Homological Dimensions under Foxby Equivalence [*†]

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Abstract

Let R and S be rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathscr{T}$ be a subcategory of the Auslander class $\mathcal{A}_C(S)$ and $\mathscr{H} = \{C \otimes_S T \mid T \in \mathscr{T}\}$. Then for any left R-module M, the $\mathscr{T}$-projective dimension of $\mathrm{Hom}_R(C, M)$ is at most the $\mathscr{H}$-projective dimension of M, and they are identical when M is in the Bass class $\mathcal{B}_C(R)$. If ${}_RC_S$ is faithful and $\mathscr{T}$ is resolving, then in a short exact sequence of left R-modules, the $\mathscr{H}$-projective dimensions of any two terms can determine an upper bound of that of the third term. Furthermore, we apply these results to the cases of $\mathscr{T}$ being the subcategories of (weak) flat modules, projective modules and $\mathcal{A}_C(S)$ respectively. Some known results are obtained as corollaries.

1 Introduction

In order to study various duality theories over commutative rings, Foxby [9] and Golod [12] introduced the so-called semidualizing modules and related Auslander and Bass classes. Then Holm and White [13] extended them to arbitrary associative rings. Let R and S be arbitrary rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$ be the Auslander and Bass classes with respect to C respectively. It was shown in [13, Theorem 1] that there exists the following Foxby equivalence:

$$\mathcal{A}_C(S) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_S -}{\rightleftarrows}} \mathcal{B}_C(R).$$

For other Foxby equivalences between some subclasses of $\mathcal{A}_C(S)$ and that of $\mathcal{B}_C(R)$, the reader is referred to [13, Theorem 1] and [19, Theorem 4.6]. Homological properties of modules under Foxby equivalences have been studied by many authors; see [1, 11, 13], [17]–[19], [23]–[27] and references therein. Let $\mathscr{T}$ be a subcategory of $\mathcal{A}_C(S)$ and $\mathscr{H}$ a subcategory of $\mathcal{B}_C(R)$ such that there exists a Foxby equivalence between $\mathscr{T}$ and $\mathscr{H}$. One of the aims in this paper is to establish the relation between the $\mathscr{T}$-projective dimension of $\mathrm{Hom}_R(C, M)$ and the $\mathscr{H}$-projective dimension of M for any left R-module M.

As fundamental invariants in homological theory, homological dimensions play a crucial role in studying the structures of modules and rings. Let R be an arbitrary ring and let $\mathscr{T}$ be a category of left R-modules. For a left R-module A, we use $\mathscr{T}\text{-}\mathrm{pd}\, A$ to denote the $\mathscr{T}$-projective dimension of A. Let

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

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be an exact sequence of left R-modules. In many cases, the following assertions hold (see [17, Introduction] for details):

(1) $\mathscr{T}\text{-}\operatorname{pd} A_2 \leq \max\{\mathscr{T}\text{-}\operatorname{pd} A_1, \mathscr{T}\text{-}\operatorname{pd} A_3\}$ with equality if $\mathscr{T}\text{-}\operatorname{pd} A_1 + 1 \neq \mathscr{T}\text{-}\operatorname{pd} A_3$.

(2) $\mathscr{T}\text{-}\operatorname{pd} A_3 \leq \max\{\mathscr{T}\text{-}\operatorname{pd} A_1 + 1, \mathscr{T}\text{-}\operatorname{pd} A_2\}$ with equality if $\mathscr{T}\text{-}\operatorname{pd} A_1 \neq \mathscr{T}\text{-}\operatorname{pd} A_2$.

(3) $\mathscr{T}\text{-}\operatorname{pd} A_1 \leq \max\{\mathscr{T}\text{-}\operatorname{pd} A_2, \mathscr{T}\text{-}\operatorname{pd} A_3 - 1\}$ with equality if $\mathscr{T}\text{-}\operatorname{pd} A_2 \neq \mathscr{T}\text{-}\operatorname{pd} A_3$.

In particular, it was shown in [17, Theorem 3.2] that if $\mathscr{T}$ contains all projective left R-modules, then the above assertions hold if and only if $\mathscr{T}$ is resolving in the sense that $\mathscr{T}$ contains all projective left R-modules, and $\mathscr{T}$ is closed under extensions and kernels of epimorphisms. Observe that if the assertions (1) and (3) hold, then $\mathscr{T}$ is closed under extensions and kernels of epimorphisms. Thus, it is natural to ask the following question: whether do the assertions (1)–(3) hold for some categories of left R-modules that do not contain all projective left R-modules? Our other aim is to give some positive answers to this question.

The paper is organized as follows. In Section 2, we give some notions and notations which will be used in the sequel. Let R and S be arbitrary rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathscr{T}$ be a subcategory of $\mathcal{A}_C(S)$ and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$. In Section 3, we prove the following results.

Theorem 1.1. (Theorem 3.2) *For any left R-module M, we have*

$$\mathscr{T}\text{-}\operatorname{pd} \operatorname{Hom}_R(C, M) \leq \mathscr{H}\text{-}\operatorname{pd} M$$

with equality if $M \in \mathcal{B}_C(R)$.

Theorem 1.2. (Theorem 3.3) *Let $\mathscr{T}$ be resolving, and let*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

be an exact sequence in $\operatorname{Mod} R$. *If either $\mathscr{H}\text{-}\operatorname{pd} A_1 < \infty$ or ${}_RC_S$ is faithful, then we have*

(1) $\mathscr{H}\text{-}\operatorname{pd} A_2 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_1, \mathscr{H}\text{-}\operatorname{pd} A_3\}$ *with equality if* $\mathscr{H}\text{-}\operatorname{pd} A_1 + 1 \neq \mathscr{H}\text{-}\operatorname{pd} A_3$.

(2) $\mathscr{H}\text{-}\operatorname{pd} A_3 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_1 + 1, \mathscr{H}\text{-}\operatorname{pd} A_2\}$ *with equality if* $\mathscr{H}\text{-}\operatorname{pd} A_1 \neq \mathscr{H}\text{-}\operatorname{pd} A_2$.

(3) $\mathscr{H}\text{-}\operatorname{pd} A_1 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_2, \mathscr{H}\text{-}\operatorname{pd} A_3 - 1\}$ *with equality if* $\mathscr{H}\text{-}\operatorname{pd} A_2 \neq \mathscr{H}\text{-}\operatorname{pd} A_3$.

As a consequence, we obtain some properties of left R-modules with finite $\mathscr{H}$-projective dimension (Propositions 3.6 and 3.7). In Section 4, we apply these results to the cases of $\mathscr{T}$ being the categories of (weak) flat left S-modules, projective left S-modules and $\mathcal{A}_C(S)$ respectively, in this case $\mathscr{H}$ corresponds to the categories of (weak) C-flat left R-modules, C-projective left R-modules and $\mathcal{B}_C(R)$ respectively (Proposition 4.1 and Theorem 4.5). Note that none of the categories of (weak) C-flat left R-modules and C-projective left R-modules contain all projective left R-modules and that the former one is not self-orthogonal in general. This means that the main results obtained in Section 3 are non-trivial complements to [17, Theorem 3.2], [2, Lemma 3.12] and [15, Theorem 3.8(2)] respectively.

2 Preliminaries

Throughout this paper, all rings are associative rings with unit. Let R be a ring. We use $\operatorname{Mod} R$ (respectively, $\operatorname{Mod} R^{op}$) to denote the category of left (respectively, right) R-modules. We use $\mathcal{F}(R)$,

$\mathcal{P}(R)$ and $\mathcal{I}(R)$ to denote the subcategories of $\operatorname{Mod} R$ consisting of flat, projective and injective modules respectively.

Let $\mathscr{X}$ be a subcategory of $\operatorname{Mod} R$ and $M \in \operatorname{Mod} R$. Recall from [7] that a homomorphism $f : X \to M$ in $\operatorname{Mod} R$ with X in $\mathscr{X}$ is called an *$\mathscr{X}$-precover* of M if $\operatorname{Hom}_R(X', f)$ is epic for any $X' \in \mathscr{X}$; and f is called an *$\mathscr{X}$-cover* of M if it is an $\mathscr{X}$-precover of M and any endomorphism $h : X \to X$ is an automorphism whenever $f = fh$. The subcategory $\mathscr{X}$ is called a *(pre)covering* in $\operatorname{Mod} R$ if each module in $\operatorname{Mod} R$ admits an $\mathscr{X}$-(pre)cover. Dually, the notions of $\mathscr{X}$-*(pre)envelopes* and *(pre)enveloping* classes are defined.

The *$\mathscr{X}$-projective dimension* $\mathscr{X}\text{-pd}\, M$ of a module $M \in \operatorname{Mod} R$ is defined as $\inf\{n \mid$ there exists an exact sequence

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all X_i in $\mathscr{X}\}$, and set $\mathscr{X}\text{-pd}\, M = \infty$ if no such integer exists, and set $\mathscr{X}\text{-pd}\, 0 = -1$. Dually, the notion of the *$\mathscr{X}$-injective dimension* $\mathscr{X}\text{-id}\, M$ of M is defined. In particular, we use $\operatorname{fd}_R M$, $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the flat, projective, and injective dimensions of M respectively.

Recall that $\mathscr{X}$ is called *self-orthogonal* if $\operatorname{Ext}_R^{\geq 1}(X_1, X_2) = 0$ for any $X_1, X_2 \in \mathscr{X}$. Also recall that $\mathscr{X}$ is called *resolving* if $\mathcal{P}(R) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and kernels of epimorphisms; dually, $\mathscr{X}$ is called *coresolving* if $\mathcal{I}(R) \subseteq \mathscr{X}$ and $\mathscr{X}$ is closed under extensions and cokernels of monomorphisms. A sequence

$$\mathbb{M} := \cdots \to M_1 \to M_2 \to M_3 \to \cdots$$

in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(\mathscr{X}, -)$*-exact* (respectively, $\operatorname{Hom}_R(-, \mathscr{X})$*-exact*) if $\operatorname{Hom}_R(X, \mathbb{M})$ (respectively, $\operatorname{Hom}_R(\mathbb{M}, X)$) is exact for any $X \in \mathscr{X}$.

Definition 2.1. ([1, 13]) Let R and S be rings.

(1) An (R, S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied.

- (a1) ${}_RC$ admits a degreewise finite R-projective resolution.
- (a2) C_S admits a degreewise finite S^{op}-projective resolution.
- (b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \operatorname{Hom}_{S^{op}}(C, C)$ is an isomorphism.
- (b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_R(C, C)$ is an isomorphism.
- (c1) $\operatorname{Ext}_R^{\geq 1}(C, C) = 0$.
- (c2) $\operatorname{Ext}_{S^{op}}^{\geq 1}(C, C) = 0$.

(2) A semidualizing bimodule ${}_RC_S$ is called *faithful* if the following conditions are satisfied.

- (f1) For any $M \in \operatorname{Mod} R$, if $\operatorname{Hom}_R(C, M) = 0$, then $M = 0$.
- (f2) For any $N \in \operatorname{Mod} S^{op}$, if $\operatorname{Hom}_{S^{op}}(C, N) = 0$, then $N = 0$.

Wakamatsu [28] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*, see [4, 21]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if it is Wakamatsu tilting ([30, Corollary 3.2]). Typical examples of semidualizing bimodules include the free module of rank one and the dualizing module over a Cohen-Macaulay local ring. Over any commutative ring, all semidualizing bimodules are faithful ([13, Proposition 3.1]). For more examples of semidualizing bimodules, the reader is referred to [13, 24, 29].

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. For convenience, we write

$$(-)_* := \mathrm{Hom}_R(C, -).$$

Thus, if M is a left R-module, then M_* has a left S-module structure via $(s \cdot f)(c) = f(cs)$ for any $s \in S$, $f \in M_*$ and $c \in C$. We also write

$${}_RC^{\perp} := \{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{\geq 1}(C, M) = 0\},$$

$${C_S}^{\top} := \{N \in \mathrm{Mod}\, S \mid \mathrm{Tor}_{\geq 1}^S(C, N) = 0\}.$$

Following [13], set

$$\mathcal{F}_C(R) := \{C \otimes_S F \mid F \text{ is flat in } \mathrm{Mod}\, S\},$$

$$\mathcal{P}_C(R) := \{C \otimes_S P \mid P \text{ is projective in } \mathrm{Mod}\, S\},$$

$$\mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in } \mathrm{Mod}\, R\}.$$

The modules in $\mathcal{F}_C(R)$, $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called *C-flat*, *C-projective* and *C-injective* respectively. When ${}_RC_S = {}_RR_R$, C-flat, C-projective and C-injective modules are exactly flat, projective and injective modules respectively. Symmetrically, the categories $\mathcal{F}_C(S^{op})$, $\mathcal{P}_C(S^{op})$ and $\mathcal{I}_C(R^{op})$ are defined.

Let $N \in \mathrm{Mod}\, S$. Then there exists the following canonical evaluation homomorphism:

$$\mu_N : N \longrightarrow (C \otimes_S N)_*$$

defined by $\mu_N(x)(c) = c \otimes x$ for any $c \in C$ and $x \in N$. The module N is called *adjoint C-coreflexive* if μ_N is an isomorphism. Symmetrically, the *adjoint C-coreflexive module* is defined in $\mathrm{Mod}\, R^{op}$.

Let $M \in \mathrm{Mod}\, R$. Then there exists the following canonical evaluation homomorphism:

$$\theta_M : C \otimes_S M_* \longrightarrow M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$. Recall from [24] that M is called *C-coreflexive* if θ_M is an isomorphism. Symmetrically, the *C-coreflexive module* is defined in $\mathrm{Mod}\, S^{op}$.

Definition 2.2. ([13])

(1) The *Auslander class* $\mathcal{A}_C(S)$ with respect to C consists of all left S-modules N satisfying the following conditions.

  (A1) $N \in {C_S}^{\top}$.

  (A2) $C \otimes_S N \in {}_RC^{\perp}$.

  (A3) N is adjoint C-coreflexive.

(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all left R-modules M satisfying the following conditions.

  (B1) $M \in {}_RC^{\perp}$.

  (B2) $M_* \in {C_S}^{\top}$.

  (B3) M is C-coreflexive.

The *Auslander class* $\mathcal{A}_C(R^{op})$ in $\mathrm{Mod}\, R^{op}$ and the *Bass class* $\mathcal{B}_C(S^{op})$ in $\mathrm{Mod}\, S^{op}$ are defined symmetrically.

For a module $M \in \operatorname{Mod} R$, we use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M)$$

to denote a minimal injective copresentation of M.

Definition 2.3. ([24]) Let $M \in \operatorname{Mod} R$. Then the left S-module $\operatorname{cTr}_C M := \operatorname{Coker} {f^0}_*$ is called the *cotranspose* of M with respect to ${}_RC_S$, and M is called *∞-C-cotorsionfree* if $\operatorname{cTr}_C M \in {C_S}^{\top}$.

We use $\mathrm{c}\mathcal{T}(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of ∞-C-cotorsionfree modules.

Lemma 2.4. ([24, Corollary 3.4 and Theorem 3.9]) *A module $M \in \mathrm{c}\mathcal{T}(R)$ if and only if M satisfies the conditions* (B2) *and* (B3) *in Definition 2.2(2), and hence*

$$\mathcal{B}_C(R) = {}_RC^{\perp} \cap \mathrm{c}\mathcal{T}(R).$$

Let N be a module in $\operatorname{Mod} S$. Bican, El Bashir and Enochs proved in [5] that N has a flat cover. We use

$$F_1(N) \xrightarrow{f_0} F_0(N) \to N \to 0$$

to denote a minimal flat presentation of N in $\operatorname{Mod} S$, where $F_0(N) \to N$ and $F_1(N) \to \operatorname{Im} f_0$ are the flat covers of N and $\operatorname{Im} f_0$ respectively.

Definition 2.5. ([26]) Let $N \in \operatorname{Mod} S$. Then the left R-module $\operatorname{acTr}_C N := \operatorname{Ker}(1_C \otimes f_0)$ is called the *adjoint cotranspose* of N with respect to ${}_RC_S$, and N is called *adjoint ∞-C-cotorsionfree* if $\operatorname{acTr}_C N \in {}_RC^{\perp}$.

We use $\mathrm{ac}\mathcal{T}(S)$ to denote the subcategory of $\operatorname{Mod} S$ consisting of adjoint ∞-C-cotorsionfree modules.

Lemma 2.6. ([26, Corollary 3.3 and Proposition 3.4]) *A module $N \in \mathrm{ac}\mathcal{T}(S)$ if and only if N satisfies the conditions* (A2) *and* (A3) *in Definition 2.2(1), and hence*

$$\mathcal{A}_C(S) = {C_S}^{\top} \cap \mathrm{ac}\mathcal{T}(S).$$

3 General results

3.1 Relative projective dimensions

We begin with the following result.

Lemma 3.1. *Let $\mathscr{T}$ be a subcategory of* $\operatorname{Mod} S$ *and* $M \in \operatorname{Mod} R$, *and set* $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$.

(1) *If $\mathscr{T} \subseteq \mathrm{ac}\mathcal{T}(S)$, then*

$$\mathscr{T}\text{-}\operatorname{pd} M_* \leq \mathscr{H}\text{-}\operatorname{pd} M.$$

(2) *If $\mathscr{T} \subseteq {C_S}^{\top}$ and $M \in \mathrm{c}\mathcal{T}(R)$, then*

$$\mathscr{H}\text{-}\operatorname{pd} M \leq \mathscr{T}\text{-}\operatorname{pd} M_*.$$

Proof. (1) Let $M \in \operatorname{Mod} R$ with $\mathscr{H}\text{-}\operatorname{pd} M = n < \infty$. Then there exists an exact sequence

$$0 \to H_n \to \cdots \to H_1 \to H_0 \to M \to 0 \tag{3.1}$$

in $\operatorname{Mod} R$ with all H_i in $\mathscr{H}$. By assumption, there exists some module $T_i \in \mathscr{T}$ such that $H_i = C \otimes_S T_i$ for any $0 \leq i \leq n$. Since $\mathscr{T} \subseteq \operatorname{ac}\mathcal{T}(S)$, we have $C \otimes_S T \in {}_RC^{\perp}$ and $(C \otimes_S T)_* \cong T$ for any $T \in \mathscr{T}$ by Lemma 2.6. Then applying the functor $(-)_*$ to the exact sequence (3.1) yields the following exact sequence

$$0 \to (C \otimes_S T_n)_* \to \cdots \to (C \otimes_S T_1)_* \to (C \otimes_S T_0)_* \to M_* \to 0$$

in $\operatorname{Mod} R$ with $(C \otimes_S T_i)_* \cong T_i$ for any $0 \leq i \leq n$, and thus $\mathscr{T}\text{-}\operatorname{pd} M_* \leq n$.

(2) Let $\mathscr{T} \subseteq {C_S}^{\top}$ and $M \in \operatorname{c}\mathcal{T}(R)$ with $\mathscr{T}\text{-}\operatorname{pd} M_* = n < \infty$. Then there exists an exact sequence

$$0 \to T_n \to \cdots \to T_1 \to T_0 \to M_* \to 0 \tag{3.2}$$

in $\operatorname{Mod} S$ with all T_i in $\mathscr{T}$. Since $M \in \operatorname{c}\mathcal{T}(R)$, we have $M_* \in {C_S}^{\top}$ and $C \otimes_S M_* \cong M$ by Lemma 2.4. Then applying the functor $C \otimes_S -$ to the exact sequence (3.2) yields the following exact sequence

$$0 \to C \otimes_S T_n \to \cdots \to C \otimes_S T_1 \to C \otimes_S T_0 \to \underbrace{C \otimes_S M_*}_{\cong M} \to 0$$

in $\operatorname{Mod} R$ with all $C \otimes_S T_i$ in $\mathscr{H}$, and thus $\mathscr{H}\text{-}\operatorname{pd} M \leq n$. □

According to Lemmas 2.4 and 2.6, we get the following result from Lemma 3.1.

Theorem 3.2. *Let $\mathscr{T}$ be a subcategory of $\mathcal{A}_C(S)$ and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$. Then for any $M \in \operatorname{Mod} R$, we have*

$$\mathscr{T}\text{-}\operatorname{pd} M_* \leq \mathscr{H}\text{-}\operatorname{pd} M$$

with equality if $M \in \operatorname{c}\mathcal{T}(R)$ (in particular, if $M \in \mathcal{B}_C(R)$).

Compare the following result with [17, Theorem 3.2].

Theorem 3.3. *Let $\mathscr{T}$ be a resolving subcategory of $\mathcal{A}_C(S)$ and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$. Let*

$$0 \to A_1 \to A_2 \to A_3 \to 0 \tag{3.3}$$

be an exact sequence in $\operatorname{Mod} R$. If either $\mathscr{H}\text{-}\operatorname{pd} A_1 < \infty$ or ${}_RC_S$ is faithful, then we have

(1) *$\mathscr{H}\text{-}\operatorname{pd} A_2 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_1, \mathscr{H}\text{-}\operatorname{pd} A_3\}$ with equality if $\mathscr{H}\text{-}\operatorname{pd} A_1 + 1 \neq \mathscr{H}\text{-}\operatorname{pd} A_3$.*

(2) *$\mathscr{H}\text{-}\operatorname{pd} A_3 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_1 + 1, \mathscr{H}\text{-}\operatorname{pd} A_2\}$ with equality if $\mathscr{H}\text{-}\operatorname{pd} A_1 \neq \mathscr{H}\text{-}\operatorname{pd} A_2$.*

(3) *$\mathscr{H}\text{-}\operatorname{pd} A_1 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_2, \mathscr{H}\text{-}\operatorname{pd} A_3 - 1\}$ with equality if $\mathscr{H}\text{-}\operatorname{pd} A_2 \neq \mathscr{H}\text{-}\operatorname{pd} A_3$.*

Proof. We first prove those three inequalities hold true.

If $\mathscr{H}\text{-}\operatorname{pd} A_1 = \infty$, then those two inequalities in (1) and (2) hold true. Now suppose $\mathscr{H}\text{-}\operatorname{pd} A_1 < \infty$. Since $\mathscr{T} \subseteq \mathcal{A}_C(S)$, we have $\mathscr{H} \subseteq \mathcal{B}_C(R)$ by [13, Theorem 1], and hence $A_1 \in \mathcal{B}_C(R)(\subseteq {}_RC^{\perp})$ by [13, Theorem 6.2]. Thus

$$\mathscr{T}\text{-}\operatorname{pd} {A_1}_* = \mathscr{H}\text{-}\operatorname{pd} A_1 < \infty$$

by Theorem 3.2. Applying the functor $(-)_*$ to the exact sequence (3.3) yields the following exact sequence

$$0 \to {A_1}_* \to {A_2}_* \to {A_3}_* \to 0 \tag{3.4}$$

in $\operatorname{Mod} S$. Since $\mathscr{T}$ is resolving, it follows from [17, Theorem 3.2] that

(i) $\mathscr{T}\text{-}\operatorname{pd} A_{2*} \leq \max\{\mathscr{T}\text{-}\operatorname{pd} A_{1*}, \mathscr{T}\text{-}\operatorname{pd} A_{3*}\}$ with equality if $\mathscr{T}\text{-}\operatorname{pd} A_{1*} + 1 \neq \mathscr{T}\text{-}\operatorname{pd} A_{3*}$.

(ii) $\mathscr{T}\text{-}\operatorname{pd} A_{3*} \leq \max\{\mathscr{T}\text{-}\operatorname{pd} A_{1*} + 1, \mathscr{T}\text{-}\operatorname{pd} A_{2*}\}$ with equality if $\mathscr{T}\text{-}\operatorname{pd} A_{1*} \neq \mathscr{T}\text{-}\operatorname{pd} A_{2*}$.

If $\mathscr{H}\text{-}\operatorname{pd} A_3 = \infty$, then the inequality in (1) holds true. Now suppose $\mathscr{H}\text{-}\operatorname{pd} A_3 < \infty$. Then in a similar way as above, we have $A_3 \in \mathcal{B}_C(R)$ and

$$\mathscr{T}\text{-}\operatorname{pd} A_{3*} = \mathscr{H}\text{-}\operatorname{pd} A_3 < \infty.$$

By [13, Theorem 6.2], we have $A_2 \in \mathcal{B}_C(R)$, and hence similarly we have

$$\mathscr{T}\text{-}\operatorname{pd} A_{2*} = \mathscr{H}\text{-}\operatorname{pd} A_2.$$

Thus the inequality in (1) follows from (i).

If $\mathscr{H}\text{-}\operatorname{pd} A_2 = \infty$, then the inequality in (2) holds true. Now suppose $\mathscr{H}\text{-}\operatorname{pd} A_2 < \infty$. Then we also have $A_2 \in \mathcal{B}_C(R)$ and

$$\mathscr{T}\text{-}\operatorname{pd} A_{2*} = \mathscr{H}\text{-}\operatorname{pd} A_2 < \infty.$$

By [13, Theorem 6.2], we have $A_3 \in \mathcal{B}_C(R)$, and hence similarly we have

$$\mathscr{T}\text{-}\operatorname{pd} A_{3*} = \mathscr{H}\text{-}\operatorname{pd} A_3.$$

Thus the inequality in (2) follows from (ii).

If either $\mathscr{H}\text{-}\operatorname{pd} A_2 = \infty$ or $\mathscr{H}\text{-}\operatorname{pd} A_3 = \infty$, then the inequality in (3) holds true. Now suppose $\mathscr{H}\text{-}\operatorname{pd} A_2 < \infty$ and $\mathscr{H}\text{-}\operatorname{pd} A_3 < \infty$. Then in a similar way as above, we have $A_2, A_3 \in \mathcal{B}_C(R)$. If $\mathscr{H}\text{-}\operatorname{pd} A_1 < \infty$, then we also have $A_1 \in \mathcal{B}_C(R)$. If ${}_RC_S$ is faithful, then $A_1 \in \mathcal{B}_C(R)$ by [13, Theorem 6.3]. In both cases, we can get the exact sequence (3.4). Since $\mathscr{T}$ is resolving, it follows from [17, Theorem 3.2] that

(iii) $\mathscr{T}\text{-}\operatorname{pd} A_{1*} \leq \max\{\mathscr{T}\text{-}\operatorname{pd} A_{2*}, \mathscr{T}\text{-}\operatorname{pd} A_{3*} - 1\}$ with equality if $\mathscr{T}\text{-}\operatorname{pd} A_{2*} \neq \mathscr{T}\text{-}\operatorname{pd} A_{3*}$.

Since $\mathscr{T}\text{-}\operatorname{pd} A_{i*} = \mathscr{H}\text{-}\operatorname{pd} A_i$ for $1 \leq i \leq 3$ by Theorem 3.2, the inequality in (3) follows from (iii).

The latter assertions in (1)–(3) follow from (i)–(iii). □

As an immediate consequence of Theorem 3.3, we get the following corollary.

Corollary 3.4. *Let $\mathscr{T}$ be a resolving subcategory of $\mathcal{A}_C(S)$, and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$. Then the following assertions hold true.*

1. *The subcategory of* $\operatorname{Mod} R$ *consisting of modules with finite $\mathscr{H}$-projective dimension is closed under extensions and cokernels of monomorphisms. Moreover, if ${}_RC_S$ is faithful, then this subcategory is closed under kernels of epimorphisms.*
2. *For any $n \geq 0$, the subcategory of* $\operatorname{Mod} R$ *consisting of modules with $\mathscr{H}$-projective dimension at most n is closed under extensions.*

We need the following observation.

Lemma 3.5. *Let $\mathscr{T}$ be a resolving subcategory of $\mathcal{A}_C(S)$, and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$.*

1. *For any* $\operatorname{Hom}_R(C, -)$*-exact exact sequence*

$$0 \to A \to B \to H \to 0$$

in $\operatorname{Mod} R$ *with $H \in \mathscr{H}$, we have $A \in \mathscr{H}$ if and only if $B \in \mathscr{H}$.*

(2) *If $\mathscr{T}$ is closed under direct summands, then so is $\mathscr{H}$.*

Proof. (1) Let $H \in \mathscr{H}(\subseteq \mathcal{B}_C(R))$. Then by Theorem 3.2, we have $H_* \in \mathscr{T}$. By assumption, we have an exact sequence

$$0 \to A_* \to B_* \to H_* \to 0$$

in $\operatorname{Mod} S$. Since $\mathscr{T}$ is resolving, we have that $A_* \in \mathscr{T}$ if and only if $B_* \in \mathscr{T}$. If $A \in \mathscr{H}(\subseteq \mathcal{B}_C(R))$, then $A_* \in \mathscr{T}$, and so $B_* \in \mathscr{T}$; moreover, we have $B \in \mathcal{B}_C(R)$ by [13, Theorem 6.2], and thus $B \cong C \otimes_S B_* \in \mathscr{H}$. Similarly, we get that if $B \in \mathscr{H}$, then $A \in \mathscr{H}$.

(2) Let $H \in \mathscr{H}(\subseteq \mathcal{B}_C(R))$ with $H \cong H_1 \oplus H_2$. Then $H_* \cong H_{1*} \oplus H_{2*}$. By Theorem 3.2, we have $H_* \in \mathscr{T}$. Since $\mathscr{T}$ is closed under direct summands, we have $H_{1*}, H_{2*} \in \mathscr{T}$, and hence $H_i \cong C \otimes_S H_{i*} \in \mathscr{H}$ for $i = 1, 2$. □

As a consequence, we get the following result.

Proposition 3.6. *Let $\mathscr{T}$ be a resolving subcategory of $\mathcal{A}_C(S)$ closed under direct summands, and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$, and let $M \in \operatorname{Mod} R$ and $n \geq 0$. If $\mathscr{H}$ is precovering in $\operatorname{Mod} R$, then the following statements are equivalent.*

(1) $\mathscr{H}\text{-pd}\, M \leq n$

(2) *There exists a $\operatorname{Hom}_R(\mathscr{H}, -)$-exact exact sequence*

$$0 \to H_n \to \cdots \to H_1 \to H_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all H_i in $\mathscr{H}$.

Proof. (2) ⇒ (1) It is trivial.

(1) ⇒ (2) We proceed by induction on n. The case for $n = 0$ is trivial. Now suppose $n \geq 1$. Then there exists an exact sequence

$$0 \to K' \to H'_0 \to M \to 0$$

in $\operatorname{Mod} R$ with $H'_0 \in \mathscr{H}$ and $\mathscr{H}\text{-pd}\, K' \leq n - 1$. It yields that any $\mathscr{H}$-precover of M is epic. Since $\mathscr{H}$ is precovering in $\operatorname{Mod} R$, there exists a $\operatorname{Hom}_R(\mathscr{H}, -)$-exact exact sequence

$$0 \to K \to H_0 \to M \to 0$$

in $\operatorname{Mod} R$ with $H_0 \in \mathscr{H}$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K' & \to & X & \to & H_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K' & \to & H'_0 & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

By [14, Lemma 2.4(1)], the the middle column in the above diagram is $\mathrm{Hom}_R(\mathscr{H},-)$-exact, and thus it splits and $X \cong K \oplus H'_0$. Applying Corollary 3.4(2) to the middle row in the above diagram yields $\mathscr{H}\text{-pd}\, X \leq n-1$.

By [24, Proposition 3.7 and Theorem 3.9] and Lemma 3.5(1), we have that $\mathscr{H}$ is a $\mathcal{P}_C(R)$-resolving subcategory of $\mathrm{Mod}\, R$ with a $\mathcal{P}_C(R)$-proper generator $\mathcal{P}_C(R)$ in the sense of [15]. Since $\mathscr{T}$ is closed under direct summands by assumption, so is $\mathscr{H}$ by Lemma 3.5(2). Notice that $\mathscr{H} \subseteq \mathcal{B}_C(R) \subseteq \mathcal{P}_C(R)^{\perp}$, thus $\mathscr{H}\text{-pd}\, K \leq n-1$ by [15, Corollory 3.9], and the assertion follows by induction. $\square$

Compare the following result with [2, Lemma 3.12] and [15, Theorem 3.8(2)].

Proposition 3.7. *Let $\mathscr{T}$ be a resolving subcategory of $\mathcal{A}_C(S)$, and set $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$, and let $M \in \mathrm{Mod}\, R$ and $n \geq 0$. Assume that $\mathscr{H}\text{-pd}\, M \leq n$ and*

$$0 \to K \to H_0 \to M \to 0 \tag{3.5}$$

is an exact sequence in $\mathrm{Mod}\, R$ with $H_0 \in \mathscr{H}$. Then $\mathscr{H}\text{-pd}\, K \leq n-1$ if any of the following conditions is satisfied.

(1) *${}_RC_S$ is faithful.*

(2) *$\mathscr{H}$ is closed under direct summands and the exact sequence (3.5) is $\mathrm{Hom}_R(\mathscr{H},-)$-exact.*

Proof. Under the condition (1), the case for $n = 0$ follows from Theorem 3.3(3). Under the condition (2), when $n = 0$, the exact sequence (3.5) splits and K is a direct summand of H_0. Since $\mathscr{H}$ is closed under direct summands, we have $K \in \mathscr{H}$.

Now suppose $n \geq 1$. By assumption, there exists an exact sequence

$$0 \to K' \to H'_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with $H'_0 \in \mathscr{H}$ and $\mathscr{H}\text{-pd}\, K' \leq n-1$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K' & \to & X & \to & H_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K' & \to & H'_0 & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Applying Corollary 3.4(2) to the middle row in the above diagram, we have $\mathscr{H}\text{-pd}\, X \leq n-1$. Under the condition (1), applying Theorem 3.3(3) to the middle column in the above diagram yields $\mathscr{H}\text{-pd}\, K \leq n-1$. Under the condition (2), by the same argument as that in the proof of Proposition 3.6, we also have $\mathscr{H}\text{-pd}\, K \leq n-1$. $\square$

3.2 Relative injective dimensions

In this subsection, we list the counterparts of all results in Subsection 3.1, and omit those proofs which are completely dual to that in Subsection 3.1.

Lemma 3.8. *Let* $\mathscr{H}$ *be a subcategory of* $\operatorname{Mod} R$ *and* $N \in \operatorname{Mod} S$*, and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$*.*

(1) *If* $\mathscr{H} \subseteq \mathrm{c}\mathcal{T}(R)$*, then*

$$\mathscr{H}\text{-}\operatorname{id} C \otimes_S N \leq \mathscr{T}\text{-}\operatorname{id} N.$$

(2) *If* $\mathscr{H} \subseteq {}_R C^{\perp}$ *and* $N \in \mathrm{ac}\mathcal{T}(S)$*, then*

$$\mathscr{T}\text{-}\operatorname{id} N \leq \mathscr{H}\text{-}\operatorname{id} C \otimes_S N.$$

Proof. (1) Let $N \in \operatorname{Mod} S$ with $\mathscr{T}\text{-}\operatorname{id} N = n < \infty$. Then there exists an exact sequence

$$0 \to N \to T^0 \to T^1 \to \cdots \to T^n \to 0 \tag{3.6}$$

in $\operatorname{Mod} S$ with all T^i in $\mathscr{T}$. By assumption, there exists some module $H^i \in \mathscr{H}$ such that $T^i = {H^i}_*$ for any $0 \leq i \leq n$. Since $\mathscr{H} \subseteq \mathrm{c}\mathcal{T}(R)$, we have $H_* \in {C_S}^{\top}$ and $C \otimes_S H_* \cong H$ for any $H \in \mathscr{H}$ by Lemma 2.4. Then applying the functor $C \otimes_S -$ to the exact sequence (3.6) yields the following exact sequence

$$0 \to C \otimes_S N \to C \otimes_S {H^0}_* \to C \otimes_S {H^1}_* \to \cdots \to C \otimes_S {H^n}_* \to 0$$

in $\operatorname{Mod} S$ with $C \otimes_S {H^i}_* \cong H^i$ for any $0 \leq i \leq n$, and thus $\mathscr{H}\text{-}\operatorname{id} C \otimes_S N \leq n$.

(2) Let $\mathscr{H} \subseteq {}_R C^{\perp}$ and $N \in \mathrm{ac}\mathcal{T}(S)$ with $\mathscr{H}\text{-}\operatorname{id} C \otimes_S N = n < \infty$. Then there exists an exact sequence

$$0 \to C \otimes_S N \to H^0 \to H^1 \to \cdots \to H^n \to 0 \tag{3.7}$$

in $\operatorname{Mod} R$ with all H^i in $\mathscr{H}$. Since $N \in \mathrm{ac}\mathcal{T}(S)$, we have $C \otimes_S N \in {}_R C^{\perp}$ and $(C \otimes_S N)_* \cong N$ by Lemma 2.6. Then applying the functor $(-)_*$ to the exact sequence (3.7) yields the following exact sequence

$$0 \to \underbrace{(C \otimes_S N)_*}_{\cong N} \to {H^0}_* \to {H^1}_* \to \cdots \to {H^n}_* \to 0$$

in $\operatorname{Mod} S$ with all ${H^i}_*$ in $\mathscr{T}$, and thus $\mathscr{T}\text{-}\operatorname{id} N \leq n$. □

According to Lemmas 2.4 and 2.6, we get the following result from Lemma 3.8.

Theorem 3.9. *Let* $\mathscr{H}$ *be a subcategory of* $\mathcal{B}_C(R)$ *and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$*. Then for any* $N \in \operatorname{Mod} S$*, we have*

$$\mathscr{H}\text{-}\operatorname{id} C \otimes_S N \leq \mathscr{T}\text{-}\operatorname{id} N$$

with equality if $N \in \mathrm{ac}\mathcal{T}(S)$ *(in particular, if* $N \in \mathcal{A}_C(S)$*).*

Compare the following result with [17, Theorem 3.9].

Theorem 3.10. *Let* $\mathscr{H}$ *be a coresolving subcategory of* $\mathcal{B}_C(R)$*, and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$*. Let*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

be an exact sequence in $\operatorname{Mod} S$*. If either* $\mathscr{T}\text{-}\operatorname{id} A_3 < \infty$ *or* ${}_R C_S$ *is faithful, then we have*

(1) $\mathscr{T}\text{-}\operatorname{id} A_2 \le \max\{\mathscr{T}\text{-}\operatorname{id} A_1, \mathscr{T}\text{-}\operatorname{id} A_3\}$ *with equality if* $\mathscr{T}\text{-}\operatorname{id} A_1 \ne \mathscr{T}\text{-}\operatorname{id} A_3 + 1$.

(2) $\mathscr{T}\text{-}\operatorname{id} A_1 \le \max\{\mathscr{T}\text{-}\operatorname{id} A_2, \mathscr{T}\text{-}\operatorname{id} A_3 + 1\}$ *with equality if* $\mathscr{T}\text{-}\operatorname{id} A_2 \ne \mathscr{T}\text{-}\operatorname{id} A_3$.

(3) $\mathscr{T}\text{-}\operatorname{id} A_3 \le \max\{\mathscr{T}\text{-}\operatorname{id} A_1 - 1, \mathscr{T}\text{-}\operatorname{id} A_2\}$ *with equality if* $\mathscr{T}\text{-}\operatorname{id} A_1 \ne \mathscr{T}\text{-}\operatorname{id} A_2$.

As an immediate consequence of Theorem 3.10, we get the following corollary.

Corollary 3.11. *Let* $\mathscr{H}$ *be a coresolving subcategory of* $\mathcal{B}_C(R)$, *and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$. *Then the following assertions hold true.*

(1) *The subcategory of* $\operatorname{Mod} S$ *consisting of modules with finite* $\mathscr{T}$*-injective dimension is closed under extensions and kernels of epimorphisms. Moreover, if* ${}_RC_S$ *is faithful, then this subcategory is closed under cokernels of monomorphisms.*

(2) *For any* $n \ge 0$, *the subcategory of* $\operatorname{Mod} S$ *consisting of modules with* $\mathscr{T}$*-injective dimension at most* n *is closed under extensions.*

Recall that a sequence in $\operatorname{Mod} S$ is called $(C \otimes_S -)$*-exact* if it is exact after applying the functor $C \otimes_S -$. It follows from [26, p.298, Observation] that a sequence in $\operatorname{Mod} S$ is $(C \otimes_S -)$-exact if and only if it is $\operatorname{Hom}_S(-, \mathcal{I}_C(S))$-exact.

Lemma 3.12. *Let* $\mathscr{H}$ *be a coresolving subcategory of* $\mathcal{B}_C(R)$, *and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$.

(1) *For any* $\operatorname{Hom}_S(-, \mathcal{I}_C(S))$*-exact (equivalently,* $(C \otimes_S -)$*-exact) exact sequence*

$$0 \to T \to A \to B \to 0 \tag{3.8}$$

in $\operatorname{Mod} S$ *with* $T \in \mathscr{T}$, *we have* $A \in \mathscr{T}$ *if and only if* $B \in \mathscr{T}$.

(2) *If* $\mathscr{H}$ *is closed under direct summands, then so is* $\mathscr{T}$.

As a consequence, we get the following result.

Proposition 3.13. *Let* $\mathscr{H}$ *be a coresolving subcategory of* $\mathcal{B}_C(R)$ *closed under direct summands, and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$, *and let* $N \in \operatorname{Mod} S$ *and* $n \ge 0$. *If* $\mathscr{T}$ *is preenveloping in* $\operatorname{Mod} S$, *then the following statements are equivalent.*

(1) $\mathscr{T}\text{-}\operatorname{id} N \le n$

(2) *There exists a* $\operatorname{Hom}_S(-, \mathscr{T})$*-exact exact sequence*

$$0 \to N \to T^0 \to T^1 \to \cdots \to T^n \to 0$$

in $\operatorname{Mod} S$ *with all* T^i *in* $\mathscr{T}$.

Compare the following result with the dual of [2, Lemma 3.12] and [15, Theorem 4.8(2)].

Proposition 3.14. *Let* $\mathscr{H}$ *be a coresolving subcategory of* $\mathcal{B}_C(R)$, *and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$, *and let* $N \in \operatorname{Mod} S$ *and* $n \ge 0$. *Assume that* $\mathscr{T}\text{-}\operatorname{id} N \le n$ *and*

$$0 \to N \to T^0 \to L \to 0 \tag{3.9}$$

is an exact sequence in $\operatorname{Mod} S$ *with* $T^0 \in \mathscr{T}$. *Then* $\mathscr{T}\text{-}\operatorname{id} L \le n - 1$ *if any of the following conditions is satisfied.*

(1) ${}_RC_S$ *is faithful.*

(2) *$\mathscr{T}$ is closed under direct summands and the exact sequence* (3.9) *is* $\mathrm{Hom}_S(-,\mathscr{T})$*-exact.*

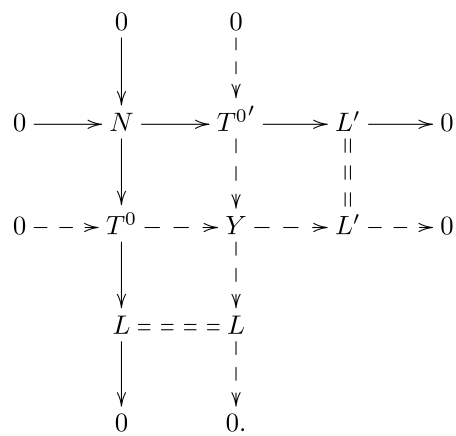
For the reader's convenience, we provide some ideas for the proofs of the last two propositions. Proposition 3.13 will be proved by induction on n. The case for $n = 0$ is trivial. When $n \geq 1$, there exists an exact sequence

$$0 \to N \to T^{0'} \to L' \to 0$$

in $\mathrm{Mod}\, S$ with $T^{0'} \in \mathscr{T}$ and $\mathscr{T}$-id $L' \leq n-1$. It yields that any $\mathscr{T}$-preenvelope of N is monic. Since $\mathscr{T}$ is preenveloping in $\mathrm{Mod}\, S$, there exists a $\mathrm{Hom}_S(-,\mathscr{T})$-exact exact sequence

$$0 \to N \to T^0 \to L \to 0$$

in $\mathrm{Mod}\, S$ with $T^0 \in \mathscr{T}$. Consider the following push-out diagram:



By [14, Lemma 2.4(2)], the the middle column in the above diagram is $\mathrm{Hom}_S(-,\mathscr{T})$-exact, and thus it splits and $Y \cong T^{0'} \oplus L$. The rest of the proof is left to the reader. In addition, the same diagram as above will be used in the proof of Proposition 3.14.

4 Applications to special categories of modules

Recall from [10] that a module $N \in \mathrm{Mod}\, S$ is called *weak flat* if $\mathrm{Tor}_1^S(X, N) = 0$ for any right S-module X admitting a degreewise finite S^{op}-projective resolution. A weak flat module is called *level* in [6]. We use $\mathcal{WF}(S)$ to denote the subclass of $\mathrm{Mod}\, S$ consisting of weak flat modules, and write

$$\mathrm{wfd}_S\, N := \mathcal{WF}(S)\text{-}\,\mathrm{pd}\, N,$$

$$\mathcal{WF}_C(R) := \{C \otimes_S G \mid G \in \mathcal{WF}(S)\}.$$

By [13, Lemma 4.1], [11, Theorem 2.2] and Lemma 2.6, we have

$$\mathcal{P}(S) \subseteq \mathcal{F}(S) \subseteq \mathcal{WF}(S) \subseteq \mathcal{A}_C(S) \subseteq \mathrm{ac}\mathcal{T}(S).$$

The assertions (1) and (2) in the following result are [25, Theorem 3.5(1)(2)].

Proposition 4.1. *For any* $M \in \mathrm{Mod}\, R$*, we have*

(1) $\mathrm{fd}_S\, M_* \leq \mathcal{F}_C(R)\text{-}\,\mathrm{pd}\, M$.

(2) $\mathrm{pd}_S M_* \leq \mathcal{P}_C(R)\text{-}\mathrm{pd}\, M$.
(3) $\mathrm{wfd}_S M_* \leq \mathcal{WF}_C(R)\text{-}\mathrm{pd}\, M$.
(4) $\mathcal{A}_C(S)\text{-}\mathrm{pd}\, M_* \leq \mathcal{B}_C(R)\text{-}\mathrm{pd}\, M$.

All equalities hold if $M \in \mathrm{c}\mathcal{T}(R)$ (*in particular, if* $M \in \mathcal{B}_C(R)$).

Proof. The assertions (1)–(3) follow from Theorem 3.2 by putting $\mathscr{T} = \mathcal{F}(S)$, $\mathcal{P}(S)$ and $\mathcal{WF}(S)$ respectively. Since $\mathcal{B}_C(R) = \{C \otimes_S T \mid T \in \mathcal{A}_C(S)\}$ by [13, Theorem 1], the assertion (4) follows from Theorem 3.2 by putting $\mathscr{T} = \mathcal{A}_C(S)$. □

Furthermore, we get the following result.

Corollary 4.2. *For any adjoint* C*-coreflexive left* S*-module* N*, we have*
(1) $\mathrm{fd}_S N \leq \mathcal{F}_C(R)\text{-}\mathrm{pd}\, C \otimes_S N$.
(2) $\mathrm{pd}_S N \leq \mathcal{P}_C(R)\text{-}\mathrm{pd}\, C \otimes_S N$.
(3) $\mathrm{wfd}_S N \leq \mathcal{WF}_C(R)\text{-}\mathrm{pd}\, C \otimes_S N$.
(4) $\mathcal{A}_C(S)\text{-}\mathrm{pd}\, N \leq \mathcal{B}_C(R)\text{-}\mathrm{pd}\, C \otimes_S N$.

All equalities hold if $N \in \mathcal{A}_C(S)$.

Proof. Let N be an adjoint C-coreflexive left S-module. Then $N \cong (C \otimes_S N)_*$ in $\mathrm{Mod}\, S$ and $C \otimes_S N \in \mathrm{Mod}\, R$. Putting $M = C \otimes_S N$ in Proposition 4.1, the assertions follow. □

Recall from [10] that a module $M \in \mathrm{Mod}\, R$ is called *weak injective* if $\mathrm{Ext}^1_R(X, M) = 0$ for any left R-module X admitting a degreewise finite R-projective resolution. A weak injective module is called *absolutely clean* in [6]. We use $\mathcal{WI}(R)$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of weak injective modules, and write

$$\mathrm{wid}_R M := \mathcal{WI}(R)\text{-}\mathrm{id}\, M,$$

$$\mathcal{WI}_C(S) := \{E_* \mid E \in \mathcal{WI}(R)\}.$$

By [13, Lemma 4.1], [11, Theorem 2.2] and Lemma 2.4, we have

$$\mathcal{I}(R) \subseteq \mathcal{WI}(R) \subseteq \mathcal{B}_C(R) \subseteq \mathrm{c}\mathcal{T}(R).$$

Recall from [20, 22] that a module $M \in \mathrm{Mod}\, R$ is called FP*-injective* (or *absolutely pure*) if $\mathrm{Ext}^1_R(X, M) = 0$ for any finitely presented left R-module X. We use $\mathcal{FI}(R)$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of FP-injective modules, and write

$$\mathrm{FP}\text{-}\mathrm{id}_R M := \mathcal{FI}(R)\text{-}\mathrm{id}\, M,$$

$$\mathcal{FI}_C(S) := \{E_* \mid E \in \mathcal{FI}(R)\}.$$

Recall that a ring R is called *left coherent* if any finitely generated left ideal of R is finitely presented.

The assertions (1) and (3) in the following result extend [25, Theorem 3.5(3)] and [18, Theorem 3.4] respectively.

Proposition 4.3. *For any* $N \in \mathrm{Mod}\, S$*, we have*
(1) $\mathrm{id}_R C \otimes_S N \leq \mathcal{I}_C(S)\text{-}\mathrm{id}\, N$.
(2) $\mathrm{wid}_R C \otimes_S N \leq \mathcal{WI}_C(S)\text{-}\mathrm{id}\, N$.

(3) *If R is a left coherent ring, then* $\mathrm{FP}\text{-}\mathrm{id}_R\, C\otimes_S N \leq \mathcal{FI}_C(S)\text{-}\mathrm{id}\, N$.

(4) $\mathcal{B}_C(R)\text{-}\mathrm{id}\, C\otimes_S N \leq \mathcal{A}_C(S)\text{-}\mathrm{id}\, N$.

All equalities hold if $N \in \mathrm{ac}\mathcal{T}(S)$ (in particular, if $N \in \mathcal{A}_C(S)$).

Proof. The assertions (1) and (2) follow from Theorem 3.9 by putting $\mathscr{H} = \mathcal{I}(R)$ and $\mathcal{WI}(R)$ respectively. Since $\mathcal{A}_C(S) = \{H_* \mid H \in \mathcal{B}_C(R)\}$ by [13, Theorem 1], the assertion (4) follows from Theorem 3.9 by putting $\mathscr{H} = \mathcal{B}_C(R)$.

Assume that R is a left coherent ring. Then a left R-module admits a degreewise finite R-projective resolution if and only if it is finitely presented. So $\mathcal{WI}(R) = \mathcal{FI}(R)$ and hence $\mathcal{WI}_C(S) = \mathcal{FI}_C(S)$. Thus the assertion (3) follows from (2). □

Furthermore, we get the following result.

Corollary 4.4. *For any C-coreflexive left R-module M, we have*

(1) $\mathrm{id}_R\, M \leq \mathcal{I}_C(S)\text{-}\mathrm{id}\, M_*$.

(2) $\mathrm{wid}_R\, M \leq \mathcal{WI}_C(S)\text{-}\mathrm{id}\, M_*$.

(3) *If R is a left coherent ring, then* $\mathrm{FP}\text{-}\mathrm{id}_R\, M \leq \mathcal{FI}_C(S)\text{-}\mathrm{id}\, M_*$.

(4) $\mathcal{B}_C(R)\text{-}\mathrm{id}\, M \leq \mathcal{A}_C(S)\text{-}\mathrm{id}\, M_*$.

All equalities hold if $M \in \mathcal{B}_C(R)$.

Proof. Let M be a C-coreflexive left R-module. Then $C\otimes_S M_* \cong M$ in $\mathrm{Mod}\, R$ and $M_* \in \mathrm{Mod}\, S$. Putting $N = M_*$ in Proposition 4.3, the assertions follow. □

Note that [27, Example 4.7(1)] shows that, in general, none of $\mathcal{F}_C(R)$, $\mathcal{P}_C(R)$ and $\mathcal{WF}_C(R)$ contain $\mathcal{P}(R)$, and hence none of them are resolving. Since the assertion (1) in the following result is a consequence of Theorem 3.3, this means that Theorem 3.3 is a non-trivial complement to [17, Theorem 3.2].

On the other hand, the class $\mathcal{P}_C(R)$ is self-orthogonal by [27, Lemma 2.5], so for any $M \in \mathrm{Mod}\, R$ and $n \geq 0$, it is easy to see that $\mathcal{P}_C(R)\text{-pd}\, M \leq n$ if and only if there exist a $\mathrm{Hom}_R(\mathcal{P}_C(R), -)$-exact exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all G_i in $\mathcal{P}_C(R)$. However, neither $\mathcal{F}_C(R)$ nor $\mathcal{WF}_C(R)$ (even when ${}_RC_S = {}_RR_R$) are self-orthogonal. Since the assertion (2) in the following result is a consequence of Propositions 3.6 and 3.7, this means that Propositions 3.6 and 3.7 are non-trivial complements to [2, Lemma 3.12] and [15, Theorem 3.8(2)].

Theorem 4.5. *Let $\mathscr{H}$ be any of the following subcategories of* $\mathrm{Mod}\, R$:

$$\text{(i) } \mathcal{F}_C(R), \quad \text{(ii) } \mathcal{P}_C(R), \quad \text{(iii) } \mathcal{WF}_C(R), \quad \text{(iv) } \mathcal{B}_C(R).$$

Then the following assertions hold true.

(1) *For any exact sequence*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\mathrm{Mod}\, R$, if either $\mathscr{H}\text{-pd}\, A_1 < \infty$ or ${}_RC_S$ is faithful, then we have

(1.1) $\mathscr{H}\text{-pd}\, A_2 \leq \max\{\mathscr{H}\text{-pd}\, A_1, \mathscr{H}\text{-pd}\, A_3\}$ *with equality if* $\mathscr{H}\text{-pd}\, A_1 + 1 \neq \mathscr{H}\text{-pd}\, A_3$.

(1.2) $\mathscr{H}\text{-}\operatorname{pd} A_3 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_1 + 1, \mathscr{H}\text{-}\operatorname{pd} A_2\}$ *with equality if* $\mathscr{H}\text{-}\operatorname{pd} A_1 \neq \mathscr{H}\text{-}\operatorname{pd} A_2$.

(1.3) $\mathscr{H}\text{-}\operatorname{pd} A_1 \leq \max\{\mathscr{H}\text{-}\operatorname{pd} A_2, \mathscr{H}\text{-}\operatorname{pd} A_3 - 1\}$ *with equality if* $\mathscr{H}\text{-}\operatorname{pd} A_2 \neq \mathscr{H}\text{-}\operatorname{pd} A_3$.

(2) *For any* $M \in \operatorname{Mod} R$ *and* $n \geq 0$*, the following assertions hold.*

(2.1) $\mathscr{H}\text{-pd}\, M \leq n$ *if and only if there exists a* $\operatorname{Hom}_R(\mathscr{H}, -)$*-exact exact sequence*

$$0 \to H_n \to \cdots \to H_1 \to H_0 \to M \to 0$$

in $\operatorname{Mod} R$ *with all* H_i *in* $\mathscr{H}$.

(2.2) *Assume that* $\mathscr{H}\text{-pd}\, M \leq n$ *and*

$$0 \to K \to H_0 \to M \to 0 \tag{4.1}$$

is an exact sequence in $\operatorname{Mod} R$ *with* $H_0 \in \mathscr{H}$*. Then* $\mathscr{H}\text{-pd}\, K \leq n-1$ *if any of the following conditions is satisfied.*

(i) ${}_RC_S$ *is faithful.*

(ii) $\mathscr{H}$ *is closed under direct summands and the exact sequence* (4.1) *is* $\operatorname{Hom}_R(\mathscr{H}, -)$*-exact.*

Proof. It is well known that the categories $\mathcal{F}(S)$ and $\mathcal{P}(S)$ are resolving. The categories $\mathcal{WF}(S)$ and $\mathcal{A}_C(S)$ are resolving by [11, Proposition 2.5(1)] and [13, Theorem 6.2] respectively. On the other hand, the categories $\mathcal{F}_C(R)$, $\mathcal{P}_C(R)$, $\mathcal{WF}_C(R)$ and $\mathcal{B}_C(R)$ are precovering in $\operatorname{Mod} R$ by [13, Proposition 5.3(1)(2)], [11, Theorem 2.12(1)] and [16, Theorem 3.3(2)] respectively. Now the assertions follow from Theorem 3.3, Propositions 3.6 and 3.7 by putting $\mathscr{T} = \mathcal{F}(S)$, $\mathcal{P}(S)$, $\mathcal{WF}(S)$ and $\mathcal{A}_C(S)$ respectively, and putting $\mathscr{H} = \{C \otimes_S T \mid T \in \mathscr{T}\}$. □

The following result is the dual of Theorem 4.5, in which the assertion (1) is a consequence of Theorem 3.10 and the assertion (2) is a consequence of Propositions 3.13 and 3.14. It means that Theorem 3.10 is a non-trivial complement to [17, Theorem 3.9] and that Propositions 3.13 and 3.14 are non-trivial complements to the dual of [2, Lemma 3.12] and [15, Theorem 4.8(2)].

Theorem 4.6. *Let* $\mathscr{T}$ *be any of the following subcategories of* $\operatorname{Mod} S$*:*

(i) $\mathcal{I}_C(S)$, (ii) $\mathcal{WI}_C(S)$, (iii) $\mathcal{FI}_C(S)$ (*when* R *is a left coherent ring*), (iv) $\mathcal{A}_C(S)$.

Then the following assertions hold true.

(1) *For any exact sequence*

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\operatorname{Mod} R$*, if either* $\mathscr{T}\text{-}\operatorname{id} A_3 < \infty$ *or* ${}_RC_S$ *is faithful, then we have*

(1.1) $\mathscr{T}\text{-}\operatorname{id} A_2 \leq \max\{\mathscr{T}\text{-}\operatorname{id} A_1, \mathscr{T}\text{-}\operatorname{id} A_3\}$ *with equality if* $\mathscr{T}\text{-}\operatorname{id} A_1 \neq \mathscr{T}\text{-}\operatorname{id} A_3 + 1$.

(1.2) $\mathscr{T}\text{-}\operatorname{id} A_1 \leq \max\{\mathscr{T}\text{-}\operatorname{id} A_2, \mathscr{T}\text{-}\operatorname{id} A_3 + 1\}$ *with equality if* $\mathscr{T}\text{-}\operatorname{id} A_2 \neq \mathscr{T}\text{-}\operatorname{id} A_3$.

(1.3) $\mathscr{T}\text{-}\operatorname{id} A_3 \leq \max\{\mathscr{T}\text{-}\operatorname{id} A_1 - 1, \mathscr{T}\text{-}\operatorname{id} A_2\}$ *with equality if* $\mathscr{T}\text{-}\operatorname{id} A_1 \neq \mathscr{T}\text{-}\operatorname{id} A_2$.

(2) *For any* $N \in \operatorname{Mod} S$ *and* $n \geq 0$*, the following assertions hold.*

(2.1) $\mathscr{T}\text{-id}\, N \leq n$ *if and only if there exists a* $\operatorname{Hom}_S(-, \mathscr{T})$*-exact exact sequence*

$$0 \to N \to T^0 \to T^1 \to \cdots \to T^n \to 0$$

in $\operatorname{Mod} S$ *with all* T^i *in* $\mathscr{T}$.

(2.2) *Assume that* $\mathscr{T}$-id $N \leq n$ *and*

$$0 \to N \to T^0 \to L \to 0 \tag{4.2}$$

is an exact sequence in Mod S *with* $T^0 \in \mathscr{T}$. *Then* $\mathscr{T}$-id $L \leq n-1$ *if any of the following conditions is satisfied.*

(i) ${}_RC_S$ *is faithful.*

(ii) $\mathscr{T}$ *is closed under direct summands and the exact sequence* (4.2) *is* $\mathrm{Hom}_S(-, \mathscr{T})$*-exact.*

Proof. It is well known that the class $\mathcal{I}(R)$ is coresolving. The categories $\mathcal{WI}(R)$ and $\mathcal{B}_C(R)$ are coresolving by [11, Proposition 2.5(2)] and [13, Theorem 6.2] respectively. On the other hand, the categories $\mathcal{I}_C(S)$, $\mathcal{WI}_C(S)$ and $\mathcal{A}_C(S)$ are preenveloping in Mod S by [13, Proposition 5.3(3)], [11, Theorem 2.12(2)] and [16, Theorem 3.3(1)] respectively. Now the assertions follow from Theorem 3.10, Propositions 3.13 and 3.14 by putting $\mathscr{H} = \mathcal{I}(R)$, $\mathcal{WI}(R)$, $\mathcal{FI}(R)$ (when R is a left coherent ring) and $\mathcal{B}_C(R)$ respectively, and putting $\mathscr{T} = \{H_* \mid H \in \mathscr{H}\}$. □

For a subcategory $\mathscr{X}$ of Mod R, we write

$$\mathrm{id}_R\, \mathscr{X} := \sup\{\mathrm{id}_R\, X \mid X \in \mathscr{X}\}.$$

For a subcategory $\mathscr{Y}$ of Mod S, we write

$$\mathrm{pd}_S\, \mathscr{Y} := \sup\{\mathrm{pd}_S\, Y \mid Y \in \mathscr{Y}\}.$$

Proposition 4.7.

(1) *Let* $\mathscr{T}$ *be a resolving subcategory of* $\mathcal{A}_C(S)$, *and set* $\mathscr{H} := \{C \otimes_S T \mid T \in \mathscr{T}\}$. *Then*

$$\sup\{\mathscr{H}\text{-}\mathrm{pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R) \textit{ with } \mathscr{H}\text{-}\mathrm{pd}\, M < \infty\} \leq \mathrm{id}_R\, \mathscr{H}.$$

(2) *Let* $\mathscr{H}$ *be a coresolving subcategory of* $\mathcal{B}_C(R)$, *and set* $\mathscr{T} := \{H_* \mid H \in \mathscr{H}\}$. *Then*

$$\sup\{\mathscr{T}\text{-}\mathrm{id}\, N \mid N \in \mathrm{ac}\mathcal{T}(S) \textit{ with } \mathscr{T}\text{-}\mathrm{id}\, N < \infty\} \leq \mathrm{pd}_S\, \mathscr{T}.$$

Proof. (1) Let $\mathrm{id}_R\, \mathscr{H} = n < \infty$ and $M \in \mathrm{c}\mathcal{T}(R)$ with $\mathscr{H}\text{-}\mathrm{pd}\, M = m < \infty$. By Theorem 3.2, $\mathscr{T}\text{-}\mathrm{pd}\, M_* = \mathscr{H}\text{-}\mathrm{pd}\, M = m$. By [17, Lemma 3.1(1)], there exists an exact sequence

$$0 \to T_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M_* \to 0 \tag{4.3}$$

in Mod S with $T_m \in \mathscr{T}$ and all P_i projective. Because $C \otimes_S M_* \cong M$ and $M_* \in {C_S}^{\top}$ by Lemma 2.4, applying the functor $C \otimes_S -$ to the exact sequence (4.3), we get the following exact sequence:

$$0 \to C \otimes_S T_m \to C \otimes_S P_{m-1} \to \cdots \to C \otimes_S P_1 \to C \otimes_S P_0 \to \underbrace{C \otimes_S M_*}_{\cong M} \to 0 \tag{4.4}$$

in Mod R with $C \otimes_S T_m$ and all $C \otimes_S P_i$ being in $\mathscr{H}$. By [13, Theorem 6.4(a)], we have

$$\mathrm{Ext}^j_R(C \otimes_S P_i, C \otimes_S T_m) \cong \mathrm{Ext}^j_S(P_i, T_m) = 0$$

for any $0 \leq i \leq m-1$ and $j \geq 1$.

Suppose $m > n$. Since $\mathrm{id}_R\, C\otimes_S T_m \leq n$ by assumption, applying the functor $\mathrm{Hom}_R(-, C\otimes_S T_m)$ to the exact sequence (4.4) yields

$$\mathrm{Ext}^1_R(K, C\otimes_S T_m) \cong \mathrm{Ext}^m_R(M, C\otimes_S T_m) = 0,$$

where $K = \mathrm{Coker}(C\otimes_S T_m \to C\otimes_S P_{m-1})$. Thus the exact sequence

$$0 \to C\otimes_S T_m \to C\otimes_S P_{m-1} \to K \to 0$$

splits, and hence $K \in \mathcal{P}_C(R)$ by [13, Proposirion 5.1(b)]. Since $\mathcal{P}(S) \subseteq \mathscr{T}$, we have $\mathcal{P}_C(R) \subseteq \mathscr{H}$. Thus $K \in \mathscr{H}$ and $\mathscr{H}$-$\mathrm{pd}\, M \leq m-1$, which is a contradiction. Consequently we conclude that $m \leq n$.

(2) It is dual to the proof of (1), so we omit it. □

The assertions (1) and (2) in the following corollary have been obtained in [25, Proposirion 3.6].

Corollary 4.8. *The following assertions hold.*

(1) $\sup\{\mathcal{F}_C(R)\text{-pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{F}_C(R)\text{-pd}\, M < \infty\} \leq \mathrm{id}_R\, \mathcal{F}_C(R)$.
(2) $\sup\{\mathcal{P}_C(R)\text{-pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{P}_C(R)\text{-pd}\, M < \infty\} \leq \mathrm{id}_R\, \mathcal{P}_C(R)$.
(3) $\sup\{\mathcal{WF}_C(R)\text{-pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{WF}_C(R)\text{-pd}\, M < \infty\} \leq \mathrm{id}_R\, \mathcal{WF}_C(R)$.
(4) $\sup\{\mathcal{B}_C(R)\text{-pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{B}_C(R)\text{-pd}\, M < \infty\} \leq \mathrm{id}_R\, \mathcal{B}_C(R)$.
(5) $\sup\{\mathcal{I}_C(S)\text{-id}\, N \mid N \in \mathrm{ac}\mathcal{T}(S)$ *with* $\mathcal{I}_C(S)\text{-id}\, N < \infty\} \leq \mathrm{pd}_S\, \mathcal{I}_C(S)$.
(6) $\sup\{\mathcal{WI}_C(S)\text{-id}\, N \mid N \in \mathrm{ac}\mathcal{T}(S)$ *with* $\mathcal{WI}_C(S)\text{-id}\, N < \infty\} \leq \mathrm{pd}_S\, \mathcal{WI}_C(S)$.
(7) $\sup\{\mathcal{A}_C(S)\text{-id}\, N \mid N \in \mathrm{ac}\mathcal{T}(S)$ *with* $\mathcal{A}_C(S)\text{-id}\, N < \infty\} \leq \mathrm{pd}_S\, \mathcal{A}_C(S)$.

Proof. The assertions (1)–(4) follow from Proposition 4.7(1) by putting $\mathscr{T} = \mathcal{F}(S)$, $\mathcal{P}(S)$, $\mathcal{WF}(S)$ and $\mathcal{A}_C(S)$ respectively, and putting $\mathscr{H} = \{C\otimes_S T \mid T \in \mathscr{T}\}$.

The assertions (5)–(7) follow from Proposition 4.7(2) by putting $\mathscr{H} = \mathcal{I}(R)$, $\mathcal{WI}(R)$ and $\mathcal{B}_C(R)$ respectively, and putting $\mathscr{T} = \{H_* \mid H \in \mathscr{H}\}$. □

For a module $M \in \mathrm{Mod}\, R$, we use $\mathrm{Add}_R\, M$ (respectively, $\mathrm{add}_R\, M$) to denote the subcategory of $\mathrm{Mod}\, R$ consisting of all direct summands of direct sums of (finite) copies of M.

Corollary 4.9. *For a left noetherian ring R, the following assertions hold.*

(1) $\sup\{\mathcal{F}_C(R)\text{-pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{F}_C(R)\text{-pd}\, M < \infty\} \leq \mathrm{id}_R\, C$.
(2) $\sup\{\mathcal{P}_C(R)\text{-pd}\, M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{P}_C(R)\text{-pd}\, M < \infty\} \leq \mathrm{id}_R\, C$.

Proof. By [25, Proposition 3.4(1)(2)], we have that $\mathcal{F}_C(R)$ consists of modules M such that M is a direct summand of a direct limit of modules in $\mathrm{add}_R\, C$ and $\mathcal{P}_C(R) = \mathrm{Add}_R\, C$. Then

$$\mathrm{id}_R\, C = \mathrm{id}_R\, \mathcal{F}_C(R) = \mathrm{id}_R\, \mathcal{P}_C(R)$$

by [3, Theorem 1.1]. Now the assertions follows from Corollary 4.8(1)(2). □

Let ${}_RC_S = {}_RR_R$. Then $\mathrm{c}\mathcal{T}(R) = \mathrm{Mod}\, R$ by Lemma 2.4. It is clear that $\mathcal{F}_C(R) = \mathcal{F}(R)$ and $\mathcal{P}_C(R) = \mathcal{P}(R)$, and hence $\mathcal{F}_C(R)\text{-pd}\, M = \mathrm{fd}_R\, M$ and $\mathcal{P}_C(R)\text{-pd}\, M = \mathrm{pd}_R\, M$ for any $M \in \mathrm{Mod}\, R$. Then we obtain immediately the following result by putting ${}_RC_S = {}_RR_R$ in Corollary 4.9.

Corollary 4.10. ([25, Corollary 3.7]) *For a left noetherian ring R, the following assertions hold.*

(1) $\sup\{\mathrm{fd}_R M \mid M \in \operatorname{Mod} R$ *with* $\mathrm{fd}_R M < \infty\} \leq \mathrm{id}_R R$.

(2) $\sup\{\mathrm{pd}_R M \mid M \in \operatorname{Mod} R$ *with* $\mathrm{pd}_R M < \infty\} \leq \mathrm{id}_R R$.

Set $(-)^+ := \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. According to Lemmas 2.4 and 2.6, the following lemma is essentially contained in the proof of [16, Proposition 3.2].

Lemma 4.11. *The following assertions hold.*

(1) *A module* $N' \in \mathrm{c}\mathcal{T}(S^{op})$ *if and only if* $N'^+ \in \mathrm{ac}\mathcal{T}(S)$.

(2) *A module* $N \in \mathrm{ac}\mathcal{T}(S)$ *if and only if* $N^+ \in \mathrm{c}\mathcal{T}(S^{op})$.

For further application, we need the following observation.

Proposition 4.12. *The following assertions hold.*

(1) *If* R *is a left noetherian ring, then for any* $N \in \operatorname{Mod} S$*, we have*

$$\mathcal{I}_C(S)\text{-}\operatorname{id} N = \mathcal{F}_C(S^{op})\text{-}\operatorname{pd} N^+.$$

(2) $\sup\{\mathcal{F}_C(S^{op})\text{-pd}\, N' \mid N' \in \mathrm{c}\mathcal{T}(S^{op})$ *with* $\mathcal{F}_C(S^{op})\text{-pd}\, N' < \infty\}$
$\leq \sup\{\mathcal{I}_C(S)\text{-id}\, N \mid N \in \mathrm{ac}\mathcal{T}(S)$ *with* $\mathcal{I}_C(S)\text{-id}\, N < \infty\}$ *with equality if* R *is a left noetherian ring.*

Proof. (1) Let R be a left noetherian ring. Then for any $N \in \operatorname{Mod} S$ and $n \geq 0$, we have

$$\begin{aligned} & \mathcal{I}_C(S)\text{-}\operatorname{id} N = n \\ \Longleftrightarrow\ & N \in \mathcal{A}_C(S) \text{ and } \mathrm{id}_R C \otimes_S N = n \text{ (by [13, Lemma 5.1(c)] and Proposition 4.3)} \\ \Longleftrightarrow\ & N^+ \in \mathcal{B}_C(S^{op}) \text{ and } \mathrm{fd}_{R^{op}} (C \otimes_S N)^+ = n \text{ (by [16, Proposition 3.2] and [8, Theorem 2.2])} \\ \Longleftrightarrow\ & \mathcal{F}_C(S^{op})\text{-}\operatorname{pd} N^+ = n. \text{ (by Proposition 4.1(1))} \end{aligned}$$

(2) Suppose

$$\sup\{\mathcal{I}_C(S)\text{-}\operatorname{id} N \mid N \in \mathrm{ac}\mathcal{T}(S) \text{ with } \mathcal{I}_C(S)\text{-}\operatorname{id} N < \infty\} = n < \infty.$$

Let $N' \in \mathrm{c}\mathcal{T}(S^{op})$ with $\mathcal{F}_C(S^{op})\text{-pd}\, N' < \infty$. Then $N'^+ \in \mathrm{ac}\mathcal{T}(S)$ and $\mathcal{I}_C(S)\text{-id}\, N'^+ = \mathcal{F}_C(S^{op})\text{-pd}\, N' < \infty$ by Lemma 4.11(1) and [17, Theorem 4.17(1)] respectively, and hence $\mathcal{F}_C(S^{op})\text{-pd}\, N' \leq n$. The first assertion follows.

Suppose that R is a left noetherian ring and

$$\sup\{\mathcal{F}_C(S^{op})\text{-}\operatorname{pd} N' \mid N' \in \mathrm{c}\mathcal{T}(S^{op}) \text{ with } \mathcal{F}_C(S^{op})\text{-}\operatorname{pd} N' < \infty\} = n < \infty.$$

Let $N \in \mathrm{ac}\mathcal{T}(S)$ with $\mathcal{I}_C(S)\text{-id}\, N < \infty$. Then $N^+ \in \mathrm{c}\mathcal{T}(S^{op})$ and $\mathcal{F}_C(S^{op})\text{-pd}\, N^+ = \mathcal{I}_C(S)\text{-id}\, N < \infty$ by Lemma 4.11(2) and the assertion (1) respectively, and hence $\mathcal{I}_C(S)\text{-id}\, N \leq n$. The proof is finished. □

As a consequence, we obtain the following corollary.

Corollary 4.13. *For a left noetherian ring* R*, the following assertions hold.*

(1) $\sup\{\mathcal{I}_C(S)\text{-id}_R N \mid N \in \mathrm{ac}\mathcal{T}(S)$ *with* $\mathcal{I}_C(S)\text{-id}_S N < \infty\} \leq \mathrm{id}_{S^{op}} C$.

(2) $\sup\{\mathrm{id}_R M \mid M \in \operatorname{Mod} R$ *with* $\mathrm{id}_R M < \infty\} \leq \mathrm{id}_{R^{op}} R$.

Proof. (1) It follows from Proposition 4.12(2) and the symmetric version of Corollary 4.9(1).

(2) Let ${}_RC_S = {}_RR_R$. Then $\mathrm{ac}\mathcal{T}(S) = \mathrm{Mod}\, S(= \mathrm{Mod}\, R)$ by Lemma 2.6. It is clear that $\mathcal{I}_C(S) = \mathcal{I}(S)(= \mathcal{I}(R))$ and $\mathcal{I}_C(S)$-id $M = \mathrm{id}_S\, M(= \mathrm{id}_R\, M)$ for any $M \in \mathrm{Mod}\, S(= \mathrm{Mod}\, R)$. Thus the assertion follows from (1). □

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