ON THE RINGS OF PROJECTIVE CHARACTERS OF ABELIAN GROUPS AND DIHEDRAL GROUPS

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Abstract. In this paper, for abelian groups and dihedral groups, we study the tensor products of their irreducible projective representations. By explicitly determining the decompositions of these tensor products, we obtain the structures of the rings of projective characters.

1. Introduction

Let G be a finite group. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be a multiplier that represents an element of order d in $H^2(G, \mathbb{C}^\times)$. In this paper, a *projective representation* (π, V, α) of G over $\mathbb{C}$ of degree n and multiplier α is a map $\pi : G \to \mathrm{GL}(V)$ such that $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$, where V is an n-dimensional vector space over $\mathbb{C}$. Changing α by a suitable coboundary, we may assume that $\alpha^d = 1$. In particular, α is *unitary*, i.e., $|\alpha(x, y)| = 1$ for any $x, y \in G$.

Denote by Rep_G^α the category of projective representations of G with multiplier α. In [3, Section 2], Cheng constructed a character theory to study Rep_G^α by exploiting the analogy between projective representations and linear representations. Let (π, V, α) be a projective representation of G. The *character* of (π, V, α) $\chi_\pi : G \to \mathbb{C}$ is defined by the equation

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)) \text{ for all } g \in G.$$

Up to isomorphism, each object in Rep_G^α is determined by its character. Let (π_i, V_i, α) $(i \in I)$ be the simple objects in Rep_G^α and their corresponding characters be χ_i. We form a free group

$$R(G, \alpha) = \oplus_{i\in I}\mathbb{Z} \cdot \chi_i.$$

If $\alpha = 1$, then $R(G) := R(G, \alpha)$ is the usual ring of characters of G. Define

$$\mathcal{R}(G, \alpha) = \oplus_{j=0}^{d-1} R(G, \alpha^j).$$

If (π, V, α) and (π', V', α') are two projective representations of G with characters χ and χ' respectively, then the tensor product $V \otimes V'$ is in the obvious way a projective representation of G with multiplier $\alpha\alpha'$, and its character is $\chi\chi'$. Therefore, the group $\mathcal{R}(G, \alpha)$ has a ring structure. Moreover, it is an algebra (graded by $\mathbb{Z}/d\mathbb{Z}$) over the ring of linear characters $R(G)$.

*Keywords: projective representations of finite groups; Schur multiplier; abelian group; dihedral group
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As a consequence of Brauer's Theorem, the space $\operatorname{Spec} R(G)$ is connected. See for example [11, Section 11.4] for a proof of this fact. The next question is: what can we say about the space $\operatorname{Spec}\mathcal{R}(G,\alpha) \to \operatorname{Spec} R(G)$?

In this paper, in the case G is an abelian group or a dihedral group and $d = 2$, we determine the structure of $R(G,\alpha)$ as an $R(G)$-module and the structure of $\mathcal{R}(G,\alpha)$ as an $R(G)$-algebra. In particular, we prove the following result.

Theorem 1.1. *With the notation as above and let $d = 2$.*

1. *If G is an abelian group, then $\mathcal{R}(G,\alpha) \cong R(G)[X]$ with $X^2 = \psi \sum_{\chi \in H^\perp} \chi$, where $H \subset G$ is a subgroup, $\psi \in \widehat{G} := \operatorname{Hom}(G,\mathbb{C}^\times)$. Both H and ψ are determined by α. Moreover, the number of irreducible projective representations (up to isomorphism) in $\operatorname{Rep}_G^\alpha$ is $|H|$ (the cardinality of H).*
2. *If $G = D_{4n}$ is a dihedral group of order $4n$, then $R(G,\alpha)$ is generated by one element X as an $R(G)$-module. This generator X can be the character of any irreducible projective representation of G with multiplier α and X^2 depends on this choice.*

The ideas of the proofs for the two cases are different. In the abelian groups case, the proof is theoretical and we also deduce properties of Schur multipliers. Moreover, we reprove some results of Mumford (Remarks 2.7 and 2.8). This is the content of Section 2.

In the dihedral groups case, the proof is computational and we obtain the structure of $\mathcal{R}(G,\alpha)$ by listing all simple objects in $\operatorname{Rep}_G^\alpha$ and decomposing their tensor products explicitly. Our method here applies to more general groups. We explain this in Section 3.2 and classify projective representations of groups of type $C_m \ltimes C_p$ (Proposition 3.1). This is the content of Section 3.

Remark 1.2. In the dihedral groups case, if $4 \nmid |G|$, then $H^2(G,\mathbb{C}^\times)$ is trivial and there is nothing to prove. If $4 \mid |G|$, then $H^2(G,\mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$. Therefore, Theorem 1.1 contains all possible cases for dihedral groups.

Remark 1.3. For general G, it is not easy to give an explicit description of the graded $R(G)$-algebra $\mathcal{R}(G,\alpha)$. For the above two types of groups we consider in this paper, their projective representations are all induced (with respect to the given multiplier) from one-dimensional linear representations of special subgroups. This makes the argument and the computation simpler. Nevertheless, in Section 2.3, for an arbitrary finite group G, we compute the characters of symmetric powers and exterior powers of a given projective representation of G and obtain a relation between $R(G,\alpha)$ and $R(G,\alpha^n)$ for n coprime to $|G|$.

Remark 1.4. In Section 3.3, we generalize the second part of Theorem 1.1 to the case of the infinite dihedral group. In order to do so, we use a twisted version of the Peter-Weyl Theorem. Since the authors could not find a reference for this result, we provide a detail proof in the Appendix.

Remark 1.5. If $d \geq 3$, $\mathcal{R}(G,\alpha)$ in general cannot be generated by one element as an $R(G)$-algebra. We would like to propose the following question: for what kind of pairs (G,α), can $R(G,\alpha)$ be generated by one element as an $R(G)$-module and can $\mathcal{R}(G,\alpha)$ be generated by one element as an $R(G)$-algebra?

Remark 1.6. One of the motivations for this paper is to generalize the results in [2], where a special projective representation of $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is studied. With the results in this paper, one may generalize the results in [2, Section 2] to all abelian groups and groups of type $C_m \ltimes C_p$. This is done in [4].

2. The case of abelian groups

2.1. **Basic properties.** In this section, G is a finite abelian group. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be a unitary cocycle. Let (π, V, α) be an irreducible projective representation of G with multiplier α. Let $(\overline{\pi}, \overline{V}, \overline{\alpha} = \alpha^{-1})$ be the projective representation defined by $\overline{V} = V$ and $\overline{\pi}(g) = \overline{\pi(g)}$, where the last $^-$ means complex conjugation. Then the tensor product $\pi \otimes \overline{\pi}$ is a projective representation of G with multiplier $\alpha\overline{\alpha} = 1$, i.e., it is a linear representation. The character of $\pi \otimes \overline{\pi}$ is $\chi_\pi \chi_{\overline{\pi}}$, where χ_π and $\chi_{\overline{\pi}}$ are the characters of π and $\overline{\pi}$ respectively. Let $\rho : G \to \mathbb{C}^\times$ be a one-dimensional linear representation that appears in $\pi \otimes \overline{\pi}$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\rho, \pi \otimes \overline{\pi}) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \overline{\chi_\pi(g) \chi_{\overline{\pi}}(g)} = \dim_{\mathbb{C}} \mathrm{Hom}(\pi \otimes \rho, \pi) \leq 1. \tag{2.1}$$

Therefore, the dimension must be one, i.e., every one-dimensional representation has multiplicity at most one in $\pi \otimes \overline{\pi}$. Define

$$\mathcal{H}(V) := \{\rho \in \widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times) \mid \dim_{\mathbb{C}} \mathrm{Hom}(\rho, \pi \otimes \overline{\pi}) = 1\}.$$

Proposition 2.1. *With the notation as above, $\mathcal{H}(V)$ is a subgroup of $\widehat{G}$. Moreover, $\overline{\mathcal{H}(V)} := \{\overline{\rho} \mid \rho \in \mathcal{H}(V)\}$ coincides with $\mathcal{H}(V)$.*

Proof. By equation (2.1), a one-dimensional linear representation ρ is an element of $\mathcal{H}(V)$ if and only if $\pi \otimes \rho$ is isomorphic to π as projective representations. From this observation, it is easy to see that $\mathcal{H}(V)$ is a subgroup of $\widehat{G}$.

Moreover, if $\rho \in \mathcal{H}(V)$, then $V \otimes \overline{\rho} \cong (V \otimes \rho) \otimes \overline{\rho} \cong V \otimes (\rho \otimes \overline{\rho}) \cong V$. Thus $\overline{\rho} \in \mathcal{H}(V)$ and the last claim follows. □

Let $H(V) \subset G$ be $\mathcal{H}(V)^\perp := \{g \in G \mid \rho(g) = 1 \text{ for any } \rho \in \mathcal{H}(V)\}$. Then

$$[G : H(V)] = (\dim_{\mathbb{C}} V)^2. \tag{2.2}$$

By [3, Propositions 2.3, 2.14], the number of irreducible projective representations in Rep_G^α is $|H(V)|$. Let (π', V', α) be another irreducible projective representation of G with multiplier α. Considering the tensor product $\overline{V} \otimes V'$ as a projective representation of G with trivial multiplier, by the same argument as above, it is easy to see that $V' \cong V \otimes \rho$ for some $\rho \in \widehat{G}$. Therefore,

$$V \otimes \overline{V} \cong V' \otimes \overline{V}'.$$

Thus $H(V) = H(V')$. This group is independent of the choice of V. From now on, we denote it by H_α. The product $(V \otimes \overline{V})|_{H_\alpha}$ decomposes as $(\dim_{\mathbb{C}} V)^2$-copies of the trivial linear representation of H_α.

Proof of the first part of Theorem 1.1. In the case $\alpha^2 = 1$, V and $\overline{V}$ are both projective representations of G with the same multiplier α. Therefore, $\overline{V} = V \otimes \psi^{-1}$ for some $\psi \in \widehat{G}$. Applying the above discussion to this case, the claim follows. □

Lemma 2.2. *With the notation as above, $\alpha|_{H_\alpha \times H_\alpha}$ is a coboundary.*

Proof. Suppose that $\alpha|_{H_\alpha\times H_\alpha}$ is not a coboundary, then the irreducible projective representations of H_α with multiplier $\alpha|_{H_\alpha\times H_\alpha}$ have dimension at least two. Let $W \subset V|_{H_\alpha}$ be such an irreducible object. Then $W \otimes \overline{W} \subset (V \otimes \overline{V})|_{H_\alpha}$ is a sub-representation. By the same argument as before, $W \otimes \overline{W} \cong \oplus_{j\in J}\chi_j$, where χ_j are different one-dimensional linear representations of H_α. This contradicts to the fact that $(V \otimes \overline{V})|_{H_\alpha}$ is a direct sum of trivial representations of H_α. □

Consider the map $f : G \times G \to \mathbb{C}^\times$ defined by

$$f(x, y) = \frac{\alpha(x, y)}{\alpha(y, x)}$$

for any $x, y \in G$. It is easy to check that f is a bi-homomorphism and thus induces a homomorphism $\lambda : G \to \widehat{G}$.

Definition 2.3. We say that $\alpha \in Z^2(G, \mathbb{C}^\times)$ is *non-degenerate* if the homomorphism $\lambda : G \to \widehat{G}$ is an isomorphism.

Proposition 2.4. *For any $h \in H_\alpha$ and $g \in G$, $\alpha(h, g) = \alpha(g, h)$. In particular, the cocycle α is non-degenerate if and only if H_α is trivial.*

Proof. A subgroup X of G is called *α-symmetric* if $\alpha(x, y) = \alpha(y, x)$ for any $x, y \in X$. Let K be a maximal α-symmetric subgroup of G such that $H_\alpha \subset K$. Such K exists since H_α is α-symmetric by Lemma 2.2.

By [3, Proposition 2.14], $|K| \cdot \dim_{\mathbb{C}} V = |G|$ and $V \cong \alpha\,\mathrm{Ind}_K^G\,\chi$. Here χ is a one-dimensional projective representation of K with multiplier $\alpha|_{K\times K}$ and $\alpha\,\mathrm{Ind}$ is the induction of projective representations with respect to α (see for example [3, Section 2.2]). Moreover, by [3, Corollary 2.11],

$$V|_K \cong \oplus_{s\in G/K}\chi^s,$$

where $\chi^s(k) = \frac{\alpha(s^{-1},k)}{\alpha(k,s^{-1})}\chi(k)$ for $k \in K$. Therefore,

$$(V \otimes \overline{V})|_{H_\alpha} = \oplus_{s,t\in G/K}\chi^s|_{H_\alpha} \otimes \overline{\chi^t}|_{H_\alpha}.$$

By the definition of H_α, $\chi^s|_{H_\alpha} \otimes \overline{\chi^t}|_{H_\alpha} = 1$ for any $s, t \in G/K$. Let t be the unity element, then one obtains that $\alpha(s^{-1}, h) = \alpha(h, s^{-1})$ for any $h \in H_\alpha$ and $s \in G/K$. The first claim follows and $H_\alpha \subset \mathrm{Ker}(\lambda)$.

Moreover, from the above discussion, one sees that $(V \otimes \overline{V})|_{\mathrm{Ker}(\lambda)}$ is a direct sum of trivial representations of $\mathrm{Ker}(\lambda)$. Since H_α is maximal with respect to this property, $\mathrm{Ker}(\lambda) \subset H_\alpha$. The second claim follows. □

2.2. **On non-degenerate cocycles.** The cocycle $\alpha \in Z^2(G, \mathbb{C}^\times)$ defines a group G_α which sits in the short exact sequence

$$1 \to \mathbb{C}^\times \to G_\alpha \to G \to 0.$$

Lemma 2.5. *With the notation as above, the following conditions are equivalent.*

(1) *$\mathbb{C}^\times = \mathrm{Cent}(G_\alpha)$, the center of G_α.*
(2) *α is non-degenerate.*
(3) *There are subgroups K_1, K_2 of G such that $G = K_1 \oplus K_2$, K_1 and K_2 are maximal α-symmetric subgroups of G, and $K_2 = \widehat{K}_1$.*

Proof. The equivalence between (1) and (2) is clear. It is also easy to see that (3) implies (1) and (2). To obtain (3) from (2), one observes that $f : G \times G \to \mathbb{C}^\times$ induces a perfect pairing $f : K \times G/K \to \mathbb{C}^\times$ for any maximal α-symmetric subgroup K. $\square$

Theorem 2.6. *With the notation as above, there is a one-to-one correspondence between*

A: *the set of linear representations of G_α with $\mathbb{C}^\times$ acting as scalar multiplication,*

B: *the set of projective representations of G with multiplier α.*

*If α is non-degenerate, then up to isomorphism, there is only one irreducible linear representation of G_α with $\mathbb{C}^\times$ acting as scalar multiplication. Every object in set **A** is a direct sum of copies of this representation.*

Proof. The correspondence is as follows. For any $\rho : G_\alpha \to \mathrm{GL}(V)$ from set **A**, define $\pi : G \to \mathrm{GL}(V)$ by $\pi(g) = \rho(1, g)$. Then (π, V) is a projective representation of G with multiplier α. Conversely, for any (π, V, α) from set **B**, define $\rho : G_\alpha \to \mathrm{GL}(V)$ by $\rho(a, g) = a\pi(g)$ for $a \in \mathbb{C}^\times$ and $g \in G$. Then ρ is an object in set **A**. These two maps are certainly inverse of each other. The claim follows.

If α is non-degenerate, then up to isomorphism, there is only one irreducible projective representation of G with multiplier α. The second claim follows from the first. $\square$

Remark 2.7. Let K be an α-symmetric subgroup of G. The set $\tilde{K} = \{(1, k) \in G_\alpha \mid k \in K\}$ is a subgroup of G_α. It is isomorphic to its image in G. We call such a subgroup $\tilde{K}$ a *level subgroup*. With this convention,

$$\dim_{\mathbb{C}} V^{\tilde{K}} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Rep}_{\tilde{K}}}(\mathrm{id}, V|_{\tilde{K}}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Rep}^\alpha_K}(\mathrm{id}, V|_K). \tag{2.3}$$

The last dimension is one if V is irreducible, since in this case $V|_K \cong \otimes_{\chi \in \widehat{K}} \chi$. Note that the discussion still holds if we replace $\mathbb{C}$ by an algebraically closed field with characteristic not dividing $|G|$, one sees that Theorem 2.6 implies [8, Proposition 3].

Remark 2.8. The above discussion is closely related to the study of theta-groups ([8], [9, Sections 23, 24]). We explain another proof of [9, Theorem 2]. Let X be an abelian variety over the complex numbers $\mathbb{C}$. Let L be an ample line bundle over X. Then

$$K(L) := \{x \in X \mid L \text{ is invariant under the translation of } x\}$$

is a finite group and $|K(L)| = (\dim_{\mathbb{C}} H^0(X, L))^2$. Since $H^0(X, L)$ is a representation of the theta-group attached to (X, L) ([9, Page 295, Definition]), this representation must be irreducible if the theta-group is non-degenerate.

2.3. **Remarks on characters of symmetric and exterior powers.** In general, it is not easy to obtain an explicit description of $\mathcal{R}(G, \alpha)$ as a graded algebra over $R(G)$. Nevertheless, for any finite group G, we may compute explicitly the characters of symmetric and exterior powers of a given projective representation and obtain a relation between $R(G, \alpha)$ and $R(G, \alpha^n)$ for n coprime to $|G|$ (cf. [11, Chap. 9.1, Exercises]).

Let $\pi : G \to \mathrm{GL}(V)$ be a projective representation of G with unitary multiplier α and character χ. The projective representation $(\pi^{\otimes k}, W := V^{\otimes k}, \alpha^k)$ of G has two natural subprojective representations $\pi^k_S : G \to \mathrm{GL}(\mathrm{Sym}^k(V))$ and $\pi^k_A : G \to \mathrm{GL}(\mathrm{Alt}^k(V))$. Let χ^k_S and χ^k_A be the characters of π^k_S and π^k_A respectively. Define

$$S_T(\chi) = \sum_{k=0}^{\infty} \chi^k_S T^k, \quad A_T(\chi) = \sum_{k=0}^{\infty} \chi^k_A T^k,$$

where T is an indeterminate.

Lemma 2.9. *Let $g \in G$. Then*

$$S_T(\chi)(g) = \frac{1}{\det(1 - \pi(g)T)}, \quad A_T(\chi)(g) = \frac{1}{\det(1 + \pi(g)T)}.$$

Proof. Let (λ_i) be the eigenvalues of $\pi(g)$. Choose a basis (e_i) of V consisting eigenvectors of $\pi(g)$. Then $(e_{i_1,i_2,\cdots,i_k})_{i_1 \leq \cdots \leq i_k}$ form a basis of $\mathrm{Sym}^k V$, where

$$e_{i_1,i_2,\cdots,i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}}.$$

Thus $(\lambda_{i_1} \cdots \lambda_{i_k})_{i_1 \leq \cdots \leq i_k}$ are eigenvalues of $\pi_S^k(g)$ on $\mathrm{Sym}^k V$, i.e.,

$$\chi_S^k(g) = \sum_{i_1 \leq \cdots \leq i_k} \lambda_{i_1} \cdots \lambda_{i_k} \tag{2.4}$$

Therefore,

$$\begin{aligned} \frac{1}{\det(1 - \pi(g)T)} &= \prod_i \frac{1}{1 - \lambda_i T} \\ &= \prod_i (\sum_{k=0}^{\infty} \lambda_i^k T^k) \\ &= \sum_{k=0}^{\infty} (\sum_{i_1 \leq \cdots \leq i_k} \lambda_{i_1} \cdots \lambda_{i_k}) T^k = S_T(\chi)(g). \end{aligned} \tag{2.5}$$

The proof for $A_T(\chi)$ is similar, using the fact that $\frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}}$ with $i_1 < \cdots < i_k$ form a basis of $\mathrm{Alt}^k V$. ☐

Let k be a positive integer. Let f be a function on G. Define function $\Psi_\alpha^k(f)$ by

$$\Psi_\alpha^k(f)(g) = \alpha(g, g^{k-1}) \alpha(g, g^{k-2}) \cdots \alpha(g, g) f(g^k) \text{ for all } g \in G.$$

Lemma 2.10. *With the above notation,*

$$\begin{aligned} S_T(\chi) &= \exp(\sum_{k=1}^{\infty} \Psi_\alpha^k(\chi) T^k / k), \\ A_T(\chi) &= \exp(\sum_{k=1}^{\infty} (-1)^{k-1} \Psi_\alpha^k(\chi) T^k / k). \end{aligned} \tag{2.6}$$

Proof. We prove the lemma for $S_T(\chi)$. The other case is similar. It suffices to show that

$$\sum_{k=1}^{\infty} \Psi_\alpha^k(\chi)(g) T^k / k = -\log\det(1 - \pi(g)T).$$

Since $\log\det(1 - \pi(g)T) = \sum_i \log(1 - \lambda_i T)$, it suffices to show that

$$\Psi_\alpha^k(\chi)(g) = \sum_i \lambda_i^k.$$

This follows from the definition of $\Psi_\alpha^k(\chi)$ and the fact

$$\pi(g)^k = \alpha(g, g^{k-1})\alpha(g, g^{k-2}) \cdots \alpha(g, g)\pi(g^k).$$

Hence the lemma follows. □

Proposition 2.11. *With the above notation,*

$$\begin{aligned} n\chi_S^n &= \sum_{k=1}^{n} \Psi_a^k(\chi)\chi_S^{n-k}, \\ n\chi_A^n &= \sum_{k=1}^{n} (-1)^{k-1}\Psi_a^k(\chi)\chi_A^{n-k}. \end{aligned} \tag{2.7}$$

Proof. We only prove the first equality. The other case is similar. By the above computation, we have

$$\sum_{k=0}^{\infty} \chi_S^k T^k = \exp(\sum_{k=1}^{\infty} \Psi_\alpha^k(\chi)T^k/k).$$

Taking derivative with respect to T on both sides, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} k\chi_S^k T^{k-1} &= \exp(\sum_{k=1}^{\infty} \Psi_\alpha^k(\chi)T^k/k) \sum_{k=1}^{\infty} \Psi_\alpha^k(\chi)T^{k-1} \\ &= \sum_{k=0}^{\infty} \chi_S^k T^k \sum_{k=1}^{\infty} \Psi_\alpha^k(\chi)T^{k-1}. \end{aligned} \tag{2.8}$$

Comparing the coefficients of T^{n-1}, the proposition follows. □

Corollary 2.12. *With the above notation,*

(1) Ψ_α^n *sends* $R(G, \alpha)$ *to* $R(G, \alpha^n)$.
(2) *Let* χ *be an irreducible projective character in* $R(G, \alpha)$. *If* $(n, |G|) = 1$, *then* $\Psi_\alpha^n(\chi)$ *is an irreducible projective character in* $R(G, \alpha^n)$.

In particular, if $(n, |G|) = 1$, *then* Ψ_α^n *induces a bijection between* $R(G, \alpha)$ *and* $R(G, \alpha^n)$.

Proof. Using induction on n, the first claim follows easily from Proposition 2.11. For (2), we see that $\Psi_\alpha^n(\chi)$ is an element in $R(G, \alpha^n)$ by (1). Therefore, to show that it is an irreducible projective character, it suffices to show that $\Psi_\alpha^n(\chi)(1) \geq 0$ and $(\Psi_\alpha^n(\chi), \Psi_\alpha^n(\chi)) = 1$. By the assumptions on χ and n, these two conditions hold. The claim follows. □

3. The case of dihedral groups

3.1. **Projective representations of groups of type $C_m \ltimes C_p$.** In this section, G is a semidirect product of the type $C_m \ltimes C_p$, where p is a prime number, C_m and C_p are cyclic groups of order m and p respectively. Fix a presentation of G

$$G = \langle a, b \mid a^m = 1, b^p = 1, bab^{-1} = a^r \rangle,$$

where $r \in \mathbb{Z}_{\geq 0}$ and $r^p \equiv 1 \pmod{m}$. In this case, G is a metacyclic group. By [6, 2.11.3 Theorem],

$$H^2(G, \mathbb{C}^\times) = \begin{cases} 0 & \text{if } p \nmid (m, r-1), \\ \mathbb{Z}/p\mathbb{Z} & \text{if } p \mid (m, r-1). \end{cases}$$

Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ such that $\alpha|_{C_m \times C_m} = 1$. Let (π, V, α) be an irreducible projective representation of G with multiplier α and $\dim_{\mathbb{C}} V > 1$. Then $\dim_{\mathbb{C}} V = p$ by [3, Corollary 3.11]. By assumption, $\pi|_{C_m}$ is a linear representation. Let $\chi : C_m \to \mathbb{C}^\times$ be a one-dimensional linear representation of C_m with $\dim_{\mathbb{C}} \mathrm{Hom}_{C_m}(\chi, \pi|_{C_m}) \geq 1$. By Frobenius reciprocity [3, Remark 2.9], $\dim_{\mathbb{C}} \mathrm{Hom}_G(\alpha \, \mathrm{Ind}_{C_m}^G \chi, \pi) \geq 1$. Since both projective representations have dimension p and π is irreducible, π is isomorphic to $\alpha \, \mathrm{Ind}_{C_m}^G \chi$. By [3, Proposition 2.8],

$$(\alpha \, \mathrm{Ind}_{C_m}^G \chi)|_{C_m} \cong \oplus_{0 \leq i \leq p-1} \chi^{b^i},$$

where χ^h is the twist of χ with respect to α, i.e., $\chi^h(g) = \frac{\alpha(g, h^{-1})}{\alpha(h^{-1}, hgh^{-1})} \chi(hgh^{-1})$ for all $g \in G$. We may choose a basis $\{v_0, v_1, \ldots, v_{p-1}\}$ of V, where the operators $\pi(a)$ and $\pi(b)$ are given by

$$\pi(a) v_i = \chi^{b^i}(a) v_i \text{ for } 0 \leq i \leq p-1, \quad \pi(b) v_i = v_{i+1}.$$

Here the subscript i is considered modulo p. More precisely, as $p \times p$ matrices,

$$\begin{aligned} \pi(a) &= \mathrm{diag}(\chi(a), \chi^b(a), \ldots, \chi^{b^{p-1}}(a)), \\ \pi(b) &= \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.1}$$

Every element of G can be written uniquely as $a^i b^j$ with $0 \leq i \leq (m-1)$ and $0 \leq j \leq (p-1)$. Fix ζ a primitive l-th root of unity, where $l = (m, 1 + r + \cdots + r^{p-1})$. Define $\alpha : G \times G \to \mathbb{C}^\times$ by

$$\alpha(a^i b^j, a^{i'} b^{j'}) = \begin{cases} 1 & \text{if } j = 0, \\ \zeta^{i'(1 + r + \cdots + r^{j-1})} & \text{otherwise.} \end{cases}$$

By [6, 2.11.1 Lemma and 2.11.3 Theorem], this α is a well-defined element in $Z^2(G, \mathbb{C}^\times)$ and it represents a generator of $H^2(G, \mathbb{C}^\times)$.

Proposition 3.1. *Let η be a primitive m-th root of unity and $\zeta = \eta^{m'}$. Here $m' = m/l$. For $0 \leq i \leq (m-1)$, let $\chi_i : C_m \to \mathbb{C}^\times$ be the one-dimensional representation of C_m defined by $\chi_i(a) = \eta^i$. Each χ_i gives us a projective representation $(\pi_i = \alpha \, \mathrm{Ind}_{C_m}^G \chi_i, V_i, \alpha)$ as defined by equation (3.1). Moreover, π_i and π_j are isomorphic if and only if*

$$j \equiv -m'(r^s + r^{s+1} + \cdots + r^{p-1}) + r^s i \pmod{m}, \tag{3.2}$$

for some $s \in \mathbb{Z}$ and $0 \leq s \leq p-1$.

Proof. We only need to check the last claim. First assume that $\pi_i \cong \pi_j$. Then $\pi_i|_{C_m} \cong \pi_j|_{C_m}$ and

$$\{\chi_i, \chi_i^b, \ldots, \chi_i^{b^{p-1}}\} = \{\chi_j, \chi_j^b, \ldots, \chi_j^{b^{p-1}}\}.$$

Suppose that $\chi_j = \chi_i^{b^s}$, then equation (3.2) holds.

On the other hand, if i and j are related by equation (3.2), then $\chi_j = \chi_i^{b^s}$.

$$
\begin{aligned}
\dim_{\mathbb{C}} \operatorname{Hom}_G(\pi_i, \pi_j) &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} \pi_i(g) \overline{\operatorname{Tr} \pi_j(g)} \\
&= \frac{1}{|G|} \sum_{k=0}^{m-1} \operatorname{Tr} \pi_i(a^k) \overline{\operatorname{Tr} \pi_j(a^k)} \\
&= \frac{1}{|G|} \sum_{k=0}^{m-1} \sum_{t=0}^{p-1} \chi_i^{b^t}(a^k) \sum_{t'=0}^{p-1} \overline{\chi_j^{b^{t'}}(a^k)} \\
&= \frac{1}{p} \sum_{t=0}^{p-1} \sum_{t'=0}^{p-1} (\frac{1}{m} \sum_{k=0}^{m-1} \chi_i^{b^t}(a^k) \overline{\chi_j^{b^{t'}}(a^k)}) \\
&= \frac{1}{p} \sum_{t=0}^{p-1} \sum_{t'=0}^{p-1} (\dim_{\mathbb{C}} \operatorname{Hom}_{C_m}(\chi_i^{b^t}, \chi_j^{b^{t'}})).
\end{aligned} \tag{3.3}
$$

The claim follows since $\dim_{\mathbb{C}} \operatorname{Hom}_{C_m}(\chi_i^{b^t}, \chi_j^{b^{t'}}) = \begin{cases} 1 \text{ if } (t'-t) \equiv s \pmod{p}, \\ 0 \text{ otherwise.} \end{cases}$ □

Remark 3.2. It is easy to check that equation (3.2) defines an equivalence relation for elements in $\mathbb{Z}/m\mathbb{Z}$. Thus it gives us a partition of $\mathbb{Z}/m\mathbb{Z}$ into m/p equivalent classes, each class contains exactly p elements. Fix one element from each equivalence class, we obtain a subset $\{i_1, i_2, \ldots, i_{m/p}\} \subset \mathbb{Z}/m\mathbb{Z}$. Then the representations $\{\pi_{i_j} := \alpha \operatorname{Ind}_{C_m}^G \chi_{i_j}\}_{j=1,2,\ldots,m/p}$ is a complete set of irreducible projective representations in $\operatorname{Rep}_G^\alpha$. Moreover, the above discussion applies to α^n as well.

We have obtained a complete and explicit classification of projective representations of groups of type $C_m \ltimes C_p$. See also [5, Section 4] and the references there for similar results. One understands the decompositions of tensor products of projective representations by explicitly computing the corresponding characters. In particular, we have the following result.

Proposition 3.3. *With the notation as in Proposition 3.1, let $\rho_i : G \to \mathrm{GL}_p(\mathbb{C})$ be the linear G-representation $\operatorname{Ind}_{C_m}^G \chi_i$. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\pi_j, \rho_i \otimes \pi_k) \leq 1$$

for any $0 \leq i, j, k \leq (m-1)$.

Proof. By [3, Proposition 2.2],

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho_k) =& \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\, \pi_i(g) \overline{\mathrm{Tr}(\pi_j \otimes \rho_k)(g)} \\
=& \frac{1}{|G|} \sum_{l=0}^{m-1} (\sum_{t=0}^{p-1} \chi_i^{b^t}(a^l)) \overline{(\sum_{t'=0}^{p-1} \chi_j^{b^{t'}}(a^l) \sum_{s=0}^{p-1} \chi_{kr^s}(a^l))} \\
=& \frac{1}{|G|} \sum_{t=0}^{p-1} \sum_{t'=0}^{p-1} \sum_{s=0}^{p-1} \sum_{l=0}^{m-1} \chi_i^{b^t}(a^l) \overline{\chi_j^{b^{t'}}(a^l) \chi_{kr^s}(a^l)} \\
=& \frac{1}{p} \sum_{t=0}^{p-1} \sum_{t'=0}^{p-1} \sum_{s=0}^{p-1} \dim_{\mathbb{C}} \mathrm{Hom}_{C_m}(\chi_i^{b^t}, \chi_j^{b^{t'}} \otimes \chi_{kr^s}).
\end{aligned}$$

For a fixed s, there exists exactly one pair (t, t') with $0 \leq t, t' \leq (p-1)$, such that $\dim_{\mathbb{C}} \mathrm{Hom}_{C_m}(\chi_i^{b^t}, \chi_j^{b^{t'}} \otimes \chi_{kr^s}) = 1$. The proposition follows easily. $\square$

3.2. **Projective representations of finite dihedral groups.** In this section, G is a dihedral group of order $4n$. Fix a presentation of G

$$G = \langle a, b \mid a^{2n} = 1, b^2 = 1, bab = a^{-1} \rangle.$$

Every element of G can be written uniquely as $a^i b^j$ with $0 \leq i \leq (2n-1)$ and $b \in \{0, 1\}$. In this case $H^2(G, \mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$. The linear representations of G are well-known ([11, Section 5.3]). Fix ζ a primitive $2n$-th root of 1.

Proposition 3.4. *There exist $(n-1)$ two-dimensional irreducible linear representations of G, which are given by*

$$\begin{aligned} \rho_l : G &\to \mathrm{GL}_2(\mathbb{C}) \\ a^i b^j &\mapsto A_l^i B^j, \end{aligned} \tag{3.4}$$

where $1 \leq l \leq (n-1)$, $A_l = \begin{pmatrix} \zeta^l & 0 \\ 0 & \zeta^{-l} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

There exist four one-dimensional representations of G, which are given by

- $\rho_0 : a \mapsto 1, b \mapsto 1.$
- $\rho_{-1} : a \mapsto -1, b \mapsto 1.$
- $\rho_{-2} : a \mapsto 1, b \mapsto -1.$
- $\rho_{-3} : a \mapsto -1, b \mapsto -1.$

The above gives us a complete list of irreducible linear representations of $G = D_{4n}$.

Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be the cocycle defined by

$$\alpha(a^i b^j, a^{i'} b^{j'}) = \zeta^{ji'}.$$

Here $0 \leq i \leq (2n-1)$ and $b \in \{0, 1\}$. Then it is a unitary cocycle that represents the non-trivial element in $H^2(G, \mathbb{C}^\times)$. Applying Proposition 3.1, we obtain the following result.

Proposition 3.5. *There exist n two-dimensional irreducible projective representations of G with multiplier α, which are given by*

$$\begin{aligned}\pi_l : G &\to \mathrm{GL}_2(\mathbb{C}) \\ a^i b^j &\mapsto C_l^i B^j,\end{aligned} \tag{3.5}$$

where $1 \le l \le n$, $C_l = \begin{pmatrix} \zeta^l & 0 \\ 0 & \zeta^{1-l} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark 3.6. Note that for the above chosen α, $\alpha^2 \neq 1$. In order to be consistent with the notation in Section 1, one may change α by the coboundary $\mu : G \to \mathbb{C}^\times$ defined by

$$\mu(a^i) = \mu(a^i b) = \eta^{-i},$$

where η is a fixed square root of ζ and $0 \le i \le (2n-1)$.

Using the classification in Propositions 3.4 and 3.5, the following results follow from direct computation.

Corollary 3.7. *With the notation as above, the decompositions of tensor products of irreducible projective representations of $G = D_{4n}$ are the following.*

(1) *For $1 \le l \le n-1$, $\rho_0 \otimes \rho_l = \rho_{-2} \otimes \rho_l = \rho_l$, $\rho_{-1} \otimes \rho_l = \rho_{-3} \otimes \rho_l = \rho_{n-l}$.*
(2) *For $1 \le l, k \le n-1$,*

$$\rho_l \otimes \rho_k = \begin{cases} \rho_0 \oplus \rho_{-1} \oplus \rho_{-2} \oplus \rho_{-3} & \text{if } l+k = n \text{ and } l-k = 0, \\ \rho_{-1} \oplus \rho_{-3} \oplus \rho_{|l-k|} & \text{if } l+k = n \text{ and } l \neq k, \\ \rho_{[l+k]} \oplus \rho_0 \oplus \rho_{-2} & \text{if } l+k \neq n \text{ and } l-k = 0, \\ \rho_{|l-k|} \oplus \rho_{[l+k]} & \text{if } l+k \neq n \text{ and } l-k \neq 0. \end{cases}$$

Here $[l+k] = l+k$ if $l+k \le n-1$, $2n-(l+k)$ if $l+k \ge n+1$.

(3) *For $1 \le l \le n$, $\rho_0 \otimes \pi_l = \rho_{-2} \otimes \pi_l = \pi_l$, $\rho_{-1} \otimes \pi_l = \rho_{-3} \otimes \pi_l = \pi_{n-l+1}$.*
(4) *For $1 \le k \le n-1$, $1 \le l \le n$,*

$$\rho_k \otimes \pi_l = \pi_{|l-k|'} \oplus \pi_{[l+k]'}.$$

Here

$$[l+k]' = \begin{cases} l+k & \text{if } l+k \le n, \\ 2n+1-(l+k) & \text{if } l+k \ge n+1; \end{cases}$$

$$|l-k|' = \begin{cases} l-k & \text{if } l-k > 0, \\ 1+k-l & \text{if } l-k \le 0. \end{cases}$$

(5) *For $1 \le k \le n$, $1 \le l \le n$,*

$$\pi_l \otimes \pi_k \sim \begin{cases} \rho_0 \oplus \rho_{-1} \oplus \rho_{-2} \oplus \rho_{-3} & \text{if } l+k = n+1 \text{ and } l-k = 0, \\ \rho_{-1} \oplus \rho_{-3} \oplus \rho_{|l-k|} & \text{if } l+k = n+1 \text{ and } l \neq k, \\ \rho_{[l+k-1]} \oplus \rho_0 \oplus \rho_{-2} & \text{if } l+k \neq n+1 \text{ and } l-k = 0, \\ \rho_{|l-k|} \oplus \rho_{[l+k-1]} & \text{if } l+k \neq n+1 \text{ and } l-k \neq 0. \end{cases}$$

In particular, the second part of Theorem 1.1 holds.

3.3. **Projective representations of the infinite dihedral group D_∞.** In this section, we apply the twisted Peter-Weyl Theorem to D_∞ and generalize Section 3.2. For completeness, we give a proof of the twisted Peter-Weyl Theorem in the appendix. Starting with Theorem A.1, as explained in [11, Section 4.3], the character theory for finite groups generalizes to a character theory for compact groups.

Let G be a compact group and $\alpha \in Z^2(G, \mathbb{C}^{|\cdot|=1})$. Let N be a closed subgroup of G. Consider α as a multiplier of N by restriction. Let (r, W, α) be a finite dimensional projective representation of N with multiplier α. Consider the induced projective representation $\pi = \alpha \operatorname{Ind}_N^G r$ on the space

$$V := \{f \in L^2(G, W) \mid f(hg) = \alpha(hg, g^{-1}) r(h) f(g) \text{ for all } h \in N, g \in G\}.$$

The map $\pi : G \to \mathrm{GL}(V)$ is defined by the equation $(\pi(g)f)(g') = \alpha(g', g) f(g'g)$.

Assume further that N is normal and of finite index in G. Then by [3, Section 2.2],

$$V|_N \cong \oplus_{g \in G/N} W^g,$$

and

$$\mathrm{Hom}_G(V, E) \cong \mathrm{Hom}_N(W, E|_N).$$

Here (r^g, W^g) is the α-twist of (r, W), i.e., $W^g = W$ and $r^g(h) = \frac{\alpha(h, g^{-1})}{\alpha(g^{-1}, ghg^{-1})} r(ghg^{-1})$, E is any projective representation of G with multiplier α.

Let us consider the group $G = D_\infty := S^1 \ltimes \mathbb{Z}/2\mathbb{Z}$. Fix a presentation of G

$$G = \langle t_\theta, \iota \mid t_\theta \in S^1, \iota^2 = 1, \iota t_\theta \iota = t_{-\theta} \rangle,$$

where t_θ represents the element $e^{i\theta} \in S^1$. Considering the Lyndon-Hochschild-Serre spectral sequence of the sequence

$$0 \to S^1 \to G \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

one obtains an element of $H^2(G, \mathbb{C}^{|\cdot|=1})$ with order two represented by the cocycle α defined by

$$\alpha(t_\theta, x) = 1 \text{ for any } x \in G, \quad \alpha(t_\theta \iota, t_\gamma) = \alpha(t_\theta \iota, t_\gamma \iota) = t_\gamma.$$

Let $\widehat{G}_\alpha$ be the set of isomorphism classes of finite dimensional irreducible projective representations of G with multiplier α. Let π be an object in $\widehat{G}_\alpha$. Note that $\alpha|_{S^1 \times S^1} = 1$. The restriction $\pi|_{S^1}$ is a linear representation. Let χ be a one-dimensional linear representation of S^1 appears in $\pi|_{S^1}$. By the discussion on induced representations above, one obtains that $\pi \cong \alpha \operatorname{Ind}_{S^1}^G \chi$. Assume that $\chi(t_\theta) = e^{in\theta}$, then by choosing an appropriate basis, $\pi_n := \alpha \operatorname{Ind}_{S^1}^G \chi$ is given by

$$\pi_n(t_\theta) = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{i(-n+1)\theta} \end{pmatrix}, \quad \pi_n(\iota) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the character theory, up to isomorphism, $\{\pi_n\}_{n \in \mathbb{Z}_{>0}}$ are *all the irreducible projective representations of G with multiplier α.*

Since α represents an element of order two in $H^2(G, \mathbb{C}^{|\cdot|=1})$, any tensor product $\pi_m \otimes \pi_n$ is equivalent to a linear representation of G. By [11, Chap 5, Section 5.5], all the irreducible linear representations of G are given by

- $\rho_0 : t_\theta \mapsto 1, \iota \mapsto 1$; $\rho_{-1} : t_\theta \mapsto 1, \iota \mapsto -1$.

- for $l \in \mathbb{Z}_{>0}$,

$$\rho_l : t_\theta \mapsto \begin{pmatrix} e^{il\theta} & 0 \\ 0 & e^{-il\theta} \end{pmatrix}, \ \iota \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following result follows from direct computation of characters.

Proposition 3.8. *With the notation as above,*

$$\pi_m \otimes \pi_n \sim \begin{cases} \rho_{m+n} \oplus \rho_{1+|m-n|} & \text{if } |m-n| \neq 1, \\ \rho_{m+n} \oplus \rho_0 \oplus \rho_{-1} & \text{if } |m-n| = 1. \end{cases}$$

For $l > 0$,

$$\rho_l \otimes \pi_n \cong \begin{cases} \pi_{l+n} \oplus \pi_{-l+n} & \text{if } l < n, \\ \pi_{l+n} \oplus \pi_{l-n-1} & \text{if } l \geq n. \end{cases}$$

At the end, let us consider the regular representation $(r_\alpha, L^2(G))$ and the projections to the π_n-isotropic part. Note that the invariant measure of G is $\mathrm{d}\,x/4\pi$. The projection $P_n : L^2(G) \to L^2(G)[\pi_n]$ from Remark A.4 is given by the formula

$$\begin{aligned} P_n f &= \frac{1}{4\pi}\int_0^{2\pi} \overline{\chi_n(t_\theta)} r_\alpha(t_\theta) \cdot f \,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \overline{\chi_n(t_\theta\iota)} r_\alpha(t_\theta\iota) \cdot f \,\mathrm{d}\,\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi} (e^{-in\theta} + e^{i(1-n)\theta}) r_\alpha(t_\theta) \cdot f \,\mathrm{d}\,\theta. \end{aligned} \tag{3.6}$$

The following result is then clear.

Proposition 3.9. *Let $G = D_\infty$. Define functions $\{f_n, f'_n, F_n, F'_n\}_{n \in \mathbb{Z}_{>0}}$ in $L^2(G)$ by*

$$\begin{aligned} f_n(x) &= \begin{cases} e^{in\theta} & \text{if } x = e^{i\theta}, \\ 0 & \text{if } x = e^{i\theta}\iota, \end{cases} \quad f'_n(x) = \begin{cases} 0 & \text{if } x = e^{i\theta}, \\ e^{in\theta} & \text{if } x = e^{i\theta}\iota; \end{cases} \\ F_n(x) &= \begin{cases} 0 & \text{if } x = e^{i\theta}, \\ e^{i(1-n)\theta} & \text{if } x = e^{i\theta}\iota, \end{cases} \quad F'_n(x) = \begin{cases} e^{i(1-n)\theta} & \text{if } x = e^{i\theta}, \\ 0 & \text{if } x = e^{i\theta}\iota. \end{cases} \end{aligned} \tag{3.7}$$

Then $\{f_n, f'_n, F_n, F'_n\}_{n \in \mathbb{Z}_{>0}}$ form an orthogonal basis of $L^2(G)$. Moreover, the two dimensional subspaces $\langle f_n, f'_n \rangle$ and $\langle F_n, F'_n \rangle$ are r_α-invariant and they are isomorphic to π_n. This gives us an explicit decomposition of the right regular projective representation.

Appendix A. A twisted Peter-Weyl Theorem

A.1. **The statement of the main result.** In this appendix, G is a compact topological group with a fixed Haar measure $\int_G \cdot \,\mathrm{d}\,g$. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be a multiplier of G. In the following, we assume that α is unitary, i.e., $|\alpha(x,y)| = 1$ for any $x, y \in G$. A *projective representation* (π, V, α) of G over $\mathbb{C}$ with multiplier α is a continuous map $\pi : G \to \mathrm{U}(V)$ such that $\pi(x)\pi(y) = \alpha(x,y)\pi(xy)$ for all $x, y \in G$, where V is a Hilbert space, $\mathrm{U}(V)$ is the space of unitary operators from V to V. Here *continuous* means that the map $(g, v) \mapsto \pi(g)v$ is a continuous map from $G \times V$ to V.

Denote by $\widehat{G}_\alpha$ the set of isomorphism classes of finite dimensional irreducible projective representations of G with multiplier α. Let (π, V, α) be an element in $\widehat{G}_\alpha$. Fix a G-invariant Hermitian inner product $\langle , \rangle$ on V, which exists by the averaging argument. Given $v, w \in V$, the function $f : g \mapsto \langle \pi(g)v, w \rangle$ is called a *matrix coefficient* of π. Let

$\mathcal{A}_\alpha(G)$ be the space spanned by all matrix coefficients of finite dimensional irreducible projective representations of G with multiplier α.

Let $L^2(G)$ be the space of measurable functions on G for which $\int_G |f(g)|^2 \,\mathrm{d}\, g < \infty$. If $f \in L^2(G)$, define $||f||_2 = (\int_G |f(g)|^2 \,\mathrm{d}\, g)^{1/2}$. Let $L^1(G)$ be the space of measurable functions on G for which $\int_G |f(g)| \,\mathrm{d}\, g < \infty$. If $f \in L^1(G)$, define $||f||_1 = \int_G |f(g)| \,\mathrm{d}\, g$. Given $f, f' \in L^2(G)$, define an inner product by

$$\langle f, f' \rangle_2 = \int_G f(g)\overline{f'(g)} \,\mathrm{d}\, g. \tag{A.1}$$

With this inner product, $L^2(G)$ is a Hilbert space. Furthermore, $ff' \in L^1(G)$ and we have the following inequalities.

$$\begin{aligned} &||ff'||_1 \leq ||f||_2 ||f'||_2, \\ &|\langle f, f' \rangle_2| \leq ||f||_2 ||f'||_2 \text{ (Schwarz inequality).} \end{aligned} \tag{A.2}$$

If $f : G \to \mathbb{C}$ and $g \in G$, define $r(g)f := r_\alpha(g)f : G \to \mathbb{C}$ by

$$(r(g)f)(g_0) = \alpha(g_0, g)f(g_0 g)$$

for all $g_0 \in G$. It is easy to check that $r(g)f \in L^2(G)$ if $f \in L^2(G)$ and $r(g)$ is an element in $\mathrm{U}(L^2(G))$. Then $r : G \to \mathrm{U}(L^2(G))$ defines a unitary projective representation of G with associated multiplier α. We call it the *right translation* or *right regular projective representation* of G on $L^2(G)$ with respect to α. It is also easy to check that $\langle , \rangle_2$ is G-invariant, i.e.,

$$\langle r(g)f, r(g)f' \rangle_2 = \langle f, f' \rangle_2.$$

Thus $(r, L^2(G), \alpha)$ decomposes as a direct sum of irreducible unitary projective representations. We prove the following result by the strategy as for linear representations with an extra attention on the multiplier α. See for example [10] for the case of linear representations.

Theorem A.1 (Peter-Weyl Theorem)**.** *Let G be a compact group, $\alpha \in Z^2(G, \mathbb{C}^\times)$ a unitary multiplier of G. Then the following claims hold.*

(1) *$\mathcal{A}_\alpha(G)$ is dense in $L^2(G)$.*
(2) *Every irreducible unitary projective representation of G is finite dimensional.*
(3) *Fix an element ρ in each class in $\widehat{G}_\alpha$ and denote by d_ρ the dimension of ρ. Then as projective representations of G with multiplier α,*

$$(r, L^2(G), \alpha) \cong \oplus_{\rho \in \widehat{G}_\alpha} \rho^{\oplus d_\rho}.$$

(4) *If $\psi \in L^2(G)$, then*

$$||\psi||_2^2 = \sum_{\rho \in \widehat{G}_\alpha} d_\rho \cdot \mathrm{Tr}(\rho_\psi \rho_\psi^*) = \sum_{\rho \in \widehat{G}_\alpha} d_\rho \cdot ||\rho_\psi||_{HS}.$$

Here $\rho_\psi = \int_G \psi(g)\rho(g)^{-1} \,\mathrm{d}\, g$; $||M||_{HS} = \sum_{i,j} m_{ij}^2$ for a matrix $M = (m_{ij})$ of finite rank.

(5) *The characters $(\chi_\rho)_{\rho \in \widehat{G}_\alpha}$ form an orthonormal basis of $\mathbb{H}_\alpha$ (Definition A.11).*

Remark A.2. We only prove the first claim of the theorem in next section, because the other claims follow from the first by the same argument as in the linear representations case. See for example [10] and [12]. Note that if we take α to be the trivial cocycle, the argument gives a proof of the classical Peter-Weyl Theorem. On the other hand, we explain some other ideas in the following remarks.

Remark A.3. One may obtain the second claim using the classical Peter-Weyl Theorem. Indeed, the unitary cocycle α gives us a compact group G_α which is an extension of G by $\mathbb{C}^{|\cdot|=1} \cong S^1$. Then the irreducible projective representations of G are irreducible linear representations of G_α with certain conditions on the subgroup $\mathbb{C}^{|\cdot|=1}$ (cf. Theorem 2.6), which are finite dimensional by the Peter-Weyl Theorem.

Remark A.4. As in [11, Chap. 2, Prop. 8], one may decompose a projective representation into a direct sum of irreducible ones and construct explicit projections for this decomposition as follows. Let H be a separable Hilbert space. Let π be a unitary projective representation of G on H with multiplier α. For each $1 \leq i, j \leq d_\rho$, define

$$P_{ij}^{\rho} = d_\rho \int_G \alpha(g, g^{-1})^{-1} r_{ji}^{\rho}(g^{-1}) \pi(g) \, \mathrm{d}\, g.$$

Here $r_{ji}^{\rho}(g)$ is the (j, i)-th entry of $\rho(g)$ in a matrix form. It is a linear map from H to H. Define

$$P^{\rho} = \sum_{i=1}^{d_\rho} P_{ii}^{\rho}.$$

These operators have the following properties.

(1) If $g \in G$, then $\pi(g) P_{ij}^{\rho} = \sum_k r_{ki}^{\rho}(g) P_{kj}^{\rho}$.
(2) $P_{ij}^{\rho} \circ P_{kl}^{\rho} = \delta_{jk} P_{il}^{\rho}$.
(3) If $\rho \neq \rho'$, then $P_{ij}^{\rho} \circ P_{kl}^{\rho'} = 0$ for all i, j, k, l.
(4) $(P_{ij}^{\rho})^* = P_{ji}^{\rho}$.
(5) $P^{\rho} \in \mathrm{Hom}_G(H, H)$.
(6) If $\rho \neq \rho'$, then $P^{\rho} \circ P^{\rho'} = 0$.
(7) $P^{\rho} \circ P^{\rho} = P^{\rho}$.

In other words, P^{ρ} is the projection from H to the ρ-isotropic part. The operators P_{ij}^{ρ} are the ones to decompose the ρ-isotropic part into a direct sum of projective representations isomorphic to ρ. Using these operators, one could give another proof of Theorem A.1(3) by the same argument as in [11, Chap. 2, Prop. 8].

Remark A.5 (Trace formula twisted by α and decomposition of $L^2(G)$). Let G be a unimodular group (not necessarily compact in this remark) and $\Gamma \subset G$ a discrete normal subgroup such that $\Gamma \backslash G$ is compact. Let $\alpha \in Z^2(\Gamma \backslash G, S^1)$ be a unitary multiplier. We may view α as an element in $Z^2(G, S^1)$ via the natural projection $G \times G \to \Gamma \backslash G \times \Gamma \backslash G$. The right regular representation r_α of G with respect to α over $L^2(\Gamma \backslash G)$ is defined by

$$(r_\alpha(h)(f))(g) = \alpha(g, h) f(gh).$$

Let $\phi : G \to \mathbb{C}$ be a continuous map with compact support. Define $R(\phi) : L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)$ by

$$(R(\phi)f)(x) = \int_G \phi(g)\alpha(x,g)f(xg)\,\mathrm{d}\,g = \int_G \phi(x^{-1}g)\alpha(x,x^{-1}g)f(g)\,\mathrm{d}\,g.$$

It is easy to check that this is well defined. Note that we may write $R(\phi) = \int_G \phi(g)r_\alpha(g)\,\mathrm{d}\,g$. Thus $R(\phi)$ sends each irreducible component of $(r_\alpha, L^2(\Gamma\backslash G), \alpha)$ to itself. Moreover,

$$\begin{aligned}(R(\phi)f)(x) &= \int_G \phi(x^{-1}g)\alpha(x,x^{-1}g)f(g)\,\mathrm{d}\,g \\ &= \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\phi(x^{-1}\gamma g)\alpha(x,x^{-1}\gamma g)f(g)\,\mathrm{d}\,g = \int_{\Gamma\backslash G} K_\phi(x,g)f(g)\,\mathrm{d}\,g,\end{aligned} \tag{A.3}$$

where $K_\phi(x,g) = \sum_{\gamma\in\Gamma}\phi(x^{-1}\gamma g)\alpha(x,x^{-1}\gamma g)$. Then $R(\phi)$ is of trace class and

$$\mathrm{Tr}(R(\phi)) = \int_{\Gamma\backslash G} K_\phi(x,x)\,\mathrm{d}\,x = \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\phi(x^{-1}\gamma x)\alpha(x,x^{-1}\gamma x)\,\mathrm{d}\,x.$$

(See for example [7, Lemma 4.1].) Let $\mathfrak{o}$ be the set of conjugacy classes of Γ. For each class in $\mathfrak{o}$, fix an element γ and denote this conjugacy class by $\mathfrak{o}_\gamma$. If γ is an element of a group H, denote by H^γ the centralizer of γ in H. With the above notation, $\mathfrak{o}_\gamma = \{\delta^{-1}\gamma\delta \mid \delta \in \Gamma^\gamma\backslash\Gamma\}$. Therefore

$$\begin{aligned}\mathrm{Tr}(R(\phi)) &= \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\phi(x^{-1}\gamma x)\alpha(x,x^{-1}\gamma x)\,\mathrm{d}\,x \\ &= \sum_{\mathfrak{o}_\gamma}\sum_{\delta\in\Gamma^\gamma\backslash\Gamma}\int_{\Gamma\backslash G}\phi(x^{-1}\delta^{-1}\gamma\delta x)\alpha(x,x^{-1}\delta^{-1}\gamma\delta x)\,\mathrm{d}\,x \\ &= \sum_{\mathfrak{o}_\gamma}\int_{\Gamma^\gamma\backslash G}\phi(x^{-1}\gamma x)\alpha(x,x^{-1}\gamma x)\,\mathrm{d}\,x \\ &= \sum_{\mathfrak{o}_\gamma}\int_{G^\gamma\backslash G}\left(\int_{\Gamma^\gamma\backslash G^\gamma}\phi(x^{-1}y^{-1}\gamma yx)\alpha(x,x^{-1}y^{-1}\gamma yx)\,\mathrm{d}\,y\right)\mathrm{d}\,x \\ &= \sum_{\mathfrak{o}_\gamma}\mathrm{vol}(\Gamma^\gamma\backslash G^\gamma)\int_{G^\gamma\backslash G}\phi(x^{-1}\gamma x)\alpha(x,x^{-1}\gamma x)\,\mathrm{d}\,x.\end{aligned} \tag{A.4}$$

If we let G be compact, Γ be the trivial subgroup $\{1\}$, then we obtain Theorem A.1(3) immediately.

A.2. **The proof of the main result.** In this section, we prove the first claim of Theorem A.1, the other claims then follow as explained in Remark A.2.

Proposition A.6. *With the same notation as in Section A.1, $\mathcal{A}_\alpha(G)$ is dense in $L^2(G)$.*

The strategy of the proof is similar as for linear representations (see for example [10]). The difference here is that we have to pay extra attention on the cocycle α. First, we prove some lemmas.

Lemma A.7. *Let $f : g \mapsto \langle \pi(g)v, w\rangle$ be a matrix coefficient of π. Then the functions $g \mapsto \alpha(g, g^{-1})\overline{f(g^{-1})}$, $g \mapsto \alpha(g, h)f(gh)$, $g \mapsto \alpha(h, g)\alpha(h^{-1}, h)^{-1}f(hg)$ are matrix coefficients of π. We call them the adjoint of f, the right translation of f, the left translation of f, respectively.*

Proof. Note that

$$\begin{aligned} \overline{f(g^{-1})} &= \overline{\langle \pi(g^{-1})v, w\rangle} = \langle w, \pi(g^{-1})v\rangle \\ &= \langle \pi(g)w, \pi(g)\pi(g^{-1})v\rangle = \alpha(g, g^{-1})^{-1}\langle \pi(g)w, v\rangle. \end{aligned} \tag{A.5}$$

This shows that $g \mapsto \alpha(g, g^{-1})\overline{f(g^{-1})}$ is a matrix coefficient. Similarly, it is easy to see that

$$\begin{aligned} f(gh) &= \alpha(g, h)^{-1}\langle \pi(g)(\pi(h)v), w\rangle, \\ f(hg) &= \alpha(h, g)^{-1}\alpha(h^{-1}, h)\langle \pi(g)v, \pi(h^{-1})w\rangle. \end{aligned} \tag{A.6}$$

The other claims follow easily. □

Denote by $\mathcal{C}(G)$ the space of continuous functions from G to $\mathbb{C}$. It is dense in $L^2(G)$.

Lemma A.8. *Let $f \in L^2(G)$. Then the map $g \mapsto r(g)f$ is a continuous map from G to $L^2(G)$.*

Proof. Let $\epsilon > 0$. Choose $\phi \in \mathcal{C}(G)$ such that $||f - \phi||_2 < \epsilon/3$. Note that G is compact, each continuous function on G is uniformly continuous. In particular, for the function $\alpha(g, -)\phi(g-)$, there exists an open neighborhood U of $1 \in G$ such that if $h^{-1}h' \in U$, then $|\alpha(g, h)\phi(gh) - \alpha(g, h')\phi(gh')| < \epsilon/3$ for all $g \in G$. Note that

$$\begin{aligned} ||r(h)f - r(h')f||_2 &\le ||r(h)f - r(h)\phi||_2 + ||r(h)\phi - r(h')\phi||_2 + ||r(h')\phi - r(h')f||_2 \\ &= 2||f - \phi||_2 + ||r(h)\phi - r(h')\phi||_2 \\ &< \epsilon. \end{aligned} \tag{A.7}$$

The continuity follows. □

Lemma A.9. *Let $f \in L^2(G)$. For every $\epsilon > 0$, there exist finitely many $g_i \in G$ and Borel sets $B_i \subset G$ such that G is the disjoint union of the B_i's and $||r(g)f - r(g_i)f||_2 < \epsilon$ for all i and $g \in B_i$.*

Proof. By Lemma A.8, there exists an open neighborhood U of 1 such that $||r(g)f - f||_2 < \epsilon$ for all $g \in U$. Note that $\{hU \mid h \in G\}$ is an open cover of G and G is compact, there exist finitely many $g_1, \ldots, g_n$ such that $G = \cup_{i=1}^n g_iU$. Let $B_i = g_iU - \cup_{j=1}^{i-1} g_jU$. It is easy to check that these objects satisfy the property in the statement. □

Lemma A.10. *Let $f \in L^2(G)$ and $f_1 \in L^1(G)$. Define $F : G \to \mathbb{C}$ by*

$$F(g') = \int_G \alpha(g', g)f(g'g)f_1(g)\,\mathrm{d}\,g.$$

Then F is an element in $L^2(G)$ and it is a limit of a sequence of functions, each of which is a finite linear combination of right translates of f.

Proof. Let $\epsilon > 0$. Choose g_i and B_i as in Lemma A.9. Set $e_i = \int_{B_i} f_1(g)\,\mathrm{d}\,g$. Then

$$\begin{aligned}||F - \sum_{i=1}^{n} e_i r(g_i) f||_2 &\leq \sum_{i=1}^{n} \int_{B_i} |f_1(g)| \cdot ||r(g)f - r(g_i)f||_2 \,\mathrm{d}\,g \\ &\leq \sum_{i=1}^{n} \int_{B_i} |f_1(g)| \epsilon \,\mathrm{d}\,g = \epsilon ||f_1||_1.\end{aligned} \tag{A.8}$$

The lemma follows. □

Definition A.11. A continuous function $f : G \to \mathbb{C}$ is called an *α-class function* if for all $g, h \in G$,

$$f(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})} f(g) = \frac{\alpha(h, h^{-1})}{\alpha(h, g)\alpha(hg, h^{-1})} f(g).$$

Let $\mathbb{H}_\alpha$ denote the closed subspace of $L^2(G)$ spanned by square-integrable α-class functions on G. The characters of finite dimensional projective representations of G with multiplier α belong to $\mathbb{H}_\alpha$.

Lemma A.12. *Let f be any integrable function on G. Set*

$$f'(g) = \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f(hgh^{-1}) \,\mathrm{d}\,h.$$

Then f' is an α-class function on G.

Proof. Note that

$$\begin{aligned} f'(i^{-1}gi) &= \int_G \frac{\alpha(h, i^{-1}gih^{-1})\alpha(i^{-1}gi, h^{-1})}{\alpha(h, h^{-1})} f(hi^{-1}gih^{-1}) \,\mathrm{d}\,h \\ &= \int_G \frac{\alpha(h'i, i^{-1}gii^{-1}(h')^{-1})\alpha(i^{-1}gi, i^{-1}(h')^{-1})}{\alpha(h'i, i^{-1}(h')^{-1})} f(h'g(h')^{-1}) \,\mathrm{d}\,h' \ (h' = hi^{-1}).\end{aligned}$$

Then to show that f' is an α-class function, it suffices to show that

$$\begin{aligned}&\alpha(i^{-1}, i)\alpha(h, gh^{-1})\alpha(g, h^{-1})\alpha(hi, i^{-1}h^{-1}) \\ =&\alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, h^{-1})\alpha(hi, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}).\end{aligned} \tag{A.9}$$

Since $\alpha(h, i)\alpha(hi, i^{-1}h^{-1}) = \alpha(h, h^{-1})\alpha(i, i^{-1}h^{-1})$, it suffices to show that

$$\begin{aligned}&\alpha(i^{-1}, i)\alpha(h, gh^{-1})\alpha(g, h^{-1})\alpha(i, i^{-1}h^{-1}) \\ =&\alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, i)\alpha(hi, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}).\end{aligned} \tag{A.10}$$

This follows from the following computation.

$$\begin{aligned} RHS &= \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, gh^{-1})\alpha(i, i^{-1}gh^{-1})\alpha(i^{-1}gi, i^{-1}h^{-1}) \\ &= \alpha(i^{-1}, gi)\alpha(g, i)\alpha(h, gh^{-1})\alpha(i, i^{-1}gi)\alpha(gi, i^{-1}h^{-1}) \\ &= \alpha(h, gh^{-1})[\alpha(i^{-1}, gi)\alpha(i, i^{-1}gi)][\alpha(g, i)\alpha(gi, i^{-1}h^{-1})] \\ &= \alpha(h, gh^{-1})\alpha(i, i^{-1})\alpha(g, h^{-1})\alpha(i, i^{-1}h^{-1}) = LHS.\end{aligned} \tag{A.11}$$

The lemma follows. □

Lemma A.13. *Let $f : G \to \mathbb{C}$ be an α-class function. Then $f'(g) = \alpha(g, g^{-1})\overline{f(g^{-1})}$ is also an α-class function.*

Proof. One needs to show that

$$f'(hgh^{-1}) = \frac{\alpha(h,h^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})} f'(g).$$

This is equivalent to

$$\frac{\alpha(hgh^{-1},hg^{-1}h^{-1})\alpha(h,g^{-1}h^{-1})\alpha(g^{-1},h^{-1})}{\alpha(h,h^{-1})} = \frac{\alpha(h,h^{-1})\alpha(g,g^{-1})}{\alpha(h,gh^{-1})\alpha(g,h^{-1})}.$$

Note that

$$\begin{aligned}
&\alpha(hgh^{-1},hg^{-1}h^{-1})\alpha(h,g^{-1}h^{-1})\alpha(g^{-1},h^{-1})\alpha(h,gh^{-1})\alpha(g,h^{-1})\\
=&\alpha(hgh^{-1},h)\alpha(hg,g^{-1}h^{-1})\alpha(g^{-1},h^{-1})\alpha(h,gh^{-1})\alpha(g,h^{-1})\\
=&\alpha(hgh^{-1},h)\alpha(hg,g^{-1})\alpha(h,h^{-1})\alpha(h,g)\alpha(hg,h^{-1})\\
=&\alpha(h,h^{-1})[\alpha(h,g)\alpha(hg,g^{-1})][\alpha(hg,h^{-1})\alpha(hgh^{-1},h)]\\
=&\alpha(h,h^{-1})\alpha(g,g^{-1})\alpha(h,h^{-1}).
\end{aligned} \tag{A.12}$$

The lemma follows. □

Lemma A.14. *Let $f : G \to \mathbb{C}$ be an α-class function. Then*

$$\frac{f(h^{-1}g)}{\alpha(h,h^{-1}g)} = \frac{f(gh^{-1})}{\alpha(gh^{-1},h)}.$$

Proof. Since $gh^{-1} = h(h^{-1}g)h^{-1}$, it suffices to prove

$$\frac{\alpha(h,h^{-1})}{\alpha(h,h^{-1}gh^{-1})\alpha(h^{-1}g,h^{-1})} = \frac{\alpha(gh^{-1},h)}{\alpha(h,h^{-1}g)}.$$

Note that

$$\begin{aligned}
&\alpha(gh^{-1},h)\alpha(h,h^{-1}gh^{-1})\alpha(h^{-1}g,h^{-1})\\
=&\alpha(h,h^{-1}g)\alpha(h^{-1}gh^{-1},h)\alpha(h^{-1}g,h^{-1}) = \alpha(h,h^{-1}g)\alpha(h^{-1},h).
\end{aligned} \tag{A.13}$$

The lemma follows. □

With the above preparation, we can now prove Proposition A.6.

Proof of Proposition A.6. Let $\overline{\mathcal{A}_\alpha(G)}$ be the closure of $\mathcal{A}_\alpha(G)$ in $L^2(G)$. Since $\mathcal{A}_\alpha(G)$ is stable under the operations in Lemma A.7, $\overline{\mathcal{A}_\alpha(G)}$ is also stable under those operations. Suppose that $\overline{\mathcal{A}_\alpha(G)} \neq L^2(G)$. Then $\overline{\mathcal{A}_\alpha(G)}^\perp \neq \{0\}$ and it is stable under the operations in Lemma A.7. Let $f_0 \in \overline{\mathcal{A}_\alpha(G)}^\perp$ and $f_0 \neq 0$. Fix U an open neighborhood of $1 \in G$. Let $\mathbb{I}_U$ be the characteristic function on U, $|U|$ the Haar measure of U, and

$$f_U(g) = |U|^{-1} \int_G \alpha(g,g_0)\mathbb{I}_U(g_0)f_0(gg_0)\,\mathrm{d}\,g_0.$$

Since $\mathbb{I}_U, f_0 \in L^2(G)$, by Schwarz inequality, we see that $f_U \in \mathcal{C}(G)$. Furthermore, $f_0 = \lim_{U\to\{1\}} f_U$ in $L^2(G)$. Because $f_0 \neq 0$, there exist U such that $f_U \neq 0$. Since $\overline{\mathcal{A}_\alpha(G)}$ is G-stable by right translation and the right translation of G on $L^2(G)$ is unitary, $\overline{\mathcal{A}_\alpha(G)}^\perp$ is also G-stable. Hence linear combinations of right translates of f_0 belong to $\overline{\mathcal{A}_\alpha(G)}^\perp$.

By Lemma A.10, $f_U \in \overline{\mathcal{A}_\alpha(G)}^\perp$. In particular, $\overline{\mathcal{A}_\alpha(G)}^\perp$ contains a nonzero continuous function. Let f_1 be such a function. We may assume that $f_1(1) \in \mathbb{R} - \{0\}$. Define

$$f_2(g) = \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f_1(hgh^{-1}) \, \mathrm{d}\, h.$$

By Lemma A.12, f_2 is an α-class function. It is easy to see that f_2 is continuous and $f_2(1) \in \mathbb{R} - \{0\}$. Moreover, for any $f' \in \overline{\mathcal{A}_\alpha(G)}$, $f''(g) = \alpha(h^{-1}, g)\alpha(h, h^{-1})^{-1}\alpha(h^{-1}g, h)f'(h^{-1}gh)$ is also an element in $\overline{\mathcal{A}_\alpha(G)}$ by Lemma A.7. Note that

$$\begin{aligned}
\langle f_2, f' \rangle_2 &= \int_G f_2(g)\overline{f'(g)} \, \mathrm{d}\, g \\
&= \int_G \int_G \frac{\alpha(h, gh^{-1})\alpha(g, h^{-1})}{\alpha(h, h^{-1})} f_1(hgh^{-1})\overline{f'(g)} \, \mathrm{d}\, h \, \mathrm{d}\, g \\
&= \int_G \int_G \frac{\alpha(h, h^{-1}g)\alpha(h^{-1}gh, h^{-1})}{\alpha(h, h^{-1})} f_1(g)\overline{f'(h^{-1}gh)} \, \mathrm{d}\, h \, \mathrm{d}\, g \\
&= \int_G \int_G f_1(g)\overline{f''(g)} \, \mathrm{d}\, g \, \mathrm{d}\, h = 0.
\end{aligned} \tag{A.14}$$

Thus $f_2 \in \overline{\mathcal{A}_\alpha(G)}^\perp$. Define $f_3(g) = f_2(g) + \alpha(g, g^{-1})\overline{f_2(g^{-1})}$. Then f_3 is in $\overline{\mathcal{A}_\alpha(G)}^\perp$ and is an α-class function by Lemma A.13. Moreover, it is easy to check that $f_3(g) = \alpha(g, g^{-1})\overline{f_3(g^{-1})}$. Define

$$K(g, h) = f_3(gh^{-1})\alpha(gh^{-1}, h)^{-1}.$$

Since

$$\alpha(hg^{-1}, g)\alpha(gh^{-1}, h) = \alpha(hg^{-1}, gh^{-1})\alpha(1, h) = \alpha(hg^{-1}, gh^{-1}),$$

one gets $K(g, h) = \overline{K(h, g)}$. Define

$$(Tf)(g) = \int_G K(g, h)f(h) \, \mathrm{d}\, h.$$

Then T is a nonzero self-adjoint Hilbert-Schmidt operator on $L^2(G)$. Hence T has a nonzero real eigenvalue γ and the eigenspace $V_\gamma \subset L^2(G)$ is finite dimensional (see for example [1, I.8.4.1 and I.8.5.5]). Let $f \in V_\gamma$. Then

$$\begin{aligned}
(T(r(g_0)f))(g) &= \int_G K(g, g_1)\alpha(g_1, g_0)f(g_1g_0) \, \mathrm{d}\, g_1 \\
&= \int_G K(g, g_1g_0^{-1})\alpha(g_1g_0^{-1}, g_0)f(g_1) \, \mathrm{d}\, g_1 \\
&= \int_G f_3(gg_0g_1^{-1}) \frac{\alpha(g_1g_0^{-1}, g_0)}{\alpha(gg_0g_1^{-1}, g_1g_0^{-1})} f(g_1) \, \mathrm{d}\, g_1 \\
&= \int_G f_3(gg_0g_1^{-1}) \frac{\alpha(g, g_0)}{\alpha(gg_0g_1^{-1}, g_1)} f(g_1) \, \mathrm{d}\, g_1 \\
&= \int_G K(gg_0, g_1)\alpha(g, g_0)f(g_1) \, \mathrm{d}\, g_1 \\
&= \alpha(g, g_0)(Tf)(gg_0) = \gamma(r(g_0)f)(g).
\end{aligned} \tag{A.15}$$

The eigenspace V_γ is stable under right translation. Now $r : G \to \mathrm{U}(V_\gamma)$ is a finite dimensional unitary projective representation of G with multiplier α. Let $W \subset V_\gamma$ be an irreducible sub projective representation and $\{e_1, \dots, e_n\}$ an orthonormal basis of W with respect to r. Then $g \mapsto \langle r(g)e_i, e_j\rangle_2 = \int_G \alpha(g_0, g)e_i(g_0 g)\overline{e_j(g_0)}\,\mathrm{d}\,g_0$ is a matrix coefficient in $\mathcal{A}_\alpha(G)$. Since $f_3 \in \overline{\mathcal{A}_\alpha(G)}^{\perp}$, we have

$$
\begin{aligned}
0 &= \int_G f_3(g)\left(\int_G \overline{\alpha(g_0, g)e_j(g_0 g)}e_j(g_0)\,\mathrm{d}\,g_0\right)\mathrm{d}\,g \\
&= \int_G \left(\int_G f_3(g)\overline{\alpha(g_0, g)e_j(g_0 g)}\,\mathrm{d}\,g\right)e_j(g_0)\,\mathrm{d}\,g_0 \\
&= \int_G \left(\int_G f_3(g_0^{-1}g)\overline{\alpha(g_0, g_0^{-1}g)}\,\overline{e_j(g)}\,\mathrm{d}\,g\right)e_j(g_0)\,\mathrm{d}\,g_0 \\
&= \int_G \left(\int_G f_3(g_0^{-1}g)\overline{\alpha(g_0, g_0^{-1}g)}e_j(g_0)\,\mathrm{d}\,g_0\right)\overline{e_j(g)}\,\mathrm{d}\,g \\
&= \int_G \left(\int_G f_3(gg_0^{-1})\overline{\alpha(gg_0^{-1}, g_0)}e_j(g_0)\,\mathrm{d}\,g_0\right)\overline{e_j(g)}\,\mathrm{d}\,g \text{ (Lemma A.14)} \\
&= \int_G (Te_j)(g)\overline{e_j(g)}\,\mathrm{d}\,g = \gamma\langle e_j, e_j\rangle_2.
\end{aligned} \tag{A.16}
$$

Hence $\gamma = 0$, which is a contradiction. Therefore, we must have $\overline{\mathcal{A}_\alpha(G)} = L^2(G)$. □

Acknowledgements Part of the work was done during a visit at Morningside Center of Mathematics at Chinese Academy of Sciences. The authors would like to thank the center for providing great working conditions. They would like to thank F. Herzig and H. Verhoek for helpful discussions. The first author would like to thank the support of Grant 020314803001 of Jiangsu Province (China).

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