ω-k-torsionfree modules and ω-left approximation dimension*

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Abstract The notion of ω-k-torsionfree modules with respect to a bimodule ω is introduced, which is characterized in terms of left $\mathrm{add}_R\,\omega$-approximations. The notion of ω-left approximation dimension is introduced, and the forms of k-syzygy modules being k-torsionfree modules are described.

Keywords: left $\mathrm{add}_R\,\omega$-approximations, ω-k-torsionfree modules, ω-left approximation dimension.

Assume that R is a ring. We use $\mathrm{mod}R$ (resp. $\mathrm{mod}R^{\mathrm{op}}$) to denote the category of modules consisting of finitely generated left (resp. right) R-modules.

It is well known that, for any positive integer k, k-syzygy modules and k-torsionfree modules[1] are two natural and interesting classes of modules in homological algebra, which played very important roles in the classification of modules and rings. Auslander and Bridger[1] showed that a k-torsionfree module is a k-syzygy module, but the converse is not true. Then it is natural to ask the question: when is the converse true? That is, what forms of k-syzygy modules are k-torsionfree modules? In ref. [1] a necessary and sufficient condition of k-syzygy modules being k-torsionfree modules was given in terms of the properties of dual modules. It was showed in refs. [2,3] that i-syzygy modules are i-torsionfree modules for any $1 \leqslant i \leqslant k$ if R is a quasi k-Gorenstein algebra. The notion of appoximations (see Definition 1) is an important research object in representation theory of algebras[4,5]. In this paper we introduce the notion of k-torsionfree modules with respect to a bimodule (see Definition 2), which is characterized in terms of the properties of approximations (Theorem 1). We also introduce the notion of left approximation dimension with respect to a bimodule (see Definition 3), and then give an answer to the above question by using obtained results and the left approximation dimension of modules (Theorem 3).

The following definition is cited from ref. [4]. However, R here is not necessarily an artin algebra.

Definition 1[4]. Let R be a ring. Assume that $\mathscr{D}$ is a full subcategory of $\mathrm{mod}R$ and $D \in \mathscr{D}$, $C \in \mathrm{mod}R$. The morphism $C \to D$ is said to be a left $\mathscr{D}$-approximation of C if $\mathrm{Hom}_R(D, X) \to \mathrm{Hom}_R(C, X)$ is epic for any $X \in \mathscr{D}$. The subcategory $\mathscr{D}$ is said to be covariantly finite in $\mathrm{mod}R$ if every module in $\mathrm{mod}R$ has a left $\mathscr{D}$-approximation. In this case, $\mathscr{D}$ is called a covariantly finite subcategory of $\mathrm{mod}R$.

In the following, R is a left noether ring, S is a right noether ring, ${}_R\omega_S$ is a given (R, S)-bimodule and the natural map $S^{\mathrm{op}} \to \mathrm{End}({}_R\omega)$ is an isomorphism. We use $(-)^{\omega}$ to denote $\mathrm{Hom}(-, {}_R\omega_S)$, and $\mathscr{P}(R)$ (resp. $\mathscr{P}(S^{\mathrm{op}})$) to denote the subcategory of $\mathrm{mod}R$ (resp.

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modS^{op}) consisting of projective left R-modules (resp. right S-modules). For any $A \in \mathrm{mod}R$ (resp. modS^{op}), let σ_A: $A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^{\omega}$ be the canonical evaluation map. If σ_A is a monomorphism, then A is called an ω-torsionless module. If σ_A is an isomorphism, then A is called an ω-reflexive module. It is seen easily that P and P^{ω} are ω-reflexive modules for any $P \in \mathscr{P}(S^{op})$. Moreover, we use add$_R A$ (resp. addA_S) to denote the subcategory of mod$R$ (resp. modS^{op}) consisting of all modules isomorphic to the direct summands of finite direct sums of copies of $_R A$ (resp. A_S).

1 Left add$_R$ ω-approximations and ω-k-torsionfree modules

We first give the following fundamental lemma.

Lemma 1. Every module in mod$R$ has a left add$_R$ ω-approximation.

Proof. Suppose $M \in \mathrm{mod}R$. By Lemma 2 of ref. [6], $M^{\omega} \in \mathrm{mod}S^{op}$. Then there is a module $P \in \mathscr{P}(S^{op})$ such that $P \xrightarrow{f} M^{\omega}$ is epic. Let h be the composition homomorphism: $M \xrightarrow{\sigma_M} M^{\omega\omega} \xrightarrow{f^{\omega}} P^{\omega}$; that is, $h = f^{\omega} \cdot \sigma_M$.

Assume that g: $M \to Q$ is any homomorphism of R-modules with $Q \in \mathrm{add}_R\omega$. Since $Q^{\omega} \in \mathscr{P}(S^{op})$, there is a homomorphism of S^{op}-modules s: $Q^{\omega} \to P$ such that $g^{\omega} = f \cdot s$ and so $g^{\omega\omega} = s^{\omega} \cdot f^{\omega}$. On the other hand, $\sigma_Q \cdot g = g^{\omega\omega} \cdot \sigma_M$ and σ_Q is an isomorphism, so $g = \sigma_Q^{-1} \cdot g^{\omega\omega} \cdot \sigma_M = \sigma_Q^{-1} \cdot s^{\omega} \cdot f^{\omega} \cdot \sigma_M = (\sigma_Q^{-1} \cdot s^{\omega}) \cdot h$ and hence h: $M \to P^{\omega}$ is a left add$_R$ ω-approximation of M. This finishes the proof.

Corollary 1. add$_R$ ω is covariantly finite in mod$R$.

Proposition 1. Let $M \in \mathrm{mod}R$. The following statements are equivalent.

(i) M is an ω-torsionless module.

(ii) There is an exact sequence $0 \to M \xrightarrow{f} X$ such that f: $M \to X$ is a left add$_R$ ω-approximation of M.

(iii) There is an exact sequence $0 \to M \xrightarrow{f} \omega^n$ such that f: $M \to \omega^n$ is a left add$_R$ ω-approximation of M, where n is a positive integer.

Proof. Note that M is a ω-torsionless module if and only if σ_M is monic. Then the equivalence between (i) and (ii) follows from the proof of Lemma 1 and the equivalence between (ii) and (iii) is trivial. This finishes the proof.

Proposition 2. Let $M \in \mathrm{mod}R$. The following statements are equivalent.

(i) M is an ω-reflexive module.

(ii) There is an exact sequence $0 \to M \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$ such that f_1: $M \to X_1$ and Im$f_2 \to X_2$ are left add$_R$ ω-approximations of M and Imf_2 respectively.

(iii) There is an exact sequence $0 \to M \xrightarrow{f_1} \omega^n \xrightarrow{f_2} \omega^m$ such that f_1: $M \to \omega^n$ and Im$f_2 \to \omega^m$ are left add$_R$ ω-approximations of M and Imf_2 respectively, where n and m are positive integers.

Proof. (i)⇒(ii) Assume that M is an ω-reflexive module; that is, σ_M is an isomorphism, and assume that $P \xrightarrow{f} M^{\omega}$ is an epimorphism with $P \in \mathscr{P}(S^{op})$. From the proof of Lemma 1, we have an exact sequence $0 \to M \xrightarrow{h} P^{\omega} \to N \to 0$, where $h = f^{\omega} \cdot \sigma_M$: $M \to P^{\omega}$ is a left add$_R$ ω-approximation of M, $N = \mathrm{Coker}h$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0\to N^{\omega} & \longrightarrow & P^{\omega\omega} & \xrightarrow{h^{\omega}} & M^{\omega} \\ & & \| & & \uparrow \sigma_M^{\omega} \\ & & P^{\omega\omega} & \xrightarrow{f^{\omega\omega}} & M^{\omega\omega\omega} \\ & & \uparrow \sigma_P & & \uparrow \sigma_{M^{\omega}} \\ & & P & \xrightarrow{f} & M^{\omega}\to 0 \end{array}$$

Since σ_M is an isomorphism, σ_M^{ω} and $\sigma_{M^{\omega}}$ are also isomorphisms. Moreover, σ_P is an isomorphism and f is epic, so h^{ω} is also epic and hence we have the following exact commutative diagram:

$$\begin{array}{ccccccc} 0\to M & \xrightarrow{h} & P^{\omega} & \longrightarrow & N & \longrightarrow & 0 \\ \downarrow \sigma_M & & \downarrow \sigma_{P^{\omega}} & & \downarrow \sigma_N & & \\ 0\to M^{\omega\omega} & \xrightarrow{h^{\omega\omega}} & P^{\omega\omega\omega} & \longrightarrow & N^{\omega\omega} & & \end{array}$$

Since σ_M and $\sigma_{P^{\omega}}$ are isomorphisms, σ_N is monic by Snake lemma; that is, N is an ω-torsionless module. By Proposition 1 there is a left $\mathrm{add}_R\omega$-approximation g: $N\to X_2$ of N with g being monic. Thus we obtain an exact sequence $0\to M\xrightarrow{h}P^{\omega}\to X_2$, which is the one we desire.

(i)$\Rightarrow$(iii) It is trivial.

(iii)$\Rightarrow$(i) Assume that there is an exact sequence

$$0\to M\xrightarrow{f_1}\omega^n\xrightarrow{f_2}\omega^m$$

such that f_1: $M\to\omega^n$ and $\mathrm{Im}f_2\to\omega^m$ are left $\mathrm{add}_R\,\omega$-approximations of M and $\mathrm{Im}f_2$ respectively. Then we have an induced exact sequence

$$(\omega^m)^{\omega}\xrightarrow{f_2^{\omega}}(\omega^n)^{\omega}\xrightarrow{f_1^{\omega}}M^{\omega}\to 0.$$

This implies that we have the following exact commutative diagram:

$$\begin{array}{ccccc} 0\to M & \xrightarrow{f_1} & \omega^n & \xrightarrow{f_2} & \omega^m \\ \downarrow \sigma_M & & \downarrow \sigma_{\omega^n} & & \downarrow \sigma_{\omega^m} \\ 0\to M^{\omega\omega} & \xrightarrow{f_1^{\omega\omega}} & (\omega^n)^{\omega\omega} & \xrightarrow{f_2^{\omega\omega}} & (\omega^m)^{\omega\omega} \end{array}$$

Since σ_{ω^n} and σ_{ω^m} are isomorphisms, by diagram chasing we know that σ_M is also an isomorphism and M is an ω-reflexive module. This finishes the proof.

Lemma 2. Let $0\to B\xrightarrow{f}Q_1\xrightarrow{g}Q_0\xrightarrow{h}A\to 0$ be an exact sequence in $\mathrm{mod}\,S^{\mathrm{op}}$ with Q_0, $Q_1\in\mathrm{add}\,\omega_S$.

(i) If $\mathrm{Ext}_S^1(\omega,\omega)=0$, then $0\to A^{\omega}\xrightarrow{h^{\omega}}Q_0^{\omega}\xrightarrow{g^{\omega}}Q_1^{\omega}\xrightarrow{f^{\omega}}B^{\omega}$ is exact if and only if $\mathrm{Ext}_S^1(A,\omega)=0$.

(ii) If $\mathrm{Ext}_S^1(\omega,\omega)=0=\mathrm{Ext}_S^2(\omega,\omega)$, then $0\to A^{\omega}\xrightarrow{h^{\omega}}Q_0^{\omega}\xrightarrow{g^{\omega}}Q_1^{\omega}\xrightarrow{f^{\omega}}B^{\omega}\to 0$ is exact if and only if $\mathrm{Ext}_S^1(A,\omega)=0=\mathrm{Ext}_S^2(A,\omega)$.

Proof. Put $K=\mathrm{Im}\,g$. Then there is a decomposition $g=i\cdot\pi$ with π: $Q_1\to K$ epic and i: $K\to Q_0$ monic. It is easy to check the sufficiency of (i) and (ii). In the following we only need to check the necessity.

(i) Assume that there is an exact sequence

$$\begin{array}{c} 0 \to A^{\omega} \xrightarrow{h^{\omega}} Q_0^{\omega} \xrightarrow{g^{\omega}} Q_1^{\omega} \xrightarrow{f^{\omega}} B^{\omega} \\ {}_{i^{\omega}}\searrow \quad \nearrow_{\pi^{\omega}} \\ K^{\omega} \end{array}$$

Then we have exact sequences $0 \to A^{\omega} \xrightarrow{h^{\omega}} Q_0^{\omega} \xrightarrow{i^{\omega}} K^{\omega}$ and $0 \to K^{\omega} \xrightarrow{\pi^{\omega}} Q_1^{\omega} \xrightarrow{f^{\omega}} B^{\omega}$. So $K^{\omega} \cong \mathrm{Im}\pi^{\omega} \cong \mathrm{Ker}f^{\omega} \cong \mathrm{Im}g^{\omega} \cong \mathrm{Im}(\pi^{\omega} \cdot i^{\omega}) \cong \mathrm{Im}i^{\omega}$ (note: π^{ω} is monic) and i^{ω} is epic. Thus we obtain a long exact sequence

$$0 \to A^{\omega} \xrightarrow{h^{\omega}} Q_0^{\omega} \xrightarrow{i^{\omega}} K^{\omega} \to 0 \to \mathrm{Ext}_S^1(A, \omega) \to \mathrm{Ext}_S^1(Q_0, \omega) = 0$$

which implies $\mathrm{Ext}_S^1(A, \omega) = 0$.

(ii) By (i), it suffices to show that $\mathrm{Ext}_S^2(A, \omega) = 0$. By the exactness of the sequence $0 \to K^{\omega} \xrightarrow{\pi^{\omega}} Q_1^{\omega} \xrightarrow{f^{\omega}} B^{\omega} \to 0$ we have $\mathrm{Ext}_S^1(K, \omega) = 0$. But $\mathrm{Ext}_S^2(A, \omega) \cong \mathrm{Ext}_S^1(K, \omega)$, so $\mathrm{Ext}_S^2(A, \omega) = 0$. This finishes the proof.

Let $A \in \mathrm{mod}R$ and $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a projective resolution of A in $\mathrm{mod}R$. Then we have an exact sequence $0 \to A^{\omega} \to P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to \mathrm{Coker}f^{\omega} \to 0$. Put $X = \mathrm{Coker}f^{\omega}$, then $X \in \mathrm{mod}S^{\mathrm{op}}$. From Lemma 2.1 and its proof in Huang and Tang's paper[1)] we have the following result.

Lemma 3. Let the natural map $R \to \mathrm{End}(\omega_S)$ be an isomorphism, $A \in \mathrm{mod}R$ and $X \in \mathrm{mod}S^{\mathrm{op}}$ as above. If $\mathrm{Ext}_S^1(\omega, \omega) = 0$, then there is an exact sequence $0 \to \mathrm{Ext}_S^1(X, \omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega}$. If $\mathrm{Ext}_S^1(\omega, \omega) = 0 = \mathrm{Ext}_S^2(\omega, \omega)$, then there is an exact sequence $0 \to \mathrm{Ext}_S^1(X, \omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}_S^2(X, \omega) \to 0$.

Proposition 3. Let k be a positive integer, $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$ and the natural map $R \to \mathrm{End}(\omega_S)$ an isomorphism. If $P_1 \xrightarrow{f} P_0 \to A \to 0$ and $Q_1 \xrightarrow{g} Q_0 \to A \to 0$ are two projective resolutions of A in $\mathrm{mod}R$, then $\mathrm{Ext}_S^i(X, \omega) \cong \mathrm{Ext}_S^i(Y, \omega)$ for any $1 \leqslant i \leqslant k$, where $X = \mathrm{Coker}f^{\omega}$ and $Y = \mathrm{Coker}g^{\omega}$.

Proof. If $k = 1$, then $\mathrm{Ext}_S^1(X, \omega) \cong \mathrm{Ker}\sigma_A \cong \mathrm{Ext}_S^1(Y, \omega)$ by Lemma 3. If $k = 2$, we further have $\mathrm{Ext}_S^2(X, \omega) \cong \mathrm{Coker}\sigma_A \cong \mathrm{Ext}_S^2(Y, \omega)$ by Lemma 3. So the conclusion holds when $k = 1, 2$. Now suppose $k \geqslant 3$. Since $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$ and there are exact sequences $0 \to A^{\omega} \to P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to X \to 0$ and $0 \to A^{\omega} \to Q_0^{\omega} \xrightarrow{g^{\omega}} Q_1^{\omega} \to Y \to 0$ with each P_i^{ω}, $Q_i^{\omega} \in \mathrm{add}\omega_S$, $\mathrm{Ext}_S^i(X, \omega) \cong \mathrm{Ext}_S^{i-2}(A^{\omega}, \omega) \cong \mathrm{Ext}_S^i(Y, \omega)$ for any $3 \leqslant i \leqslant k$. So $\mathrm{Ext}_S^i(X, \omega) \cong \mathrm{Ext}_S^i(Y, \omega)$ for any $1 \leqslant i \leqslant k$. This finishes the proof.

Definition 2. Let k be a positive integer, $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$ and the natural map $R \to \mathrm{End}(\omega_S)$ an isomorphism and A and X as above. If $\mathrm{Ext}_S^i(X, \omega) = 0$ for any $1 \leqslant i \leqslant k$, then A is called an ω-k-torsionfree module.

1) Huang Zhaoyong, Tang Gaohua, Selforthogonal modules over coherent rings, Journal of Pure and Applied Algebra, to appear.

Remark. We know from Proposition 3 that the above definition is well-defined; that is, it does not depend on the choice of a projective resolution of the given module.

For the sake of relating convenience, we denote X above by $\mathrm{Tr}_{\omega}A$.

Lemma 4. Let $M \in \mathrm{mod}R$ and the natural map $R \to \mathrm{End}(\omega_S)$ be an isomorphism.

(i) If $\mathrm{Ext}_S^1(\omega, \omega) = 0$, then M is an ω-torsionless module if and only if M is an ω-1-torsionfree module.

(ii) If $\mathrm{Ext}_S^1(\omega, \omega) = 0 = \mathrm{Ext}_S^2(\omega, \omega)$, then M is an ω-reflexive module if and only if M is an ω-2-torsionfree module.

Proof. Assume that $P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ is an exact sequence in $\mathrm{mod}R$ with $P_1, P_0 \in \mathscr{P}(R)$. Then we have an exact sequence $0 \to M^{\omega} \xrightarrow{g^{\omega}} P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to \mathrm{Tr}_{\omega}M \to 0$ and the following commutative diagram:

$$\begin{array}{ccccccccc} & & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \to & 0 \\ & & \downarrow \sigma_{P_1} & & \downarrow \sigma_{P_0} & & \downarrow \sigma_M & & \\ 0 \to & (\mathrm{Tr}_{\omega}M)^{\omega} \to & P_1^{\omega\omega} & \xrightarrow{f^{\omega\omega}} & P_0^{\omega\omega} & \xrightarrow{g^{\omega\omega}} & M^{\omega\omega} & & \end{array}$$

Since σ_{P_0} and σ_{P_1} are isomorphisms, by diagram chasing it is easy to see that σ_M is monic if and only if the lower row of the above diagram is exact, and σ_M is isomorphic if and only if the lower row of the above diagram is exact and $g^{\omega\omega}$ is epic. Then our conclusions follow from Lemma 2. This finishes the proof.

The following is the main result of this section.

Theorem 1. Let $M \in \mathrm{mod}R$ and k be a positive integer. If the natural map $R \to \mathrm{End}(\omega_S)$ is an isomorphism and $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$, then the following statements are equivalent.

(i) M is an ω-k-torsionfree module.

(ii) There is an exact sequence $0 \to M \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k$ such that each $\mathrm{Im}f_i \to X_i$ is a left $\mathrm{add}_R\,\omega$-approximation of $\mathrm{Im}f_i$, $1 \leqslant i \leqslant k$.

(iii) There is an exact sequence $0 \to M \xrightarrow{f_1} \omega^{n_1} \xrightarrow{f_2} \cdots \xrightarrow{f_k} \omega^{n_k}$ such that each $\mathrm{Im}f_i \to \omega^{n_i}$ is a left $\mathrm{add}_R\,\omega$-approximation of $\mathrm{Im}f_i$, $1 \leqslant i \leqslant k$.

Proof. We proceed by induction on k. When $k = 1, 2$, the conclusions follow from Proposition 1, Proposition 2 and Lemma 4. Now suppose $k \geqslant 3$.

(i)$\Rightarrow$(ii) By Proposition 2 and Lemma 4, there is an exact sequence $0 \to M \xrightarrow{f_1} X_1 \to N \to 0$ such that $f_1 : M \to X_1$ is a left $\mathrm{add}_R\,\omega$-approximation of M. Then we have an exact sequence $0 \to N^{\omega} \to X_1^{\omega} \xrightarrow{f_1^{\omega}} M^{\omega} \to 0$ and so $\mathrm{Ext}_S^i(N^{\omega}, \omega) \cong \mathrm{Ext}_S^{i+1}(M^{\omega}, \omega)$ for any $i \geqslant 1$. Note that M is an ω-k-torsionfree module; that is, $\mathrm{Ext}_S^i(\mathrm{Tr}_{\omega}M, \omega) = 0$ for any $1 \leqslant i \leqslant k$, so $\mathrm{Ext}_S^i(M^{\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$ and hence $\mathrm{Ext}_S^{i+2}(\mathrm{Tr}_{\omega}N, \omega) \cong \mathrm{Ext}_S^i(N^{\omega}, \omega) = 0$ for any $1 \leqslant i \leqslant k-3$.

On the other hand, M is clearly an ω-reflexive module and we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & M & \longrightarrow & X_1 & \longrightarrow & N & \to & 0 \\ & & \downarrow \sigma_M & & \downarrow \sigma_{X_1} & & \downarrow \sigma_N & & \\ 0 & \to & M^{\omega\omega} & \to & X_1^{\omega\omega} & \longrightarrow & N^{\omega\omega} & \to & 0 \end{array}$$

where σ_M and σ_{X_1} are isomorphisms, so σ_N is also an isomorphism and N is an ω-reflexive module. By Lemma 4(ii), $\mathrm{Ext}_S^1(\mathrm{Tr}_\omega N, \omega) = 0 = \mathrm{Ext}_S^2(\mathrm{Tr}_\omega N, \omega)$ and thus $\mathrm{Ext}_S^i(\mathrm{Tr}_\omega N, \omega) = 0$ for any $1 \leqslant i \leqslant k-1$ and N is an ω-$(k-1)$-torsionfree module. Now our conclusion follows easily from induction hypothesis.

(ii)⇒(iii) It is trivial.

(iii)⇒(i) Assume that there is an exact sequence

$$0 \to M \xrightarrow{f_1} \omega^{n_1} \xrightarrow{f_2} \cdots \xrightarrow{f_k} \omega^{n_k}$$

such that each $\mathrm{Im} f_i \to \omega^{n_i}$ is a left $\mathrm{add}_R\, \omega$-approximation of $\mathrm{Im} f_i$. Put $N = \mathrm{Im} f_2$. By induction hypothesis, N is an ω-$(k-1)$-torsionfree module. So N is an ω-reflexive module and $\mathrm{Ext}_S^i(N^\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k-3$.

Consider the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 \to M & \xrightarrow{f_1} & \omega^{n_1} & \xrightarrow{g_1} & N \to 0 \\ \downarrow \sigma_M & & \downarrow \sigma_{\omega^{n_1}} & & \downarrow \sigma_N \\ 0 \to M^{\omega\omega} & \xrightarrow{f_1^{\omega\omega}} & (\omega^{n_1})^{\omega\omega} & \xrightarrow{g_1^{\omega\omega}} & N^{\omega\omega} \end{array}$$

Note that σ_N and $\sigma_{\omega^{n_1}}$ are isomorphisms, so σ_M is an isomorphism and $g_1^{\omega\omega}$ is epic, which yield that M is an ω-reflexive module and $\mathrm{Ext}_S^1(M^\omega, \omega) = 0$. On the other hand, it follows from the exact sequence $0 \to N^\omega \to (\omega^{n_1})^\omega \to M^\omega \to 0$ that $\mathrm{Ext}_S^i(M^\omega, \omega) \cong \mathrm{Ext}_S^{i-1}(N^\omega, \omega) = 0$ for any $2 \leqslant i \leqslant k-2$. So $\mathrm{Ext}_S^i(M^\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k-2$ and $\mathrm{Ext}_S^i(\mathrm{Tr}_\omega M, \omega) = 0$ for any $3 \leqslant i \leqslant k$, but M is an ω-reflexive module. Thus M is an ω-k-torsionfree module. This finishes the proof.

In case ${}_R\omega_S = {}_RR_R$, an ω-k-torsionfree module defined above is just a k-torsionfree module defined in ref. [1]. A module $M \in \mathrm{mod} R$ is called a k-syzygy module if there is an exact sequence $0 \to M \to P_1 \to \cdots \to P_k$ with each $P_i \in \mathscr{P}(R)$, $1 \leqslant i \leqslant k$.

Corollary 2. Let $M \in \mathrm{mod} R$ and k be a positive integer. The following statements are equivalent.

(i) M is a (usual) k-torsionfree module.

(ii) There is an exact sequence $0 \to M \xrightarrow{f_1} P_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} P_k$ such that each $\mathrm{Im} f_i \to P_i$ is a left $\mathscr{P}(R)$-approximation of $\mathrm{Im} f_i$, $1 \leqslant i \leqslant k$.

(iii) There is an exact sequence $0 \to M \xrightarrow{f_1} F_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} F_k$ such that each F_i is a finitely generated free left R-module and each $\mathrm{Im} f_i \to F_i$ is a left $\mathscr{P}(R)$-approximation of $\mathrm{Im} f_i$, $1 \leqslant i \leqslant k$.

The following lemma is completely similar to Lemma 1 of ref. [3], we omit the proof.

Lemma 5. Let $A \in \mathrm{mod} S^{\mathrm{op}}$. The following statements are equivalent.

(i) $\mathrm{Ext}_S^i(A, \omega) = 0$ for any $1 \leqslant i \leqslant k$.

(ii) If $P_{k+2} \to P_{k+1} \to \cdots \to P_1 \to A \to 0$ is exact with $P_i \in \mathscr{P}(S^{op})$ $(1 \leqslant i \leqslant k)$, then the induced sequence $0 \to A^\omega \to P_1^\omega \to \cdots \to P_{k+1}^\omega \to P_{k+2}^\omega$ is exact.

Of the condition of "the natural map $R \to \mathrm{End}(\omega_S)$ is an isomorphism and $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$" in Definition 2 or Theorem 1, we give a characterization as follows.

Theorem 2. Let k be a positive integer. The following statements are equivalent.

(i) There is an exact sequence $0 \to R \xrightarrow{f_1} X_1 \to \cdots \to X_{k+1} \xrightarrow{f_{k+2}} X_{k+2}$ with each $\mathrm{Im} f_i \to X_i$ a left $\mathrm{add}_R\, \omega$-approximation of $\mathrm{Im} f_i$, $1 \leqslant i \leqslant k+2$.

(ii) The natural map $R \to \mathrm{End}(\omega_S)$ is an isomorphism and $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$.

Proof. By Lemma 1 we have the following complex

$$0 \to R \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_{k+1}} X_{k+1} \xrightarrow{f_{k+2}} X_{k+2},$$

with each $\mathrm{Im} f_i \to X_i$ being a left $\mathrm{add}_R\, \omega$-approximation of $\mathrm{Im} f_i$, $1 \leqslant i \leqslant k+2$. Then there is an induced exact sequence

$$X_{k+2}^\omega \xrightarrow{f_{k+2}^\omega} X_{k+1}^\omega \xrightarrow{f_{k+1}^\omega} \cdots \to X_1^\omega \xrightarrow{f_1^\omega} R^\omega \to 0.$$

Since the natural map $S^{op} \to \mathrm{End}({}_R\omega)$ is an isomorphism, each $X_i^\omega \in \mathscr{P}(S^{op})$ and each $X_i (\in \mathrm{add}_R\, \omega)$ is an ω-reflexive module, $1 \leqslant i \leqslant k+2$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to R & \longrightarrow & X_1 & \longrightarrow \cdots \to & X_{k+1} & \to & X_{k+2} \\ \downarrow \sigma_R & & \downarrow \sigma_{X_1} & & \downarrow \sigma_{X_{k+1}} & & \downarrow \sigma_{X_{k+2}} \\ 0 \to R^{\omega\omega} & \longrightarrow & X_1^{\omega\omega} & \to \cdots \to & X_{k+1}^{\omega\omega} & \to & X_{k+2}^{\omega\omega} \end{array}$$

It is clear that σ_R is an isomorphism if and only if the natural map $R \to \mathrm{End}(\omega_S)$ is an isomorphism. By Lemma 5, the lower row of the above diagram is exact if and only if $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$. On the other hand, each σ_{X_i} is an isomorphism, then it follows easily from Proposition 2 that the upper row of the above diagram is exact if and only if the natural map $R \to \mathrm{End}(\omega_S)$ is an isomorphism and $\mathrm{Ext}_S^i(\omega, \omega) = 0$ for any $1 \leqslant i \leqslant k$. This finishes the proof.

A module $T_S \in \mathrm{mod}\, S^{op}$ is called a Wakamatsu tilting module[7] if the natural map $S^{op} \to \mathrm{End}({}_{\mathrm{End}(T_S)} T)$ is an isomorphism and $\mathrm{Ext}_S^i(T, T) = 0$ for any $i \geqslant 1$.

Corollary 3. The following statements are equivalent.

(i) There is an exact sequence $0 \to R \xrightarrow{f_1} X_1 \to \cdots \to X_i \xrightarrow{f_i} \cdots$ with each $\mathrm{Im} f_i \to X_i$ being a left $\mathrm{add}_R\, \omega$-approximation of $\mathrm{Im} f_i$, $i \geqslant 1$.

(ii) ω_S is a Wakamatsu tilting module.

Proof. It is trivial by Theorem 2.

2 ω-left approximation dimension

In this section we introduce a new kind of dimension: ω-left approximation dimension. We give an answer to the question mentioned at the beginning of this paper by using the results obtained in the last section and the ω-left approximation dimension of modules.

We first give two lemmas.

Lemma 6. Let $\mathscr{D}$ be a full subcategory of $\mathrm{mod}\, R$ and $M \in \mathrm{mod}\, R$. If there are exact sequences

$$0 \to M \xrightarrow{f_1} X_1 \to K_1 \to 0, \quad 0 \to M \xrightarrow{f'_1} X'_1 \to K'_1 \to 0,$$

where both f_1 and f'_1 are left $\mathscr{D}$-approximations of M, then $K_1 \oplus X'_1 \cong K'_1 \oplus X_1$.

Proof. Consider the following push-out diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & M & \xrightarrow{f_1} & X_1 & \to & K_1 & \to & 0 \\ & & {\scriptstyle f'_1}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X'_1 & \xrightarrow{g_1} & X & \longrightarrow & K_1 & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & K'_1 & = & K'_1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since f_1 is a left $\mathscr{D}$-approximation of M, there is a homomorphism $\alpha: X_1 \to X'_1$ such that $\alpha \cdot f_1 = f'_1 = 1_{X'_1} \cdot f'_1$. Note that the above diagram is a push-out diagram, so by the universal property (cf. ref. [8], p.42), there is a homomorphism $\beta: X \to X'_1$ such that $\beta \cdot g_1 = 1_{X'_1}$, which implies that the middle row of the above diagram splits and $X \cong K_1 \oplus X'_1$. Similarly we show that the middle column of the above diagram splits and $X \cong K'_1 \oplus X_1$. Consequently $K_1 \oplus X'_1 \cong K'_1 \oplus X_1$. This finishes the proof.

Remark. Let $\mathscr{D}$ be a full subcategory of mod$R$ and $D \in \mathscr{D}$, $C \in$ mod$R$. The morphism $D \to C$ is said to be a right $\mathscr{D}$-approximation of C if $\mathrm{Hom}_R(X, D) \to \mathrm{Hom}_R(X, C)$ is epic for any $X \in \mathscr{D}$[4]. We have a dual result of Lemma 6 as follows: Let $M \in$ mod$R$. If there are exact sequences

$$0 \to N_1 \to Y_1 \xrightarrow{h_1} M \to 0, \quad 0 \to N'_1 \to Y'_1 \xrightarrow{h'_1} M \to 0,$$

where both h_1 and h'_1 are right $\mathscr{D}$-approximations of M, then $N_1 \oplus Y'_1 \cong N'_1 \oplus Y_1$ (note: the proof of this result is dual to that of Lemma 6, in which we use the pull-back diagram instead of the push-out diagram and then use the universal property of a pull-back diagram). This dual result is a generalization of Schanuel's Lemma (cf. Theorem 3.62 of ref. [8]), and Theorem 2 of ref. [3] is a special case of it.

Lemma 7. Let $\mathscr{D}$ be a full subcategory of mod$R$ and $M \in$ mod$R$. If the left $\mathscr{D}$-approximations of M exist and one of them is monic, then all of the left $\mathscr{D}$-approximations of M are monic.

Proof. It is trivial by Definition 1.

Let $M \in$ mod$R$. By Lemma 1, there is a complex

$$\eta: 0 \to M \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_i} X_i \to \cdots$$

such that each $\mathrm{Im} f_i \to X_i$ is a left $\mathrm{add}_R\, \omega$-approximation of $\mathrm{Im} f_i$, $i \geq 1$. For any positive integer k, we use η_k to denote the kth trancated complex of η; that is, η_k is the complex

$$\eta_k: 0 \to M \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_k} X_k.$$

Assume that there is another complex

$$\eta' : 0 \to M \xrightarrow{f'_1} X'_1 \to \cdots \xrightarrow{f'_i} X'_i \to \cdots$$

such that each $\mathrm{Im} f'_i \to X'_i$ is a left $\mathrm{add}_R\, \omega$-approximation of $\mathrm{Im} f'_i$, $i \geqslant 1$. We use η'_k to denote the kth trancated complex of η'. It follows easily from Lemmas 6 and 7 that η_k is exact if and only if η'_k is exact.

Definition 3. Let η and η_k be as above. We define $\mathrm{l.apd}_\omega(M) = k$ if k is the largest positive integer such that η_k is exact. Set $\mathrm{l.apd}_\omega(M) = \infty$ if η is exact. $\mathrm{l.apd}_\omega(M)$ is called the ω-left approximation dimension of M.

Remark. By the above argument, the definition of ω-left approximation dimension is well-defined; that is, it does not depend on the choice of η.

By Theorem 1 of last section, we have the following result.

Theorem 1′. Under the assumptions of Theorem 1, the following statements are equivalent.

(i) M is an ω-k-torsionfree module.

(ii) $\mathrm{l.apd}_\omega(M) = k$.

So, in case ${}_R\omega_S = {}_RR_R$, we know that the (usual) k-torsionfree modules are just those k-syzygy modules with R-left approximation dimension k. This gives an answer to the question in Introduction. That is, we have

Theorem 3. Let $M \in \mathrm{mod} R$ and k be a positive integer. The following statements are equivalent.

(i) M is a (usual) k-torsionfree module.

(ii) $\mathrm{l.apd}_R(M) = k$.

By Corollary 3, we further give a characterization of Wakamatsu tilting modules as follows.

Corollary 4. The following statements are equivalent.

(i) $\mathrm{l.apd}_\omega(R) = \infty$.

(ii) ω_S is a Wakamatsu tilting module.

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