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The socle of the last term in a minimal injective resolution

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Abstract Let Λ and Γ be left and right Noetherian rings and ${}_\Lambda U$ a generalized tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. For a non-negative integer k, if ${}_\Lambda U$ is $(k-2)$-Gorenstein with the injective dimensions of ${}_\Lambda U$ and U_Γ being k, then the socle of the last term in a minimal injective resolution of ${}_\Lambda U$ is non-zero.

Keywords generalized tilting modules, (quasi) k-Gorenstein modules, socle, minimal injective resolution, injective dimension

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1 Introduction

Let Λ be a left and right Noetherian ring with finite left and right self-injective dimensions. The following question still remains open.

Question. *Is the socle of the last term in a minimal injective resolution of* ${}_\Lambda\Lambda$ *non-zero?*

If Λ is Artinian, then the question is trivially true. Hoshino showed in [4, Theorem 4.5] that the question is true if the left and right self-injective dimensions of Λ are at most 2. Let Λ be an Auslander-Gorenstein ring. Then Fuller and Iwanaga showed in [3, Proposition 1.1] that this question is also true; in this case, Iwanaga and Sato further showed in [9, Theorem 6] that this socle is essential in the last term. In this paper, we also deal with this open question, and give some partial answer to it.

As a natural generalization of Auslander's k-Gorenstein rings, we introduced in [7] the notion of k-Gorenstein modules such that a left and right Noetherian ring Λ is k-Gorenstein if and only if it is k-Gorenstein as a Λ-module, and the characterizations of k-Gorenstein modules are very similar to that of k-Gorenstein rings (cf. [6, 7]).

Motivated by these results, in this paper we prove the following

Theorem. *Let* Λ *and* Γ *be left and right Noetherian rings and* ${}_\Lambda U$ *a generalized tilting module with* $\Gamma = \mathrm{End}({}_\Lambda U)$. *For a non-negative integer* k, *if* ${}_\Lambda U$ *is* $(k-2)$-*Gorenstein with the injective dimensions of* ${}_\Lambda U$ *and* U_Γ *being* k, *then the socle of the last term in a minimal injective resolution of* ${}_\Lambda U$ *is non-zero.*

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To prove the above theorem, we will prove that there exists a module N in $\mathrm{mod}\,\Gamma^{op}$ such that $\mathrm{Ext}^k_\Gamma(N,U) \neq 0$ and any simple quotient module of $\mathrm{Ext}^k_\Gamma(N,U)$ can be embedded into the last term in a minimal injective resolution of ${}_\Lambda U$. As an immediate consequence of the theorem above, we have that if Λ is a $(k-2)$-Gorenstein ring with left and right self-injective dimensions k, then the socle of the last term in a minimal injective resolution of ${}_\Lambda\Lambda$ is non-zero. This generalizes the above results of Hoshino and Fuller-Iwanaga. Furthermore, at the end of this paper, we get that the last conclusion mentioned above also holds true for a left quasi $(k-2)$-Gorenstein ring with left and right self-injective dimensions k.

2 Preliminaries

In this section, we give some definitions in our terminology and collect some facts which are often used in the rest of this paper.

Let Λ be a ring. We use $\mathrm{mod}\,\Lambda$ to denote the category of finitely generated left Λ-modules, and $\mathrm{Mod}\,\Lambda$ to denote the category of left Λ-modules.

Definition 2.1 [10]. *A module ${}_\Lambda U$ in $\mathrm{mod}\,\Lambda$ is called a generalized tilting module if ${}_\Lambda U$ is self-orthogonal* (*that is,* $\mathrm{Ext}^i_\Lambda({}_\Lambda U, {}_\Lambda U) = 0$ *for any* $i \geqslant 1$), *and possessing an exact sequence*:

$$0 \to {}_\Lambda\Lambda \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

such that: (1) *all terms U_i are isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda U$, and* (2) *after applying the functor* $\mathrm{Hom}_\Lambda(\cdot, U)$ *the sequence is still exact.*

Let Λ and Γ be rings. A bimodule ${}_\Lambda T_\Gamma$ is said to be *faithfully balanced* if $\Lambda = \mathrm{End}(T_\Gamma)$ and $\Gamma = \mathrm{End}({}_\Lambda T)$; and it is said to be *self-orthogonal* if $\mathrm{Ext}^i_\Lambda({}_\Lambda T, {}_\Lambda T) = 0$ and $\mathrm{Ext}^i_\Gamma(T_\Gamma, T_\Gamma) = 0$ for any $i \geqslant 1$. For a bimodule ${}_\Lambda U_\Gamma$ with ${}_\Lambda U$ in $\mathrm{mod}\,\Lambda$ and U_Γ in $\mathrm{mod}\,\Gamma^{op}$, by [10, Corollary 3.2], we have that ${}_\Lambda U_\Gamma$ is faithfully balanced and self-orthogonal if and only if ${}_\Lambda U$ is generalized tilting with $\Gamma = \mathrm{End}({}_\Lambda U)$, if and only if U_Γ is generalized tilting with $\Lambda = \mathrm{End}(U_\Gamma)$.

Let U and A be in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$) and i a non-negetive integer. We say that the *grade* of A with respect to U, written $\mathrm{grade}_U\, A$, is at least i if $\mathrm{Ext}^j_\Lambda(A,U) = 0$ (resp. $\mathrm{Ext}^j_\Gamma(A,U) = 0$) for any $0 \leqslant j < i$. We say that the *strong grade* of A with respect to U, written $\mathrm{s.grade}_U\, A$, is at least i if $\mathrm{grade}_U\, B \geqslant i$ for all submodules B of A (cf. [5]).

Definition 2.2 [7]. *For a non-negative integer k, a module $U \in \mathrm{mod}\,\Lambda$ with $\Gamma = \mathrm{End}({}_\Lambda U)$ is called k-Gorenstein if* $\mathrm{s.grade}_U\, \mathrm{Ext}^i_\Gamma(N,U) \geqslant i$ *for any $N \in \mathrm{mod}\,\Gamma^{op}$ and $1 \leqslant i \leqslant k$. Similarly, we may define the notion of k-Gorenstein modules in* $\mathrm{mod}\,\Gamma^{op}$.

For a module T in $\mathrm{Mod}\,\Lambda$ (resp. $\mathrm{Mod}\,\Lambda^{op}$), we use $\text{add-lim}_\Lambda\, T$ (resp. $\text{add-lim}_\Lambda\, T$) to denote the full subcategory of $\mathrm{Mod}\,\Lambda$ (resp. $\mathrm{Mod}\,\Lambda^{op}$) consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_\Lambda T$ (resp. T_Λ).

Definition 2.3 [6]. *Let Λ be a ring and T in $\mathrm{Mod}\,\Lambda$. For a module A in $\mathrm{Mod}\,\Lambda$, if there exists an exact sequence $\cdots \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0$ in $\mathrm{Mod}\,\Lambda$ with $T_i \in \text{add-lim}_\Lambda\, T$ for any $i \geqslant 0$, then we define $T\text{-lim.dim}_\Lambda(A) = \inf\{n \mid$ there exists an exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0$ in $\mathrm{Mod}\,\Lambda$ with $T_i \in \text{add-lim}_\Lambda\, T$ for any $0 \leqslant i \leqslant n\}$. We set $T\text{-lim.dim}_\Lambda(A)$ infinite if no such an integer exists. For Λ^{op}-modules, we may define such a dimension similarly.*

From now on, both Λ and Γ are left and right Noetherian rings and ${}_\Lambda U$ is a generalized tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. We always assume that

$$0 \to {}_\Lambda U \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

is a minimal injective resolution of ${}_\Lambda U$, and

$$0 \to U_\Gamma \to E'_0 \to E'_1 \to \cdots \to E'_i \to \cdots$$

is a minimal injective resolution of U_Γ.

The following result gives some equivalent characterizations of k-Gorenstein modules, among which the equivalence of (1) and $(1)^{op}$ is due to [10, Theorem 7.5], and the other implications are contained in [6, Theorem II].

Theorem 2.4. *The following statements are equivalent for a non-negative integer* k:

(1) ${}_\Lambda U$ *is* k-*Gorenstein*;

(2) U-lim.dim$_\Lambda(E_i) \leqslant i$ *for any* $0 \leqslant i \leqslant k-1$;

(3) $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(\text{-}, U), U)$ *preserves monomorphisms in* $\mathrm{mod}\,\Lambda$ *for any* $0 \leqslant i \leqslant k-1$;

$(1)^{op}$ U_Γ *is* k-*Gorenstein*;

$(2)^{op}$ U-lim.dim$_\Gamma(E'_i) \leqslant i$ *for any* $0 \leqslant i \leqslant k-1$;

$(3)^{op}$ $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Gamma(\text{-}, U), U)$ *preserves monomorphisms in* $\mathrm{mod}\,\Gamma^{op}$ *for any* $0 \leqslant i \leqslant k-1$.

Recall from [9] that a left and right Noetherian ring Λ is called *Auslander's* k-*Gorenstein* if for any $1 \leqslant i \leqslant k$, the i-th term in a minimal injective resolution of ${}_\Lambda\Lambda$ has flat dimension at most $i-1$. Notice that if putting ${}_\Lambda T = {}_\Lambda\Lambda$ (resp. $T_\Lambda = \Lambda_\Lambda$), then the dimension defined in Definition 2.3 is just the flat dimension of modules. So, by Theorem 2.4, we have that Λ is a k-Gorenstein ring if and only if it is k-Gorenstein as a Λ-module.

Suppose that $A \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$). We call $\mathrm{Hom}_\Lambda(A, {}_\Lambda U_\Gamma)$ (resp. $\mathrm{Hom}_\Gamma(A, {}_\Lambda U_\Gamma)$) the *dual module* of A with respect to U, and denote either of these modules by A^*. For a homomorphism f between Λ-modules (resp. Γ^{op}-modules), we put $f^* = \mathrm{Hom}(f, {}_\Lambda U_\Gamma)$. Let $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. A is called U-*torsionless* (resp. U-*reflexive*) if σ_A is a monomorphism (resp. an isomorphism). Under the assumption of ${}_\Lambda U$ being generalized tilting with $\Gamma = \mathrm{End}({}_\Lambda U)$ (more generally, ${}_\Lambda U_\Gamma$ being faithfully balanced), it is easy to see that any projective module in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$) is U-reflexive.

The first three statements in the following result is the U-dual version of parts of [2, Theorem 4.7].

Theorem 2.5 [6, Theorem 5.4]**.** *The following statements are equivalent for a non-negative integer* k:

(1) s.grade$_U\mathrm{Ext}^{i+1}_\Gamma(N, U) \geqslant i$ *for any* $N \in \mathrm{mod}\,\Gamma^{op}$ *and* $1 \leqslant i \leqslant k$;

(2) U-lim.dim$_\Lambda(E_i) \leqslant i+1$ *for any* $0 \leqslant i \leqslant k-1$;

(3) grade$_U\mathrm{Ext}^i_\Lambda(M, U) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k$;

(4) $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(\text{-}, U), U)$ *preserves monomorphisms* $X \to Y$ *with both* X *and* Y U-*torsionless in* $\mathrm{mod}\,\Lambda$ *for any* $0 \leqslant i \leqslant k-1$.

If one of the above equivalent conditions holds, then ${}_\Lambda U$ *is called a quasi* k-*Gorenstein module. We may define* U_Γ *to be a quasi* k-*Gorenstein module similarly.*

Remark. As already pointed out in [6], contrary to the notion of k-Gorenstein modules, that of quasi k-Gorenstein modules is not left-right symmetric.

Lemma 2.6 [7, Lemma 2.2]**.** *Let* k *a non-negative integer. For a module* M *in* $\mathrm{mod}\,\Lambda$, *if* grade$_U\, M \geqslant k$ *and* grade$_U\mathrm{Ext}^k_\Lambda(M, U) \geqslant k+1$, *then* $\mathrm{Ext}^k_\Lambda(M, U) = 0$.

Lemma 2.7 [5, Lemma 2.7]**.** *The following statements are equivalent*:

(1) grade$_U\, \mathrm{Ext}^2_\Lambda(M, U) \geqslant 1$ *for any* $M \in \mathrm{mod}\,\Lambda$;

(2) M^* *is* U-*reflexive for any* $M \in \mathrm{mod}\,\Lambda$;

$(1)^{op}$ grade$_U\, \mathrm{Ext}^2_\Gamma(N, U) \geqslant 1$ *for any* $N \in \mathrm{mod}\,\Gamma^{op}$;

$(2)^{op}$ N^* *is* U-*reflexive for any* $N \in \mathrm{mod}\,\Gamma^{op}$.

Let $M \in \mathrm{mod}\,\Lambda$ and

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

be a projective presentation of M in $\mathrm{mod}\,\Lambda$. Then we have an exact sequence:

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to X \to 0,$$

where $X = \mathrm{Coker}\, f^*$.

Lemma 2.8 [8, Lemma 2.1]**.** *Let* M *and* X *be as above. Then we have the following exact sequences*:

$$0 \to \mathrm{Ext}^1_\Gamma(X, U) \to M \xrightarrow{\sigma_M} M^{**} \to \mathrm{Ext}^2_\Gamma(X, U) \to 0,$$

$$0 \to \mathrm{Ext}^1_\Lambda(M,U) \to X \xrightarrow{\sigma_X} X^{**} \to \mathrm{Ext}^2_\Lambda(M,U) \to 0.$$

Let M and X be as above. For a non-negative integer k, recall from [5] that M is called *U-k-torsionfree* if $\mathrm{Ext}^i_\Gamma(X,U)=0$ for any $1 \leqslant i \leqslant k$. M is called *U-k-syzygy* if there exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ with all X_i in $\mathrm{add}\,_\Lambda U$, where $\mathrm{add}\,_\Lambda U$ denotes the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of all modules isomorphic to direct summands of a finite direct sum of copies of $_\Lambda U$. Putting $_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$, then, in this case, the notions of U-k-torsionfree modules and U-k-syzygy modules are just that of k-torsionfree modules and k-syzygy modules respectively (cf. [2] for the definitions of k-torsionfree modules and k-syzygy modules). By Lemma 2.8, it is easy to see that a module in $\mathrm{mod}\,\Lambda$ is U-torsionless (resp. U-reflexive) if and only if it is U-1-torsionfree (resp. U-2-torsionfree). We may define U-k-torsionfree modules and U-k-syzygy modules in $\mathrm{mod}\,\Gamma^{op}$ similarly.

We use $\mathcal{T}^k_U(\mathrm{mod}\,\Lambda)$ (resp. $\mathcal{T}^k_U(\mathrm{mod}\,\Gamma^{op})$) and $\Omega^k_U(\mathrm{mod}\,\Lambda)$ (resp. $\Omega^k_U(\mathrm{mod}\,\Gamma^{op})$) to denote the full subcategory of $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Gamma^{op}$) consisting of U-k-torsionfree modules and U-k-syzygy modules, respectively. It was pointed out in [5] that $\mathcal{T}^k_U(\mathrm{mod}\,\Lambda) \subseteq \Omega^k_U(\mathrm{mod}\,\Lambda)$ and $\mathcal{T}^k_U(\mathrm{mod}\,\Gamma^{op}) \subseteq \Omega^k_U(\mathrm{mod}\,\Gamma^{op})$.

In the following result, the equivalence of (1) and $(1)^{op}$ was proved in [7, Lemma 3.3], and the latter assertion is contained in [5, Theorem 3.1].

Lemma 2.9. *The following statements are equivalent for a non-negative integer k:*

(1) $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Lambda(M,U) \geqslant i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leqslant i \leqslant k-1$;

$(1)^{op}$ $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Gamma(N,U) \geqslant i$ *for any* $N \in \mathrm{mod}\,\Gamma^{op}$ *and* $1 \leqslant i \leqslant k-1$.

If one of the above equivalent conditions holds, then for any $1 \leqslant i \leqslant k$, $\Omega^i_U(\mathrm{mod}\,\Lambda) = \mathcal{T}^i_U(\mathrm{mod}\,\Lambda)$ *and* $\Omega^i_U(\mathrm{mod}\,\Gamma^{op}) = \mathcal{T}^i_U(\mathrm{mod}\,\Gamma^{op})$.

The following result gives a sufficient condition when a k-syzygy module is U-k-syzygy.

Lemma 2.10. (1) *If $_\Lambda U$ is quasi $(k-1)$-Gorenstein, then each k-syzygy module in* $\mathrm{mod}\,\Gamma^{op}$ *is in* $\Omega^k_U(\mathrm{mod}\,\Gamma^{op})$.

$(1)^{op}$ *If U_Γ is quasi $(k-1)$-Gorenstein, then each k-syzygy module in* $\mathrm{mod}\,\Lambda$ *is in* $\Omega^k_U(\mathrm{mod}\,\Lambda)$.

Proof. The case for $k=0$ is trivial. Since Γ is U-reflexive (and certainly is U-1-syzygy), it is easy to see that each 1-syzygy module in $\mathrm{mod}\,\Gamma^{op}$ is U-1-syzygy. The case for $k=1$ follows. Now suppose $k \geqslant 2$. If $_\Lambda U$ is quasi $(k-1)$-Gorenstein, then by Theorem 2.5, we have that $\mathrm{s.grade}_U \mathrm{Ext}^{i+1}_\Gamma(N,U) \geqslant i$ for any $N \in \mathrm{mod}\,\Gamma^{op}$ and $1 \leqslant i \leqslant k-1$. It follows from the symmetric result of [7, Lemma 3.2] that each k-syzygy module in $\mathrm{mod}\,\Gamma^{op}$ is in $\Omega^k_U(\mathrm{mod}\,\Gamma^{op})$. This prove (1). We may prove $(1)^{op}$ similarly.

3 Main results

For a Λ-module (resp. Γ^{op}-module) X, we use $\mathrm{l.id}_\Lambda(X)$ (resp. $\mathrm{r.id}_\Gamma(X)$) to denote its left (resp. right) injective dimension. In the following, k is a non-negative integer.

In this section, we prove the following theorem, which is the main result in this paper.

Theorem 3.1. *If $_\Lambda U$ is $(k-2)$-Gorenstein with* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) = k$, *then* $\mathrm{Soc}(E_k) \neq 0$, *where* $\mathrm{Soc}(E_k)$ *is the socle of* E_k.

We use

$$0 \to {}_\Lambda\Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$$

to denote a minimal injective resolution of $_\Lambda\Lambda$. Hoshino showed in [4, Theorem 4.5] that if $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda) = k \leqslant 2$, then $\mathrm{Soc}(I_k) \neq 0$. Recall that Λ is called *Auslander-Gorenstein* if Λ is a k-Gorenstein ring for all k and $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda) < \infty$. Fuller and Iwanaga showed in [3, Proposition 1.1] that if Λ is Auslander-Gorenstein with $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda) = k$, then $\mathrm{Soc}(I_k) \neq 0$ (in fact, it was showed in [9, Theorem 6] that in this case $\mathrm{Soc}(I_k)$ is essential in I_k).

By Theorem 3.1, we have the following corollary, which generalizes the above results of Hoshino and Fuller-Iwanaga.

Corollary 3.2. *If Λ is a $(k-2)$-Gorenstein ring with* $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda) = k$, *then* $\mathrm{Soc}(I_k) \neq 0$.

In the following, we prove Theorem 3.1. We begin with the following result, which is the U-dual version of [4, Lemma 4.2].

Lemma 3.3. *Let n be a positive integer. If* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) \leqslant n$, *then, for any* $N \in \mathrm{mod}\,\Gamma^{op}$, $\mathrm{grade}_U \mathrm{Ext}^n_\Gamma(N, U) \geqslant 1$; *moreover, if* $n \geqslant 2$, *then* $\mathrm{grade}_U \mathrm{Ext}^n_\Gamma(N, U) \geqslant 2$.

Proof. Let N be any module in $\mathrm{mod}\,\Gamma^{op}$ and

$$\cdots \to Q_i \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

a projective resolution of N in $\mathrm{mod}\,\Gamma^{op}$. Put $X = \mathrm{Coker}(Q^*_{n-1} \to Q^*_n)$. Since $\mathrm{r.id}_\Gamma(U) \leqslant n$, by Lemma 2.8 we have the following exact sequence:

$$0 \to \mathrm{Ext}^n_\Gamma(N, U) \to X \xrightarrow{\sigma_X} X^{**} \to \mathrm{Ext}^{n+1}_\Gamma(N, U) = 0. \tag{1}$$

We then get the following exact sequence:

$$\begin{aligned} 0 \to X^{***} \xrightarrow{\sigma^*_X} X^* \to [\mathrm{Ext}^n_\Gamma(N, U)]^* \to \mathrm{Ext}^1_\Lambda(X^{**}, U) \to \mathrm{Ext}^1_\Lambda(X, U) \\ \to \mathrm{Ext}^1_\Lambda(\mathrm{Ext}^n_\Gamma(N, U), U) \to \mathrm{Ext}^2_\Lambda(X^{**}, U). \end{aligned} \tag{2}$$

Since both Q_n and Q_{n-1} are U-reflexive, it is easy to see that $X^* \cong \mathrm{Ker}(Q_n \to Q_{n-1})$. Since $\mathrm{r.id}_\Gamma(U) \leqslant n$, $\mathrm{Ext}^i_\Gamma(X^*, U) = 0$ for any $i \geqslant 1$ and we get an exact sequence:

$$0 \to X^{**} \to Q^*_{n+1} \to Q^*_{n+2} \to \cdots$$

in $\mathrm{mod}\,\Lambda$ with Q^*_i in $\mathrm{add}_\Lambda U$ for any $i \geqslant n+1$. Since $\mathrm{l.id}_\Lambda(U) \leqslant n$, $\mathrm{Ext}^i_\Lambda(X^{**}, U) = 0$ for any $i \geqslant 1$. By [1, Proposition 20.14], σ^*_X is epic. So, by the exact sequence (2), we have that $[\mathrm{Ext}^n_\Gamma(N, U)]^* = 0$ and $\mathrm{grade}_U \mathrm{Ext}^n_\Gamma(N, U) \geqslant 1$.

Put $Y = \mathrm{Coker}(Q_n \to Q_{n-1})$. If $n \geqslant 2$, then Y is 1-syzygy and hence it is in $\Omega^1_U(\mathrm{mod}\,\Gamma^{op})$ by Lemma 2.10. So σ_Y is monic, and it follows from Lemma 2.8 that $\mathrm{Ext}^1_\Lambda(X, U) \cong \mathrm{Ker}\,\sigma_Y = 0$. Hence, by the exact sequence (2), we have that $\mathrm{Ext}^1_\Lambda(\mathrm{Ext}^n_\Gamma(N, U), U) = 0$ and $\mathrm{grade}_U \mathrm{Ext}^n_\Gamma(N, U) \geqslant 2$.

The following result is a generalization of Lemma 3.3.

Lemma 3.4. *If* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) = k$ *and* $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Gamma(N, U) \geqslant i$ *for any* $N \in \mathrm{mod}\,\Gamma^{op}$ *and* $1 \leqslant i \leqslant k-2$, *then* $\mathrm{grade}_U \mathrm{Ext}^k_\Gamma(N, U) \geqslant k$ *for any* $N \in \mathrm{mod}\,\Gamma^{op}$.

Proof. The case for $k = 0$ is trivial. The cases for $k = 1$ and 2 follow from Lemma 3.3.

Now suppose that $k \geqslant 3$, N is any module in $\mathrm{mod}\,\Gamma^{op}$ and

$$\cdots \to Q_i \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

is a projective resolution of N in $\mathrm{mod}\,\Gamma^{op}$. Put $X = \mathrm{Coker}(Q^*_{k-1} \to Q^*_k)$. By using the same argument as that in the proof of Lemma 3.3, we have the following exact sequence:

$$0 \to \mathrm{Ext}^k_\Gamma(N, U) \to X \xrightarrow{\sigma_X} X^{**} \to 0. \tag{3}$$

Since $\mathrm{grade}_U \mathrm{Ext}^i_\Gamma(N, U) \geqslant i$ for any $N \in \mathrm{mod}\,\Gamma^{op}$ and $1 \leqslant i \leqslant k-2$, it follows from Lemma 2.9 that $\Omega^i_U(\mathrm{mod}\,\Gamma^{op}) = \mathcal{T}^i_U(\mathrm{mod}\,\Gamma^{op})$ for any $1 \leqslant i \leqslant k-1$. Notice that $\mathrm{Coker}(Q_k \to Q_{k-1})$ is $(k-1)$-syzygy, so it is in $\Omega^{k-1}_U(\mathrm{mod}\,\Gamma^{op})$ by Lemma 2.10. Thus $\mathrm{Coker}(Q_k \to Q_{k-1})$ is in $\mathcal{T}^{k-1}_U(\mathrm{mod}\,\Gamma^{op})$ and therefore $\mathrm{Ext}^i_\Lambda(X, U) = 0$ for any $1 \leqslant i \leqslant k-1$.

On the other hand, by using the same argument as that in the proof of Lemma 3.3, we have that $\mathrm{Ext}^i_\Lambda(X^{**}, U) = 0$ for any $i \geqslant 1$ and σ^*_X is epic. Then from the long exact sequence induced by the exact sequence (3), we get easily that $\mathrm{grade}_U \mathrm{Ext}^k_\Gamma(N, U) \geqslant k$.

Lemma 3.5. $\mathrm{Hom}_\Lambda(S, E_i) \cong \mathrm{Ext}^i_\Lambda(S, U)$ *for any simple* Λ*-module* S *and* $i \geqslant 0$.

Proof. It is easy to verify.

For a module A in $\mathrm{mod}\,\Lambda$, we set $\mathrm{grade}_U A = \infty$ if $\mathrm{grade}_U A \geqslant i$ for all non-negative integers i. The following result is the U-dual version of [4, Theorem 4.5].

Lemma 3.6. *If* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) = k \leqslant 2$, *then* $\mathrm{Soc}(E_k) \neq 0$.

Proof. If $k = 0$, then, by [8, Proposition 2.8], E_0 is an embedding injective cogenerator for $\bmod \Lambda$ and hence $\mathrm{Soc}(E_0) \neq 0$.

If $k = 1$, then there exists a module N in $\bmod \Gamma^{op}$ such that $\mathrm{Ext}_\Gamma^1(N, U) \neq 0$. Let S be any simple quotient module of $\mathrm{Ext}_\Gamma^1(N, U)$. By Lemma 3.3, $\mathrm{grade}_U \mathrm{Ext}_\Gamma^1(N, U) \geqslant 1$, so $S^* = 0$. Suppose $\mathrm{Ext}_\Lambda^1(S, U) = 0$. Notice that $\mathrm{l.id}_\Lambda(U) = 1$, so $\mathrm{grade}_U S = \infty$ and hence $S = 0$ by [8, Corollary 2.5], a contradiction. Thus $\mathrm{Ext}_\Lambda^1(S, U) \neq 0$. By Lemma 3.5, we have that $\mathrm{Hom}_\Lambda(S, E_1) \neq 0$, so S is isomorphic to a submodule of E_1 and $\mathrm{Soc}(E_1) \neq 0$.

If $k = 2$, then there exists a module N in $\bmod \Gamma^{op}$ such that $\mathrm{Ext}_\Gamma^2(N, U) \neq 0$. Let X be a maximal submodule of $\mathrm{Ext}_\Gamma^2(N, U)$ and

$$0 \to X \to \mathrm{Ext}_\Gamma^2(N, U) \to S \to 0$$

an exact sequence in $\bmod \Lambda$. Then S is simple. By Lemma 3.3, $\mathrm{grade}_U \mathrm{Ext}_\Gamma^2(N, U) \geqslant 2$. So $S^* = 0$ and $\mathrm{Ext}_\Lambda^1(S, U) \cong X^*$.

Let

$$P_1 \to P_0 \to S \to 0$$

be a projective presentation of S in $\bmod \Lambda$. Put $Y = \mathrm{Coker}(P_0^* \to P_1^*)$. Then, by Lemma 2.8, we have an exact sequence:

$$0 \to \mathrm{Ext}_\Lambda^1(S, U) \to Y \xrightarrow{\sigma_Y} Y^{**} \to \mathrm{Ext}_\Lambda^2(S, U) \to 0. \tag{4}$$

By the symmetric result of Lemma 3.3, we have that $\mathrm{grade}_U \mathrm{Ext}_\Lambda^i(M, U) \geqslant i$ for any $M \in \bmod \Lambda$ and $1 \leqslant i \leqslant 2$. It then follows from the symmetric result of [5, Proposition 4.3] that $\mathcal{T}_U^2(\bmod \Gamma^{op})$ (= {the U-reflexive modules in $\bmod \Gamma^{op}$}) is extension closed (that is, the middle term B of any short sequence $0 \to A \to B \to C \to 0$ is in $\mathcal{T}_U^2(\bmod \Gamma^{op})$ provided that the end terms A and C are in $\mathcal{T}_U^2(\bmod \Gamma^{op})$). On the other hand, both $\mathrm{Ext}_\Lambda^1(S, U)$ ($\cong X^*$) and Y^{**} are U-reflexive by Lemma 2.7. So, if $\mathrm{Ext}_\Lambda^2(S, U) = 0$, then, by the exact sequence (4), Y is U-reflexive and $\mathrm{Ext}_\Lambda^1(S, U) = 0$. Notice that $\mathrm{l.id}_\Lambda(U) = 2$, so we in fact get that $\mathrm{grade}_U S = \infty$. Then $S = 0$ by [8, Corollary 2.5], a contradiction. Consequently $\mathrm{Ext}_\Lambda^2(S, U) \neq 0$. By Lemma 3.5, we have that $\mathrm{Hom}_\Lambda(S, E_2) \neq 0$, so S is isomorphic to a submodule of E_2 and $\mathrm{Soc}(E_2) \neq 0$.

Theorem 3.7. *If* ${}_\Lambda U$ *is quasi* $(k-2)$*-Gorenstein with* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) = k$ *and each* $(k-2)$*-syzygy module in* $\bmod \Lambda$ *is* U*-*$(k-2)$*-syzygy, then* $\mathrm{Soc}(E_k) \neq 0$.

Proof. The case for $k \leqslant 2$ follows from Lemma 3.6. Now suppose $k \geqslant 3$. Since $\mathrm{r.id}_\Gamma(U) = k$, there exists a module N in $\bmod \Gamma^{op}$ such that $\mathrm{Ext}_\Gamma^k(N, U) \neq 0$. Let X be a maximal submodule of $\mathrm{Ext}_\Gamma^k(N, U)$ and

$$0 \to X \to \mathrm{Ext}_\Gamma^k(N, U) \to S \to 0 \tag{5}$$

an exact sequence in $\bmod \Lambda$. Then S is simple.

Since ${}_\Lambda U$ is quasi $(k-2)$-Gorenstein, $\mathrm{s.grade}_U \mathrm{Ext}_\Gamma^{i+1}(N, U) \geqslant i$ for any $1 \leqslant i \leqslant k-2$ by Theorem 2.5. So $\mathrm{s.grade}_U \mathrm{Ext}_\Gamma^k(N, U) \geqslant k-2$ and $\mathrm{grade}_U X \geqslant k-2$. On the other hand, it follows from Lemma 3.4 that $\mathrm{grade}_U \mathrm{Ext}_\Gamma^k(N, U) \geqslant k$. Then, by the exact sequence (5), we have that $\mathrm{grade}_U S \geqslant k-1$.

Let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to S \to 0$$

be a projective resolution of S in $\bmod \Lambda$.

We claim that $\mathrm{Ext}_\Lambda^k(S, U) \neq 0$. Otherwise, if $\mathrm{Ext}_\Lambda^k(S, U) = 0$, then by Lemma 2.8, we have an exact sequence:

$$0 \to \mathrm{Ext}_\Lambda^{k-1}(S, U) \to H \xrightarrow{\sigma_H} H^{**} \to 0, \tag{6}$$

where $H = \mathrm{Coker}(P_{k-2}^* \to P_{k-1}^*)$.

Since ${}_\Lambda U$ is quasi $(k-2)$-Gorenstein, by Theorem 2.5, we have that $\mathrm{grade}_U \mathrm{Ext}_\Lambda^i(M, U) \geqslant i$ for any $M \in \bmod \Lambda$ and $1 \leqslant i \leqslant k-2$. It follows from Lemma 2.9 that $\Omega_U^i(\bmod \Lambda) = \mathcal{T}_U^i(\bmod \Lambda)$ for any $1 \leqslant i \leqslant k-1$. Notice that $\mathrm{Coker}(P_{k-1} \to P_{k-2})$ is a $(k-2)$-syzygy module. Then by assumption, it is in $\Omega_U^{k-2}(\bmod \Lambda)$ and hence in $\mathcal{T}_U^{k-2}(\bmod \Lambda)$. So $\mathrm{Ext}_\Gamma^i(H, U) = 0$ for any $1 \leqslant i \leqslant k-2$.

By using an argument similar to that in the proof of Lemma 3.3, we get that $\mathrm{Ext}^i_\Gamma(H^{**}, U) = 0$ for any $i \geqslant 1$ and σ^*_H is epic. So, from the long exact sequence induced by the exact sequence (6), it is easy to get that $\mathrm{grade}_U\,\mathrm{Ext}^{k-1}_\Lambda(S, U) \geqslant k-1$. Since $\mathrm{grade}_U\,\mathrm{Ext}^k_\Gamma(N, U) \geqslant k$, $\mathrm{Ext}^{k-2}_\Lambda(X, U) \cong \mathrm{Ext}^{k-1}_\Lambda(S, U)$ by the exact sequence (5). So $\mathrm{grade}_U\,\mathrm{Ext}^{k-2}_\Lambda(X, U) \geqslant k-1$. It then follows from Lemma 2.6 that $\mathrm{Ext}^{k-2}_\Lambda(X, U) = 0$ and thus $\mathrm{Ext}^{k-1}_\Lambda(S, U) = 0$ and $\mathrm{grade}_U\,S \geqslant k+1$. Therefore $\mathrm{grade}_U\,S = \infty$ for $\mathrm{l.id}_\Lambda(U) = k$. It follows from [8, Corollary 2.5] that $S = 0$, a contradiction. The claim is proved.

Now by Lemma 3.5, we have that $\mathrm{Hom}_\Lambda(S, E_k) \cong \mathrm{Ext}^k_\Lambda(S, U) \neq 0$, which implies that S is isomorphic to a submodule of E_k and $\mathrm{Soc}(E_k) \neq 0$.

Corollary 3.8. *If* ${}_\Lambda U$ *is quasi* $(k-2)$*-Gorenstein and* U_Γ *is quasi* $(k-3)$*-Gorenstein with* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) = k$, *then* $\mathrm{Soc}(E_k) \neq 0$.

Proof. If U_Γ is quasi $(k-3)$-Gorenstein, then, by Lemma 2.10, each $(k-2)$-syzygy module in $\mathrm{mod}\,\Lambda$ is U-$(k-2)$-syzygy. Now our conclusion follows from Theorem 3.7.

Putting $k = 3$, the following is an immediate consequence of Corollary 3.8.

Corollary 3.9. *If* ${}_\Lambda U$ *is quasi* 1*-Gorenstein with* $\mathrm{l.id}_\Lambda(U) = \mathrm{r.id}_\Gamma(U) = 3$, *then* $\mathrm{Soc}(E_3) \neq 0$.

Proof of Theorem 3.1. Notice that ${}_\Lambda U$ is $(k-2)$-Gorenstein if and only if U_Γ is $(k-2)$-Gorenstein by Theorem 2.4. So we get Theorem 3.1 from Corollary 3.8.

We call Λ a *left quasi k-Gorenstein ring* if ${}_\Lambda\Lambda$ is quasi k-Gorenstein. It is clear that a k-Gorenstein ring is left quasi k-Gorenstein. Putting ${}_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$, by Theorem 3.7, we have the following result, which is a generalization of Corollary 3.2.

Corollary 3.10. *If* Λ *is a left quasi* $(k-2)$*-Gorenstein ring with* $\mathrm{l.id}_\Lambda(\Lambda) = \mathrm{r.id}_\Lambda(\Lambda) = k$, *then* $\mathrm{Soc}(I_k) \neq 0$.

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