to F_i(N) \overset{\pi_i(N)}{\to} \cdots \to F_1(N) \overset{\pi_1(N)}{\to} F_0(N) \overset{\pi_0(N)}{\to} N \to 0$$

a *minimal flat resolution* of N_R, where $\pi_0(N) :\ F_0(N) \to N$ is a flat cover of N and $\pi_i(N) :\ F_i(N) \to \mathrm{Ker}\, \pi_{i-1}(N)$ is a flat cover of $\mathrm{Ker}\, \pi_{i-1}(N)$ for any $i \geqslant 1$. We denote the right flat dimension of N by $\mathrm{r.fd}_R(N)$. It is easy to verify that $\mathrm{r.fd}_R(N) \leqslant n$ if and only if $F_{n+1}(N) = 0$.

3 Main results

In this section, we give the proof of the main results mentioned in Section 1, by establishing the relation between the flat dimensions of the corresponding terms in the minimal injective resolutions of R_R and $T_t(R)_{T_t(R)}$.

From now on, assume that M_R is finitely generated, faithful and projective with $S = \mathrm{End}_R(M)$ and ${}_SM$ is finitely generated projective. We set $I^{-1}({}^*R) = 0$ and $\mathrm{r.fd}_S\, 0 = -1$.

Lemma 3.1. *Let X, Y and f be as in Section 2. Then $(F_0(Y) \otimes_S M, F_0(Y))_1 \oplus (F_0(X), 0)_0 \xrightarrow{\psi_0} (X, Y)_f \to 0$ is an exact sequence in $\mathrm{Mod}\,\Lambda$ with $(F_0(Y) \otimes_S M, F_0(Y))_1 \oplus (F_0(X), 0)_0$ flat and $\psi_0 = ((f(\pi_0(Y) \otimes_S 1_M), \pi_0(X)), \pi_0(Y))$. Moreover, if f_1 is the associated homomorphism of $\mathrm{Ker}\,\psi_0$, then $0 \to \mathrm{Ker}\, f \to \mathrm{Coker}\, f_1 \to F_0(X) \to \mathrm{Coker}\, f \to 0$ is an exact sequence in $\mathrm{Mod}\, R$.*

Proof. Since M is left S-flat, we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}\,\pi_0(Y) \otimes_S M \arrow[r] \arrow[d, "f_1"] & F_0(Y) \otimes_S M \arrow[r, "\pi_0(Y)\otimes_S 1_M"] \arrow[d, "\binom{1_{F_0(Y)\otimes_S M}}{0}"] & Y \otimes_S M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}\, h \arrow[r] & (F_0(Y) \otimes_S M) \oplus F_0(X) \arrow[r, "h"] & X \arrow[r] & 0, \end{tikzcd}$$

where $h = (f(\pi_0(Y) \otimes_S 1_M), \pi_0(X))$ and f_1 is established by diagram-chasing. Then by Lemma 2.1, we have that $((F_0(Y) \otimes_S M), F_0(Y))_1 \oplus (F_0(X), 0)_0 \in \mathrm{Mod}\,\Lambda$ is flat. The last assertion follows from the snake lemma. $\square$

By [19, Corollary 3],

$$I^0(\Lambda) = {}_1(I^0(R), {}^*(I^0(R))) \oplus {}_1(I^0(M), {}^*(I^0(M)))$$

and

$$I^i(\Lambda) = {}_1(I^i(R), {}^*(I^i(R))) \oplus {}_1(I^i(M), {}^*(I^i(M))) \oplus {}_0(0, I^{i-1}({}^*R)) \qquad (i \geqslant 1)$$

give a minimal injective resolution of Λ_Λ. In the following, we will construct a flat resolution of $I^i(\Lambda_\Lambda)$ for any $i \geqslant 0$, and then consider the Auslander-type condition of the triangular matrix ring $T_t(R)$.

Proposition 3.2. (1) *Let I_R be injective and $\xi_I : {}^*I \otimes_S M \to I$ be defined by $\xi_I(\alpha \otimes x) = \alpha(x)$ for any $\alpha \in {}^*I$ and $x \in M$ be the natural homomorphism. Then*

$$\begin{aligned} &F_0 = (F_0({}^*I) \otimes_S M, F_0({}^*I))_1, \\ &\mathrm{Ker}\,\psi_0 = (\mathrm{Ker}\,\xi_I(\pi_0({}^*I) \otimes_S 1_M), \mathrm{Ker}\,\pi_0({}^*I))_{f_1}, \\ &F_i = (F_i({}^*I) \otimes_S M, F_i({}^*I))_1 \oplus (F_0(\mathrm{Ker}\, h_{i-1}), 0)_0, \\ &\mathrm{Ker}\,\psi_i = (\mathrm{Ker}\, h_i, \mathrm{Ker}\,\pi_i({}^*I))_{f_{i+1}} \qquad (i \geqslant 1) \end{aligned}$$

give a flat resolution of the injective right Λ-module ${}_1(I_R, {}^*I_R)$:

$$\cdots \to F_n \xrightarrow{\psi_n} F_{n-1} \xrightarrow{\psi_{n-1}} \cdots \to F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} {}_1(I_R, {}^*I_R) \to 0,$$

where $h_0 = \xi_I(\pi_0({}^*I) \otimes_S 1_M)$ and $h_i = (f_i(\pi_i({}^*I) \otimes_S 1_M), \pi_0(\mathrm{Ker}\, h_{i-1}))$ and f_i is established by diagram-chasing as in Lemma 3.1 for $i \geqslant 1$. In particular, $\mathrm{r.fd}_\Lambda\, {}_1(I_R, {}^*I_R) \leqslant k$ if and only if $\mathrm{r.fd}_R\, \mathrm{Ker}\,\xi_I \leqslant k-1$ and $\mathrm{r.fd}_S\, {}^*I_R \leqslant k$.

(2) *If $\mathrm{Hom}_R({}_SM_R, R)$ is finitely generated right S-projective, then $F_0 = (F_0(E) \otimes_S M, F_0(E))_1$ and $F_i = (F_i(E) \otimes_S M, F_i(E))_1 \oplus (F_{i-1}(E) \otimes_S M, 0)_0$ $(i \geqslant 1)$ give a flat resolution of ${}_0(0, E_S)$ in $\mathrm{Mod}\,\Lambda$. In particular, $\mathrm{r.fd}_\Lambda\, {}_0(0, E_S) \leqslant k$ if and only if $\mathrm{r.fd}_S\, E \leqslant k-1$.*

Proof. (1) We proceed by induction on i. Since M_R is finitely generated, faithful and projective with $S = \mathrm{End}_R(M)$, by [1, Proposition 20.11], it is not difficult to verify that ξ_I is epic. Thus we have the following exact sequence:

$$0 \to \mathrm{Ker}\,\xi_I \to {}^*I \otimes_S M \xrightarrow{\xi_I} I \to 0.$$

Then F_0 and $\operatorname{Ker}\psi_0$ are given by the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\pi_0(^*I)\otimes_S M \arrow[r] \arrow[d, "f_1"] & F_0(^*I)\otimes_S M \arrow[r, "\pi_0(^*I)\otimes_S 1_M"] \arrow[d, equal] & {}^*I\otimes_S M \arrow[r] \arrow[d, "\xi_I"] & 0 \\ 0 \arrow[r] & \operatorname{Ker}\xi_I(\pi_0(^*I)\otimes_S 1_M) \arrow[r] & F_0(^*I)\otimes_S M \arrow[r, "\xi_I(\pi_0(^*I)\otimes_S 1_M)"] & I \arrow[r] & 0, \end{tikzcd}$$

where f_1 is established by diagram-chasing. By the snake lemma, we have an exact sequence $0\to \operatorname{Ker}\pi_0(^*I)\otimes_S M \xrightarrow{f_1} \operatorname{Ker}\xi_I(\pi_0(^*I)\otimes_S M)\to \operatorname{Ker}\xi_I\to 0$. Then by using Lemma 3.1 iteratively and the induction hypothesis, we get the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\pi_i(^*I)\otimes_S M \arrow[r] \arrow[d, "f_{i+1}"] & F_i(^*I)\otimes_S M \arrow[r, "\pi_i(^*I)\otimes_S 1_M"] \arrow[d, "\binom{1_{F_i(^*I)\otimes_S M}}{0}"] & \operatorname{Ker}\pi_{i-1}(^*I)\otimes_S M \arrow[r] \arrow[d, "f_i"] & 0 \\ 0 \arrow[r] & \operatorname{Ker}h_i \arrow[r] & (F_i(^*I)\otimes_S M)\oplus F_0(\operatorname{Ker}h_{i-1}) \arrow[r, "h_i"] & \operatorname{Ker}h_{i-1} \arrow[r] & 0, \end{tikzcd}$$

where $h_0=\xi_I(\pi_0(^*I)\otimes_S 1_M)$, $h_i=(f_i(\pi_i(^*I)\otimes_S 1_M),\pi_0(\operatorname{Ker}h_{i-1}))$ for any $i\geqslant 1$, and the induced homomorphism f_{i+1} is monic. By Lemma 3.1, for any $i\geqslant 1$ we have

$$F_i=(F_i(^*I)\otimes_S M, F_i(^*I))_1\oplus(F_0(\operatorname{Ker}h_{i-1}),0)_0 \quad \text{and} \quad \operatorname{Ker}\psi_i=(\operatorname{Ker}h_i,\operatorname{Ker}\pi_i(^*I))_{f_{i+1}}.$$

Moreover, by the snake lemma we get an exact sequence: $0\to\operatorname{Coker}f_{i+1}\to F_0(\operatorname{Ker}h_{i-1})\to\operatorname{Coker}f_i\to 0$ for any $i\geqslant 1$. So $\operatorname{Coker}f_i$ is right R-flat if and only if $\mathrm{r.fd}_R\operatorname{Ker}\xi_I\leqslant i-1$. In addition, by Lemma 2.1, ${}_1(I_R,{}^*I_R)$ is right Λ-flat if and only if ${}^*I_R$ is right S-flat and $\operatorname{Ker}\xi_I=0$. So for any $k\geqslant 0$, we have that $\mathrm{r.fd}_\Lambda\,{}_1(I_R,{}^*I_R)\leqslant k$ if and only if $\operatorname{Ker}\psi_{k-1}$ is right Λ-flat, if and only if $\operatorname{Coker}f_k$ is right R-flat and $\operatorname{Ker}\pi_{k-1}(^*I_R)$ is right S-flat, if and only if $\mathrm{r.fd}_R\operatorname{Ker}\xi_I\leqslant k-1$ and $\mathrm{r.fd}_S\,{}^*I_R\leqslant k$.

(2) By Lemma 3.1, we have that $F_0=(F_0(E)\otimes_S M,F_0(E))_1$ and there exists an exact sequence:

$$0\to\operatorname{Ker}\pi_0(E)\otimes_S M\xrightarrow{f_1}F_0(E)\otimes_S M\to E\otimes_S M\to 0.$$

By using an argument similar to that in (1), we have that $F_i=(F_i(E)\otimes_S M,F_i(E))_1\oplus(F_{i-1}(E)\otimes_S M,0)_0$ for any $i\geqslant 1$, and that $\operatorname{Coker}f_i$ is right R-flat if and only if $\mathrm{r.fd}_R\,E\otimes_S M\leqslant i-1$. Thus for any $k\geqslant 1$, $\mathrm{r.fd}_\Lambda\,{}_0(0,E_S)\leqslant k$ if and only if $\mathrm{r.fd}_S\,E\leqslant k$ and $\operatorname{Coker}f_k$ is right R-flat, if and only if $\mathrm{r.fd}_S\,E\leqslant k$ and $\mathrm{r.fd}_R\,E\otimes_S M\leqslant k-1$. So, it suffices to prove that $\mathrm{r.fd}_R\,E\otimes_S M\leqslant k-1$ if and only if $\mathrm{r.fd}_S\,E\leqslant k-1$.

Let $\cdots\to F_i\xrightarrow{f_i}F_{i-1}\to\cdots\to F_1\xrightarrow{f_1}F_0\xrightarrow{f_0}E\to 0$ be a flat resolution of E_S in $\operatorname{Mod}S$. Since ${}_SM$ is projective, $\cdots\to F_i\otimes_S M\xrightarrow{f_i\otimes_S 1_M}F_{i-1}\otimes_S M\to\cdots\to F_1\otimes_S M\xrightarrow{f_1\otimes_S 1_M}F_0\otimes_S M\xrightarrow{f_0\otimes_S 1_M}E\otimes_S M\to 0$ is a flat resolution of $E\otimes_S M$ in $\operatorname{Mod}R$. So, if $\mathrm{r.fd}_S\,E\leqslant k-1$, then $\mathrm{r.fd}_R\,E\otimes_S M\leqslant k-1$.

Conversely, assume that $\mathrm{r.fd}_R\,E\otimes_S M\leqslant k-1$. Then $\operatorname{Coker}f_k\otimes_S M$ is right R-flat. So $\operatorname{Coker}f_k\otimes_S M$ is a direct limit of a direct system of finitely generated projective right R-modules $\{Q_i\}_{i\in I}$, i.e.,

$$\operatorname{Coker}f_k\otimes_S M=\varinjlim_{i\in I}Q_i,$$

where I is a direct index set. Because M_R is finitely generated projective and $S=\operatorname{End}_R(M)$, by [1, Proposition 20.10] and [10, Lemma 1.2.5], we have

$$\operatorname{Coker}f_k\cong\operatorname{Hom}_R({}_SM_R,\operatorname{Coker}f_k\otimes_S M)\cong\operatorname{Hom}_R\left({}_SM_R,\varinjlim_{i\in I}Q_i\right)\cong\varinjlim_{i\in I}\operatorname{Hom}_R({}_SM_R,Q_i).$$

By the assumption, $\operatorname{Hom}_R({}_SM_R,R)$ is finitely generated right S-projective, so $\operatorname{Coker}f_k$ is right S-flat and $\mathrm{r.fd}_S\,E\leqslant k-1$.

In addition, by Lemma 2.1, ${}_0(0,E_S)$ is right Λ-flat if and only if E_S is right S-flat and $E\otimes_S M=0$. Using essentially the same argument as that in proving the case for $k\geqslant 1$, we have that $E\otimes_S M=0$ if and only if $E=0$. Consequently, we conclude that $\mathrm{r.fd}_\Lambda\,{}_0(0,E_S)\leqslant k$ if and only if $\mathrm{r.fd}_S\,E\leqslant k-1$ for any $k\geqslant 0$. $\square$

Proposition 3.3. *If* $\mathrm{Hom}_R({}_SM_R, R)$ *is finitely generated right* S*-projective, then for any* $k, i \geqslant 0$, $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$ *if and only if the following conditions are satisfied*:

(1) $\mathrm{r.fd}_R \operatorname{Ker} \xi_{I^i(R)} \leqslant k-1$;

(2) $\mathrm{r.fd}_S\, {}^*(I^i(R)) \leqslant k$;

(3) $\mathrm{r.fd}_S\, I^{i-1}({}^*R) \leqslant k-1$.

Proof. Since M_R is finitely generated projective, $\mathrm{r.fd}_{\Lambda\,1}(I^i(R), {}^*(I^i(R))) \leqslant k$ yields that $\mathrm{r.fd}_{\Lambda\,1}(I^i(M), {}^*(I^i(M))) \leqslant k$. Note that $\mathrm{r.fd}_\Lambda\, I^0(\Lambda) \leqslant k$ if and only if $\mathrm{r.fd}_{\Lambda\,1}(I^0(R), {}^*(I^0(R))) \leqslant k$. So, by Proposition 3.2(1), $\mathrm{r.fd}_\Lambda\, I^0(\Lambda) \leqslant k$ if and only if $\mathrm{r.fd}_R \operatorname{Ker} \xi_{I^0(R)} \leqslant k-1$ and $\mathrm{r.fd}_S\, {}^*(I^0(R)) \leqslant k$. The case for $i=0$ follows. Now suppose $i \geqslant 1$. Note that $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$ if and only if $\mathrm{r.fd}_{\Lambda\,1}(I^i(R), {}^*(I^i(R))) \leqslant k$ and $\mathrm{r.fd}_{\Lambda\,0}(0, I^{i-1}({}^*R)) \leqslant k$. So, by Proposition 3.2, we have that $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$ if and only if $\mathrm{r.fd}_R \operatorname{Ker} \xi_{I^i(R)} \leqslant k-1$, $\mathrm{r.fd}_S\, {}^*(I^i(R)) \leqslant k$ and $\mathrm{r.fd}_S\, I^{i-1}({}^*R) \leqslant k-1$. □

Proposition 3.4. *If* $\mathrm{Hom}_R({}_SM_R, R)$ *is finitely generated right* S*-projective, then for any* $i \geqslant 0$, $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$ *yields that* $\mathrm{r.fd}_R\, I^i(R) \leqslant k$ *and* $\mathrm{r.fd}_S\, I^{i-1}(S) \leqslant k-1$.

Proof. For any $i \geqslant 0$, by (1) and (2) of Proposition 3.3, we have $\mathrm{r.fd}_R \operatorname{Ker} \xi_{I^i(R)} \leqslant k-1$ and $\mathrm{r.fd}_R\, {}^*(I^i(R)) \otimes_S M \leqslant k$. In addition, we have the following exact sequence:

$$0 \to \operatorname{Ker} \xi_{I^i(R)} \to {}^*(I^i(R)) \otimes_S M \xrightarrow{\xi_{I^i(R)}} I^i(R) \to 0,$$

which implies $\mathrm{r.fd}_R\, I^i(R) \leqslant k$.

On the other hand, since M_R is finitely generated projective, the condition (3) in Proposition 3.3 is also satisfied when ${}^*R$ is replaced by ${}^*M$. It follows that $\mathrm{r.fd}_S\, I^{i-1}(S) \leqslant k-1$. □

Proposition 3.5. *For any* $i \geqslant 0$, $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$ *if and only if* $\mathrm{r.fd}_\Gamma\, I^i(\Gamma) \leqslant k$, *where*

$$\Gamma = \begin{pmatrix} R & 0 & 0 \\ M & S & 0 \\ M & S & S \end{pmatrix}.$$

Proof. Let $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Lambda$. Then we have

$$\Gamma = \begin{pmatrix} \Lambda & 0 \\ e\Lambda & e\Lambda e \end{pmatrix}, \quad \Lambda e = e\Lambda e \cong S.$$

Since Λ can be embedded in

$$\mathrm{End}_S(e\Lambda) \cong \begin{pmatrix} \mathrm{End}_S(M) & \mathrm{Hom}_S(M,S) \\ M & S \end{pmatrix},$$

$e\Lambda$ is a faithful right Λ-module. It is trivial that $e\Lambda$ is finitely generated projective as a right Λ-module and a left S-module. Notice that $S \cong \mathrm{End}_\Lambda(e\Lambda)$ and $\mathrm{Hom}_\Lambda(e\Lambda, \Lambda)_S \cong \Lambda e_S \cong S_S$, so by Proposition 3.3, we get that $\mathrm{r.fd}_\Gamma\, I^i(\Gamma) \leqslant k$ if and only if the following conditions are satisfied.

(1) $\mathrm{r.fd}_\Lambda \operatorname{Ker} \eta_{I^i(\Lambda)} \leqslant k-1$, where $\eta_E : \mathrm{Hom}_\Lambda(e\Lambda, E) \otimes_{e\Lambda e} e\Lambda \to E$ defined by $\eta_E(\alpha \otimes x) = \alpha(x)$ for any $\alpha \in \mathrm{Hom}_\Lambda(e\Lambda, E)$ and $x \in e\Lambda$ is the natural homomorphism for an injective right R-module E.

(2) $\mathrm{r.fd}_S \mathrm{Hom}_\Lambda(e\Lambda, I^i(\Lambda))_S \leqslant k$.

(3) $\mathrm{r.fd}_S\, I^{i-1}(\mathrm{Hom}_\Lambda(e\Lambda, \Lambda)_S) \leqslant k-1$.

Assume that $\mathrm{r.fd}_\Gamma\, I^i(\Gamma) \leqslant k$. By Proposition 3.4, $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$. It remains to show that $\mathrm{r.fd}_\Lambda\, I^i(\Lambda) \leqslant k$ implies $\mathrm{r.fd}_\Gamma\, I^i(\Gamma) \leqslant k$ for any $i \geqslant 0$. To do this, it suffices to show that the conditions (1)–(3) above are satisfied.

Note that $\mathrm{Hom}_\Lambda(e\Lambda, \Lambda)_S \cong S_S$. If $L \in \operatorname{Mod} \Lambda$ is flat, then L is a direct limit of a direct system of finitely generated projective right Λ-modules $\{P_i\}_{i \in I}$, i.e., $L = \varinjlim_{i \in I} P_i$, where I is a directed set. So

$$\mathrm{Hom}_\Lambda(e\Lambda, L) \cong \mathrm{Hom}_\Lambda\left(e\Lambda, \varinjlim_{i \in I} P_i\right) \cong \varinjlim_{i \in I} \mathrm{Hom}_\Lambda(e\Lambda, P_i)$$

is right S-flat. Then it is not difficult to verify that $\mathrm{r.fd}_\Lambda I^i(\Lambda) \leqslant k$ yields $\mathrm{r.fd}_S \mathrm{Hom}_\Lambda(e\Lambda, I^i(\Lambda))_S \leqslant k$. Thus Condition (2) is satisfied.

Using $\mathrm{Hom}_\Lambda(e\Lambda, \Lambda)_S \cong S_S$ again, then Condition (3) is satisfied by Proposition 3.4.

By [19, Corollary 3],

$$I^0(\Lambda) = {}_1(I^0(R), {}^*(I^0(R))) \oplus {}_1(I^0(M), {}^*(I^0(M))),$$

and

$$I^i(\Lambda) = {}_1(I^i(R), {}^*(I^i(R))) \oplus {}_1(I^i(M), {}^*(I^i(M))) \oplus {}_0(0, I^{i-1}({}^*R))$$

give a minimal injective resolution of Λ_Λ. So, to verify Condition (1), it suffices to show that $\mathrm{r.fd}_\Lambda \mathrm{Ker}\, \eta_E \leqslant k-1$ for any injective right Λ-module E, where E is of the form:

(a) ${}_0(0, I_S)$ with I_S injective, or

(b) ${}_1(I_R, {}^*(I_R))$ with I_R injective.

If E is of the form (a), then it is not difficult to verify that $e\Lambda \cong (M, S)_\psi$ (see [8, p. 2]) and $Ee = {}_0(0, I_S)e = {}_0(0, I_S) = E$. So $\mathrm{Hom}_\Lambda(e\Lambda, E) \otimes_S e\Lambda \cong Ee \otimes_S e\Lambda \cong E \otimes_S (M, S)_\psi \cong E$, where $\psi : S \otimes_S M \to M$ is the natural isomorphism. Moreover, $S \cong \mathrm{End}_\Lambda(e\Lambda)$ and $e\Lambda$ is finitely generated projective as a right Λ-module and a left S-module. By [1, Proposition 20.11], it is easy to see that η_E is epic. Because $e\Lambda e \cong S$ and $\mathrm{Hom}_\Lambda(e\Lambda, E) \otimes_S e\Lambda \cong E$, $\mathrm{Ker}\, \eta_E = 0$.

If E is of the form (b), then $E = {}_1(I_R, {}^*(I_R))$ and $\mathrm{Ker}\, \eta_E \cong {}_0(\mathrm{Ker}\, \xi_{I_R}, 0)$. On the other hand, by Lemma 3.1, we have $\mathrm{r.fd}_\Lambda\, {}_0(X_R, 0) = \mathrm{r.fd}_R X$. So $\mathrm{r.fd}_\Lambda \mathrm{Ker}\, \eta_{I^i(\Lambda)} \leqslant k-1$ if and only if $\mathrm{r.fd}_R \mathrm{Ker}\, \xi_{I^i(R)} \leqslant k-1$. Because $\mathrm{r.fd}_\Lambda I^i(\Lambda) \leqslant k$, $\mathrm{r.fd}_\Lambda \mathrm{Ker}\, \eta_E \leqslant k-1$. □

We are now in a position to state the main result in this section.

Theorem 3.6. *If R is a left and right Noetherian ring and t is a positive integer, then $T_t(R)$ is a left and right Noetherian ring, and* $\mathrm{r.fd}_{T_t(R)} I^i(T_t(R)) = \max\{\mathrm{r.fd}_R I^i(R),\ \mathrm{r.fd}_R I^{i-1}(R) + 1\}$ *for any $i \geqslant 0$.*

Proof. The first assertion follows from Proposition 2.4. We will prove the second assertion by induction on t. The case for $t = 1$ is trivial, and the case for $t = 2$ follows from [8, Theorem 3.10].

Now assume $t \geqslant 3$. By Lemma 2.3,

$$T_t(R) = \begin{pmatrix} T_{t-1}(R) & 0 \\ {}_R R^{(t-1)}_{T_{t-1}(R)} & R \end{pmatrix}$$

with $R^{(t-1)}_{T_{t-1}(R)}$ faithful and finitely generated projective and $\mathrm{End}_{T_{t-1}(R)}(R^{(t-1)}) \cong R$. By Proposition 3.5, $\mathrm{r.fd}_{T_t(R)} I^i(T_t(R)) \leqslant k$ if and only if $\mathrm{r.fd}_{T_{t-1}(R)} I^i(T_{t-1}(R)) \leqslant k$. So we have that $\mathrm{r.fd}_{T_t(R)} I^i(T_t(R)) = \mathrm{r.fd}_{T_2(R)} I^i(T_2(R)) = \max\{\mathrm{r.fd}_R I^i(R), \mathrm{r.fd}_R I^{i-1}(R) + 1\}$ by the induction hypothesis. □

As an immediate consequence of Theorem 3.6, we get the main theorem mentioned in Section 1.

Theorem 3.7. *If R is a left and right Noetherian ring and $n, k \geqslant 0$, $t \geqslant 1$, then R is $G_n(k)$ if and only if so is $T_t(R)$.*

Proof. By Theorem 3.6, we have that $\mathrm{r.fd}_R I^i(R) \leqslant \mathrm{r.fd}_{T_t(R)} I^i(T_t(R))$ for any $i \geqslant 0$, so the sufficiency is trivial. Conversely, if R is $G_n(k)$, then by Theorem 3.6, $\mathrm{r.fd}_{T_t(R)} I^i(T_t(R)) = \max\{\mathrm{r.fd}_R I^i(R), \mathrm{r.fd}_R I^{i-1}(R) + 1\} \leqslant i + k$ for any $0 \leqslant i \leqslant n-1$ and $T_t(R)$ is $G_n(k)$. □

We recall some notions introduced by Iyama in [20]. Let R be a left and right Noetherian ring and $l, n \geqslant 0$. R is said to satisfy the $(l, n)^{\mathrm{op}}$-*condition* if $\mathrm{r.fd}\, I_i(R) \leqslant l-1$ for any $0 \leqslant i \leqslant n-1$. It is easy to see that R is $G_n(k)$ if and only if R satisfies the $(k+i, i)^{\mathrm{op}}$-condition for any $1 \leqslant i \leqslant n$. In addition, if $\mathrm{r.fd}_R I^i(R) < \mathrm{r.fd}_R I^n(R)$ for any $0 \leqslant i \leqslant n-1$, then n is called a *dominant number* of R_R. As another application of Theorem 3.6, we get the following.

Corollary 3.8. *If R is a left and right Noetherian ring, then for any $l, n \geqslant 0$, $t \geqslant 1$, we have the following*:

(1) *If R satisfies the $(l, n)^{\mathrm{op}}$-condition, then $T_t(R)$ satisfies the $(l+1, n)^{\mathrm{op}}$-condition. Conversely, if $T_t(R)$ satisfies the $(l, n)^{\mathrm{op}}$-condition, then so does R*;

(2) *If n is a dominant number of R, then $n+1$ is a dominant number of $T_t(R)$.*

Proof. (1) If R satisfies the $(l,n)^{\mathrm{op}}$-condition, then by Theorem 3.6, we have $\mathrm{r.fd}_{T_t(R)}\, I^i(T_t(R)) = \max\{\mathrm{r.fd}_R\, I^i(R),\ \mathrm{r.fd}_R\, I^{i-1}(R)+1\} \leqslant l$ for any $0 \leqslant i \leqslant n-1$, which implies that $T_t(R)$ satisfies the $(l+1,n)^{\mathrm{op}}$-condition. Conversely, by Theorem 3.6, we have that $\mathrm{r.fd}_R\, I^i(R) \leqslant \mathrm{r.fd}_{T_t(R)}\, I^i(T_t(R))$ for any $i \geqslant 0$, so it is trivial that $T_t(R)$ satisfies the $(l,n)^{\mathrm{op}}$-condition implies so does R.

(2) If n is a dominant number of R, then $\mathrm{r.fd}_R\, I^i(R) < \mathrm{r.fd}_R\, I^n(R)$ for any $0 \leqslant i \leqslant n-1$. So by Theorem 3.6, for any $0 \leqslant i \leqslant n$, we have that $\mathrm{r.fd}_{T_t(R)}\, I^{n+1}(T_t(R)) = \max\{\mathrm{r.fd}_R\, I^{n+1}(R), \mathrm{r.fd}_R\, I^n(R)+1\} \geqslant \mathrm{r.fd}_R\, I^n(R)+1 > \max\{\mathrm{r.fd}_R\, I^i(R), \mathrm{r.fd}_R\, I^{i-1}(R)+1\} = \mathrm{r.fd}_{T_t(R)}\, I^i(T_t(R))$, which implies that $n+1$ is a dominant number of $T_t(R)$. □

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