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On the Flatness and Injectivity of Dual Modules (II) *

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Abstract: For a commutative ring R and an injective cogenerator E in the category of R-modules, we characterize QF rings, IF rings and semihereditary rings by using the properties of the dual modules with respect to E.

Key words: QF rings; flatness; injectivity; dual modules.

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1. Introduction

Throughout this paper, R will denote an associate, commutative ring with identity and all modules are unital. E always denotes a certain injective cogenerator in the category of R-modules.

Let M be an R-module. In [4] we introduce the notion of the dual module $\mathrm{Hom}_R(M,E)$ with respect to E, and denote it by M^e. It is shown that the flatness of M^e is equivalent to the FP-injectivity or the injectivity of M if and only if R is a coherent ring or a noether ring respectively. The FP-injectivity, the injectivity of M and the projectivity of M^e are pairwisely equivalent if and only if R is an artin ring (see [4]).

Recall that R is called a QF ring if R is an artin ring and for each ideal I of R, $I = 0 :_R (0 :_R I)$ (see [5]). Such rings have been extensively studied, many properties equivalent to this definition have been obtained. For example, the following statements are equivalent:

(1) R is a QF ring;

(2) R is a noether ring and for each ideal I of R, $I = 0 :_R (0 :_R I)$;

(3) R is an artin ring (or a noether ring) and R is a cogenerator in the category of R-modules;

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(4) R is an artin ring (or a noether ring) and R is selfinjective;

(5) Any projective R-module is injective;

(6) Any injective R-module is projective.

Also recall that R is called an IF ring if every injective R-module is flat (see [7]). It is shown that R is an IF ring if and only if R is a coherent ring and R is self FP-injective. It is clear that the notion of IF rings is a generalization of that of QF rings and Von Neumann regular rings.

In this paper we will introduce the notions of E-artin rings, E-coherent rings and f-cogenerators, and characterize QF rings, IF rings and semihereditary rings by using the properties of dual modules.

2. Main results

Proposition 1 *The following statements are equivalent.*

(1) R is a QF ring;

(2) R is a noether ring and R^e is flat;

(3) R is an artin ring and R^e is flat;

(4) M^e is a projective module for any flat module M;

(4)′ M^e is a projective module for any projective module M;

(4)″ M^e is a projective module for any free module M;

(5) M^e is a submodule of a projective module for any flat module M;

(5)′ M^e is a submodule of a projective module for any projective module M;

(5)″ M^e is a submodule of a projective module for any free module M.

Proof (2) $\Leftrightarrow$ (1) Suppose that R is a noether ring and R^e is flat. Then R is an FP-injective R-module by [4, Corollary 2]. So R is selfinjective and hence R is a QF ring. The converse implication is trivial.

(1) $\Rightarrow$ (3) Suppose R is a QF ring. Then R is an artin ring and any injective R-module is projective. Because $R^e \cong E$ is injective, R^e is projective.

(3) $\Rightarrow$ (4) Suppose M is a flat module. Then there is a free module $R^{(I)}$ where I is a set such that $R^{(I)} \to M \to 0$ is exact. Since M is flat, this exact sequence is pure. So $0 \to M^e \to [R^{(I)}]^e \cong (R^e)^I$ is exact and splits by [4, Lemma 1]. It follows that M^e is a direct summand of $(R^e)^I$. Since R is an artin ring and R^e is flat, R^e is projective. So $(R^e)^I$ is also projective, it follows that M^e is projective.

(4) $\Rightarrow$ (4)′ $\Rightarrow$ (4)″ $\Rightarrow$ (5)″ and (4) $\Rightarrow$ (5) $\Rightarrow$ (5)′ $\Rightarrow$ (5)″ are trivial.

(5)″ $\Rightarrow$ (1) Suppose M is an injective R-module. There is a free module F such that $F \to M^e \to 0$ is exact, so $0 \to M^{ee} \to F^e$ is exact. By (5)″ F^e is a submodule of a projective module P. It is known [4, Corollary 1] that M is a submodule of M^{ee}, we get that M is isomorphic to a submodule of P. So M is projective and R is a QF ring. □

We know from Proposition 1 that R is a QF ring if and only if P^e is (a submodule of) a projective module for any projective module P. It is natural to ask what properties R possesses if Q^e is (a submodule of) a projective module for any injective module Q. [4, Theorem 3] says that R is an artin ring if and only if Q^e is projective for any injective module Q.

Definition 1 *R is called an E-artin ring if Q^e is a submodule of a projective module for any injective module Q.*

Remark 1 An artin ring is an E-artin ring by [4, Theorem 3]. If R is a hereditary ring, then R is artinian if and only if R is E-artinian.

Recall that an R-module A is called FP-injective if $\mathrm{Ext}^1_R(B, A) = 0$ for any finitely presented R-module B. A ring R is called self FP-injective if R is FP-injective as an R-module (see [7]).

Lemma 1 *Let R be an E-artin ring. Then the direct product of a family of projective modules can be embedded in a projective module.*

Proof Suppose $\{P_i\}_{i\in I}$ is a family of projective modules where I is a set. Then each P_i^e is injective. By [3, Corollary 2.1.12] $\bigoplus_{i\in I} P_i^e$ is FP-injective, which implies that the exact sequence $0 \to \bigoplus_{i\in I} P_i^e \to \prod_{i\in I} P_i^e$ is pure, and so $(\prod_{i\in I} P_i^e)^e \to (\bigoplus_{i\in I} P_i^e)^e \cong \prod_{i\in I} P_i^{ee} \to 0$ splits. It follows that $\prod_{i\in I} P_i^{ee}$ is a direct summand of $(\prod_{i\in I} P_i^e)^e$. Because $\prod_{i\in I} P_i^e$ is injective and R is an E-artin ring, $(\prod_{i\in I} P_i^e)^e$ is a submodule of a projective module P. Since $\prod_{i\in I} P_i$ is a submodule of $\prod_{i\in I} P_i^{ee}$, $\prod_{i\in I} P_i$ can be embedded in P. □

Lemma 2 *The following statements are equivalent.*

(1) R is an E-artin ring;

(2) M^e is a submodule of a projective module for any FP-injective module M;

(3) $\mathrm{Hom}_R(B, C)$ is a submodule of a projective module for any injective module (or FP-injective module) B and any injective module C;

(4) P^{ee} is a submodule of a projective module for any flat module P;

(4)′ P^{ee} is a submodule of a projective module for any projective module P;

(4)″ P^{ee} is a submodule of a projective module for any free module P.

Proof (1) $\Rightarrow$ (2) Suppose M is an FP-injective module. Then the exact sequence $0 \to M \to E(M)$ is pure where $E(M)$ is the injective envelope of M. It follows from [4, Lemma 1] that M^e is a direct summand of $[E(M)]^e$. We know from (1) and Definition 1 that M^e is a submodule of a projective module.

(2) $\Rightarrow$ (3) Suppose B is an FP-injective module and C is an injective module. Since E is an injective cogenerator in the category of R-modules, C is a direct summand of E^I for some set I. So $\mathrm{Hom}_R(B, C)$ is a direct summand of $\mathrm{Hom}_R(B, E^I) \cong (B^e)^I$. Since B^e is a submodule of a projective module P_1 by (2), $(B^e)^I$ is a submodule of P_1^I. It is known from Lemma 1 that P_1^I can be embedded in some projective module, and we are done.

(3) $\Rightarrow$ (4) If P is a flat module, then P^e is injective. By (3) P^{ee} is a submodule of a projective module.

(4) $\Rightarrow$ (4)′ $\Rightarrow$ (4)″ are trivial.

(4)″ $\Rightarrow$ (1) For any injective module Q, Q is a direct summand of E^I for some set I, so Q^e is a direct summand of $(E^I)^e \cong [R^{(I)}]^{ee}$. By (4)″ Q^e is a submodule of a projective module, it follows that R is an E-artin ring. □

Remark 2 Suppose both E and E' are injective cogenerators in the category of R-modules. By Lemma 2, R is an E-artin ring if and only if R is an E'- artin ring.

Definition 2 *An R-module C is called an f-cogenerator in the category of R-modules if C cogenerates every finitely presented R-module.*

Lemma 3 *Let C be an f-cogenerator in the category of R-modules. Then*

$$0:_R(0:_C I) = I$$

for any finitely generated ideal I of R.

Proof It is clear that $I \subseteq 0:_R(0:_C I)$. We only need to prove that

$$0:_R(0:_C I) \subseteq I.$$

Let $s \in R - I$. Since C is an f-cogenerator and R/I is finitely presented, C cogenerates R/I. Then we have a nonzero homomorphism $h: R/I \to C$ such that $h(s+I) \neq 0$. Suppose that $g: R \to R/I$ is the natural epimorphism. Then

$$0 = hg(I) = hg(1)I.$$

So $hg(1) \in 0:_C I$. But

$$hg(1)s = hg(s) = h(s+I) \neq 0.$$

It follows that $s \notin 0:_R(0:_C I)$. So

$$0:_R(0:_C I) \subseteq I. \qquad \square$$

For any R-modules M and N, recall from [1, p. 109] that

$$\mathrm{Rej}_M(N) = \bigcap\{\mathrm{Ker} f \mid f \in \mathrm{Hom}_R(M,N)\}.$$

Lemma 4 *Let R be a coherent ring. Then R is an f-cogenerator in the category of R-modules if and only if R is self FP-injective.*

Proof ($\Rightarrow$) Suppose that both I_1 and I_2 are finitely generated ideals of R. Since R is a coherent ring, it follows from [3, Theorem 2.3.2] that $I_1 \cap I_2$, $0:_R I_1$ and $0:_R I_2$ are finitely generated ideals of R. Then from Lemma 3 we get that

$$\begin{aligned} 0:_R(0:_R(I_1 \cap I_2)) = I_1 \cap I_2 &= [0:_R(0:_R I_1)] \cap [0:_R(0:_R I_2)] \\ &= 0:_R(0:_R I_1 + 0:_R I_2). \end{aligned}$$

So

$$\begin{aligned} 0:_R(I_1 \cap I_2) = 0:_R[0:_R(0:_R(I_1 \cap I_2))] &= 0:_R[0:_R(0:_R I_1 + 0:_R I_2)] \\ &= 0:_R I_1 + 0:_R I_2. \end{aligned}$$

By [5, Theorem 1] R is self FP-injective.

($\Leftarrow$) Let M be a finitely presented R-module and let $0 \neq x \in M$. We claim that there is a nonzero homomorphism $h: Rx \to R$ with $h(x) \neq 0$. Otherwise, suppose $(Rx)^* = \mathrm{Hom}_R(Rx, R) = 0$. Since Rx is finitely presented, there is an exact sequence

$F_1 \to F_0 \to Rx \to 0$ with F_0 and F_1 finitely generated free modules. Then $0 \to F_0^* \to F_1^* \to A \to 0$ is exact where A =Coker$(F_0^* \to F_1^*)$. Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} F_1 & \longrightarrow & F_0 & \longrightarrow & Rx & \longrightarrow 0 \\ \downarrow \sigma_{F_1} & & \downarrow \sigma_{F_0} & & \downarrow \varphi & \\ F_1^{**} & \longrightarrow & F_0^{**} & \longrightarrow & Ext_R^1(A,R) & \longrightarrow 0 \end{array}$$

where σ_{F_0}, σ_{F_1} are the canonical evaluation homomorphisms, φ is an induced homomorphism. It is known that $\sigma_{F_0}, \sigma_{F_1}$ are isomorphisms, so φ is also an isomorphism and hence $\text{Ext}_R^1(A,R) \cong Rx \neq 0$, which contradicts that R is self FP-injective since A is finitely presented.

Since Rx and M are finitely presented, a nonzero homomorphism $h: Rx \to R$ can be extended to a homomorphism $\bar{h}: M \to R$ with $\bar{h}(x) = h(x) \neq 0$. Thus $\text{Rej}_M(R) = 0$, and R cogenerates M by [1, Corollary 8.13]. The proof is finished. □

We now in a position to give the main result.

Theorem 1 *The following statements are equivalent.*

(1) R is a QF ring;

(2) R is an artin ring and R^e is flat;

(2)′ R is a noether ring and R^e is flat;

(3) M^e is a projective module for any free (projective, flat) module M;

(3)′ M^e is a submodule of a projective module for any free (projective, flat) module M;

(4) R is an E-artin ring and R is self FP-injective;

(5) R is an artin ring and R is an f-cogenerator in the category of R-modules;

(5)′ R is a noether ring and R is an f-cogenerator in the category of R-modules;

(6) R is an artin ring and some injective cogenerator is flat;

(6)′ R is a noether ring and some injective cogenerator is flat;

(7) R is an artin ring and $E(R/m)$ is flat for each $m \in \text{Max}(R)$, where $\text{Max}(R)$ is the maximal spectrum of R;

(7)′ R is a noether ring and $E(R/m)$ is flat for each $m \in \text{Max}(R)$.

Proof (1) ⇔ (2) ⇔ (2)′ ⇔ (3) ⇔ (3)′ See Proposition 1.

(1) ⇔ (5) ⇔ (5)′ follow easily from Lemma 4.

(1) ⇒ (4), (1) ⇒ (6) ⇒ (6)′ ⇒ (7)′ and (1) ⇒ (7) ⇒ (7)′ are trivial.

(4) ⇒ (1) Suppose that Q is an injective module. We know that Q is a submodule of E^I for some set I. Since R is self FP-injective, $R^{(I)}$ is FP-injective. By Lemma 2 E^I is a submodule of a projective module P because $E^I \cong [R^{(I)}]^e$. It follows that Q is a submodule of P. Hence Q is projective and then R is a QF ring.

(7)′ ⇒ (1) Suppose that R is a noether ring and $E(R/m)$ is flat for each $m \in \text{Max}(R)$. Let $E_1 = \bigoplus_{m\in \text{Max}(R)} E(R/m)$. Then E_1 is flat. It follows from [8, Theorem 9.51] that $\text{Hom}_R(\text{Ext}_R^1(R/I,R), E_1) \cong \text{Tor}_1^R(\text{Hom}_R(R,E_1), R/I)$ for any ideal I of R. Since

$$\text{Tor}_1^R(\text{Hom}_R(R,E_1), R/I) \cong \text{Tor}_1^R(E_1, R/I) = 0,$$

$$\mathrm{Hom}_R(\mathrm{Ext}^1_R(R/I,R),E_1)=0.$$

It is known [1, Corollary 18.16] that E_1 is an injective cogenerator in the category of R-modules, so $\mathrm{Ext}^1_R(R/I,R)=0$ and hence R is selfinjective and R is a QF ring. □

Definition 3 *R is called an E-coherent ring if Q^e is a submodule of a flat module for any injective module Q.*

Remark 3 By [4, Theorem 1], a coherent ring is an E-coherent ring. If R is a semihereditary ring, then R is coherent if and only if R is E-coherent.

Remark 4 We can get similar conclusions about E-coherent rings to that about E-artin rings in Lemmas 1 and 2, which we omit.

Theorem 2 *The following statements are equivalent.*

(1) R is an IF ring;

(2) R is a coherent ring and R^e is flat;

(3) M^e is a flat module for any free (projective, flat) module M;

(3)′ M^e is a submodule of a flat module for any free (projective, flat) module M;

(4) R is an E-coherent ring and R is self FP-injective;

(5) R is a coherent ring and R is an f-cogenerator in the category of R-modules;

(6) R is a coherent ring and some injective cogenerator is flat;

(7) R is a coherent ring and $E(R/m)$ is flat for each $m\in\mathrm{Max}(R)$.

Proof The proof is similar to that of Theorem 1, and is omitted here. □

Theorem 3 *Consider the following conditions.*

(1) R is a semihereditary ring;

(2) M^e is an FP-injective module for any finitely presented module M;

(3) M^{ee} is a flat module for any finitely presented module M;

(4) M^{eee} is an FP-injective module for any finitely presented module M.

In general (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). If R is self FP-injective, then the above conditions are equivalent.

Proof (2) $\Rightarrow$ (1) Suppose K is a finitely generated submodule of a projective module. Then K is a submodule of some finitely generated free module R^n. So we have an exact sequence $0\to K\to R^n\to M\to 0$ with M finitely presented, and hence $0\to M^e\to (R^n)^e\to K^e\to 0$ is exact. Since M^e is FP-injective by (2), the latter exact sequence is pure. So it splits by [4, Lemma 1]. Because $(R^n)^e\cong E^n$ is injective, M^e is also injective. Then M is flat by [6, Theorem 1.4]. It follows from [8, Theorem 3.57] that $0\to K\to R^n\to M\to 0$ splits. Thus K is projective and R is a semihereditary ring.

(2) $\Rightarrow$ (3) If (2) holds, then R is a semihereditary ring. So R is a coherent ring. If M is a finitely presented module, then M^e is FP-injective by (2). It follows from [4, Theorem 1] that M^{ee} is flat.

(3) $\Rightarrow$ (4) It follows from [6, Theorem 1.4].

(4) $\Rightarrow$ (2) Suppose M is a finitely presented module. Then M^{eee} is FP-injective by (4). Since M^e is a direct summand of M^{eee} by [9, Exercise 23, p. 46], M^e is FP-injective.

Now let R be self FP-injective. We will show that (1) implies (2).

Let R be a semihereditary ring. Since R is self FP-injective, R is an IF ring. Suppose that M is a finitely presented module. Then there is an exact sequence $0 \to K \to F \to M \to 0$ with K a finitely generated module and F a finitely generated projective module. K^c is flat by Theorem 2, so the exact sequence $0 \to M^c \to F^c \to K^c \to 0$ is pure. It follows that M^c is a pure submodule of the injective module F^c. So M^c is FP-injective. This completes the proof of this theorem. □

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关于对偶模的平坦性和内射性 (II)

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摘 要: 对交换环 R 和 R- 模范畴上的一个内射余生成元 E, 我们用相对于 E 的对偶模的性质刻画了 QF 环, IF 环和半遗传环.

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An Extension and a Correction Concerning Raney's Lemma *

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Abstract: We give an extension of Raney's lemma and correct a generalization of Raney's lemma in R.L.Graham et al's Concrete Mathematics.

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1. An extension of Raney's lemma

Consider a sequence $< a_1, a_2, \cdots, a_m >$ of real numbers with $\sum_{i=1}^m a_i > 0$.

We arrange $< a_1, a_2, \cdots, a_m >$ on a circle in clockwise direction, and let $(a_1, a_2, \cdots, a_m)$ denote this circle arrangement of length m. For given $a_i, i = 1, 2, \cdots, m$, define $a_{i_1} = a_i, a_{i_2} = a_{i+1}, \cdots, a_{i_m} = a_{i-1}$ with $a_j = a_k$ if $j \equiv k (\mathrm{mod} m)$. If $\sum_{j=1}^k a_{i_j} > 0$ for all $k, k = 1, 2, \cdots, m$, we call a_i an initial point of $(a_1, a_2, \cdots, a_m)$.

Now, we prove the existence of initial point in $(a_1, a_2, \cdots, a_m)$ by induction on m.

If $m = 1$, a_1 is an initial point.

For given $(a_1, a_2, \cdots, a_m, a_{m+1})$ of length $m+1$, if $a_i \geq 0, i = 1, 2, \cdots, m+1$, since $\sum_{i=1}^{m+1} a_i > 0$, if there exists $a_k > 0, a_k$ is an initial point; If there exists $a_i < 0$, we consider the following algorithm.

If $\langle a_{k_1}, a_{k_2}, \cdots, a_{k_l} \rangle$ satisfies

$$a_{k_1} a_{k_m} < 0, \quad a_{k_1} a_{k_{l+1}} < 0, \quad a_{k_1} a_{k_j} \geq 0$$

for all $j = 1, 2, \cdots, l$, then we ignore the sequence structure of $\langle a_{k_1}, a_{k_2}, \cdots, a_{k_l} \rangle$ while regarding it as a big point with value $\sum_{j=1}^l a_{k_j}$. The circle arrangement can be partitioned into the union of such subsequences. Denote these big points by $A_1, A_2, \cdots$, beginning from any chosen big point A_1. So, we obtain a circle arrangement $(A_1, A_2, \cdots)$ with $A_{i_1} A_{i_2} < 0$. Since

$$\sum A_i = \sum_{j=1}^{m+1} a_j > 0,$$

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there exist two consecutive big points A_{k_1}, A_{k_2} such that

$$A_{k_1} > 0, \quad A_{k_1} + A_{k_2} > 0.$$

Regarding $\langle A_{k_1}, A_{k_2}\rangle$ as a new big point B with value $A_{k_1} + A_{k_2}$ and replacing $\langle A_{k_1}, A_{k_2}\rangle$ by B in $(A_1, A_2, \cdots)$, we obtain a new circle arrangement of length $\leq m$. There is an initial point in this new circle arrangement by induction assumption. Obviously, if $A_i \neq B$ is an initial point, then the first element of the subsequence expressed by A_i is an initial point of $(a_1, a_2, \cdots, a_{m+1})$; if B is an initial point, then the first element of the subsequence expressed by A_{k_1} is an initial point of $(a_1, a_2, \cdots, a_{m+1})$.

Summing up the above discussion, we obtain the following

Theorem 1 *There exists an initial point in circle arrangement $(a_1, a_2, \cdots, a_m)$ of real numbers with $\sum_{i=1}^{m} a_i > 0$.*

If $a_i's$ $(i = 1, 2, \cdots, m)$ are integers with $\sum_{i=1}^{m} a_i > 0$, and a_{k_1}, a_{k_s} $(s > 1)$ are two initial points in $(a_1, a_2, \cdots, a_m)$, then

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{s-1} a_{k_j} + \sum_{j=s}^{m} a_{k_j} \geq 1 + 1 = 2,$$

so,there exists only one initial point in $(a_1, a_2, \cdots, a_m)$ of integers with $\sum_{i=1}^{m} a_i = 1$, viz., exactly one of the cyclic shifts

$$\langle a_1, a_2, \cdots, a_m\rangle, \langle a_2, \cdots, a_m, a_1\rangle, \cdots, \langle a_m, a_1, \cdots, a_{m-1}\rangle$$

has all of its partial sums positive.This is the conclusion of Raney's lemma (see[1]). Hence, Theorem 1 can be regarded as an extension of Raney's lemma.

Remark We point out that Theorem 1 can be extented to the setting of ordered semigroup, the details omitted here.

2. A correction of a generalization of Raney's lemma

Consider circle arrangement $(a_1, a_2, \cdots, a_m)$ of integers with $a_i \leq 1$ for all i, and $\sum_{i=1}^{m} a_i = l > 0$.

Theorem 1 tells us that there exist initial points in $(a_1, a_2, \cdots, a_m)$, if a_{r_1}, a_{r_s} $(s > 1)$ are two consecutive initial points, that is, a_{r_k} $(1 < k < s)$ is not initial point,we assert that $\sum_{i=1}^{s-1} a_{r_i} = 1$. Otherwise, $\sum_{i=1}^{s-1} a_{r_i} > 1$. Since $a_{r_1} = 1, \sum_{i=2}^{s-1} a_{r_i} \geq 1$. Now, let

$$S = \{k | \sum_{j=2}^{k} a_{r_j} = 0, \quad \text{and} \quad 2 \leq k < s - 1\},$$

$$h = \max S + 1, \ \text{ if } \ S \neq \emptyset; \ = 2, \ \text{ if } \ S = \emptyset.$$

Obviously, a_{r_h} is an initial point, contradicting the consecutivity of a_{r_1} and a_{r_s} (since $1 < h < s$). Hence, $\sum_{i=1}^{s-1} a_{r_i} = 1$. Since $\sum_{i=1}^{m} a_i = l$, there are exactly l initial points in $(a_1, a_2, \cdots, a_m)$.

Untying $(a_1, a_2, \cdots, a_m)$ at a_i, we obtain a line arrangement or sequence

$$\langle a_{i_1}, a_{i_2}, \cdots, a_{i_m} \rangle.$$

Let $p = \min\{q | a_{i_1} = a_{i_{1+q}} \text{ for all } i = 1, 2, \cdots . m\}$, then

$$(a_1, a_2, \cdots, a_m) = (a_1, \cdots, a_p, a_1, \cdots, a_p, \cdots, a_1 . \cdots, a_p)$$

consists of $\frac{m}{p}$ sequences $\langle a_1, a_2, \cdots, a_p \rangle$, l initial points in $(a_1, a_2, \cdots, a_m)$ produce $\frac{l}{\frac{m}{p}} = \frac{lp}{m}$ different sequences.

Summing up the above discussion, we have the following

Theorem 2 *If $(a_1, a_2, \cdots, a_m)$ is any circle arrangement of integers with $a_i \le 1$ for all i, and with $\sum_{i=1}^{m} a_i = l > 0$, then there are exactly l initial points, but exactly $\frac{lp}{m}$ of the cyclic shifts*

$$\langle a_1, a_2, \cdots, a_m \rangle, \langle a_2, \cdots, a_m, a_1 \rangle, \cdots, \langle a_m, a_1, \cdots, a_{m-1} \rangle$$

have all positive partial sums.

This is a correction of the generalization of Raney's lemma (see [1]) which says: If $\langle x_1, x_2, \cdots, x_m \rangle$ is any sequence of integers with $x_i \le 1$ for all j, and with $x_1 + x_2 + \cdots + x_m = l \ge 0$, then exactly l of the cyclic shifts

$$\langle x_1, x_2, \cdots, x_m \rangle, \langle x_2, \cdots, x_m, x_1 \rangle, \cdots, \langle x_m, x_1, \cdots, x_{m-1} \rangle$$

have all positive partial sums.

For example, for given sequence $\langle -2, 1, 1, 1, -2, 1, 1, 1 \rangle, m = 8, l = 2, p = 4$, there is exactly $\frac{2 \times 4}{8} = 1$, but not two, cyclic shift $\langle 1, 1, 1, -2, 1, 1, 1, -2 \rangle$ which has all partial sums positive. Of course, there are 2 initial points in the circle arrangement $(-2, 1, 1, 1, -2, 1, 1, 1)$.

References:

[1] GRAHAM R L, KNUTH D F, PATASHNIK O. *Concrete Mathematics* [M]. Addison-Wesley Publishing Company, 1992, 345, 348.

关于 Raney 引理的修正与扩展

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摘 要: 本文对 Raney 引理进行了扩展，并对 R.L.Graham 等人的著作 Concrete Mathematics 中涉及的一个广义 Raney 引理进行了修正.