TENSOR PRODUCTS OF STRONG COTILTING MODULES AND COHEN–MACAULAY ALGEBRAS

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Abstract. Let Λ and Γ be finite-dimensional algebras over a field k, and let U be a Λ-module and V a Γ-module. We prove that both U and V are strong cotilting modules if and only if $U \otimes_k V$ is a strong cotilting $\Lambda \otimes_k \Gamma$-module. As a consequence, if both Λ and Γ are Cohen–Macaulay algebras, then so is $\Lambda \otimes_k \Gamma$, and the converse is false in general. It answers affirmatively an open question posed by Auslander and Reiten. On the other hand, another open question posed by Auslander and Reiten is: for a Cohen–Macaulay algebra with finitistic dimension d, is it Iwanaga–Gorenstein if the category of Cohen–Macaulay modules and that of d-syzygy modules coincide? It was answered negatively by Chan, Iyama and Marczinzik recently, but we prove that a Cohen–Macaulay algebra with finitistic dimension d is Iwanaga–Gorenstein if and only if the category of Cohen–Macaulay modules and that of $(d+1)$-syzygy modules coincide. Finally, we prove that for Cohen–Macaulay algebras Λ and Γ, if $\Lambda \otimes_k \Gamma$ is CM-finite (respectively, CM-free), then so are both Λ and Γ, and the converse is false in general. In particular, a conjecture posed by Hu, Luo, Xiong and Zhou is disproved.

1. Introduction

Cohen-Macaulay rings and Gorenstein rings, as well as Cohen-Macaulay modules, whose importance is well established in commutative Noetherian ring theory, were extended to artin algebras by Auslander and Reiten in [2], and were developed further by them in [3]. The following definition is motivated by the fact that a commutative Notherian complete local ring is Cohen–Macaulay if and only if it has a dualizing module.

Definition 1.1. ([3]) An artin algebra Λ is *Cohen–Macaulay* if there is a (Λ, Λ)-bimodule W such that

$$- \otimes_\Lambda W : \mathcal{P}^{<\infty}(\Lambda) \to \mathcal{I}^{<\infty}(\Lambda) \quad \text{and} \quad \mathrm{Hom}_\Lambda(W, -) : \mathcal{I}^{<\infty}(\Lambda) \to \mathcal{P}^{<\infty}(\Lambda)$$

are quasi-inverse equivalences, where $\mathcal{P}^{<\infty}(\Lambda)$ and $\mathcal{I}^{<\infty}(\Lambda)$ are the categories of finitely generated Λ-modules with finite projective and injective dimensions, respectively. Such a W is called a *dualizing bimodule*. The category of *Cohen–Macaulay modules* with respect to W is defined as

$$\mathrm{CM}_W(\Lambda) := \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^{\geq 1}_\Lambda(M, W) = 0\}.$$

The algebra Λ is *Iwanaga-Gorenstein* if ${}_\Lambda\Lambda_\Lambda$ is a dualizing bimodule, which is equivalent to that the left and right self-injective dimensions of Λ are finite.

In the following, both Λ and Γ are finite-dimensional k-algebras with k a field. It is clear that Iwanaga–Gorenstein algebras are Cohen–Macaulay. It was proved that two algebras Λ and Γ are Iwanaga–Gorenstein if and only if so is their tensor product $\Lambda \otimes_k \Gamma$ [3, Proposition 2.2(b)]. On the other hand, if Λ is a Gorenstein algebra with left and right self-injective dimensions being d, then the category of Cohen–Macaulay modules and that of d-syzygy modules coincide [3, Proposition 3.1(b)]. Based on these observations, Auslander and Reiten posed the following two questions.

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Question 1.2. ([3, p.229 and p.231])

(1) Is the tensor product of Cohen–Macaulay algebras again Cohen–Macaulay?
(2) Let Λ be a Cohen–Macaulay algebra with finitistic injective dimension d. If the category of Cohen–Macaulay modules and that of d-syzygy modules coincide, is then Λ Iwanaga–Gorenstein?

Hu, Luo, Xiong and Zhou proved in that if k a splitting field for Λ or Γ, and if either Λ or Γ is Gorenstein, then both Λ and Γ are Cohen–Macauly free if and only if so is $\Lambda \otimes_k \Gamma$ [17, Corollary 4.8]. They conjectured that this result holds true in general.

Conjecture 1.3. ([17, p.1033]) *Both Λ and Γ are Cohen–Macauly free if and only if so is $\Lambda \otimes_k \Gamma$.*

Our aims are to study Question 1.2 and Conjecture 1.3.

The paper is organized as follows. In Section 2, we give some terminology and preliminary results. In particular, we extend the top criterion for determining a module to be strong cotilting from the basic case described in [8, Proposition 0.2] to the general case (Proposition 2.2), which plays a crucial role in the sequel.

Following Definition 1.1, it is clear that Λ is Cohen–Macaulay if and only if it has a dualizing bimodule. For a (Λ, Λ)-bimodule W, define

$$\lambda_W : \Lambda \to \mathrm{End}(W_\Lambda) \quad \text{via} \quad \lambda_W(a)(w) = aw$$

and

$$\rho_W : \Lambda^{\mathrm{op}} \to \mathrm{End}({}_\Lambda W) \quad \text{via} \quad \rho_W(a^{op})(w) = wa$$

for any $a \in \Lambda$ and $w \in W$. The bimodule W is *faithfully balanced* if both maps are isomorphisms. In [2], the so-called *strong cotilting modules* are introduced; see Definition 2.1 for the definition. It is known from [3, Proposition 1.3] that W is dualizing if and only if it is strong cotilting on both sides and faithfully balanced. In Section 3, we prove the following result.

Theorem 1.4. (Theorems 3.3 and 3.6)

(1) For any $U \in \mathrm{mod}\,\Lambda$ and $V \in \mathrm{mod}\,\Gamma$, both U and V are strong cotilting modules if and only if $U \otimes_k V$ is a strong cotilting module over $\Lambda \otimes_k \Gamma$.
(2) Both Λ and Γ have strong cotilting modules if and only if so has $\Lambda \otimes_k \Gamma$; more precisely, if U_Λ and V_Γ are basic strong cotilting modules, then any strong cotilting $(\Lambda \otimes_k \Gamma)$-module W satisfies $\mathrm{add}\, W = \mathrm{add}(U \otimes_k V)$.

In this result, assertion (2) does not follow directly from assertion (1), because a strong cotilting module over $\Lambda \otimes_k \Gamma$ need not be of tensor-product form. Let K/k be a field extension. We prove that a Λ-module U is strong cotilting if and only if $U \otimes_k K$ is strong cotilting $\Lambda \otimes_k K$-module (Theorem 3.10). If K/k is finite, then it follows directly from Theorem 1.4(1). When K/k is infinite, we prove this result separately.

In Section 4, we prove the following result based on Theorem 1.4.

Theorem 1.5. (Theorems 4.1 and 4.7 and Corollary 4.2)

(1) For bimodules ${}_\Lambda U_\Lambda$ and ${}_\Gamma V_\Gamma$, both of them are dualizing if and only if $U \otimes_k V$ is dualizing for $\Lambda \otimes_k \Gamma$. In this case, for any $0 \neq M \in \mathrm{mod}\,\Lambda$ and $0 \neq N \in \mathrm{mod}\,\Gamma$, it holds that both M and N are Cohen–Macaulay if and only if $M \otimes_k N$ is a Cohen–Macaulay $\Lambda \otimes_k \Gamma$-module.
(2) If both Λ and Γ are Cohen–Macauly algebras, then so is $\Lambda \otimes_k \Gamma$.

Auslander and Reiten proved in [2, Proposition 6.8] that if ${}_\Lambda U_\Lambda$ is dualizing, then $\begin{pmatrix} U & 0 \\ U & U \end{pmatrix} \left(\cong U \otimes_k \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} \right)$ is dualizing for $\begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix} \left(\cong \Lambda \otimes_k \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} \right)$. Theorem

1.5(1) generalizes this result, and Theorem 1.5(2) answers Question 1.2(1) affirmatively[1]. We give an example to illustrate that the converse of Theorem 1.5(2) does not hold true, even for k to be a split basic field (Example 4.4). However, this converse holds true if either of Λ or Γ is a central simple algebra (Proposition 4.5).

Though Question 1.2(2) was answered negatively by Chan, Iyama and Marczinzik in [8, Proposition 4.8 and Example 4.9], somehow surprisingly, in this question, if "d-syzygy" is replaced by "$(d+1)$-syzygy", then the answer is positive.

Theorem 1.6. (Theorem 4.9) *Let Λ be a Cohen–Macaulay algebra with a dualizing bimodule W, and let* $\operatorname{fidim}\Lambda = d$. *Then the following statements are equivalent.*

(1) Λ is Iwanaga–Gorenstein.
(2) W is a $(d+1)$-syzygy module.
(3) The category of Cohen–Macaulay modules and that of $(d+1)$-syzygy modules coincide.
(4) For some $n \geq d+1$, the category of Cohen–Macaulay modules and that of n-syzygy modules coincide.

In Section 5, we introduce the notions for Cohen–Macaulay algebras to be relative Cohen–Macaulay finite and Cohen–Macaulay free, respectively; see Definition 5.1(2). When the algebra is Gorenstein, relative Cohen–Macaulay finite and Cohen–Macaulay free algebras are exactly Cohen–Macaulay finite and Cohen–Macaulay free algebras, respectively. We prove the following result.

Theorem 1.7. (Theorems 5.2 and 5.7) *Let Λ and Γ be algebras.*

(1) If $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay finite, then so are both Λ and Γ.
(2) Suppose that both Λ and Γ are Cohen–Macaulay algebras.
  (2.1) If $\Lambda \otimes_k \Gamma$ is relative Cohen–Macaulay finite (respectively, Cohen–Macaulay free), then so are both Λ and Γ.
  (2.2) Over a perfect field k, $\Lambda \otimes_k \Gamma$ is relative CM-free if and only if so are both Λ and Γ.

Moreover, we give examples to illustrate that the converse of both assertions in Theorem 1.7 does not hold true in general; see Examples 5.3 and 5.4. In particular, the latter example disproves Conjecture 1.3.

2. Preliminaries

Throughout this paper, all algebras are finite-dimensional algebras over a field k, all modules are finitely generated right modules unless otherwise stated, and write $D := \operatorname{Hom}_k(-, k)$. For an algebra Λ, we use $Z(\Lambda)$, $\operatorname{rad}\Lambda$, $\operatorname{gldim}\Lambda$ and $\Lambda^e := \Lambda \otimes_k \Lambda^{\mathrm{op}}$ to denote the center, radical, global dimension and enveloping algebra of Λ, respectively, and use $\operatorname{mod}\Lambda$ to denote the category of finitely generated right Λ-modules. We use $D^b(\operatorname{mod}\Lambda)$ to denote the bounded derived category of $\operatorname{mod}\Lambda$ and use $X[i]$ to denote the ith shift of a complex X. We write $\operatorname{proj}\Lambda$ for the full subcategory of $\operatorname{mod}\Lambda$ consisting of projective modules, and write $K^b(\operatorname{proj}\Lambda)$ for the homotopy category of bounded complexes of projective modules. For an object $X \in K^b(\operatorname{proj}\Lambda)$, denote by $\operatorname{thick}(X)$ the smallest full triangulated subcategory which contains X and is closed under direct summands.

Let $M \in \operatorname{mod}\Lambda$. We use $\operatorname{pd}_\Lambda M$ and $\operatorname{id}_\Lambda M$ to denote the projective and injective dimensions of M, respectively, and use $\operatorname{rad} M$, $\operatorname{soc} M$ and $\operatorname{top} M$ to denote the radical, socle and top of M, respectively. Recall from [13] that a module in $\operatorname{mod}\Lambda$ is *Gorenstein projective* if it is a cycle in an exact complex of finitely generated projective right Λ-modules which remains exact

[1]Note that Chan, Iyama and Marczinzik claimed in the abstract of [8] that the answer to Question 1.2(1) is positive; however, in the main body of their paper there is neither a statement of this claim nor any relevant proof, instead gave some specific examples below [8, Proposition 0.2]. In fact, we have not found these in the available literature.

after applying the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$. We write $\mathrm{Gproj}\,\Lambda$ for the subcategory of $\mathrm{mod}\,\Lambda$ consisting of Gorenstein projective modules, and write ${}^{\perp}M := \{A \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}_\Lambda^{\geq 1}(A, M) = 0\}$. We use

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to \cdots$$

to denote a minimal injective coresolution of M. The algebra Λ is *Iwanaga–Gorenstein* if $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda < \infty$. For any $n \geq 1$, the algebra Λ is *Auslander n-Gorenstein* if $\mathrm{pd}_\Lambda E^i(\Lambda) \leq i$ for any $0 \leq i \leq n-1$, which is left–right symmetric [14, Theorem 3.7]; and it is *quasi n-Gorenstein* if $\mathrm{pd}_\Lambda E^i(\Lambda) \leq i+1$ for any $0 \leq i \leq n-1$ [18]. More generally, following [19, Definition 3.3], we say that Λ is $G_n(m)$ with $m \geq 0$ if $\mathrm{pd}_\Lambda E^i(\Lambda) \leq i+m$ for any $0 \leq i \leq n-1$. With this convention, $G_n(0)$ is precisely Auslander n-Gorenstein, while $G_n(1)$ is quasi n-Gorenstein.

We use $\mathrm{add}\,M$ to denote the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of direct summands of direct sums of finite copies of M. Two modules M and N in $\mathrm{mod}\,\Lambda$ are *additively equivalent* if $\mathrm{add}\,M = \mathrm{add}\,N$. A module is *basic* if its indecomposable direct summands are pairwise non-isomorphic. Thus each module has, up to isomorphism, a unique basic module in its additive-equivalence class. A module $G \in \mathrm{mod}\,\Lambda$ is a *generator* if $\Lambda \in \mathrm{add}\,G$; and it is a *projective generator* if it is also projective. For modules $P, Q \in \mathrm{proj}\,\Lambda$, a morphism $f : P \to Q$ is called *radical* if it belongs to the radical of the additive category $\mathrm{proj}\,\Lambda$; equivalently,

$$\mathrm{Im}\, f \subseteq Q\,\mathrm{rad}\,\Lambda = \mathrm{rad}\,Q;$$

see [6, Chapter I, Section 2]. In particular, a radical morphism with nonzero codomain cannot be a split epimorphism. For any $n \geq 1$, we use $\Omega_\Lambda^n M$ to denote the nth syzygy of M, in particular, set $\Omega_\Lambda^0 M := M$. We write $\Omega^n(\mathrm{mod}\,\Lambda) := \mathrm{add}\{\Omega_\Lambda^n M \mid M \in \mathrm{mod}\,\Lambda\}$.

For a subcategory $\mathcal{X}$ of $\mathrm{mod}\,\Lambda$, we use $\widehat{\mathcal{X}}$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of all modules M for which there exists an exact sequence

$$0 \to X_n \to \cdots \to X_0 \to M \to 0$$

in $\mathrm{mod}\,\Lambda$ with $n \geq 0$ and all X_i in $\mathcal{X}$. A full subcategory of $\mathrm{mod}\,\Lambda$ is *resolving* if it contains $\mathrm{proj}\,\Lambda$ and is closed under direct summands, extensions and kernels of epimorphisms. Dually, a *coresolving* subcategory contains all injective modules and is closed under direct summands, extensions and cokernels of monomorphisms.

We write

$$\mathcal{P}^{<\infty}(\Lambda) := \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{pd}_\Lambda X < \infty\}, \qquad \mathcal{I}^{<\infty}(\Lambda) := \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{id}_\Lambda X < \infty\},$$

and

$$\mathrm{fidim}\,\Lambda := \sup\{\mathrm{id}_\Lambda X \mid X \in \mathrm{mod}\,\Lambda \text{ with } \mathrm{id}_\Lambda X < \infty\}.$$

We say that k is a *splitting field* for a finite-dimensional k-algebra Λ if $\mathrm{End}(S_\Lambda) \cong k$ for any simple Λ-module S; equivalently, $\Lambda/\mathrm{rad}\,\Lambda$ is a direct product of full matrix algebras over k [17, Lemma 2.3(2)]. The algebra Λ is *basic* if

$$\Lambda/\mathrm{rad}\,\Lambda \cong \Upsilon_1 \times \cdots \times \Upsilon_m$$

with all Υ_i division k-algebras, and it is *split basic* if all Υ_i are equal to k, or equivalently if $\Lambda/\mathrm{rad}\,\Lambda \cong k^m$. Thus a basic algebra is split basic precisely when k is a splitting field for it.

Definition 2.1. ([2, 3]) Let $U \in \mathrm{mod}\,\Lambda$.

(1) The module U is *cotilting* if the following conditions are satisfied.
  (C1) $\mathrm{id}_\Lambda U < \infty$;
  (C2) $\mathrm{Ext}_\Lambda^{\geq 1}(U, U) = 0$;
  (C3) $D\Lambda \in \widehat{\mathrm{add}\,U}$.

  A cotilting module U is *strong*, or Ext-*maximal*, if $\widehat{\mathrm{add}\,U} = \mathcal{I}^{<\infty}(\Lambda)$.

(2) The module U is *tilting* if the following conditions are satisfied.

(T1) $\mathrm{pd}_\Lambda U < \infty$;
(T2) $\mathrm{Ext}_\Lambda^{\geq 1}(U, U) = 0$;
(T3) There exists an exact sequence

$$0 \to \Lambda \to U^0 \to \cdots \to U^s \to 0$$

in $\mathrm{mod}\,\Lambda$ with $s \geq 0$ and all U^i in $\mathrm{add}\,U$.

On additive-equivalence classes of cotilting modules, we use the order

$$[U] \geq [V] \quad \Longleftrightarrow \quad \mathrm{Ext}_\Lambda^{\geq 1}(U, V) = 0.$$

The equivalence of Definition 2.1 with maximality, and the fact that a maximal element is the maximum element, follow from the dual of [21, Theorem 3.1]. The following top criterion plays a crucial role in the sequel.

Proposition 2.2. *Let U be a cotilting module in $\mathrm{mod}\,\Lambda$ and put $\Gamma := \mathrm{End}(U_\Lambda)$. Then U_Λ is strong if and only if any simple module in $\mathrm{mod}\,\Gamma^{op}$ occurs in $\mathrm{top}_\Gamma U := U/((\mathrm{rad}\,\Gamma)U)$.*

Proof. For a basic cotilting module, this is [8, Proposition 1.2]; it is the cotilting counterpart of [11, Proposition 7.1]. Let U^{b} be a basic representative of U, and let $e \in \Gamma$ correspond to the summand U^{b}. Since $\mathrm{add}\,U = \mathrm{add}\,U^{\mathrm{b}}$, the idempotent e is full. Hence

$$e\Gamma e \cong \mathrm{End}(U^{\mathrm{b}}{}_\Lambda), \qquad eU \cong U^{\mathrm{b}},$$

and the Morita equivalence $e(-)$ bijects simple modules. Since e is full, it also preserves radicals:

$$e((\mathrm{rad}\,\Gamma)U) = (\mathrm{rad}(e\Gamma e))(eU).$$

Consequently, we have $e(\mathrm{top}_\Gamma U) \cong \mathrm{top}_{e\Gamma e}(eU)$. Strongness is invariant under additive equivalence. The basic case therefore gives the assertion. □

The next Künneth formulas are cited from [7, Chapter XI, Theorem 3.1].

Lemma 2.3. *For any $M, M' \in \mathrm{mod}\,\Lambda$ and $N, N' \in \mathrm{mod}\,\Gamma$, there are natural isomorphisms*

$$\mathrm{Hom}_{\Lambda\otimes_k\Gamma}(M \otimes_k N, M' \otimes_k N') \cong \mathrm{Hom}_\Lambda(M, M') \otimes_k \mathrm{Hom}_\Gamma(N, N'),$$

$$\mathrm{Ext}^r_{\Lambda\otimes_k\Gamma}(M \otimes_k N, M' \otimes_k N') \cong \bigoplus_{p+q=r} \mathrm{Ext}^p_\Lambda(M, M') \otimes_k \mathrm{Ext}^q_\Gamma(N, N').$$

If $X \in \mathrm{mod}\,\Lambda$ and $Y \in \mathrm{mod}\,\Gamma$, then

$$\mathrm{id}_{\Lambda\otimes_k\Gamma}(X \otimes_k Y) = \mathrm{id}_\Lambda X + \mathrm{id}_\Gamma Y.$$

The analogous equality holds for projective dimension of modules.

We shall repeatedly use restriction of scalars. Let

$$\mathrm{Res}_\Lambda^{\Lambda\otimes_k\Gamma} : \mathrm{mod}\,\Lambda \otimes_k \Gamma \longrightarrow \mathrm{mod}\,\Lambda$$

be restriction along the homomorphism $\Lambda \to \Lambda \otimes_k \Gamma$ via $a \mapsto a \otimes 1$ for any $a \in \Lambda$. Thus, for any $Z \in \mathrm{mod}\,\Lambda \otimes_k \Gamma$, the restriction $\mathrm{Res}_\Lambda^{\Lambda\otimes_k\Gamma} Z$ has the same underlying vector space as Z, and its right Λ-action is $z \cdot a := z(a \otimes 1)$. We use the analogous notation for restriction to Γ.

Lemma 2.4. *With the notation above, the following assertions hold and the same statements hold after interchanging Λ and Γ.*

(1) *The functor $\mathrm{Res}_\Lambda^{\Lambda\otimes_k\Gamma}$ is exact and faithful. It preserves finite direct sums and direct summands. In particular, it sends nonzero modules to nonzero modules.*
(2) *It sends projective $\Lambda \otimes_k \Gamma$-modules to projective Λ-modules and injective $\Lambda \otimes_k \Gamma$-modules to injective Λ-modules. Consequently, we have*

$$\mathrm{pd}_\Lambda \mathrm{Res}_\Lambda^{\Lambda\otimes_k\Gamma} Z \leq \mathrm{pd}_{\Lambda\otimes_k\Gamma} Z \quad \text{and} \quad \mathrm{id}_\Lambda \mathrm{Res}_\Lambda^{\Lambda\otimes_k\Gamma} Z \leq \mathrm{id}_{\Lambda\otimes_k\Gamma} Z$$

for any $Z \in \mathrm{mod}\,\Lambda \otimes_k \Gamma$.

(3) *If $X \in \operatorname{mod}\Lambda$ and Y is a finite-dimensional k-vector space, then*

$$\operatorname{Res}_{\Lambda}^{\Lambda\otimes_k\Gamma}(X \otimes_k Y) \cong X^{\dim_k Y}$$

whenever $X \otimes_k Y$ is given a $\Lambda \otimes_k \Gamma$-module structure whose restricted Λ-action is on the first factor. In particular, this applies when $Y \in \operatorname{mod}\Gamma$.

(4) *For any $r \geq 0$, we have*

$$\operatorname{Res}_{\Lambda}^{\Lambda\otimes_k\Gamma}(\Omega^r_{\Lambda\otimes_k\Gamma}(\operatorname{mod}\Lambda \otimes_k \Gamma)) \subseteq \Omega^r_{\Lambda}(\operatorname{mod}\Lambda).$$

Assertion (1) remains valid without a finite-dimensionality assumption on the second tensor factor. The argument for assertion (3) also applies to an arbitrary k-vector space Y: if Ξ is a basis of Y, then the right-hand side in part (3) is the direct sum $X^{(\Xi)}$.

Proof. Restriction does not change the underlying vector spaces or linear maps, thus kernels and cokernels are unchanged, which proves exactness and faithfulness. The assertions about finite direct sums, summands, and nonzero modules follow immediately.

Choosing a k-basis of Γ, as a right Λ-module, we have $\Lambda \otimes_k \Gamma \cong \Lambda^{\dim_k \Gamma}$. By assertion (1), the restriction sends projective to projectives. Moreover,

$$\operatorname{Res}_{\Lambda}^{\Lambda\otimes_k\Gamma} D(\Lambda \otimes_k \Gamma) \cong \operatorname{Res}_{\Lambda}^{\Lambda\otimes_k\Gamma}(D\Lambda \otimes_k D\Gamma) \cong (D\Lambda)^{\dim_k \Gamma}.$$

Every injective $\Lambda \otimes_k \Gamma$-module is a direct summand of a direct sum of finite copies of $D(\Lambda \otimes_k \Gamma)$. Thus the restriction of injectives are injectives. Restricting finite projective or injective resolutions now gives the two dimension inequalities in assertion (2).

If $y_1, \ldots, y_s$ is a basis of Y, the map

$$X^s \to \operatorname{Res}_{\Lambda}^{\Lambda\otimes_k\Gamma}(X \otimes_k Y) \ \text{ via } \ (x_1, \ldots, x_s) \mapsto \sum_{j=1}^{s} x_j \otimes y_j$$

is a Λ-module isomorphism. This proves assertion (3).

Finally, restrict the first r steps of a projective resolution of a $\Lambda \otimes_k \Gamma$-module. Assertion (2) shows that the resulting terms are projective over Λ, and exactness shows that its rth kernel is the restriction of the original rth kernel. Since restriction also preserves finite direct sums and direct summands, the definition of the additive syzygy category gives assertion (4). $\square$

The following two lemmas are useful in studying Question 1.2(1).

Lemma 2.5. *For any self-injective algebra Σ, each faithful finite-dimensional left Σ-module L is a generator.*

Proof. Choose a k-basis $\ell_1, \ldots, \ell_s$ of L. Since L is faithful, the map

$$\Sigma \longrightarrow L^s \ \text{ via } \ r \longmapsto (r\ell_1, \ldots, r\ell_s)$$

is injective. Since ${}_\Sigma\Sigma$ is injective, this monomorphism splits, and thus ${}_\Sigma\Sigma \in \operatorname{add} L$. $\square$

Lemma 2.6. *Let Δ and Θ be algebras. Suppose that a semisimple left Δ-module L and a semisimple left Θ-module P contain every simple module over their respective algebras. Then every simple left $\Delta \otimes_k \Theta$-module is a quotient of a finite direct sum of copies of $L \otimes_k P$.*

Proof. Put $\overline{\Delta} := \Delta/\operatorname{rad}\Delta$, $\overline{\Theta} := \Theta/\operatorname{rad}\Theta$ and $\Sigma := \overline{\Delta} \otimes_k \overline{\Theta}$. Since L contains every simple left Δ-module, it is faithful over $\overline{\Delta}$; similarly, P is faithful over $\overline{\Theta}$. Thus the action maps

$$\overline{\Delta} \longrightarrow \operatorname{End}_k(L) \ \text{ and } \ \overline{\Theta} \longrightarrow \operatorname{End}_k(P)$$

are injective. Tensoring these maps and using $\operatorname{End}_k(L) \otimes_k \operatorname{End}_k(P) \cong \operatorname{End}_k(L \otimes_k P)$ shows that $L \otimes_k P$ is faithful over Σ.

Moreover, we have

$$D(\overline{\Delta} \otimes_k \overline{\Theta}) \cong D\overline{\Delta} \otimes_k D\overline{\Theta} \cong \overline{\Delta} \otimes_k \overline{\Theta}$$

as left modules. Thus Σ is self-injective, even though it need not be semisimple. By Lemma 2.5, we have that $L \otimes_k P$ is a Σ-generator.

Put

$$I := (\operatorname{rad}\Delta) \otimes_k \Theta + \Delta \otimes_k (\operatorname{rad}\Theta)$$

and choose $r, s \geq 1$ with $(\operatorname{rad}\Delta)^r = 0$ and $(\operatorname{rad}\Theta)^s = 0$. Then $I^{r+s-1} = 0$. Thus I is nilpotent and therefore $I \subseteq \operatorname{rad}(\Delta \otimes_k \Theta)$. Moreover, we have $(\Delta \otimes_k \Theta)/I \cong \Sigma$. Consequently every simple $\Delta \otimes_k \Theta$-module is annihilated by I and is a simple Σ-module. Since $L \otimes_k P$ is a Σ-generator, every such simple is a quotient of a finite direct sum of copies of $L \otimes_k P$. $\square$

We also need the following observation.

Proposition 2.7. *If U is a strong cotilting Λ-module, then*

$$\operatorname{fidim}\Lambda = \operatorname{id}_\Lambda U.$$

Proof. Let $\operatorname{id}_\Lambda U = d$ and let $X \in \mathcal{I}^{<\infty}(\Lambda)$. Strongness gives an exact sequence

$$0 \to U_r \to \cdots \to U_0 \to X \to 0$$

in $\operatorname{mod}\Lambda$ with all U_i in $\operatorname{add} U$. After applying the functor D yields a finite coresolution of DX by modules of projective dimension at most d, and hence $\operatorname{id}_\Lambda X = \operatorname{pd}_{\Lambda^{\mathrm{op}}} DX \leq d$. Thus $\operatorname{fidim}\Lambda \leq d = \operatorname{id}_\Lambda U$. The reverse inequality holds because $U \in \mathcal{I}^{<\infty}(\Lambda)$. $\square$

3. Tensor products of strong cotilting modules

We begin with the following lemma.

Lemma 3.1. *Let $M \in \operatorname{mod}\Lambda$ with $\operatorname{pd}_\Lambda M < \infty$ and $\operatorname{Ext}^{\geq 1}_\Lambda(M, M) = 0$. Then M is tilting if and only if $\Lambda \in \operatorname{thick}(M)$ in $K^b(\operatorname{proj}\Lambda)$.*

Proof. Suppose that M is tilting. Divide a finite exact sequence

$$0 \to \Lambda \to M_0 \to \cdots \to M_s \to 0$$

in $\operatorname{mod}\Lambda$ with all M_i in $\operatorname{add} M$ into short exact sequences. The associated triangles show successively that $\Lambda \in \operatorname{thick}(M)$.

Conversely, assume that $\Lambda \in \operatorname{thick}(M)$ and put $\Delta := \operatorname{End}(M_\Lambda)$. The functor $\Phi := \mathbf{R}\operatorname{Hom}_\Lambda(-, M)$ induces a contravariant triangle equivalence

$$\operatorname{thick}(M)^{\mathrm{op}} \to K^b(\operatorname{proj}\Delta^{op}).$$

Indeed, on $\operatorname{add} M$ this is the usual equivalence

$$\operatorname{Hom}_\Lambda(-, M) : (\operatorname{add} M)^{\mathrm{op}} \to \operatorname{proj}\Delta^{op}.$$

Note that $\Phi(\Lambda) = \operatorname{Hom}_\Lambda(\Lambda, M)$ is concentrated in degree zero. The assumption $\Lambda \in \operatorname{thick}(M)$ shows that the left Δ-module $\operatorname{Hom}_\Lambda(\Lambda, M)$ has finite projective dimension. Choose an exact sequence

$$0 \to Q_s \to \cdots \to Q_0 \to \operatorname{Hom}_\Lambda(\Lambda, M) \to 0$$

in $\operatorname{mod}\Delta^{\mathrm{op}}$ with all Q_i projective. For each i, choose $M_i \in \operatorname{add} M$ such that $Q_i \cong \operatorname{Hom}_\Lambda(M_i, M)$. The equivalence Φ gives an exact sequence

$$0 \to \Lambda \to M_0 \to \cdots \to M_s \to 0.$$

This implies that M is tilting. $\square$

We first prove the tensor-product property for cotilting modules, in which, the sufficiency is the cotilting counterpart of [10, Theorem 1.1].

Proposition 3.2. *Let $U \in \operatorname{mod}\Lambda$ and $V \in \operatorname{mod}\Gamma$. Then $U \otimes_k V$ is cotilting over $\Lambda \otimes_k \Gamma$ if and only if U and V are cotilting over Λ and Γ, respectively. In this case, we have*

$$\operatorname{id}_{\Lambda\otimes_k\Gamma}(U \otimes_k V) = \operatorname{id}_\Lambda U + \operatorname{id}_\Gamma V.$$

Proof. Suppose first that U and V are cotilting over Λ and Γ, respectively. Lemma 2.3 gives the dimension formula and $\mathrm{Ext}^{\geq 1}_{\Lambda\otimes_k\Gamma}(U\otimes_k V, U\otimes_k V)=0$. Tensoring finite $\mathrm{add}\,U$- and $\mathrm{add}\,V$-resolutions of $D\Lambda$ and $D\Gamma$ gives a finite $\mathrm{add}(U\otimes_k V)$-resolution of $D\Lambda\otimes D\Gamma\cong D(\Lambda\otimes_k\Gamma)$. This proves the sufficiency.

Conversely, suppose that $U\otimes_k V$ is cotilting over $\Lambda\otimes_k\Gamma$. It follows from Lemma 2.4(2)(3) that

$$\mathrm{id}_\Lambda U=\mathrm{id}_\Lambda \mathrm{Res}^{\Lambda\otimes_k\Gamma}_\Lambda(U\otimes_k V)\leq \mathrm{id}_{\Lambda\otimes_k\Gamma}(U\otimes_k V)<\infty,$$

where $\mathrm{Res}^{\Lambda\otimes_k\Gamma}_\Lambda(U\otimes_k V)\cong U^{\dim_k V}$ and $\dim_k V\geq 1$. Similarly, we get $\mathrm{id}_\Gamma V<\infty$.

For any $i\geq 1$, the Ext-Künneth decomposition contains the direct summand $\mathrm{Ext}^i_\Lambda(U,U)\otimes_k \mathrm{End}_\Gamma(V)$, thus the self-orthogonality of $U\otimes_k V$ implies that of U. Similarly, we get that V is self-orthogonal.

It remains to verify the cogenerator axiom. Restrict a finite $\mathrm{add}(U\otimes_k V)$-resolution of $D(\Lambda\otimes_k\Gamma)$ to $D\Lambda$. By Lemma 2.4(1)(3), it becomes a finite $\mathrm{add}\,U$-resolution of $\mathrm{Res}^{\Lambda\otimes_k\Gamma}_\Lambda D(\Lambda\otimes_k\Gamma)\cong (D\Lambda)^{\dim_k\Gamma}$. After applying the functor D, the associated triangles yield $\Lambda^{\dim_k\Gamma}\in\mathrm{thick}(DU)$ in $K^b(\mathrm{proj}\,\Lambda^{\mathrm{op}})$. Since this subcategory is closed under direct summands, we have $\Lambda\in\mathrm{thick}(DU)$. Furthermore, we have

$$\mathrm{pd}_{\Lambda^{\mathrm{op}}} DU=\mathrm{id}_\Lambda U<\infty$$

and

$$\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(DU,DU)\cong \mathrm{Ext}^i_\Lambda(U,U)=0$$

for any $i\geq 1$. Lemma 3.1 shows that DU is a tilting Λ^{op}-module. Dualizing its tilting coresolution of Λ gives $D\Lambda\in\widehat{\mathrm{add}\,U}$. Thus U is cotilting. Similarly, we get that V is cotilting. □

We now prove the first main theorem.

Theorem 3.3. *Let $U\in\mathrm{mod}\,\Lambda$ and $V\in\mathrm{mod}\,\Gamma$. Then $U\otimes_k V$ is strong cotilting over $\Lambda\otimes_k\Gamma$ if and only if U and V are strong cotilting over Λ and Γ, respectively.*

Proof. By Proposition 3.2, we have that $U\otimes_k V$ is cotilting if and only if both U and V are cotilting. We may therefore compare strongness under this common cotilting assumption. Put

$$\Delta:=\mathrm{End}(U_\Lambda),\ \Theta:=\mathrm{End}(V_\Gamma)\ \text{ and }\ \Pi:=\mathrm{End}((U\otimes_k V)_{\Lambda\otimes_k\Gamma}).$$

Lemma 2.3 identifies Π with $\Delta\otimes_k\Theta$.

Assume first that U and V are strong. It follows from Proposition 2.2 that $L:=\mathrm{top}_\Delta U$ and $P:=\mathrm{top}_\Theta V$ contain all simple left modules over Δ and Θ, respectively. Lemma 2.6 shows that every simple left Π-module is a quotient of $(L\otimes_k P)^s$ for some s. The canonical epimorphisms $U\to L$ and $V\to P$ tensor to a Π-epimorphism $U\otimes_k V\to L\otimes_k P$. Hence every simple left Π-module occurs in $\mathrm{top}_\Pi(U\otimes_k V)$, and therefore $U\otimes_k V$ is strong by Proposition 2.2.

Conversely, assume that $U\otimes_k V$ is strong. Let S be a simple left Δ-module and choose a simple left Θ-module Q. The Π-module $S\otimes_k Q$ has a simple quotient Z. By Lemma 2.4(1)(3), restriction to Δ gives an epimorphism

$$S^{\dim_k Q}\cong \mathrm{Res}_\Delta(S\otimes_k Q)\to\mathrm{Res}_\Delta Z\to 0,$$

and $\mathrm{Res}_\Delta Z\neq 0$. Since the source is semisimple, we have $\mathrm{Res}_\Delta Z\cong S^a$ for some $a\geq 1$. Strongness and Proposition 2.2 give a Π-epimorphism

$$(U\otimes_k V)^r\to Z.$$

Lemma 2.4(1) and (3), followed by a projection $S^a\to S$, yields an epimorphism

$$U^{r\dim_k V}\to S.$$

One component $U \to S$ is nonzero and hence surjective. Thus S occurs in $\operatorname{top}_\Delta U$. Since S is arbitrary, we have that U is strong by Proposition 2.2. Similarly, we get that V is strong. □

Corollary 3.4. *If Λ and Γ have strong cotilting modules, then*

$$\operatorname{fidim}(\Lambda \otimes_k \Gamma) = \operatorname{fidim} \Lambda + \operatorname{fidim} \Gamma.$$

Proof. It follows from Theorem 3.3, Propositions 2.7 and 3.2. □

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} \Lambda$. A morphism $f : M \to X$ with $X \in \mathcal{X}$ is a *left $\mathcal{X}$-approximation* of M if every morphism from M to a module in $\mathcal{X}$ factors through f. The subcategory $\mathcal{X}$ is *covariantly finite* if every module has such a left approximation. *Right approximations* and *contravariantly finite subcategories* are defined dually; see [2].

Lemma 3.5. *If $\mathcal{I}^{<\infty}(\Lambda \otimes_k \Gamma)$ is covariantly finite, then both $\mathcal{I}^{<\infty}(\Lambda)$ and $\mathcal{I}^{<\infty}(\Gamma)$ are also covariantly finite.*

Proof. Suppose that $\mathcal{I}^{<\infty}(\Lambda \otimes_k \Gamma)$ is covariantly finite. We now prove that $\mathcal{I}^{<\infty}(\Lambda)$ is covariantly finite. To do it, let $M \in \operatorname{mod} \Lambda$. We first consider a functor

$$F_\Lambda : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda \otimes_k \Gamma$$

where $F_\Lambda(X) := X \otimes_k D\Gamma$ for any $X \in \operatorname{mod} \Lambda$. Choose a left $\mathcal{I}^{<\infty}(\Lambda \otimes_k \Gamma)$-approximation $f : F_\Lambda(M) \to Y$. Now choose $0 \neq \eta \in D\Gamma$ and $\lambda \in D(D\Gamma)$ such that $\lambda(\eta) = 1$. There are natural Λ-linear maps

$$\iota_X : X \to \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} F_\Lambda(X) \text{ via } x \mapsto x \otimes \eta,$$

$$\pi_X : \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} F_\Lambda(X) \to X \text{ via } x \otimes \varphi \mapsto \lambda(\varphi) x,$$

and $\pi_X \iota_X = 1_X$. By considering the composition of $\iota_M : M \to \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} F_\Lambda(M)$ and $\operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} f : \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} F_\Lambda(M) \to \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} Y$, we get a morphism $g : M \to \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} Y$. By Lemma 2.4(2)(3), we have

$$\operatorname{id}_\Lambda \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma} Y \leq \operatorname{id}_{\Lambda \otimes_k \Gamma} Y < \infty.$$

In the following, we show that g is a left $\mathcal{I}^{<\infty}(\Lambda)$-approximation of M. Indeed, for any $h \in \operatorname{Hom}_\Lambda(M, X)$ with $X \in \mathcal{I}^{<\infty}(\Lambda)$, we have a morphism $h \otimes 1_{D\Gamma} : F_\Lambda(M) \to F_\Lambda(X)$. Moreover, by Lemma 2.3, we have $\operatorname{id}_{\Lambda \otimes_k \Gamma} F_\Lambda(X) = \operatorname{id}_\Lambda X < \infty$. Thus there exists a morphism $q : Y \to F_\Lambda(X)$ such that $h \otimes 1_{D\Gamma} = qf$. It follows that

$$\begin{aligned}
\pi_X \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma}(q)\, g &= \pi_X \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma}(q) \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma}(f)\, \iota_M \\
&= \pi_X \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma}(qf)\, \iota_M \\
&= \pi_X \operatorname{Res}_\Lambda^{\Lambda \otimes_k \Gamma}(h \otimes 1_{D\Gamma})\, \iota_M \\
&= \pi_X\, \iota_X\, h \qquad \text{(by the naturality of } \iota) \\
&= h.
\end{aligned}$$

Thus g is a left $\mathcal{I}^{<\infty}(\Lambda)$-approximation, and therefore $\mathcal{I}^{<\infty}(\Lambda)$ is covariantly finite. Similarly, we get that $\mathcal{I}^{<\infty}(\Gamma)$ is covariantly finite. □

Compare the following result with Theorem 3.3.

Theorem 3.6. *The algebra $\Lambda \otimes_k \Gamma$ has a strong cotilting module if and only if both Λ and Γ have strong cotilting modules. More precisely, if U_Λ and V_Γ are basic strong cotilting modules, then every strong cotilting $(\Lambda \otimes_k \Gamma)$-module W satisfies*

$$\operatorname{add} W = \operatorname{add}(U \otimes_k V).$$

Thus a basic representative of W is isomorphic to a basic representative of $U \otimes_k V$. Moreover,

$$\operatorname{id}_{\Lambda \otimes_k \Gamma} W = \operatorname{id}_\Lambda U + \operatorname{id}_\Gamma V.$$

Proof. The covariant Auslander–Reiten correspondence states that an algebra Σ has a strong cotilting module if and only if $\mathcal{I}^{<\infty}(\Sigma)$ is covariantly finite [2, Theorem 5.5 and Proposition 6.3]. Suppose that $\Lambda \otimes_k \Gamma$ has a strong cotilting module. Then both $\mathcal{I}^{<\infty}(\Lambda)$ and $\mathcal{I}^{<\infty}(\Gamma)$ are covariantly finite by Lemma 3.5, and so both Λ and Γ have strong cotilting modules.

Conversely, suppose that U_Λ and V_Γ are strong. Then $U \otimes_k V$ is strong by Theorem 3.3. A strong cotilting module is the maximum of the cotilting poset. Thus its additive-equivalence class is unique by the dual of [21, Theorem 3.1], and therefore any strong cotilting module W over $\Lambda \otimes \Gamma$ has the stated additive closure. The last equality follows from Proposition 3.2. □

The assertion of Theorem 3.6 must be formulated up to additive equivalence.

Example 3.7. Let $k := \mathbb{R}$ and let

$$\mathbb{H} := \mathbb{R}\langle \mathbf{i}, \mathbf{j} \rangle / (\mathbf{i}^2 + 1, \mathbf{j}^2 + 1, \mathbf{ij} + \mathbf{ji})$$

be the Hamilton quaternion algebra. It has the real basis

$$1, \quad \mathbf{i}, \quad \mathbf{j}, \quad \mathbf{k} = \mathbf{ij}.$$

Note that $\mathbb{H}$ is a central simple $\mathbb{R}$-algebra with the center $Z(\mathbb{H}) = \mathbb{R}$; see [28, Section 1.1 and Corollary 7.1.2].

Put $\Lambda = \Gamma := \mathbb{H}$. Since $\mathbb{H}$ is a division algebra, we have that $\mathbb{H}_\mathbb{H}$ is the basic strong cotilting module. Since quaternionic conjugation is an anti-involution, it gives an $\mathbb{R}$-algebra isomorphism $\mathbb{H} \cong \mathbb{H}^{\mathrm{op}}$. The canonical map

$$\mu : \mathbb{H} \otimes_\mathbb{R} \mathbb{H}^{\mathrm{op}} \longrightarrow \mathrm{End}_\mathbb{R}(\mathbb{H}) \text{ via } \mu(a \otimes b^{\mathrm{op}})(x) = axb$$

is an isomorphism by the standard central-simple-algebra isomorphism [15, Proposition 2.4.8]. Since $\dim_\mathbb{R} \mathbb{H} = 4$, a choice of the above basis gives $\mathrm{End}_\mathbb{R}(\mathbb{H}) \cong M_4(\mathbb{R})$. Therefore

$$\mathbb{H} \otimes_\mathbb{R} \mathbb{H} \cong \mathbb{H} \otimes_\mathbb{R} \mathbb{H}^{\mathrm{op}} \cong \mathrm{End}_\mathbb{R}(\mathbb{H}) \cong M_4(\mathbb{R}).$$

Let $S = \mathbb{R}^{1\times 4}$ be the row module. The algebra $M_4(\mathbb{R})$ is semisimple and has, up to isomorphism, the unique simple right module S. Hence S is its basic strong cotilting module. Decomposition into matrix rows gives

$$M_4(\mathbb{R}) = \bigoplus_{r=1}^{4} e_{rr} M_4(\mathbb{R}) \cong S^{\oplus 4}.$$

Consequently, we have

$$\mathbb{H} \otimes_\mathbb{R} \mathbb{H} \cong S^{\oplus 4}$$

as right $M_4(\mathbb{R})$-modules.

Since every nonzero finite-dimensional right $\mathbb{H}$-module has real dimension divisible by four, every module $X \otimes_\mathbb{R} Y$ with nonzero right $\mathbb{H}$-modules X and Y has real dimension divisible by 16, and thus S is not of this form. However, we have $\mathrm{add}\, S = \mathrm{add}(\mathbb{H} \otimes_\mathbb{R} \mathbb{H})$ by Theorem 3.6.

For an arbitrary extension K/k, write $\Lambda_K := \Lambda \otimes_k K$ and $M_K := M \otimes_k K$. For any field L, write $D_L := \mathrm{Hom}_L(-, L)$; in particular, $D := D_k$. We now make precise in which sense scalar extension is a special tensor case.

Corollary 3.8. *Let K/k be a finite field extension.*

(1) For any $U \in \mathrm{mod}\, \Lambda$, we have that

$$U \text{ is (strong) cotilting} \quad \Longleftrightarrow \quad U_K \text{ is (strong) cotilting.}$$

In this case, it holds $\mathrm{id}_{\Lambda_K} U_K = \mathrm{id}_\Lambda U$.

(2) The algebra Λ_K has a strong cotilting module if and only if so has Λ. If U_Λ is a basic strong cotilting module, then any strong cotilting Λ_K-module is additively equivalent to U_K.

Proof. Regard K as the second finite-dimensional k-algebra and take its regular module K_K. This module is strong cotilting and has injective dimension zero. Assertion (1) follows from Proposition 3.2 and Theorem 3.3; and assertion (2) follows from Theorem 3.6. $\square$

Lemma 3.9. *If U is a cotilting Λ-module, then U_K is a cotilting Λ_K-module,*

$$\operatorname{id}_{\Lambda_K} U_K = \operatorname{id}_\Lambda U \quad \text{and} \quad \operatorname{End}_{\Lambda_K}(U_K) \cong \operatorname{End}_\Lambda(U) \otimes_k K.$$

Proof. Let $P_\bullet \to X$ be a projective resolution of a module X in $\operatorname{mod}\Lambda$. Then $P_\bullet \otimes_k K$ is a projective resolution of Λ_K-module X_K. For any i there is a natural isomorphism

$$\operatorname{Hom}_{\Lambda_K}(P_i \otimes_k K, Y_K) \cong \operatorname{Hom}_\Lambda(P_i, Y) \otimes_k K.$$

Taking cohomology gives

$$\operatorname{Ext}^i_{\Lambda_K}(X_K, Y_K) \cong \operatorname{Ext}^i_\Lambda(X, Y) \otimes_k K.$$

There is also a natural isomorphism of right Λ_K-modules

$$D\Lambda \otimes_k K \cong D_K(\Lambda_K).$$

Every finitely generated injective Λ-module is a direct summand of a finite direct sum of copies of $D\Lambda$. So scalar extension sends injective Λ-modules to injective Λ_K-modules, and hence

$$\operatorname{id}_{\Lambda_K} U_K \le \operatorname{id}_\Lambda U.$$

If $\operatorname{id}_\Lambda U = 0$, then $\operatorname{id}_{\Lambda_K} U_K = 0$. If $\operatorname{id}_\Lambda U = d \ge 1$, then there exists a module $X_0 \in \operatorname{mod}\Lambda$ such that $\operatorname{Ext}^d_\Lambda(X_0, U) \ne 0$. Faithfulness of K over k and the above Ext isomorphism give

$$\operatorname{Ext}^d_{\Lambda_K}((X_0)_K, U_K) \ne 0.$$

It follows that $\operatorname{id}_{\Lambda_K} U_K \ge d = \operatorname{id}_\Lambda U$, and thus

$$\operatorname{id}_{\Lambda_K} U_K = \operatorname{id}_\Lambda U.$$

This equality is also a special case of [22, Proposition 2.1(ii)].

Taking degree zero in the above Ext isomorphism gives

$$\operatorname{End}_{\Lambda_K}(U_K) \cong \operatorname{End}_\Lambda(U) \otimes_k K,$$

while the positive degrees give self-orthogonality of U_K. Finally, a finite $\operatorname{add} U$-resolution of $D\Lambda$ extends to a finite $\operatorname{add} U_K$-resolution of $D_K(\Lambda_K)$. Thus U_K is cotilting. $\square$

Corollary 3.8 is a direct tensor-product statement when K/k is finite. Lemma 3.9 extends ordinary cotilting and the endomorphism-ring identity to an arbitrary extension, but it does not by itself extend the strong property. Indeed, we know that

$$\widehat{\operatorname{add} U} = \mathcal{I}^{<\infty}(\Lambda)$$

is a statement about all finitely generated Λ-modules. In contrast,

$$\widehat{\operatorname{add} U_K} = \mathcal{I}^{<\infty}(\Lambda_K)$$

also involves Λ_K-modules which need not be obtained by scalar extension. Equivalently, the top criterion for strong cotilting involves all simple modules over $\operatorname{End}_{\Lambda_K}(U_K)$, and such simple modules need not descend to $\operatorname{End}_\Lambda(U)$. Moreover, if K/k is infinite, then (K, K) is not an object in the finite-dimensional tensor setting of Theorem 3.3. Thus neither direction below is formal from the preceding tensor theorem, and a separate proof is needed.

Theorem 3.10. *Let K/k be an arbitrary field extension and let U be a cotilting Λ-module. Then*

$$U \text{ is strong over } \Lambda \quad \Longleftrightarrow \quad U_K \text{ is strong over } \Lambda_K.$$

Proof. Put $\Delta := \operatorname{End}(U_\Lambda)$ and $L := \operatorname{top}_\Delta U$. By Lemma 3.9, we have $\operatorname{End}((U_K)_{\Lambda_K}) \cong \Delta_K$.

Assume that U is strong. The module L contains every simple left Δ-module and is faithful over $\overline{\Delta} := \Delta/\operatorname{rad}\Delta$. Thus the action map

$$\overline{\Delta} \to \operatorname{End}_k(L)$$

is injective. After tensoring with K, it follows that L_K is faithful over $\Sigma := \overline{\Delta} \otimes_k K$. The K-algebra Σ is Frobenius, since

$$D_K\Sigma \cong D_k\overline{\Delta} \otimes_k K \cong \Sigma$$

as left Σ-modules. Thus L_K is a Σ-generator by Lemma 2.5.

The ideal $(\operatorname{rad}\Delta) \otimes_k K$ is nilpotent and hence is contained in $\operatorname{rad}\Delta_K$. Every simple Δ_K-module is therefore annihilated by this ideal and is a simple Σ-module. Consequently, it is a quotient of a finite direct sum of copies of L_K, and hence it is also a quotient of a finite direct sum of copies of U_K. It follows from Proposition 2.2 that U_K is strong.

Conversely, assume that U_K is strong and let S be a simple left Δ-module. The nonzero finite-dimensional Δ_K-module S_K has a simple quotient X. Since

$$(\operatorname{rad}\Delta) \otimes_k K \subseteq \operatorname{rad}\Delta_K,$$

there is a natural epimorphism

$$L_K = \frac{U_K}{((\operatorname{rad}\Delta) \otimes_k K)U_K} \to \frac{U_K}{(\operatorname{rad}\Delta_K)U_K} = \operatorname{top}_{\Delta_K} U_K.$$

Since U_K is strong, we have that X occurs in the last semisimple module. Composing with the projection onto this simple summand gives an epimorphism $L_K \to X$.

After restriction to Δ, a k-basis Ξ of K gives

$$S_K \cong S^{(\Xi)} \quad \text{and} \quad L_K \cong L^{(\Xi)},$$

Here $S^{(\Xi)}$ denotes the direct sum of copies of S indexed by Ξ, and similarly for $L^{(\Xi)}$. The arbitrary-basis form of Lemma 2.4(1)(3) shows that restriction of the epimorphism $L_K \to X$ gives

$$L^{(\Xi)} \to \operatorname{Res}_\Delta X \to 0 \quad \text{and} \quad 0 \neq \operatorname{Res}_\Delta X \cong S^{(\mathcal{I})}$$

for a nonempty set $\mathcal{I}$. If S is not a direct summand of L, then $\operatorname{Hom}_\Delta(L, S) = 0$, and hence

$$\operatorname{Hom}_\Delta(L^{(\Xi)}, S^{(\mathcal{I})}) = 0,$$

contrary to the displayed epimorphism. Thus every simple Δ-module occurs in L, and therefor U is strong by Proposition 2.2. □

Remark 3.11. Theorem 3.10 concerns a fixed module defined over k. When K/k is infinite and Λ_K has a strong cotilting module which is not known to descend, the theorem does not imply that Λ has one. The existence equivalence in Corollary 3.8 uses finite-dimensionality of the second tensor factor.

4. Tensor products of Cohen–Macaulay algebras

In this section, we study Question 1.2 based on the results obtained in Section 3,

4.1. **Dualizing bimodules.** Let Λ be a Cohen–Macaulay algebra. When W and W' are dualizing bimodules, their underlying right modules are strong cotilting. By the uniqueness of the maximum cotilting additive class, we have $\operatorname{add} W = \operatorname{add} W'$, and hence $\mathrm{CM}_W(\Lambda) = \mathrm{CM}_{W'}(\Lambda)$. Thus this category is independent of the chosen dualizing bimodule.

Theorem 4.1. *For bimodules ${}_\Lambda U_\Lambda$ and ${}_\Gamma V_\Gamma$, it holds that $U \otimes_k V$ is a dualizing bimodule for $\Lambda \otimes_k \Gamma$ if and only if U and V are dualizing bimodules for Λ and Γ, respectively.*

Proof. Apply Theorem 3.3 to the right modules U_Λ and V_Γ, and apply them again to the right Λ^{op}-module ${}_\Lambda U$ and the right Γ^{op}-module ${}_\Gamma V$. Since

$$(\Lambda \otimes_k \Gamma)^{\mathrm{op}} \cong \Lambda^{\mathrm{op}} \otimes_k \Gamma^{\mathrm{op}},$$

it follows that $U \otimes_k V$ is strong cotilting on both sides if and only if U and V are strong cotilting on both sides.

The Hom formula in Lemma 2.3 gives

$$\operatorname{End}_{\Lambda\otimes_k\Gamma}((U \otimes_k V)_{\Lambda\otimes_k\Gamma}) \cong \operatorname{End}_\Lambda(U_\Lambda) \otimes_k \operatorname{End}_\Gamma(V_\Gamma),$$
$$\operatorname{End}_{(\Lambda\otimes_k\Gamma)^{\mathrm{op}}}({}_{\Lambda\otimes_k\Gamma}(U \otimes_k V)) \cong \operatorname{End}_{\Lambda^{\mathrm{op}}}({}_\Lambda U) \otimes_k \operatorname{End}_{\Gamma^{\mathrm{op}}}({}_\Gamma V).$$

Under these identifications, the two action maps are $\lambda_U \otimes \lambda_V$ and $\rho_U \otimes \rho_V$. We use the following elementary homological observation. Let $f : P \to P'$ and $g : Q \to Q'$ be linear maps between nonzero k-vector spaces. Then

$$f \otimes g \text{ is an isomorphism} \quad \Longleftrightarrow \quad f \text{ and } g \text{ are isomorphisms.}$$

Indeed, suppose that $f \otimes g$ is an isomorphism. From the factorization

$$f \otimes g = (f \otimes 1_{Q'})(1_P \otimes g),$$

the map $1_P \otimes g$ is a monomorphism and $f \otimes 1_{Q'}$ is an epimorphism. Exactness gives

$$P \otimes_k \operatorname{Ker} g \cong \operatorname{Ker}(1_P \otimes g) = 0, \quad (\operatorname{Coker} f) \otimes_k Q' \cong \operatorname{Coker}(f \otimes 1_{Q'}) = 0.$$

Since P and Q' are nonzero, faithfulness implies $\operatorname{Ker} g = 0$ and $\operatorname{Coker} f = 0$. Thus g is a monomorphism and f is an epimorphism.

Using the other factorization

$$f \otimes g = (1_{P'} \otimes g)(f \otimes 1_Q),$$

we similarly obtain

$$(\operatorname{Ker} f) \otimes_k Q = 0 \ \text{ and } \ P' \otimes_k \operatorname{Coker} g = 0.$$

Hence f is a monomorphism and g is an epimorphism. Therefore both f and g are isomorphisms. The converse is immediate.

Applying this observation to (λ_U, λ_V) and (ρ_U, ρ_V) shows that $\lambda_U \otimes \lambda_V$ and $\rho_U \otimes \rho_V$ are isomorphisms if and only if $\lambda_U, \lambda_V, \rho_U$, and ρ_V are isomorphisms. Therefore $U \otimes_k V$ is faithfully balanced if and only if both U and V are faithfully balanced. The proof is finished. □

The following result answers Question 1.2(1) affirmatively.

Corollary 4.2. *Let Λ and Γ be Cohen–Macaulay with dualizing bimodules U and V, respectively. Then $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay with dualizing bimodule $U \otimes_k V$, and*

$$\operatorname{fidim} \Lambda \otimes_k \Gamma = \operatorname{fidim} \Lambda + \operatorname{fidim} \Gamma = \operatorname{id}_\Lambda U + \operatorname{id}_\Gamma V.$$

This number is also $\operatorname{fidim}(\Lambda \otimes_k \Gamma)^{\mathrm{op}}$.

Proof. The first assertion follows from Theorem 4.1. Auslander and Reiten proved

$$\operatorname{fidim} \Lambda = \operatorname{id}_\Lambda U = \operatorname{id}_{\Lambda^{\mathrm{op}}} U = \operatorname{fidim} \Lambda^{\mathrm{op}}$$

for a dualizing bimodule ${}_\Lambda U_\Lambda$ [3, Proposition 1.6]. Similarly, we have

$$\operatorname{fidim} \Gamma = \operatorname{id}_\Gamma V = \operatorname{id}_{\Gamma^{\mathrm{op}}} V = \operatorname{fidim} \Gamma^{\mathrm{op}}$$

and

$$\operatorname{fidim} \Lambda \otimes_k \Gamma = \operatorname{id}_{\Lambda\otimes_k\Gamma} U \otimes_k V = \operatorname{id}_{(\Lambda\otimes_k\Gamma)^{\mathrm{op}}} U \otimes_k V = \operatorname{fidim}(\Lambda \otimes_k \Gamma)^{\mathrm{op}}.$$

Now applying Lemma 2.3 yields the desired equalities. □

The preceding theorem proves the reverse implication when the dualizing bimodule is of the form $U \otimes_k V$. For an arbitrary dualizing bimodule over the tensor product, we have the following necessary conditions.

Proposition 4.3. *If $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay, then Λ, Γ, Λ^{op}, and Γ^{op} have strong cotilting modules. Moreover, we have*

$$\operatorname{fidim} \Lambda \otimes_k \Gamma = \operatorname{fidim} \Lambda + \operatorname{fidim} \Gamma,$$
$$\operatorname{fidim} \Lambda \otimes_k \Gamma^{\mathrm{op}} = \operatorname{fidim} \Lambda^{\mathrm{op}} + \operatorname{fidim} \Gamma^{\mathrm{op}},$$

and consequently

$$\operatorname{fidim} \Lambda + \operatorname{fidim} \Gamma = \operatorname{fidim} \Lambda^{\mathrm{op}} + \operatorname{fidim} \Gamma^{\mathrm{op}}.$$

Proof. Let X be a dualizing $\Lambda \otimes_k \Gamma$-bimodule. Then $X_{\Lambda \otimes_k \Gamma}$ is strong cotilting. Therefore Theorem 3.6 gives strong cotilting modules over Λ and Γ. Next, view ${}_{\Lambda \otimes_k \Gamma}X$ as a right $(\Lambda \otimes_k \Gamma)^{\mathrm{op}}$-module. It is strong cotilting over $(\Lambda \otimes_k \Gamma)^{\mathrm{op}}$. Since

$$(\Lambda \otimes_k \Gamma)^{\mathrm{op}} \cong \Lambda^{\mathrm{op}} \otimes_k \Gamma^{\mathrm{op}},$$

Theorem 3.6 gives strong cotilting modules over Λ^{op} and Γ^{op}. Applying Corollary 3.4 to $\Lambda \otimes_k \Gamma$ and to $(\Lambda \otimes_k \Gamma)^{\mathrm{op}}$ yields the first equalities. Finally, we have

$$\operatorname{fidim} \Lambda \otimes_k \Gamma = \operatorname{fidim} (\Lambda \otimes_k \Gamma)^{\mathrm{op}}$$

by [3, Proposition 1.6], and hence the last equality follows. □

The next example shows that the converse to tensor closure of Cohen–Macaulay algebras in Corollary 4.2 fails, even for split basic algebras.

Example 4.4. For every field k, there are split basic finite-dimensional k-algebras Λ and Γ such that neither Λ nor Γ is Cohen–Macaulay. However, $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay.

Proof. We compose paths from left to right. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "a_1"] & 2 \arrow[d, "a_2"] \\ 4 \arrow[u, "a_4"] & 3 \arrow[l, "a_3"] \end{tikzcd}$$

and put

$$\Lambda := kQ/\langle a_1a_2a_3,\ a_2a_3a_4,\ a_3a_4a_1a_2\rangle.$$

Write

$$X_1 := \frac{P_1}{\operatorname{soc} P_1} = \begin{matrix}1\\2\end{matrix}, \quad X_2 := P_2 = \begin{matrix}2\\3\\4\end{matrix}, \quad X_3 := P_3 = \begin{matrix}3\\4\\1\\2\end{matrix}, \quad X_4 := P_4 = \begin{matrix}4\\1\\2\\3\end{matrix},$$

and $U := X_1 \oplus X_2 \oplus X_3 \oplus X_4$. It follows from [8, Example 2.8] that U_Λ is a basic strong cotilting module of injective dimension three.

Put $\Gamma := \operatorname{End}_\Lambda(U)$. The dual of Miyashita's double-centralizer theorem [25, Theorem 1.5] yields

$${}_\Gamma U \text{ is cotilting} \quad \text{and} \quad \operatorname{End}({}_\Gamma U) \cong \Lambda^{\mathrm{op}},$$

where the second isomorphism is induced by the right Λ-action. Moreover, we have

$$\operatorname{top}_{\Lambda^{\mathrm{op}}} U \cong S_1 \oplus S_2 \oplus S_3 \oplus S_4.$$

Proposition 2.2, applied over Γ^{op}, shows that U is strong as a left Γ-module. Thus ${}_\Gamma U$ is a strong cotilting left Γ-module.

On the other hand, by an easy computation, we have

$$\Gamma^{\mathrm{op}} \cong kQ_\Gamma/I_\Gamma,$$

where Q_Γ is the quiver

$$\begin{tikzcd} 1 \arrow[dr, "\alpha"'] & 2 \arrow[l, "\beta"'] \arrow[r, "\gamma"] & 4 \arrow[dl, "\varepsilon"] \\ & 3 \arrow[u, "\delta"] & \end{tikzcd}$$

and

$$I_\Gamma = \langle \alpha\delta,\ \delta\beta,\ \beta\alpha - \gamma\varepsilon,\ \varepsilon\delta\gamma \rangle.$$

An *anti-involution* of a k-algebra Σ is a k-linear bijection $\iota : \Sigma \to \Sigma$ such that

$$\iota(xy) = \iota(y)\iota(x) \quad \text{and} \quad \iota^2 = 1.$$

We use the term in the sense of [28, Definition 3.2.1]. There is an anti-involution ι_Λ of Λ which fixes the vertices 1 and 3, interchanges 2 and 4, and satisfies

$$\iota_\Lambda(a_1) = a_4, \quad \iota_\Lambda(a_4) = a_1, \quad \iota_\Lambda(a_2) = a_3 \quad \text{and} \quad \iota_\Lambda(a_3) = a_2.$$

Indeed, it interchanges the first two defining relations of Λ and fixes the third. We also define an anti-involution $\iota_{\Gamma^{\mathrm{op}}}$ of Q_Γ by fixing 1 and 4, interchanging 2 and 3, and satisfies

$$\iota_{\Gamma^{\mathrm{op}}}(\alpha) = \beta, \quad \iota_{\Gamma^{\mathrm{op}}}(\gamma) = \varepsilon \quad \text{and} \quad \iota_{\Gamma^{\mathrm{op}}}(\delta) = \delta.$$

It interchanges $\alpha\delta$ and $\delta\beta$ and fixes the other two relations. The same linear map on the underlying vector space is an anti-involution of Γ; denote it by ι_Γ.

Let V be the vector space U with the following (Λ, Γ)-bimodule structure:

$$a \cdot u = u\iota_\Lambda(a) \quad \text{and} \quad u \cdot e = \iota_\Gamma(e)u.$$

Then ${}_\Lambda V$ and V_Γ are strong cotilting modules. Since ι_Λ and ι_Γ are bijective, the endomorphism rings are

$$\operatorname{End}(V_\Gamma) = \operatorname{End}({}_\Gamma U) \cong \Lambda^{\mathrm{op}} \qquad \text{and} \qquad \operatorname{End}({}_\Lambda V) = \operatorname{End}(U_\Lambda) = \Gamma.$$

Here the first isomorphism sends a^{op} to the endomorphism $u \mapsto ua$, and the second sends e to $u \mapsto eu$. We use these identifications in the tensor calculation below.

After composing with the anti-involutions, the two action maps can be written as the algebra isomorphisms

$$\Lambda \xrightarrow{\sim} \operatorname{End}(V_\Gamma) \qquad \text{via} \qquad a \mapsto (u \mapsto u\iota_\Lambda(a)),$$

and

$$\Gamma^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}({}_\Lambda V) \qquad \text{via} \qquad e^{\mathrm{op}} \mapsto (u \mapsto \iota_\Gamma(e)u).$$

Set $Z := U \otimes_k V$. We define the right and left $\Lambda \otimes_k \Gamma$-actions by

$$(u \otimes_k v)(a \otimes e) := ua \otimes ve \quad \text{and} \quad (a \otimes e)(u \otimes v) := eu \otimes av, \quad \text{respectively.}$$

Since U_Λ and V_Γ are strong, we have that $Z_{\Lambda \otimes_k \Gamma}$ is strong by Theorem 3.3. Next, view ${}_\Gamma U$ as a right Γ^{op}-module and ${}_\Lambda V$ as a right Λ^{op}-module. Both modules are strong. Apply Theorem 3.3 to these two right modules, and then use

$$\Gamma^{\mathrm{op}} \otimes_k \Lambda^{\mathrm{op}} \cong (\Lambda \otimes_k \Gamma)^{\mathrm{op}}.$$

It follows that ${}_{\Lambda \otimes_k \Gamma} Z$ is strong.

It remains to check the two action maps. The Hom formula in Lemma 2.3 and the identifications above give

$$\operatorname{End}(Z_{\Lambda \otimes_k \Gamma}) \cong \operatorname{End}(U_\Lambda) \otimes_k \operatorname{End}(V_\Gamma) \cong \Gamma \otimes_k \Lambda^{\mathrm{op}}.$$

Under this isomorphism, the left action map is

$$\Lambda \otimes_k \Gamma \longrightarrow \Gamma \otimes_k \Lambda^{\mathrm{op}}, \qquad a \otimes e \mapsto e \otimes \iota_\Lambda(a)^{\mathrm{op}}.$$

It is the flip followed by the algebra isomorphism

$$\Lambda \longrightarrow \Lambda^{\mathrm{op}} \qquad \text{via} \qquad a \mapsto \iota_\Lambda(a)^{\mathrm{op}}.$$

Hence it is an isomorphism.

Similarly, we have

$$\mathrm{End}({}_{\Lambda\otimes_k\Gamma}Z) \cong \mathrm{End}({}_{\Gamma}U) \otimes_k \mathrm{End}({}_{\Lambda}V) \cong \Lambda^{\mathrm{op}} \otimes_k \Gamma.$$

Using $(\Lambda \otimes_k \Gamma)^{\mathrm{op}} \cong \Lambda^{\mathrm{op}} \otimes_k \Gamma^{\mathrm{op}}$, the right action map becomes

$$\Lambda^{\mathrm{op}} \otimes_k \Gamma^{\mathrm{op}} \longrightarrow \Lambda^{\mathrm{op}} \otimes_k \Gamma \qquad \text{via} \qquad a^{\mathrm{op}} \otimes e^{\mathrm{op}} \mapsto a^{\mathrm{op}} \otimes \iota_\Gamma(e).$$

This is an isomorphism because

$$\Gamma^{\mathrm{op}} \longrightarrow \Gamma \qquad \text{via} \qquad e^{\mathrm{op}} \mapsto \iota_\Gamma(e)$$

is an algebra isomorphism. Thus Z is faithfully balanced and is strong cotilting on both sides. It follows from [3, Proposition 1.3] that Z is a dualizing bimodule for $\Lambda \otimes_k \Gamma$, and thus $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay.

It remains to show that neither Λ nor Γ is Cohen–Macaulay. The algebra Λ is a split basic bound quiver algebra. The presentation of Γ^{op} above shows that Γ is also split basic. Suppose that Λ has a dualizing bimodule Y. Then Y_Λ and U_Λ are strong cotilting, and thus $\mathrm{add}\, Y = \mathrm{add}\, U$. Consequently, $\mathrm{End}(Y_\Lambda)$ is Morita equivalent to $\mathrm{End}(U_\Lambda) = \Gamma$. On the other hand, faithful balancedness gives $\Lambda \cong \mathrm{End}(Y_\Lambda)$. Thus Λ and Γ are Morita equivalent. Since both are basic, they must be isomorphic, contrary to

$$\dim_k \Lambda = 14 \neq 12 = \dim_k \Gamma.$$

Thus Λ is not Cohen–Macaulay.

If Γ has a dualizing bimodule Y, then the same argument applied to the strong cotilting modules Y_Γ and V_Γ would show that Γ is Morita equivalent to

$$\mathrm{End}(V_\Gamma) \cong \Lambda^{\mathrm{op}}.$$

Again both algebras are basic and have different dimensions, which is a contradiction. Thus Γ is not Cohen–Macaulay. □

The construction of bimodule $U \otimes_k V$ in the proof is different as in Theorem 4.1, thus there is no contradiction with that theorem. We next consider a case in which reflection does hold.

Recall that a *central simple k-algebra* is a finite-dimensional simple k-algebra whose center is k [15, Section 2.1]. It is *split* if it is isomorphic to $M_m(k)$ for some $m \geq 1$ [15, Definition 2.2.4]. We first consider the case in which either Λ or Γ is central simple.

Proposition 4.5. *Let Γ be a central simple k-algebra. Then $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay if and only if Λ is Cohen–Macaulay.*

Proof. Suppose that Λ is Cohen–Macaulay. It follows from [28, Theorem 7.1.1] that the central simple algebra Γ is semisimple, and hence its regular bimodule is dualizing. Thus $\Lambda \otimes_k \Gamma$ is Cohen–Macaulay by Corollary 4.2.

Conversely, let X be a dualizing $\Lambda \otimes_k \Gamma$-bimodule. The canonical map

$$\theta : \Gamma^e \longrightarrow \mathrm{End}_k(\Gamma) \;\; \text{via} \;\; \theta(a \otimes b^{\mathrm{op}})(s) = asb,$$

is an isomorphism by [15, Proposition 2.4.8]. Moreover, we have $\mathrm{End}({}_{\Gamma^e}\Gamma) = Z(\Gamma) = k$.

Put $d := \dim_k \Gamma$. After choosing a k-basis of Γ, the isomorphism θ gives

$$\Gamma^e \cong M_d(k), \qquad {}_{\Gamma^e}\Gamma \cong k^d \;\; \text{and} \;\; {}_{\Gamma^e}\Gamma^e \cong \Gamma^d.$$

Thus Γ is the unique simple Γ^e-module and is a projective generator. The Morita theory [27, Section 9.6] gives mutually quasi-inverse equivalences

$$\mathrm{Hom}_{\Gamma^e}(\Gamma, -) : \Gamma^e\text{-}\mathrm{mod} \longrightarrow k\text{-}\mathrm{mod} \;\; \text{and} \;\; - \otimes_k \Gamma : k\text{-}\mathrm{mod} \longrightarrow \Gamma^e\text{-}\mathrm{mod}\,.$$

The Λ^e-action on X commutes with the Γ^e-action. Consequently, $W := \mathrm{Hom}_{\Gamma^e}(\Gamma, X)$ is a Λ^e-module and

$$W \otimes_k \Gamma \xrightarrow{\sim} X \;\; \text{via} \;\; f \otimes s \mapsto f(s)$$

is $\Lambda^e \otimes_k \Gamma^e (= (\Lambda \otimes_k \Gamma)^e)$-linear. Since an equivalence reflects zero objects,

$$X \neq 0 \Longrightarrow W \neq 0, \qquad X \cong W \otimes_k \Gamma.$$

It follows from Theorem 4.1 that W is dualizing for Λ. □

4.2. **Cohen–Macaulay modules and Ext-support.** We next record all nonvanishing degrees in the Ext–Künneth formula. For a cotilting Λ-module U and a Λ-module M, put

$$\mathrm{Supp}_U(M) := \{i \geq 0 \mid \mathrm{Ext}^i_\Lambda(M, U) \neq 0\}.$$

For any $M \neq 0$, put

$$\delta_U(M) := \max \mathrm{Supp}_U(M).$$

For subsets $P, Q \subseteq \mathbb{N}$, write $P + Q := \{p + q \mid p \in P \text{ and } q \in Q\}$.

Proposition 4.6. *Let U and V be cotilting modules over Λ and Γ, respectively.*

(1) For any $0 \neq M \in \mathrm{mod}\,\Lambda$, the set $\mathrm{Supp}_U(M)$ is finite and nonempty and

$$\delta_U(M) = \min\{n \geq 0 \mid \Omega^n_\Lambda M \in {}^\perp U\}.$$

(2) For any $0 \neq M \in \mathrm{mod}\,\Lambda$ and $0 \neq N \in \mathrm{mod}\,\Gamma$, we have

$$\mathrm{Supp}_{U \otimes_k V}(M \otimes_k N) = \mathrm{Supp}_U(M) + \mathrm{Supp}_V(N)$$

and

$$\delta_{U \otimes_k V}(M \otimes_k N) = \delta_U(M) + \delta_V(N).$$

Proof. (1) The finiteness follows from $\mathrm{id}_\Lambda U < \infty$. Suppose $\mathrm{Ext}^{\geq 0}_\Lambda(M, U) = 0$. Choose an exact sequence

$$0 \to U_r \to \cdots \to U_0 \to D\Lambda \to 0$$

in $\mathrm{mod}\,\Lambda$ with all U_i in $\mathrm{add}\,U$. Then $\mathrm{Ext}^{\geq 0}_\Lambda(M, D\Lambda) = 0$. In particular, we have

$$0 = \mathrm{Hom}_\Lambda(M, D\Lambda) \cong DM,$$

contrary to $M \neq 0$. Thus the support is nonempty.

For any $j \geq 1$, dimension shifting yields

$$\mathrm{Ext}^j_\Lambda(\Omega^n_\Lambda M, U) \cong \mathrm{Ext}^{j+n}_\Lambda(M, U).$$

Therefore $\Omega^n_\Lambda M \in {}^\perp U$ exactly when all Ext groups in degrees greater than n vanish. The least such n is the maximum of the finite nonempty support.

(2) It follows from Lemma 2.3 that

$$\mathrm{Ext}^r_{\Lambda \otimes_k \Gamma}(M \otimes_k N, U \otimes_k V) \cong \bigoplus_{p+q=r} \mathrm{Ext}^p_\Lambda(M, U) \otimes_k \mathrm{Ext}^q_\Gamma(N, V).$$

A tensor product of two k-vector spaces is nonzero exactly when both factors are nonzero. This proves the support equality. Taking maxima gives the formula for δ. □

As a consequence of Proposition 4.6, we get the following result.

Theorem 4.7. *Let Λ and Γ have dualizing bimodules U and V, respectively. For any $0 \neq M \in \mathrm{mod}\,\Lambda$ and $0 \neq N \in \mathrm{mod}\,\Gamma$, we have*

$$M \otimes_k N \in \mathrm{CM}_{U \otimes_k V}(\Lambda \otimes_k \Gamma) \iff M \in \mathrm{CM}_U(\Lambda) \text{ and } N \in \mathrm{CM}_V(\Gamma).$$

Proof. By Proposition 4.6, we have

$$\begin{aligned} M \otimes_k N \in \mathrm{CM}_{U \otimes_k V}(\Lambda \otimes_k \Gamma) &\iff \delta_{U \otimes_k V}(M \otimes_k N) = 0 \\ &\iff \delta_U(M) = 0 \quad \text{and} \quad \delta_V(N) = 0 \\ &\iff M \in \mathrm{CM}_U(\Lambda) \quad \text{and} \quad N \in \mathrm{CM}_V(\Gamma). \end{aligned}$$

□

4.3. **The modified version of Question 1.2(2).** Though [8, Proposition 4.8 and Example 4.9] answers Question 1.2(2) negatively, we will modify this section slightly such that the corresponding answer is positive.

Let Λ be a Cohen–Macaulay algebra with dualizing bimodule W with $\operatorname{id}_\Lambda W = d$. Then

$$\Omega^n_\Lambda(\operatorname{mod}\Lambda) \subseteq \operatorname{CM}_W(\Lambda) \tag{4.1}$$

for any $n \geq d$.

Proposition 4.8. *Let Λ be a Cohen–Macaulay algebra with dualizing bimodule W, and put $d := \operatorname{fidim}\Lambda = \operatorname{id}_\Lambda W$. Then the following statements are equivalent.*

(1) $\operatorname{CM}_W(\Lambda) = \Omega^d_\Lambda(\operatorname{mod}\Lambda)$.

(2) $\Omega^d_\Lambda(\operatorname{mod}\Lambda)$ *is closed under extensions.*

(3) $\Omega^d_\Lambda(\operatorname{mod}\Lambda)$ *is a resolving subcategory of* $\operatorname{mod}\Lambda$.

Proof. It is trivial that $\operatorname{CM}_W(\Lambda) = {}^\perp W$ is resolving, thus the implications (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2) hold.

(2) $\Longrightarrow$ (1) Let

$$0 \to X \to Y \to Z \to 0$$

be exact with $Y, Z \in \Omega^d_\Lambda(\operatorname{mod}\Lambda)$. Choose $0 \to \Omega_\Lambda Z \to P \to Z \to 0$ with P projective. The pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
 & & \Omega_\Lambda Z \arrow[r, equal] \arrow[d, dashed] & \Omega_\Lambda Z \arrow[d] & \\
0 \arrow[r, dashed] & X \arrow[r, dashed] \arrow[d, equal] & X \oplus P \arrow[r, dashed] \arrow[d, dashed] & P \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] \arrow[d, dashed] & Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

gives an exact sequence

$$0 \longrightarrow \Omega_\Lambda Z \longrightarrow X \oplus P \longrightarrow Y \longrightarrow 0.$$

Since $\Omega_\Lambda Z \in \Omega^{d+1}_\Lambda(\operatorname{mod}\Lambda) \subseteq \Omega^d_\Lambda(\operatorname{mod}\Lambda)$, we have $X \oplus P \in \Omega^d_\Lambda(\operatorname{mod}\Lambda)$. So X belongs to this additively closed category, and hence $\Omega^d_\Lambda(\operatorname{mod}\Lambda)$ is resolving. It follows from [4, Theorem 1.2] that $\Omega^d_\Lambda(\operatorname{mod}\Lambda)$ is contravariantly finite. Thus we have

$$\Omega^d_\Lambda(\operatorname{mod}\Lambda) = \min\{\mathcal{X} \mid \mathcal{X} \text{ is contravariantly finite resolving and } \Omega^d_\Lambda(\operatorname{mod}\Lambda) \subseteq \mathcal{X}\}.$$

Under the order-reversing Auslander–Reiten correspondence [2, Theorem 5.5], in the form stated in [8, Theorem 1.1], it therefore corresponds to the maximum among the additive-equivalence classes of cotilting modules of injective dimension at most d. The module W is the maximum of all cotilting modules and has injective dimension d. Therefore its additive class is the same maximum. Consequently $\Omega^d_\Lambda(\operatorname{mod}\Lambda) = {}^\perp W = \operatorname{CM}_W(\Lambda)$. $\square$

The following result shows that if "d-syzygy" is replaced by "$(d+1)$-syzygy" in Question 1.2(2), then the answer is positive.

Theorem 4.9. *Let Λ be Cohen–Macaulay with a dualizing bimodule W, and let $\operatorname{fidim}\Lambda = d$. Then the following statements are equivalent.*

(1) Λ *is Iwanaga–Gorenstein.*

(2) $W \in \Omega^{d+1}_\Lambda(\operatorname{mod}\Lambda)$.

(3) $\operatorname{CM}_W(\Lambda) = \Omega^n_\Lambda(\operatorname{mod}\Lambda)$ *for some $n \geq d+1$.*

(4) $\operatorname{CM}_W(\Lambda) = \Omega^n_\Lambda(\operatorname{mod}\Lambda)$ *for any $n \geq d$.*

Moreover, if $\mathrm{CM}_W(\Lambda) = \Omega^d_\Lambda(\operatorname{mod}\Lambda)$*, then all the above and below statements are equivalent.*

(5) $\Omega^d_\Lambda(\operatorname{mod}\Lambda) = \Omega^{d+1}_\Lambda(\operatorname{mod}\Lambda)$*.*

(6) The exact category $\mathrm{CM}_W(\Lambda)$*, with the inherited exact structure, is Frobenius; that is, it has enough projective and injective objects, and these two classes coincide.*

(7) $\Lambda_\Lambda \in \operatorname{add} W$*.*

(8) $\operatorname{id}_\Lambda \Lambda \leq d$*.*

(9) $\operatorname{Ext}^{d+1}_\Lambda(M, \Lambda) = 0$ *for any* $M \in \operatorname{mod}\Lambda$*.*

Proof. The implication (4) $\Longrightarrow$ (3) is trivial. If assertion (3) holds, then

$$W \in \mathrm{CM}_W(\Lambda) = \Omega^n_\Lambda(\operatorname{mod}\Lambda) \subseteq \Omega^{d+1}_\Lambda(\operatorname{mod}\Lambda)$$

for some $n > d$, thus assertion (2) follows. This proves (3) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1) By (2) and the definition of the additive syzygy category, there are a module X, a projective module P_0, and a module W' such that

$$W \oplus W' \cong P_0 \oplus \Omega^{d+1}_\Lambda X.$$

Adding the identity sequence of P_0 to the relevant part of a projective resolution of X gives

$$0 \to W \oplus W' \xrightarrow{j} P \to Y \to 0$$

in $\operatorname{mod}\Lambda$ with P projective and $Y \in \Omega^d_\Lambda(\operatorname{mod}\Lambda)$. By (4.1), we have $Y \in \mathrm{CM}_W(\Lambda)$, and hence $\operatorname{Ext}^1_\Lambda(Y, W) = 0$. Consider the following pushout diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d, dashed] & & \\
& W' \arrow[r, equal, dashed] \arrow[d, "\binom{0}{1}"'] & W' \arrow[d, dashed] & & \\
0 \arrow[r] & W \oplus W' \arrow[r] \arrow[d, "(1\ 0)"'] & P \arrow[r] \arrow[d, dashed] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, dashed] & W \arrow[r, dashed] \arrow[d] & Q \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[r, dashed] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

Since the bottom short exact sequence is split, we can get that W is a direct summand of P, which shows that W_Λ is projective.

Let W^{b} be a basic representative of the additive class of W. It is projective. A basic cotilting module has as many nonisomorphic indecomposable summands as there are simple modules [2, p.141]. This is also the number of nonisomorphic indecomposable projective modules. Thus W^{b} contains all of them, and W_Λ is a projective generator, and therefore $\operatorname{add} W_\Lambda = \operatorname{proj}\Lambda$.

Put $\Delta := \operatorname{End}(W_\Lambda)$. Since W_Λ is a finitely generated projective generator, the classical Morita theorem applies. The natural (Δ, Λ)-bimodule ${}_\Delta W_\Lambda$ is a projective generator on both sides. Its left Δ-action is evaluation:

$$f \cdot w = f(w) \text{ for any } f \in \Delta \text{ and } w \in W.$$

Let $X \in \operatorname{mod}\Lambda$. The right Δ-action on $\operatorname{Hom}_\Lambda(W, X)$ is

$$(h \cdot f)(w) = h(f(w)).$$

Thus the equivalences between right-module categories are

$$\operatorname{Hom}_\Lambda(W, -) : \operatorname{mod}\Lambda \to \operatorname{mod}\Delta, \qquad - \otimes_\Delta W : \operatorname{mod}\Delta \to \operatorname{mod}\Lambda.$$

Since W_Λ is projective and is a generator, its two evaluation maps are isomorphisms, and thus ${}_\Delta W$ is a finitely generated projective generator. By the balance condition, the homomorphism

$$\lambda : \Lambda \xrightarrow{\sim} \Delta \text{ via } a \mapsto (w \mapsto aw)$$

is an algebra isomorphism. If we restrict scalars along λ, then the action of $a \in \Lambda$ on W is

$$a \cdot w = \lambda(a)(w) = aw.$$

Thus the restricted left Λ-module is exactly ${}_\Lambda W$. Since ${}_\Delta W$ is a projective generator, it follows that ${}_\Lambda W$ is also a projective generator. Together with $\operatorname{add} W_\Lambda = \operatorname{proj}\Lambda$, this gives

$$\Lambda_\Lambda \in \operatorname{add} W_\Lambda \text{ and } {}_\Lambda\Lambda \in \operatorname{add}({}_\Lambda W).$$

Since W is strong cotilting on both sides, we have

$$\operatorname{id}_\Lambda \Lambda \leq \operatorname{id}_\Lambda W < \infty \text{ and } \operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq \operatorname{id}_{\Lambda^{\mathrm{op}}} W < \infty.$$

Thus Λ is Iwanaga–Gorenstein, and assertion (1) follows.

(1) $\Longrightarrow$ (4) The regular bimodule of the Iwanaga–Gorenstein algebra Λ is dualizing by [8, Example 1.3(2)]. Uniqueness of the strong cotilting additive class gives $\operatorname{add} W = \operatorname{add}\Lambda$. It follows from Proposition 2.7 that $\operatorname{id}_\Lambda \Lambda = \operatorname{fidim}\Lambda = d$. For any $n \geq d$, it is easy to get that $\Omega^n_\Lambda(\operatorname{mod}\Lambda) \subseteq {}^\perp\Lambda = \mathrm{CM}_W(\Lambda)$ by dimension shifting. On the other hand, we have

$$\mathrm{CM}_W(\Lambda) = {}^\perp\Lambda = \operatorname{Gproj}\Lambda \subseteq \Omega^n_\Lambda(\operatorname{mod}\Lambda)$$

for any $n \geq 1$ by (1) and [13, Corollary 11.5.3]. Thus assertion (4) follows.

Now suppose $\mathrm{CM}_W(\Lambda) = \Omega^d_\Lambda(\operatorname{mod}\Lambda)$. It is trivial that (4) $\Longrightarrow$ (5) and (8) $\Longleftrightarrow$ (9).

(5) $\Longrightarrow$ (2) By (5), we have

$$W \in \mathrm{CM}_W(\Lambda) = \Omega^d_\Lambda(\operatorname{mod}\Lambda) = \Omega^{d+1}_\Lambda(\operatorname{mod}\Lambda),$$

and assertion (2) follows.

Let $X \in {}^\perp W = \mathrm{CM}_W(\Lambda)$. By [2, Theorem 5.5], there exist exact sequences

$$0 \to X_1 \to P \to X \to 0 \quad\text{and}\quad 0 \to X \to W_0 \to X_2 \to 0,$$

in $\operatorname{mod}\Lambda$ with where P projective, $W_0 \in \operatorname{add} W$ and $X_1, X_2 \in {}^\perp W$. The first sequence shows that the inherited exact structure has enough projectives. If X is projective in this exact structure, that sequence splits, and hence $X \in \operatorname{proj}(\Lambda)$. Thus the projective objects are exactly $\operatorname{proj}\Lambda$. Since $\operatorname{Ext}^1_\Lambda({}^\perp W, W) = 0$, the objects of $\operatorname{add} W$ are injective in the inherited exact structure. On the other hand, if X is injective there, the second sequence splits, and hence $X \in \operatorname{add} W$. Thus $\mathrm{CM}_W(\Lambda)$ has enough injectives and its injective objects are exactly $\operatorname{add} W$. This proves (6) $\Longleftrightarrow$ $\operatorname{add} W = \operatorname{proj}\Lambda$ $\Longleftrightarrow$ (1).

Assertion (7) implies that the projective generator Λ_Λ is also Ext-injective in $\mathrm{CM}_W(\Lambda)$. Thus any indecomposable projective lies in $\operatorname{add} W$. A basic representative W^{b} has precisely as many indecomposable summands as there are indecomposable projectives [2, p.141], and hence all its summands are projective, which implies $\operatorname{add} W = \operatorname{proj}\Lambda$. This proves (7) $\Longleftrightarrow$ $\operatorname{add} W = \operatorname{proj}\Lambda \Longleftrightarrow$ (1).

(9) $\Longrightarrow$ (7) $\Longrightarrow$ (8) By dimension shifting, we have

$$\operatorname{Ext}^1_\Lambda(\Omega^d_\Lambda M, \Lambda) \cong \operatorname{Ext}^{d+1}_\Lambda(M, \Lambda).$$

Note that $\mathrm{CM}_W(\Lambda) = \Omega^d_\Lambda(\operatorname{mod}\Lambda)$. Thus, if assertion (9) holds true, then Λ_Λ is Ext-injective in $\mathrm{CM}_W(\Lambda)$, which implies that $\Lambda_\Lambda \in \operatorname{add} W$ and assertion (7) follows. On the other hand, if assertion (7) holds true, then $\operatorname{id}_\Lambda \Lambda \leq \operatorname{id}_\Lambda W = \operatorname{fidim}\Lambda = d$ by Proposition 2.7, and hence assertion (8) follows. □

We use [8, Example 4.9] to illustrate that $\Omega^{d+1}_\Lambda(\operatorname{mod}\Lambda)$ cannot be replaced by $\Omega^d_\Lambda(\operatorname{mod}\Lambda)$ in Theorem 4.9(2).

Example 4.10. For $n \geq 4$, let

$$\Sigma_n := kQ/I, \qquad Q : 1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1,$$

where the arrows from i to $i+1$ and back are a_i and b_i, and

$$I = \langle a_1b_1,\ (b_{n-2}a_{n-2})^2,\ b_ia_i - a_{i+1}b_{i+1}\ (1 \leq i \leq n-3)\rangle.$$

By [8, Proposition 4.8 and Example 4.9], we have that Σ_n is Cohen–Macaulay, $\operatorname{fidim}\Sigma_n = 2$, and Σ_n is not Iwanaga–Gorenstein. It is Auslander 2-Gorenstein. Thus [8, Corollary 2.2] gives

$$\mathrm{CM}(\Sigma_n) = \Omega^2_{\Sigma_n}(\operatorname{mod}\Sigma_n).$$

For the right injectives $I_i = D(\Sigma_n e_i)$, one may choose a dualizing bimodule whose underlying right module is

$$W_n := \Omega^2_{\Sigma_n}(I_1) \oplus I_2 \oplus \cdots \oplus I_{n-1}.$$

It follows from Theorem 4.9 that

$$W_n \in \Omega^2_{\Sigma_n}(\operatorname{mod}\Sigma_n) \setminus \Omega^3_{\Sigma_n}(\operatorname{mod}\Sigma_n), \qquad \Omega^2_{\Sigma_n}(\operatorname{mod}\Sigma_n) \neq \Omega^3_{\Sigma_n}(\operatorname{mod}\Sigma_n).$$

The first explicit counterexample had already been announced in [24, pp. 210–211]; the cited results of Chan, Iyama, and Marczinzik give a family together with complete proofs. In the other direction, [8, Example 4.10] gives a contracted E_6 algebra, say Δ, of finitistic dimension two for which the underlying right module of a chosen dualizing bimodule does not belong to $\Omega^2_\Delta(\operatorname{mod}\Delta)$. It follows from Proposition 4.8 that its second syzygy category is not closed under extensions.

We next apply the Auslander n-Gorenstein condition to tensor products. The level-one case of the result below follows from the tensor formula for dominant dimension [26, Lemma 6], and a path-algebra case is contained in [5, Corollary 4.1.2].

Lemma 4.11. *Let $n \geq 1$ and let $m, m_1, m_2 \geq 0$.*

(1) If Λ is $G_n(m_1)$ and Γ is $G_n(m_2)$, then $\Lambda \otimes_k \Gamma$ is $G_n(m_1 + m_2)$.
(2) If $\Lambda \otimes_k \Gamma$ is $G_n(m)$, then both so are Λ and Γ.

Proof. By the dual version of [20, Lemma 5.3], we have that

$$0 \to \Lambda \otimes_k \Gamma \to E^0(\Lambda) \otimes_k E^0(\Gamma) \to (E^0(\Lambda) \otimes_k E^1(\Gamma)) \oplus (E^1(\Lambda) \otimes_k E^0(\Gamma)) \to \cdots$$
$$\to \oplus_{p+q=i} E^p(\Lambda) \otimes_k E^q(\Gamma) \to \cdots$$

is a minimal injective coresolution of $\Lambda \otimes_k \Gamma$ in $\operatorname{mod}\Lambda \otimes_k \Gamma$. For any $p, q \geq 0$, we have

$$\begin{aligned}\max\{\operatorname{pd}_\Lambda E^p(\Lambda), \operatorname{pd}_\Gamma E^q(\Gamma)\} &\leq \operatorname{pd}_{\Lambda\otimes_k\Gamma} E^p(\Lambda) \otimes_k E^q(\Gamma)\\ &= \operatorname{pd}_\Lambda E^p(\Lambda) + \operatorname{pd}_\Gamma E^q(\Gamma) \quad \text{(by Lemma 2.3)},\end{aligned}$$

and hence

$$\begin{aligned}\max\{\operatorname{pd}_\Lambda E^i(\Lambda), \operatorname{pd}_\Gamma E^i(\Gamma)\} &\leq \operatorname{pd}_{\Lambda\otimes_k\Gamma} \oplus_{p+q=i} E^p(\Lambda) \otimes_k E^q(\Gamma)\\ &= \max\{\operatorname{pd}_\Lambda E^p(\Lambda) + \operatorname{pd}_\Gamma E^q(\Gamma) \mid p+q=i\}.\end{aligned}$$

Now both assertions follow easily. $\square$

Note that an algebra is Auslander n-Gorenstein if and only if it is $G_n(0)$. By putting $m_1 = m_2 = m = 0$ in Lemma 4.11, we get the following result.

Proposition 4.12. *For any $n \geq 1$, it holds that $\Lambda \otimes_k \Gamma$ is Auslander n-Gorenstein if and only if so both are Λ and Γ.*

The following result provides certain criterion for judging Cohen–Macaulay algebras to be Gorenstein.

Theorem 4.13. *Let Λ and Γ be Cohen–Macaulay with dualizing bimodules U and V, respectively, and let $\operatorname{fidim}\Lambda = d$ and $\operatorname{fidim}\Gamma = e$.*

(1) If

$$\mathrm{CM}_{U\otimes_k V}(\Lambda \otimes_k \Gamma) = \Omega^{d+e}_{\Lambda\otimes_k\Gamma}(\operatorname{mod}\Lambda \otimes_k \Gamma),$$

then it holds that

(1.1)

$$\mathrm{CM}_U(\Lambda)=\Omega_\Lambda^{d+e}(\operatorname{mod}\Lambda)=\Omega_\Lambda^{d}(\operatorname{mod}\Lambda)\ \textit{and}\ \mathrm{CM}_V(\Gamma)=\Omega_\Gamma^{d+e}(\operatorname{mod}\Gamma)=\Omega_\Gamma^{e}(\operatorname{mod}\Gamma).$$

(1.2) if $e>0$, then Λ is Iwanaga–Gorenstein; and if $d>0$, then Γ is Iwanaga–Gorenstein.

(2) If $d,e>0$, then

$$\mathrm{CM}_{U\otimes_k V}(\Lambda\otimes_k\Gamma)=\Omega_{\Lambda\otimes_k\Gamma}^{d+e}(\operatorname{mod}\Lambda\otimes_k\Gamma)\quad\Longleftrightarrow\quad \textit{both } \Lambda \textit{ and } \Gamma \textit{ are Iwanaga–Gorenstein.}$$

Proof. (1) Let $X\in\mathrm{CM}_U(\Lambda)$. Since $\Gamma\in\mathrm{CM}_V(\Gamma)$, we have

$$\begin{aligned}X\in\mathrm{CM}_U(\Lambda)&\Longrightarrow X\otimes_k\Gamma\in\mathrm{CM}_{U\otimes_k V}(\Lambda\otimes_k\Gamma)\quad(\text{by Theorem 4.7})\\&\Longrightarrow X\otimes_k\Gamma\in\Omega_{\Lambda\otimes_k\Gamma}^{d+e}(\operatorname{mod}\Lambda\otimes_k\Gamma)\quad(\text{by assumption})\\&\Longrightarrow X^{\dim_k\Gamma}\cong\operatorname{Res}_\Lambda^{\Lambda\otimes_k\Gamma}(X\otimes_k\Gamma)\in\Omega_\Lambda^{d+e}(\operatorname{mod}\Lambda)\quad(\text{by Lemma 2.4(3)(4)})\\&\Longrightarrow X\in\Omega_\Lambda^{d+e}(\operatorname{mod}\Lambda).\end{aligned}$$

Together with (4.1), this yields

$$\mathrm{CM}_U(\Lambda)\subseteq\Omega_\Lambda^{d+e}(\operatorname{mod}\Lambda)\subseteq\Omega_\Lambda^{d}(\operatorname{mod}\Lambda)\subseteq\mathrm{CM}_U(\Lambda),$$

and thus $\mathrm{CM}_U(\Lambda)=\Omega_\Lambda^{d+e}(\operatorname{mod}\Lambda)=\Omega_\Lambda^{d}(\operatorname{mod}\Lambda)$. Similarly, we get $\mathrm{CM}_V(\Gamma)=\Omega_\Gamma^{d+e}(\operatorname{mod}\Gamma)=\Omega_\Gamma^{e}(\operatorname{mod}\Gamma)$.

If $e>0$, then $d+e>d$. It follows from Theorem 4.9 that Λ is Iwanaga–Gorenstein. Similarly, if $d>0$, then Γ is Iwanaga–Gorenstein.

(2) Suppose $d,e>0$. If $\mathrm{CM}_{U\otimes_k V}(\Lambda\otimes_k\Gamma)=\Omega_{\Lambda\otimes_k\Gamma}^{d+e}(\operatorname{mod}\Lambda\otimes_k\Gamma)$, then both Λ and Γ are Iwanaga–Gorenstein by (1.2). Conversely, if both Λ and Γ are Iwanaga–Gorenstein, then $\operatorname{id}_\Lambda\Lambda=d$ and $\operatorname{id}_\Gamma\Gamma=e$ by [3, Proposition 1.6]. It follows from [3, Proposition 2.2] that $\Lambda\otimes_k\Gamma$ is Iwanaga–Gorenstein of self-injective dimension $d+e$, and thus $\mathrm{CM}_{U\otimes_k V}(\Lambda\otimes_k\Gamma)=\Omega_{\Lambda\otimes_k\Gamma}^{d+e}(\operatorname{mod}\Lambda\otimes_k\Gamma)$ by [3, Proposition 3.1(b)] or Theorem 4.9. □

The positivity assumption on both finitistic dimensions in the last assertion of Theorem 4.13 is necessary. If one factor is self-injective, the degree-d equality may persist even when the other factor is not Iwanaga–Gorenstein.

Proposition 4.14. *Let Λ be Cohen–Macaulay with* $\operatorname{fidim}\Lambda=d\geq 1$*, and assume that Λ is Auslander d-Gorenstein but not Iwanaga–Gorenstein. If Γ is self-injective, then $\Lambda\otimes_k\Gamma$ is Cohen–Macaulay and Auslander d-Gorenstein with* $\operatorname{fidim}\Lambda\otimes_k\Gamma=d$ *but not Iwanaga–Gorenstein, and satisfies*

$$\mathrm{CM}(\Lambda\otimes_k\Gamma)=\Omega_{\Lambda\otimes_k\Gamma}^{d}(\operatorname{mod}(\Lambda\otimes_k\Gamma)).$$

Proof. If Γ is self-injective, then $\Lambda\otimes_k\Gamma$ is Cohen–Macaulay of finitistic dimension d by Corollary 4.2. On the other hand, we have that $\Lambda\otimes_k\Gamma$ is Auslander d-Gorenstein by Proposition 4.12. Hence [8, Corollary 2.2] gives the displayed degree-d equality. Finally, it follows from [3, Proposition 2.2] that $\Lambda\otimes_k\Gamma$ is not Iwanaga–Gorenstein. □

The following example illustrates the preceding results.

Example 4.15. Let Σ_n and W_n be as in Example 4.10, and put $\Gamma_m:=k[\varepsilon]/(\varepsilon^m)$ for $m\geq 2$. The algebra Γ_m is symmetric. Proposition 4.14 gives the explicit two-parameter family

$$\Sigma_n\otimes_k\Gamma_m\qquad(n\geq 4\ \text{and}\ \ m\geq 2)$$

of non-Iwanaga–Gorenstein Cohen–Macaulay algebras satisfying the degree-two equality. For each n and m, one may choose a dualizing bimodule whose underlying right module is $W_n\otimes_k$

Γ_m; moreover, we have

$$W_n \otimes \Gamma_m \in \Omega^2_{\Sigma_n \otimes \Gamma_m}(\mathrm{mod}(\Sigma_n \otimes \Gamma_m)),$$
$$W_n \otimes \Gamma_m \notin \Omega^3_{\Sigma_n \otimes \Gamma_m}(\mathrm{mod}(\Sigma_n \otimes \Gamma_m))$$

by Theorem 4.9. Thus the positivity of both d and e in Theorem 4.13 cannot be omitted.

On the other hand, let $n, r \geq 4$ and consider $\Sigma_n \otimes_k \Sigma_r$. Both factors satisfy the degree-two equality. However, we have

$$\mathrm{CM}(\Sigma_n \otimes \Sigma_r) \neq \Omega^4_{\Sigma_n \otimes \Sigma_r}(\mathrm{mod}(\Sigma_n \otimes \Sigma_r)).$$

The module $W_n \otimes \Sigma_r$ is Cohen–Macaulay by Theorem 4.7. If $W_n \otimes \Sigma_r \in \Omega^4_{\Sigma_n \otimes \Sigma_r}(\mathrm{mod}(\Sigma_n \otimes \Sigma_r))$, then Lemma 2.4(3)(4) yields

$$W_n^{\dim_k \Sigma_r} \in \Omega^4_{\Sigma_n}(\mathrm{mod}\,\Sigma_n) \subseteq \Omega^3_{\Sigma_n}(\mathrm{mod}\,\Sigma_n).$$

Closure under direct summands contradicts Example 4.10.

Finally, put

$$\Sigma := k[x,y]/(x,y)^2 \qquad \text{and} \qquad \Pi := kQ \text{ with } Q: \ 1 \longrightarrow 2 \longleftarrow 3.$$

Then $\mathcal{P}^{<\infty}(\Sigma) = \mathrm{proj}\,\Sigma$, $\mathcal{I}^{<\infty}(\Sigma) = \mathrm{add}\,D\Sigma$ and $\mathrm{fidim}\,\Sigma = 0$. Thus $D\Sigma$ is dualizing and

$$\mathrm{CM}_{D\Sigma}(\Sigma) = \mathrm{mod}\,\Sigma = \Omega^0_\Sigma(\mathrm{mod}\,\Sigma).$$

Since Π is hereditary, the regular bimodule Π is dualizing by [8, Example 1.3(2)], and

$$\mathrm{CM}_\Pi(\Pi) = {}^\perp\Pi = \mathrm{proj}\,\Pi = \Omega^1_\Pi(\mathrm{mod}\,\Pi).$$

It follows from Corollary 4.2 that $\Sigma \otimes_k \Pi$ is Cohen–Macaulay with dualizing bimodule $D\Sigma \otimes_k \Pi$. Nevertheless, we claim

$$\mathrm{CM}_{D\Sigma\otimes\Pi}(\Sigma \otimes \Pi) \neq \Omega^1_{\Sigma\otimes\Pi}(\mathrm{mod}(\Sigma \otimes \Pi)).$$

Indeed, equality would imply

$$\begin{aligned} D\Sigma \otimes \Pi &\in \Omega^1_{\Sigma\otimes\Pi}(\mathrm{mod}(\Sigma\otimes\Pi)), \\ (D\Sigma)^{\dim_k \Pi} &\cong \mathrm{Res}^{\Sigma\otimes\Pi}_\Sigma(D\Sigma\otimes\Pi) \in \Omega^1_\Sigma(\mathrm{mod}\,\Sigma) \qquad \text{(by Lemma 2.4(3)(4))}, \\ D\Sigma &\in \Omega^1_\Sigma(\mathrm{mod}\,\Sigma). \end{aligned}$$

Then Σ is self-injective by Theorem 4.9, which is a contradiction with the fact that Σ is not self-injective.

5. CM-finiteness and CM-freeness

In this section, we study Conjecture 1.3.

Definition 5.1. (1) ([9]) An algebra Λ is *Cohen–Macaulay finite* (*CM-finite* for short) if $\mathrm{Gproj}\,\Lambda$ has only finitely many isomorphism classes of indecomposable objects. It is *Cohen–Macaulay free* (*CM-free* for short) if $\mathrm{Gproj}\,\Lambda = \mathrm{proj}\,\Lambda$.

(2) A Cohen–Macaulay algebra Λ with a dualizing bimodule W is *relative CM-finite* if $\mathrm{CM}_W(\Lambda)$ has only finitely many isomorphism classes of indecomposable objects. It is *relative CM-free* if $\mathrm{CM}_W(\Lambda) = \mathrm{proj}\,\Lambda$.

By the independence noted after Definition 1.1, both notions in Definition 5.1(2) do not depend on the chosen dualizing bimodule. If Λ is a Gorenstein algebra, then Λ is a dualizing bimodule. By [13, Corollary 11.5.3], we have $\mathrm{CM}_\Lambda(\Lambda) = {}^\perp\Lambda = \mathrm{Gproj}\,\Lambda$. Thus, in this case, relative CM-finite and relative CM-free algebras are exactly CM-finite and CM-free algebras, respectively.

5.1. **CM-finiteness.** It was proved in [29, Theorem 3.7(1)] that if S is a commutative Artin ring and Σ/S is a separable Frobenius extension with Σ an S-algebra, then Σ is CM-finite if and only if so is S. More generally, under the Krull–Remak–Schmidt hypothesis, Liu [23, Proposition 6.6 and Remark 6.7(1)] obtained ascent under separability and descent under splitting. In particular, CM-finiteness is invariant under separable split Frobenius extensions. Hence Liu's result applies to $\Lambda \longrightarrow \Lambda \otimes_k \Gamma$ when Γ is a separable Frobenius algebra.

The following result proves reflection from the tensor product to both factors, in which neither that a factor is Frobenius nor that the canonical extension is separable.

Theorem 5.2. *Let Λ and Γ be algebras.*

(1) If $\Lambda \otimes_k \Gamma$ is CM-finite, then so are both Λ and Γ.
(2) Suppose that Λ and Γ are Cohen–Macaulay algebras. If $\Lambda \otimes_k \Gamma$ is relative CM-finite, then so are both Λ and Γ.

Proof. Let

$$\mathcal{C}_\Lambda \subseteq \operatorname{mod}\Lambda, \qquad \mathcal{C}_\Gamma \subseteq \operatorname{mod}\Gamma \qquad \text{and} \qquad \mathcal{C}_{\Lambda\otimes\Gamma} \subseteq \operatorname{mod}(\Lambda \otimes_k \Gamma)$$

be additive subcategories closed under direct summands. Assume that

$$\begin{aligned} X \in \mathcal{C}_\Lambda \quad &\Longrightarrow \quad X \otimes_k \Gamma \in \mathcal{C}_{\Lambda\otimes\Gamma}, \\ Y \in \mathcal{C}_\Gamma \quad &\Longrightarrow \quad \Lambda \otimes_k Y \in \mathcal{C}_{\Lambda\otimes\Gamma}. \end{aligned} \tag{$*$}$$

We claim that if $\mathcal{C}_{\Lambda\otimes\Gamma}$ has only finitely many non-isomorphism indecomposable modules, then so have both $\mathcal{C}_\Lambda$ and $\mathcal{C}_\Gamma$.

Suppose that $Z_1, \ldots, Z_t$ are all non-isomorphism indecomposable modules in $\mathcal{C}_{\Lambda\otimes\Gamma}$. Let X be an indecomposable object of $\mathcal{C}_\Lambda$. By $(*)$ and the Krull–Schmidt theorem, there are indices $i_1, \ldots, i_s \in \{1, \ldots, t\}$ such that

$$X \otimes_k \Gamma \cong \bigoplus_{j=1}^{s} Z_{i_j}.$$

Put $d := \dim_k \Gamma > 0$. It follows from Lemma 2.4(1)(3) that

$$X^{\oplus d} \cong \operatorname{Res}_\Lambda^{\Lambda\otimes\Gamma}(X \otimes_k \Gamma) \cong \bigoplus_{j=1}^{s} \operatorname{Res}_\Lambda Z_{i_j}.$$

Since X is indecomposable and $d > 0$, Krull–Schmidt uniqueness implies that X is isomorphic to an indecomposable direct summand of $\operatorname{Res}_\Lambda Z_i$ for some i. Each of the finitely many finitely generated modules $\operatorname{Res}_\Lambda Z_i$ has only finitely many indecomposable direct summands. Thus $\mathcal{C}_\Lambda$ has only finitely many non-isomorphism indecomposable modules. The assertion for $\mathcal{C}_\Gamma$ follows similarly. The claim is proved.

(1) Put

$$\mathcal{C}_\Lambda := \operatorname{Gproj}\Lambda, \qquad \mathcal{C}_\Gamma := \operatorname{Gproj}\Gamma \qquad \text{and} \qquad \mathcal{C}_{\Lambda\otimes\Gamma} := \operatorname{Gproj}(\Lambda \otimes_k \Gamma).$$

By [17, Proposition 2.6(1)], both implications in $(*)$ hold true. The assertion follows.

(2) Choose dualizing bimodules U and V for Λ and Γ, respectively, and set

$$\mathcal{C}_\Lambda := \operatorname{CM}_U(\Lambda), \qquad \mathcal{C}_\Gamma := \operatorname{CM}_V(\Gamma) \qquad \text{and} \qquad \mathcal{C}_{\Lambda\otimes\Gamma} := \operatorname{CM}_{U\otimes_k V}(\Lambda \otimes_k \Gamma).$$

By Theorem 4.7, both implications in $(*)$ hold true. The assertion follows. □

The following example illustrates that the converse of both assertions in Theorem 5.2 does not hold true in general, even over an algebraically closed field.

Example 5.3. Let k be an algebraically closed field and put

$$\Lambda = \Gamma := k[\varepsilon]/(\varepsilon^2).$$

This is a representation-finite algebra. Since Λ is self-injective, any module is Cohen–Macaulay. Thus both Λ and Γ are CM-finite. Their tensor product is the symmetric algebra

$$\Lambda \otimes_k \Gamma = k[x, y]/(x^2, y^2).$$

All modules are Cohen–Macaulay because $\Lambda \otimes_k \Gamma$ is self-injective. However, $\Lambda \otimes_k \Gamma$ is representation-infinite, and therefore is not CM-finite.

5.2. **CM-freeness.** Recall that a field is *perfect* if any finite algebraic extension is separable; equivalently, the field has characteristic zero, or it has characteristic $p > 0$ and its Frobenius map is surjective.

The following example disproves Conjecture 1.3, even when Λ and Γ are Gorenstein.

Example 5.4. Let k be an imperfect field of characteristic $p > 0$. By definition, the Frobenius map $k \to k$ via $x \mapsto x^p$, is not surjective. Choose $a \in k \setminus k^p$, and let α be a root of $t^p - a$ in an algebraic closure of k. We use $k(\alpha)$ to denote the smallest subfield of that algebraic closure containing k and α. Since a is not a pth power in k, we have

$$K := k(\alpha) \cong k[t]/(t^p - a) \qquad \text{and} \qquad [K : k] = p.$$

Take $\Lambda = \Gamma := K$. Then both Λ and Γ are CM-free. On the other hand, we have

$$\begin{aligned} K \otimes_k K &\cong K \otimes_k k[t]/(t^p - a) \\ &\cong K[t]/(t^p - a). \end{aligned}$$

In characteristic p, the equality $a = \alpha^p$ gives

$$t^p - a = (t - \alpha)^p.$$

If ε is the residue class of $t - \alpha$, then

$$K \otimes_k K \cong K[t]/((t - \alpha)^p) \cong K[\varepsilon]/(\varepsilon^p).$$

Here $\varepsilon \neq 0$ and $\varepsilon^p = 0$. Thus this algebra is not semisimple. However, it is a symmetric algebra, in particular, it is self-injective. It follows from [13, Corollary 11.5.3 and Theorem 12.3.1] that $\operatorname{Gproj}(K \otimes_k K) = \operatorname{mod} K \otimes_k K$. Notice that the simple $K \otimes_k K$-module $K[\varepsilon]/(\varepsilon)$ is not projective, so $K \otimes_k K$ is not CM-free in either sense.

Example 5.4 leads to the following modified form of Conjecture 1.3.

Conjecture 5.5. *For a perfect field k, both Λ and Γ are CM-free if and only if so is $\Lambda \otimes_k \Gamma$.*

In Conjecture 5.5, the necessity holds true [17, Corollary 4.8], and the sufficiency remains open. Note that this conjecture is not a formal consequence of [17, Corollary 4.9].

Lemma 5.6. *Let Λ be Cohen–Macaulay with dualizing bimodule W. Then Λ is relative CM-free if and only if* $\operatorname{gldim} \Lambda < \infty$. *In this case, we have* $\operatorname{gldim} \Lambda = \operatorname{id}_\Lambda W$.

Proof. Assume that Λ is relative CM-free (i.e., $\mathrm{CM}_W(\Lambda) = \operatorname{proj} \Lambda$) and $\operatorname{id}_\Lambda W = d$. Then for any $M \in \operatorname{mod} \Lambda$, dimension shifting gives $\operatorname{Ext}^{\geq 1}_\Lambda(\Omega^d_\Lambda M, W) = 0$, and thus $\Omega^d_\Lambda M \in {}^\perp W = \mathrm{CM}_W(\Lambda) = \operatorname{proj} \Lambda$. It implied $\operatorname{pd}_\Lambda M \leq d$, and hence $\operatorname{gldim} \Lambda \leq d$. Consequently, $\operatorname{gldim} \Lambda = d$.

Conversely, suppose $\operatorname{gldim} \Lambda < \infty$. Then Λ_Λ is strong cotilting. Since W is also strong, we have $\operatorname{add} W = \operatorname{add} \Lambda$, and hence

$$\mathrm{CM}_W(\Lambda) = {}^\perp W = {}^\perp \Lambda.$$

Since $\operatorname{gldim} \Lambda < \infty$, it is easy to see that ${}^\perp \Lambda = \operatorname{proj} \Lambda$. Thus $\mathrm{CM}_W(\Lambda) = \operatorname{proj} \Lambda$ and Λ is relative CM-free. ∎

Recall that an algebra Σ is called *separable* if it is projective as a Σ^e-module; equivalently, the multiplication map $\Sigma^e \to \Sigma$ splits as a Σ-bimodule map; see [27, Section 10.2]. If k is perfect, then any semisimple algebra is separable [27, Proposition 10.7].

Note that relative CM-free modules are exactly CM-free modules over Gorenstein algebras. Assertion (2) in the following result proves Conjecture 5.5 for Gorenstein algebras.

Theorem 5.7. *Let Λ and Γ be Cohen–Macaulay algebras.*

(1) If $\Lambda \otimes_k \Gamma$ is relative CM-free, then so are both Λ and Γ.

(2) Suppose that

$$(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Gamma/\operatorname{rad}\Gamma)$$

is semisimple. If both Λ and Γ are relative CM-free, then so are both $\Lambda \otimes_k \Gamma$, and

$$\operatorname{gldim}\Lambda \otimes_k \Gamma = \operatorname{gldim}\Lambda + \operatorname{gldim}\Gamma.$$

Consequently, over a perfect field, $\Lambda \otimes_k \Gamma$ is relative CM-free if and only if so are both Λ and Γ.

Proof. (1) Let U and V be dualizing bimodules for Λ and Γ, respectively. Assume that $\Lambda \otimes_k \Gamma$ is relative CM-free and let $X \in \mathrm{CM}_U(\Lambda)$. Since $\Gamma \in \mathrm{CM}_V(\Gamma)$, we have

$$\begin{aligned} X \in \mathrm{CM}_U(\Lambda) &\Longrightarrow X \otimes_k \Gamma \in \mathrm{CM}_{U\otimes_k V}(\Lambda \otimes_k \Gamma) \quad \text{(by Theorem 4.7)} \\ &\Longrightarrow X \otimes_k \Gamma \in \operatorname{proj}\Lambda \otimes_k \Gamma \quad \text{(since } \Lambda \otimes_k \Gamma \text{ is relative CM-free)} \\ &\Longrightarrow X^{\dim_k \Gamma} \cong \operatorname{Res}^{\Lambda}_{\Lambda} \otimes_k \Gamma(X \otimes_k \Gamma) \in \operatorname{proj}\Lambda \quad \text{(by Lemma 2.4(2)(3))} \\ &\Longrightarrow X \in \operatorname{proj}\Lambda. \end{aligned}$$

Thus Λ is relative CM-free. Similarly, we get that Γ is also relative CM-free..

(2) Suppose that both Λ and Γ are relative CM-free. Then $a := \operatorname{gldim}\Lambda < \infty$ and $b := \operatorname{gldim}\Gamma < \infty$ by Lemma 5.6. Put

$$I := (\operatorname{rad}\Lambda) \otimes_k \Gamma + \Lambda \otimes_k (\operatorname{rad}\Gamma).$$

Then the ideal I of $\Lambda \otimes_k \Gamma$ is nilpotent, and so $I \subseteq \operatorname{rad}(\Lambda \otimes_k \Gamma)$. Since

$$(\Lambda \otimes_k \Gamma)/I \cong (\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Gamma/\operatorname{rad}\Gamma)$$

is semisimple by assumption, we have $\operatorname{rad}(\Lambda \otimes_k \Gamma) \subseteq I$. Therefore $I = \operatorname{rad}(\Lambda \otimes_k \Gamma)$. Note that for any Artin algebra, its global dimension is equal to the projective dimension of the semisimple quotient. Thus we have

$$\begin{aligned} &\operatorname{gldim}\Lambda \otimes_k \Gamma = \operatorname{pd}_{\Lambda\otimes_k\Gamma}((\Lambda \otimes_k \Gamma)/\operatorname{rad}(\Lambda \otimes_k \Gamma)) \\ &= \operatorname{pd}_\Lambda(\Lambda/\operatorname{rad}\Lambda) + \operatorname{pd}_\Gamma(\Gamma/\operatorname{rad}\Gamma) \quad \text{(by Lemma 2.3)} \\ &= \operatorname{gldim}\Lambda + \operatorname{gldim}\Gamma = a + b < \infty, \end{aligned}$$

which implies that $\Lambda \otimes_k \Gamma$ is relative CM-free.

If k is perfect, then any finite-dimensional semisimple k-algebra is separable by [27, Proposition 10.7]. Put $\overline{\Lambda} := \Lambda/\operatorname{rad}\Lambda$ and $\overline{\Gamma} := \Gamma/\operatorname{rad}\Gamma$. Tensoring splittings of the two multiplication maps gives a bimodule splitting of

$$(\overline{\Lambda} \otimes_k \overline{\Gamma})^e \longrightarrow \overline{\Lambda} \otimes_k \overline{\Gamma}.$$

So $\overline{\Lambda} \otimes_k \overline{\Gamma}$ is separable and hence semisimple. Now the last assertion follows from assertions (1) and (2). □

Remark 5.8. (1) Theorem 5.7 gives the tensor-product property for relative CM-freeness. It does not settle Conjecture 5.5, because outside the Iwanaga–Gorenstein case the category $\mathrm{CM}_W(\Lambda)$ need not be the category of Gorenstein projective modules.

(2) Example 5.4 shows that the semisimplicity hypothesis in Theorem 5.7(2) cannot be omitted: there both semisimple quotients are K, but their tensor product is not semisimple. The assumption of perfect-field is suitable. The global-dimension equality in Theorem 5.7(2) is [1, Theorem 16]. See also [12, Corollary 18].

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