$\mathbb{W}^t$-approximation representations over quasi k-Gorenstein algebras

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Received April 8, 1999

Abstract The notions of quasi k-Gorenstein algebras and $\mathbb{W}^t$-approximation representations are introduced. The existence and uniqueness (up to projective equivalences) of $\mathbb{W}^t$-approximation representations over quasi k-Gorenstein algebras are established. Some applications of $\mathbb{W}^t$-approximation representations to homologically finite subcategories are given.

Keywords: $\mathbb{W}^t$-approximation representations, quasi k-Gorenstein algebras, contravariant finiteness, covariant finiteness.

Assume that Λ is an algebra. We use modΛ to denote the category of modules consisting of finitely generated (left) Λ-modules. The subcategories of modΛ mean the full subcategories and closed under isomorphism classes and summands. The modules discussed in this paper are finitely generated modules.

Representation theory of algebras is a new branch of algebras emerging at the beginning of the 1970s, whose elementary content is to study the structure of rings and algebras. Such a theory has been developed rapidly and is being perfected progressively during the last twenties years. From 1974 to 1977, Auslander and Reiten published their series of papers "Representation theory of artin algebras I—VI", in which they applied homological methods to studying indecomposable modules, especially in ref. [1], they introduced an important notion of almost split sequences, which laid the theoretical basis of representation theory of algebras. Suppose Λ is an artin algebra; that is, Λ is an algebra over a commutative artin ring R and Λ is finitely generated as an R-module. In order to prove the existence of almost split sequences over subcategories of modΛ (as well as the existence of preprojective partitions and preinjective partitions), Auslander and Smalø introduced the important notions of contravariantly finite subcategories, covariantly finite subcategories and functorially finite subcategories in 1980 (c.f. references [2,3]).

The following definition is cited from ref. [2]. However, Λ here is not necessarily an artin algebra.

Definition 1[2]. Let Λ be an algebra. Assume that $\mathscr{C} \supset \mathscr{D}$ are subcategories of modΛ and $C \in \mathscr{C}$, $D \in \mathrm{add}\mathscr{D}$, where add$\mathscr{D}$ is the subcategory of modΛ consisting of all Λ-modules isomorphic to summands of finite sums of modules in $\mathscr{D}$. The morphism $D \to C$ is said to be a right $\mathscr{D}$-approximation of C if $\mathrm{Hom}_\Lambda(X, D) \to \mathrm{Hom}_\Lambda(X, C) \to 0$ is exact for all $X \in \mathrm{add}\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be contravariantly finite in $\mathscr{C}$ if every C in $\mathscr{C}$ has a right $\mathscr{D}$-approximation. Dually, the morphism $C \to D$ is said to be a left $\mathscr{D}$-approximation of C if $\mathrm{Hom}_\Lambda(D, X) \to \mathrm{Hom}_\Lambda(C, X) \to 0$ is exact for all $X \in \mathrm{add}\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be covariantly finite in $\mathscr{C}$ if every C in $\mathscr{C}$ has a left $\mathscr{D}$-approximation. The subcategory $\mathscr{D}$ is said to be functorially finite in $\mathscr{C}$ if it is both contravariantly finite and covariantly finite in $\mathscr{C}$. The notions of contravariantly finite subcategories, covariantly finite subcategories and functorially finite subcategories are referred to as homologically finite subcategories.

Ever since the notion of homologically finite subcategories is introduced, it has been a very important research object in representation theory of algebras. The rather thoroughgoing research of many mathematicians indicated that the notion of homologically finite subcategories played a very important role in determining the representation type of an algebra Λ and the existence of almost split sequences over subcategories of modΛ as well as in studying tilting theory, etc. (c.f. refs. [2—7] and the references therein).

For any positive integer k, Auslander introduced the notion of k-Gorenstein algebras in noncommutative case, and proved that k-Gorenstein algebras are left-right symmetric (see Theorem 3.7 of ref. [8]). Auslander and Reiten studied in ref. [9] the homological finiteness of categories of modules over k-Gorenstein algebras. In this paper, we will introduce the notion of quasi k-Gorenstein algebras, and give examples to demonstrate that quasi k-Gorenstein algebras contain properly k-Gorenstein algebras and quasi k-Gorenstein algebras are not left-right symmetric unlike the case of k-Gorenstein algebras. We will define the notion of $\mathbb{W}^t$-approximation representations over quasi k-Gorenstein algebras, and will prove that there is a $\mathbb{W}^t$-approximation representation for any module over quasi k-Gorenstein algebras (where $1 \leqslant t \leqslant k$) and such a representation is unique up to projective equivalences. From the existence of $\mathbb{W}^t$-approximation representations, we may deduce the homological finiteness of $\mathbb{W}^t$ (that is, the subcategory of modΛ consisting of the modules M with $\mathrm{Ext}_\Lambda^i(M, \Lambda) = 0$ for any $1 \leqslant i \leqslant t$) and $\mathscr{P}^k(\Lambda)$ (that is, the subcategory of modΛ consisting of the modules with projective dimension not more than k) and the existence of almost split sequences over $\mathscr{P}^k(\Lambda)$.

For any module $A \in \mathrm{mod}\Lambda$ (resp. $\mathrm{mod}\Lambda^{op}$), we use $l.\mathrm{pd}_\Lambda(A)$ ($r.\mathrm{pd}_\Lambda(A)$), $l.\mathrm{fd}_\Lambda(A)$ ($r.\mathrm{fd}_\Lambda(A)$) and $l.\mathrm{id}_\Lambda(A)$ ($r.\mathrm{id}_\Lambda(A)$) to denote left (right) projective dimension, flat dimension and injective dimension of A respectively.

1 Quasi k-Gorenstein algebras

Assume that Λ is a Noetherian algebra; that is, Λ is an algebra over a commutative Noetherian ring R and Λ is finitely generated as an R-module. In the following, assume that k is a positive integer and

$$0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$$

is a minimal injective resolution of Λ as a left Λ-module.

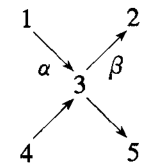
Definition 2. Λ is called a quasi k-Gorenstein algebra if $l.\mathrm{fd}_\Lambda(I_i) \leqslant i+1$ for any $0 \leqslant i \leqslant k-1$.

Assume that

$$0 \to \Lambda \to I'_0 \to I'_1 \to \cdots \to I'_i \to \cdots$$

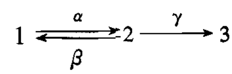
is a minimal injective resolution of Λ as a right Λ-module.

Example 1. A Noetherian algebra is called a k-Gorenstein algebra if $l.\mathrm{fd}_\Lambda(I_i) \leqslant i$ for any $0 \leqslant i \leqslant k-1$. It is clear that a k-Gorenstein algebra is a quasi k-Gorenstein algebra. But the converse does not hold in general. For example, let K be a field and $\Lambda = KT/(\beta\alpha)$, where T is the following quiver:



Then $l.\mathrm{fd}_\Lambda(I_0) = l.\mathrm{fd}_\Lambda(I_1) = 1 = r.\mathrm{fd}_\Lambda(I'_0) = r.\mathrm{fd}_\Lambda(I'_1)$. So Λ is a quasi 2-Gorenstein algebra. But Λ is even not a 1-Gorenstein algebra.

Example 2[1]. Let K be a field and T the following quiver:



(1) If $\Lambda = KT/(\alpha\beta\alpha)$, then $l.\mathrm{fd}_\Lambda(I_0) = 1$ and $r.\mathrm{fd}_\Lambda(I'_0) \geqslant 2$. (2) If $\Lambda = KT/(\gamma\alpha, \beta\alpha)$, then $l.\mathrm{fd}_\Lambda(I_0) = 2$ and $r.\mathrm{fd}_\Lambda(I'_0) = 1$. So quasi k-Gorenstein algebras are not left-right symmetric. However, k-Gorenstein algebras are left-right symmetric by Theorem 3.7 of reference [8].

Example 3. In a completely similar proof to that of Theorem 3.10 of ref. [8], we may prove that an algebra Λ is quasi k-Gorenstein if and only if $\begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$ is quasi k-Gorenstein. So we may claim that quasi k-Gorenstein algebras are sufficiently rich.

Let $A \in \mathrm{mod}\Lambda^{op}$ and i a positive integer. We denote grade $A \geqslant i$ if $\mathrm{Ext}^j_\Lambda(A, \Lambda) = 0$ for any $0 \leqslant j \leqslant i-1$. A is called an i-torsion-free module if $\mathrm{Ext}^j_\Lambda(\mathrm{Tr}A, \Lambda) = 0$ (where $\mathrm{Tr}A$ denotes the transpose of A[1]) for any $1 \leqslant j \leqslant i$. A is called an i-th syzygy module if there is an exact sequence $0 \to A \to P_0 \to \cdots \to P_{i-1}$ with the P_i's projective Λ^{op}-modules. We use $\mathscr{X}_i$ and $\Omega^i(\mathrm{mod}\Lambda^{op})$ to denote the subcategories of $\mathrm{mod}\Lambda^{op}$ consisting of i-torsion-free modules and i-th syzygy modules respectively. Let $\mathscr{C}$ be a subcategory of $\mathrm{mod}\Lambda^{op}$ and $0 \to A \to B \to C \to 0$ any exact sequence of Λ^{op}-modules. If $A, C \in \mathscr{C}$ implies that $B \in \mathscr{C}$, then $\mathscr{C}$ is said to be extension-closed.

The following theorem gives a good description of quasi k-Gorenstein algebras.

Theorem A[10][1]. *Let Λ be a Noetherian algebra. The following statements are equivalent.*

(1) *Λ is a quasi k-Gorenstein algebra.*

(2) *$\Omega^i(\mathrm{mod}\Lambda^{op})$ is extension-closed for any $1 \leqslant i \leqslant k$.*

(3) *$\mathscr{X}_i$ is extension-closed for any $1 \leqslant i \leqslant k$.*

(4) *grade $\mathrm{Ext}^i_\Lambda(X, \Lambda) \geqslant i$ for any $1 \leqslant i \leqslant k$ and $X \in \mathrm{mod}\Lambda$.*

(5) *$\mathrm{Ext}^{i-1}_\Lambda(\mathrm{Ext}^i_\Lambda(Y, \Lambda), \Lambda) = 0$ for any $1 \leqslant i \leqslant k$ and $Y \in \mathrm{mod}\Lambda$.*

2 Two lemmas

For any positive integer n, an exact sequence of Λ (or Λ^{op})-modules $X_1 \to X_2 \to \cdots \to X_n$ is said to be dual exact if the induced sequence $X_n^* \to \cdots \to X_2^* \to X_1^*$ is exact, where $X_i^* = \mathrm{Hom}_\Lambda(X_i, \Lambda)$, $1 \leqslant i \leqslant n$.

In the following, Λ is a Noetherian algebra.

1) Huang Zhaoyong, Extension closure of k-torsion-free modules, Communications in Algebra, 1999, 27(3) (to appear).

Lemma 1. *For $A \in \operatorname{mod}\Lambda^{op}$ and a positive integer n, the following statements are equivalent.*

(1) $\operatorname{Ext}^i_\Lambda(A, \Lambda) = 0$ *for any* $1 \leqslant i \leqslant n$.

(2) *Any exact sequence* $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ *with the* P_i*'s projective is dual exact.*

(3) *Any exact sequence* $P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$ *with the* P_i*'s projective is dual exact.*

Proof. (1)⇒(2) The case of $n = 1$ is clear. Suppose $n \geqslant 2$ and suppose $0 \to K \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to A \to 0$ is an exact sequence with the P_i's projective. Then $\operatorname{Ext}^1_\Lambda(\operatorname{Im} d_i, \Lambda) \cong \operatorname{Ext}^{i+1}_\Lambda(A, \Lambda) = 0$ for any $1 \leqslant i \leqslant n-1$, and hence it is easy to see that $0 \to A^* \to P_0^* \to P_1^* \to \cdots \to P_{n-1}^* \to K^* \to 0$ is exact.

(2)⇒(3). It is trivial.

(3)⇒(1). Suppose $n = 1$ and suppose the exact sequence

$$\begin{array}{ccccccccc} P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & & \searrow^{\pi} & & \nearrow_{i} & & & \\ & & & & K & & & & \end{array}$$

with the P_i's projective is dual exact. Consider the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^* & \longrightarrow & P_0^* & \xrightarrow{i^*} & K^* & \longrightarrow & \operatorname{Ext}^1_\Lambda(A, \omega) \longrightarrow 0 \\ & & \| & & \| & & \downarrow{\scriptstyle \pi^*} & & \\ 0 & \longrightarrow & A^* & \longrightarrow & P_0^* & \xrightarrow{d_1^*} & P_1^* & \xrightarrow{d_2^*} & P_2^* \end{array}$$

Since $0 \to K^* \xrightarrow{\pi^*} P_1^* \xrightarrow{d_2^*} P_2^*$ is also exact and π^* is a monomorphism, $\operatorname{Im} i^* \cong \operatorname{Im}(\pi^* \cdot i^*) = \operatorname{Im} d_1^* = \operatorname{Ker} d_2^* = \operatorname{Im} \pi^* \cong K^*$. So i^* is an epimorphism and hence $\operatorname{Ext}^1_\Lambda(A, \Lambda) = 0$. By using induction on n, we can get our conclusion.

Lemma 2. *Let* ${}_\Lambda\omega_\Lambda$ *be a* Λ-Λ *bimodule with* $\operatorname{Ext}^1_\Lambda(\omega, \omega) = 0$. *Then for any* $M \in \operatorname{mod}\Lambda$, *there is an exact sequence*

$$0 \to F \to E \to M \to 0$$

with $F = \omega^{(n)}$ *and* $\operatorname{Ext}^1_\Lambda(E, \omega) = 0$, *where n is the number of generators of* $\operatorname{Ext}^1_\Lambda(M, \omega)$ in modEnd $(M)^{op}$.

Proof. For the case $n = 0$, putting $F = 0$ and $E = M$, we get the desired exact sequence. Now suppose $M \in \operatorname{mod}\Lambda$ and $n \geqslant 1$. Let $e_1, \cdots, e_n$ be a set of generators of $\operatorname{Ext}^1_\Lambda(M, \omega)$ in mod-End$(M)^{op}$, where each e_i is represented by an extension

$$0 \to \omega \xrightarrow{f_i} E_i \xrightarrow{g_i} M \to 0.$$

We have the following pull-back diagram:

$$\begin{array}{ccccccccc} \eta: 0 & \longrightarrow & \omega^{(n)} & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \omega^{(n)} & \xrightarrow{f'} & \oplus_{i=1}^n E_i & \xrightarrow{g'} & M^{(n)} & \longrightarrow & 0 \end{array}$$

where α is the i-th injection $(1 \leqslant i \leqslant n)$, $f' = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & f_n \end{pmatrix}$, $g' = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & g_n \end{pmatrix}$.

Consider the following push-out diagram:

$$\begin{array}{ccccccccc} \eta: 0 & \longrightarrow & \omega^{(n)} & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \pi \downarrow & & \downarrow & & \| & & \\ \eta': 0 & \longrightarrow & \omega & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

If π is the i-th projection, it is easy to see that η' is just e_i. So in the long exact sequence (induced by η): $0 \to \mathrm{Hom}_\Lambda(M, \omega) \to \mathrm{Hom}_\Lambda(E, \omega) \to \mathrm{Hom}_\Lambda(\omega^{(n)}, \omega) \xrightarrow{\delta} \mathrm{Ext}^1_\Lambda(M, \omega) \to \mathrm{Ext}^1_\Lambda(E, \omega) \to 0$, the connected homomorphism δ is surjective. Thus $\mathrm{Ext}^1_\Lambda(E, \omega) = 0$ and therefore η is the desired exact sequence.

Corollary 1. *Let ${}_\Lambda\omega_\Lambda$ be a Λ-Λ bimodule with $\mathrm{Ext}^1_\Lambda(\omega, \omega) = 0$. The subcategory of* modΛ *consisting of the E with $\mathrm{Ext}^1_\Lambda(E, \omega) = 0$ is contravariantly finite.*

3 $\mathbb{W}^t$-Approximation representations

Definition 3. Let $M \in \mathrm{mod}\Lambda$. M is called a W^t-module if $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for any $1 \leqslant i \leqslant t$ (where t is a positive integer).

In the following, Λ is a quasi k-Gorenstein algebra. In this section, we will introduce the notion of $\mathbb{W}^t$-approximation representations and show that $\mathbb{W}^t$-approximation representations of any Λ-module exist uniquely (up to projective equivalences). We will also give some applications of $\mathbb{W}^t$-approximation representations.

Theorem 1. *Let M be in* modΛ. *Then for any $1 \leqslant t \leqslant k$, there is a short exact sequence*

$$0 \to K_t(M) \xrightarrow{f_t} E_t(M) \xrightarrow{g_t} M \to 0$$

such that $l.pd_\Lambda(K_t(M)) \leqslant t-1$ and $E_t(M)$ is a W^t-module.

Proof. We proceed by induction on t. The case of $t = 1$ follows from Lemma 2.

Now suppose $t \geqslant 2$ and suppose there is an exact sequence $0 \to K_{t-1}(M) \xrightarrow{f_{t-1}} E_{t-1}(M) \xrightarrow{g_{t-1}} M \to 0$ with the properties that $l.\mathrm{pd}_\Lambda(K_{t-1}(M)) \leqslant t-2$ and $E_{t-1}(M)$ is a W^{t-1}-module. So $\mathrm{Ext}^t_\Lambda(M, \Lambda) \cong \mathrm{Ext}^t_\Lambda(E_{t-1}(M), \Lambda)$. Let $0 \to K \to P_{t-2} \to \cdots \to P_0 \to E_{t-1}(M) \to 0$ be an exact sequence with the P_i's projective. Then $\mathrm{Ext}^1_\Lambda(K, \Lambda) \cong \mathrm{Ext}^t_\Lambda(E_{t-1}(M), \Lambda)$ and hence $\mathrm{Ext}^t_\Lambda(M, \Lambda) \cong \mathrm{Ext}^1_\Lambda(K, \Lambda)$.

By Lemma 2, there is an exact sequence $0 \to \Lambda^{(n)} \to E_1(K) \to K \to 0$ with the properties that

$Ext_{\Lambda}^{1}(E_1(K),\Lambda)=0$ and n is the number of generators of $\mathrm{Ext}_{\Lambda}^{1}(K,\Lambda)$ over $\mathrm{End}(K)^{op}$.

Consider the following diagram (3.1):

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & \Lambda^{(n)} & \longrightarrow & E_1(K) & \longrightarrow & K & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & P_{t-2} & & \\ & & & & & & \uparrow & & \\ & & & & & & \vdots & & \\ & & & & & & \uparrow & & \\ & & & & & & P_0 & & \\ & & & & & & \uparrow & & \\ & & & & & & E_{t-1}(M) & & \\ & & & & & & \uparrow & & \\ & & & & & & 0 & & \end{array}$$

Diagram (3.1)

Since $E_{t-1}(M)$ is a W^{t-1}-module, from Lemma 1 we may obtain the following exact diagram (3.2) by using the functor $\mathrm{Hom}_{\Lambda}(-,\Lambda)$ to act on Diagram (3.1):

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ (\Lambda^{(n)})^{*} & \longleftarrow & E_1(K)^{*} & \longleftarrow & K^{*} & \longleftarrow & 0 \\ & & & & \uparrow & & \\ & & & & P_{t-2}^{*} & & \\ & & & & \uparrow & & \\ & & & & \vdots & & \\ & & & & \uparrow & & \\ & & & & P_0^{*} & & \\ & & & & \uparrow & & \\ & & & & E_{t-1}(M)^{*} & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

Diagram (3.2)

Put $X=\mathrm{Im}(E_1(K)^{*}\rightarrow(\Lambda^{(n)})^{*})$. Since grade $\mathrm{Ext}_{\Lambda}^{1}(K,\Lambda)=$ grade $\mathrm{Ext}_{\Lambda}^{t}(M,\Lambda)\geqslant t$ by Theorem A, it follows from the exactness of $0\rightarrow X\rightarrow(\Lambda^{(n)})^{*}\rightarrow\mathrm{Ext}_{\Lambda}^{1}(K,\Lambda)\rightarrow 0$ that $X^{*}\cong\Lambda^{(n)}$ and X is a W^{t-2}-module.

Let

$$0\rightarrow Y\rightarrow Q_0\rightarrow\cdots\rightarrow Q_{t-2}\rightarrow X\rightarrow 0$$

be an exact sequence with the Q_i's projective. Then we obtain an exact commutative diagram (3.3) with the middle $t-1$ rows splitting:

Since X is a W^{t-2}-module, from Lemma 1 we know that $0\rightarrow X^{*}\rightarrow Q_{t-2}^{*}\rightarrow\cdots\rightarrow Q_0^{*}\rightarrow Y^{*}$ is

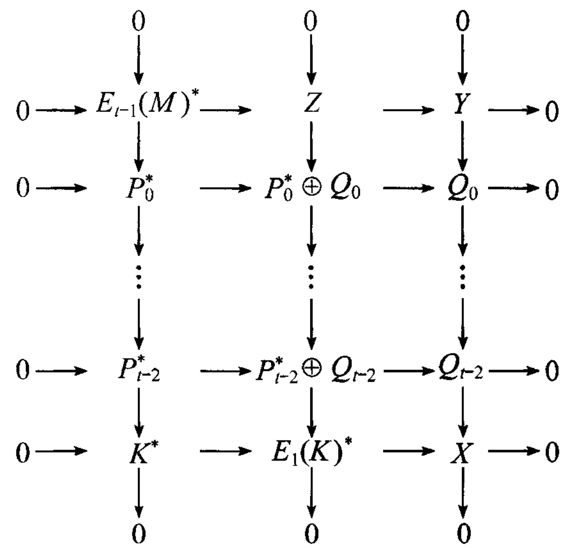


Diagram (3.3)

exact. Moreover, we have known that $X^* \cong \Lambda^{(n)}$ and $0 \to K \to P_{t-2} \to \cdots \to P_0 \to E_{t-1}(M) \to 0$ is exact. So using the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$ to act on Diagram (3.3) will yield the following exact commutative diagram (3.4):

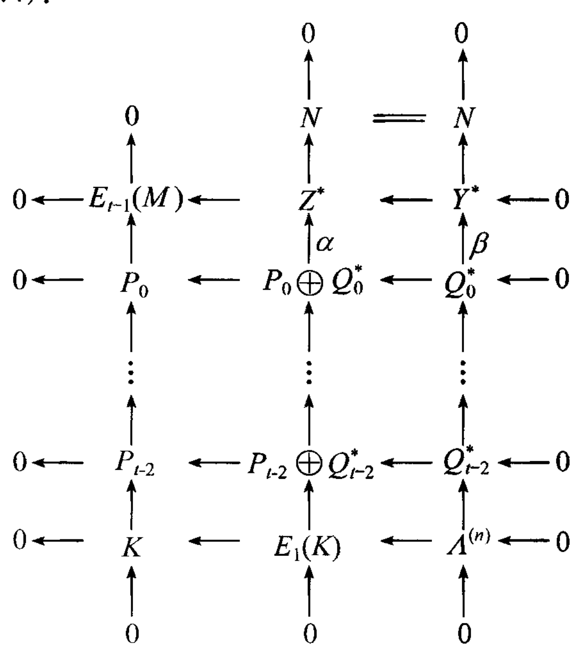


Diagram (3.4)

where $N = \mathrm{Coker}\alpha$ (note: by the snake lemma we have $\mathrm{Coker}\alpha \cong \mathrm{Coker}\beta$).

Now we put $E_t(M) = \mathrm{Im}\alpha$, and put $H = \mathrm{Im}\beta$. Then $l.\mathrm{pd}_\Lambda(H) \leqslant t-1$. From the middle column of Diagram (3.4), we have $\mathrm{Ext}^t_\Lambda(E_t(M),\Lambda) \cong \mathrm{Ext}^1_\Lambda(E_1(K),\Lambda) = 0$. In addition, we know that $Z \cong E_t(M)^*$, then from the middle columns of Diagram (3.3) and Diagram (3.4) we know that the exact sequence $0 \to E_1(K) \to P_{t-2} \oplus Q^*_{t-2} \to \cdots \to P_0 \oplus Q^*_0 \to E_t(M) \to 0$ is dual exact[1]. So by Lemma 1, $E_t(M)$ is a W^{t-1}-module and hence a W^t-module.

By the snake lemma, from Diagram (3.4) we have an exact sequence $0 \to H \to E_t(M) \xrightarrow{\sigma_t} E_{t-1}(M) \to 0$. Now put $g_t = g_{t-1} \cdot \sigma_t$ and $K_t = \mathrm{Ker} g_t$. Then we obtain an exact sequence:

$$0 \to K_t(M) \xrightarrow{f_t} E_t(M) \xrightarrow{g_t} M \to 0, \tag{1}$$

where f_t is the canonical inclusion. From the following exact commutative diagram (3.5):

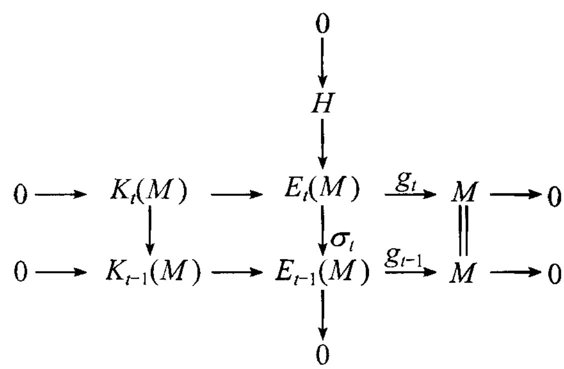


Diagram (3.5)

we obtain an exact sequence $0 \to H \to K_t(M) \to K_{t-1}(M) \to 0$. It is clear that $l.\mathrm{pd}_\Lambda(K_t(M)) \leqslant t-1$. So the exact sequence (1) is desired. This finishes the proof.

For any positive integer t, we use $\mathbb{W}^t$ to denote the subcategory of modΛ consisting of W^t-modules. From Theorem 1 we have the following corollary.

Corollary 2 *For any* $1 \leqslant t \leqslant k$, *the exact sequence* $0 \to K_t(M) \xrightarrow{f_t} E_t(M) \xrightarrow{g_t} M \to 0$ *in Theorem* 1 *is a right* $\mathbb{W}^t$*-approximation of the* Λ*-module M and* $\mathbb{W}^t$ *is contravariantly finite in* modΛ.

Proof. Let E be a W^t-module. Since $l.\mathrm{pd}_\Lambda(K_t(M)) \leqslant t-1$, it is easy to see that $\mathrm{Ext}^1_\Lambda(E, K_t(M)) = 0$. So $0 \to K_t(M) \xrightarrow{f_t} E_t(M) \xrightarrow{g_t} M \to 0$ is a right $\mathbb{W}^t$-approximation of M.

From Theorem 1 and Corollary 2, we may give the following definition.

Definition 4. For any $1 \leqslant t \leqslant k$, we call the exact sequence $0 \to K_t(M) \xrightarrow{f_t} E_t(M) \xrightarrow{g_t} M \to 0$ in Theorem 1 a $\mathbb{W}^t$-approximation representation of the Λ-module M.

In the following, we first give an application of $\mathbb{W}^t$-approximation representations.

Auslander and Reiten proved the following conclusion[4]: Let Γ be an artin algebra. If $\mathscr{P}^\infty(\Gamma)$ is contravariantly finite (where $\mathscr{P}^\infty(\Gamma)$ denotes the subcategory of modΓ consisting of the modules with finite projective dimension), then the finitistic dimension conjecture holds over Γ. Therefore it is interesting to determine when $\mathscr{P}^k(\Gamma)$ and $\mathscr{P}^\infty(\Gamma)$ are contravariantly finite, which may be useful for comprehending this conjecture.

1) Huang Zhaoyong, Faithfully balanced selforthogonal bimodules and homologically finite subcategories, Ph. D. Thesis, Beijing Normal University, 1998.

Of the homological finiteness of $\mathscr{P}^k(\Gamma)$ and $\mathscr{P}^\infty(\Gamma)$, the following conclusions are known:

(1) It is easy to see that $\mathscr{P}^0(\Gamma)$ is functorially finite;

(2) $\mathscr{P}^1(\Gamma)$ is covariantly finite[4];

(3) when Γ is of finite representation type, $\mathscr{P}^k(\Gamma)$ is functorially finite, where $k \in \mathbb{N} \cup \{\infty\}$ ($\mathbb{N}$ denotes natural numbers)[4];

(4) if Γ is stably equivalent to a hereditary algebra, then $\mathscr{P}^k(\Gamma)$ is contravariantly finite, where $k \in \mathbb{N} \cup \{\infty\}$[4, 5];

(5) if Γ is a quasi 1-Gorenstein algebra, then $\mathscr{P}^1(\Gamma)$ is contravariantly finite[7].

However, in ref. [7] there is a counterexample to explain why $\mathscr{P}^k(\Gamma)$ and $\mathscr{P}^\infty(\Gamma)$ are not always contravariantly finite. So, it is natural to raise a question: When are $\mathscr{P}^k(\Gamma)$ and $\mathscr{P}^\infty(\Gamma)$ contravariantly finite? Further, when are they homologically finite?

In the following two corollaries, we discuss the above question. Λ is also a quasi k-Gorenstein algebra.

Corollary 3 *If $l.\,id_\Lambda(\Lambda) = r.\,id_\Lambda(\Lambda) \leqslant k$, then $\mathscr{P}^k(\Lambda)$ is covariantly finite.*

Proof. By Theorem 1, we may assume that $0 \to K_k(M) \xrightarrow{f_k} E_k(M) \xrightarrow{g_k} M \to 0$ is a $\mathbb{W}^k$-approximation representation of a left Λ-module M. Since $E_k(M)$ is a W^k-module and $l.\,\mathrm{id}_\Lambda(\Lambda) \leqslant k$, $\mathrm{Ext}^i_\Lambda(E_k(M), \Lambda) = 0$ for any $i \geqslant 1$. So by Proposition 3.1 of ref. [11], there is an exact sequence $0 \to E_k(M) \to P \to X \to 0$ with P projective and $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0$ (for any $i \geqslant 1$).

Consider the following push-out diagram (3.6):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_k(M) & \xrightarrow{f_k} & E_k(M) & \xrightarrow{g_k} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_k(M) & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Diagram (3.6)

It is clear that $l.\mathrm{pd}_\Lambda(Y) \leqslant k$. Since $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0$ for any $i \geqslant 1$, it is easy to see that $\mathrm{Ext}^1_\Lambda(X, Y') = 0$ for any $Y' \in \mathrm{mod}\Lambda$ with $l.\mathrm{pd}_\Lambda(Y') \leqslant k$. So the third column of Diagram (3.6), namely the exact sequence $0 \to M \to Y \to X \to 0$ is a left $\mathscr{P}^k(\Lambda)$-approximation of M.

Corollary 4. *Let Λ be an artin algebra. If $l.\,id_\Lambda(\Lambda) = r.\,id_\Lambda(\Lambda) \leqslant k$, then $\mathscr{P}^k(\Lambda)$ is functorially finite and $\mathscr{P}^k(\Lambda)$ has almost split sequences.*

Proof. By using duality, it is easy to know from Corollary 2.8 of ref. [9] that $\mathscr{P}^k(\Lambda)$ is contravariantly finite. Then by Corollary 3, $\mathscr{P}^k(\Lambda)$ is functorially finite. It is clear that $\mathscr{P}^k(\Lambda)$ is extension closed. So $\mathscr{P}^k(\Lambda)$ has almost split sequences by the former assertion and Theorem 2.4 of reference [3].

Now we turn our attention to $\mathbb{W}^t$-approximation representations. We will mainly tackle the problem of uniqueness of $\mathbb{W}^t$- approximation representations.

Theorem 2. *For any* $1 \leqslant t \leqslant k$, *if*

$$0 \to K_t(M) \to E_t(M) \to M \to 0,$$
$$0 \to K'_t(M) \to E'_t(M) \to M \to 0$$

are both $\mathbb{W}^t$*-approximation representations of a* Λ*-module* M, *then* $K_t(M) \oplus E'_t(M) \cong K'_t(M) \oplus E_t(M)$.

Proof. Consider the following pull-back diagram (3.7): Diagram (3.7)

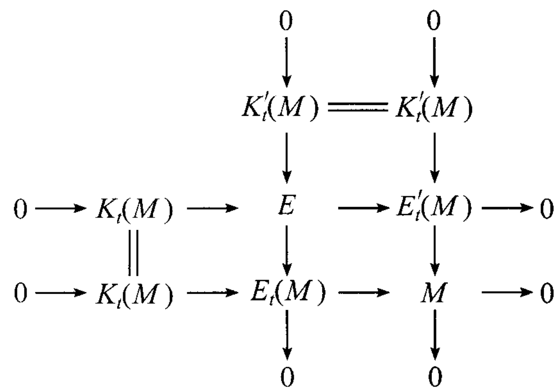


Diagram (3.7)

Noting that $l.\mathrm{pd}_\Lambda(K'_t(M)) \leqslant t-1$, we may assume that $0 \to P_{t-1} \to \cdots \to P_0 \to K'_t(M) \to 0$ is a projective resolution of $K'_t(M)$. Since $\mathrm{Ext}^i_\Lambda(E_t(M), \Lambda) = 0$ for any $1 \leqslant i \leqslant t$, $\mathrm{Ext}^i_\Lambda(E_t(M), P) = 0$ for any projective left module P and any $1 \leqslant i \leqslant t$. So $\mathrm{Ext}^1_\Lambda(E_t(M), K'_t(M)) \cong \mathrm{Ext}^t_\Lambda(E_t(M), P_{t-1}) = 0$ and hence the second column of Diagram (3.7) splits, which shows that $E \cong K'_t(M) \oplus E_t(M)$. Similarly, we may show that $E \cong K_t(M) \oplus E'_t(M)$. This finishes the proof.

Theorem 2 describes the uniqueness of the modules $K_t(M)$ and $E_t(M)$ in $\mathbb{W}^t$-approximation representations of a Λ-module M. If Λ is an artin algebra, the following theorem illustrates the uniqueness of $\mathbb{W}^t$-approximation representations up to projective equivalences; that is, in $\mathbb{W}^t$-approximation representations of a Λ-module M, both $K_t(M)$ and $E_t(M)$ are unique up to projective equivalences.

Theorem 3. *Let* Λ *be an artin algebra, for any* $1 \leqslant i \leqslant t$, *if*

$$0 \to K_t(M) \to E_t(M) \to M \to 0,$$
$$0 \to K'_t(M) \to E'_t(M) \to M \to 0$$

are both $\mathbb{W}^t$*-approximation representations of a* Λ*-module* M, *then there are projective modules* P, P', Q *and* Q' *such that* $K_t(M) \oplus P \cong K'_t(M) \oplus P'$ *and* $E_t(M) \oplus Q \cong E'_t(M) \oplus Q'$ *and* $P \oplus Q' \cong P' \oplus Q$.

In order to prove Theorem 3, we first give the following lemma.

Lemma 3. *Let* n *be a positive integer and* E *a left* Λ*-module. If* E *is a* $\mathbb{W}^n$*-module, then the following statements hold.*

(1) *If* $l.pd_\Lambda(K) \leqslant n-1$ *for some left* Λ*-module* K, *then* $\mathrm{Ext}^1_\Lambda(E, K) = 0$.

(2) *If* $l.pd_\Lambda(E) \leqslant n-1$, *then* E *is projective*.

Proof. Consider a projective resolution of K, it is easy to verify our assertion (1). Aassertion (2) follows easily from assertion (1).

Proof of Theorem 3. Let

$$\begin{aligned} K_t(M) &= K_1 \oplus \cdots \oplus K_m, \\ K'_t(M) &= K'_1 \oplus \cdots \oplus K'_p, \\ E_t(M) &= E_1 \oplus \cdots \oplus E_q, \\ E'_t(M) &= E'_1 \oplus \cdots \oplus E'_n \end{aligned}$$

be the finite indecomposable decompositions of $K_t(M)$, $K'_t(M)$, $E_t(M)$ and $E'_t(M)$, respectively. By Theorem 2, $K_t(M) \oplus E'_t(M) \cong K'_t(M) \oplus E_t(M)$. So we have an isomorphism:

$$K_1 \oplus \cdots \oplus K_m \oplus E'_1 \oplus \cdots \oplus E'_n \cong K'_1 \oplus \cdots \oplus K'_p \oplus E_1 \oplus \cdots \oplus E_q. \quad (2)$$

Since Λ is an artin algebra, By ref. [12] Corollary 15.21 of ref. [12], $K_t(M)$, $K'_t(M)$, $E_t(M)$ and $E'_t(M)$ are modules of finite length. It follows from isomorphism (2) and Krull-Schmidt Theorem (see Theorem 12.9 of ref. [12]) that $m + n = p + q$ and the indecomposable modules from each side of the isomorphism (2) are pairwise isomorphic. Then by Lemma 3(2), it is easy to get our assertions. This finishes the proof.

Now we give some properties of $\mathbb{W}^t$-approximation representations.

Lemma 4. *For any* $1 \leqslant t \leqslant k$, *a* $\mathbb{W}^t$*-approximation representation of a* Λ*-module* M

$$0 \to K_t(M) \to E_t(M) \to M \to 0$$

may induce isomorphisms:

$$\mathrm{Ext}^i_\Lambda(M, \Lambda) \cong \mathrm{Ext}^{i-1}_\Lambda(K_t(M), \Lambda) \text{ for any } 2 \leqslant i \leqslant t.$$

$$\mathrm{Ext}^i_\Lambda(M, \Lambda) \cong \mathrm{Ext}^i_\Lambda(E_t(M), \Lambda) \text{ for any } i \geqslant t+1.$$

Proof. It is trivial from the properties that $l.\mathrm{pd}_\Lambda(K_t(M)) \leqslant t-1$ and $E_t(M)$ is a $\mathbb{W}^t$-module.

Theorem 4. *For any left* Λ*-module* M, *the following statements hold*.

(1) *Suppose* $1 \leqslant t \leqslant k-1$. *If* $\mathrm{Ext}^{t+1}_\Lambda(M, \Lambda) = 0$, *then a* $\mathbb{W}^t$*-approximation representation of* M *is a* $\mathbb{W}^{t+1}$*-approximation representation of* M.

(2) *Suppose* $2 \leqslant t \leqslant k$. *If* $K_t(M)$ *is projective*, *then* $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ *for any* $2 \leqslant i \leqslant t$.

(3) *Let* $f: M \to N$ *be a homomorphism of left modules and let* $1 \leqslant s \leqslant t \leqslant k$. *Then there is an exact commutative diagram*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_t(M) & \longrightarrow & E_t(M) & \xrightarrow{g_t} & M & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & K_s(N) & \longrightarrow & E_s(N) & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Proof. Assertion (1) follows from Definition 4. It is easy to get assertion (2) from Lemma 4.

(3) Since $l.\mathrm{pd}_\Lambda(K_s(N)) \leqslant s-1 \leqslant t-1$ and $E_t(M)$ is a W^t-module, by Lemma 3(1) we have $\mathrm{Ext}^1_\Lambda(E_t(M), K_s(N)) = 0$, which induces an exact sequence $0 \to \mathrm{Hom}_\Lambda(E_t(M), K_s(N)) \to \mathrm{Hom}_\Lambda(E_t(M), E_s(N)) \to \mathrm{Hom}_\Lambda(E_t(M), N) \to 0$ from the exact sequence $0 \to K_s(N) \to E_s(N) \to$

$N \to 0$. So the homomorphism $f \cdot g_t$: $E_t(M) \to N$ can be lifted to a homomorphism $g: E_t(M) \to E_s(N)$, which induces the desired exact commutative diagram. This finishes the proof.

Let $M \in \mathscr{P}^{k-1}(\Lambda)$. By Lemma 3, for any $1 \leqslant t \leqslant k-1$, the module $E_t(M)$ in the $\mathbb{W}^t$-approximation representation of M: $0 \to K_t(M) \to E_t(M) \to M \to 0$ is a projective module. So the above representation is just a projective representation of M. We will give some characterizations and applications of such a projective representation in other paper.

Acknowledgement This paper is part of the author's dissertation. The author would like to express his heartfelt gratitude to his supervisor Prof. Liu Shaoxue for his guidance and to Profs. Zhang Yingbo, Xi Changchang and Xiao Jie and Postdoctor Deng Bangming for their suggestion and help.

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