RANK TWO BREUIL MODULES: BASIC STRUCTURES

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Abstract. In this paper, we classify certain reducible rank two Breuil modules with descent data and compute $Ext^1(\mathcal{M},\mathcal{M})$ for Breuil modules $\mathcal{M}$ of certain type.

1. Introduction and notation

In [6], the author proves a modularity theorem for some potentially Barsotti-Tate Galois representations of $G_{\mathbb{Q}}$. One of the key ingredients in the proof is to use Breuil modules with descent data to study the local universal deformation ring at prime p. In order to generalize some of the results in [6] to totally real fields case, we need to understand the structures of Breuil modules with descent data over general p-adic fields. This is the motivation for writing this paper. In this paper, we prove some results about rank two Breuil modules. The content of this paper is the following.

In section 2, we review the classification of rank one Breuil modules with certain descent data from [3]. The key proposition is Proposition 2.3. It shows that rank one Breuil modules with descent data are determined by three invariants. We also review some basic facts about these invariants.

In the next two sections, we consider reducible rank two Breuil modules. In section 3, under some assumptions on the base fields (see the last paragraph of this section), we classify all of the (rank two) extensions, in the category of Breuil modules with descent data, of the rank one modules with descent data. The main classification result is Theorem 3.9.

In section 4, we compute $Ext^1(\mathcal{M},\mathcal{M})$ for a reducible rank two Breuil module $\mathcal{M}$ of type J (see Definition 4.1). As mentioned at the beginning, this computation is the motivation for this paper. The main result is Theorem 4.2, which plays an important role in [4].

Let us first recall the definition of Breuil modules with descent data. (See for example [1] and [5].) Let k be a finite extension of $\mathbb{F}_p$ of degree r, $W(k)$ the ring of Witt vectors. Let $K_0 = W(k)[1/p]$, K be a totally and tamely ramified extension of K_0 of degree e. Fix a subfield F of K_0, and assume that there is a uniformizer π of $\mathcal{O}_K$ such that $\pi^e \in F$. Then K/F is tamely ramified, K_0/F is unramified. Assume that K/F is Galois. (This condition will be satisfied in our choice of K in this paper.) Write $G = Gal(K/F)$. Let $S = Hom_{\mathbb{F}_p}(k, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/r\mathbb{Z}$. Fix $\tau_0 \in S$, let $\tau_i = \tau_0 \circ Frob^{-i}$, where $Frob$ is the arithmetic Frobenius. Let E be a finite extension of $\mathbb{F}_p$ such that the image of τ_i is a subset of E. Let $\mathcal{S} = k \otimes_{\mathbb{F}_p} E[u]/u^{ep}$.

Let $\omega : G \to k^\times$ be the map defined by $\omega(g) = g(\pi)/\pi \pmod{\pi}$. We see that $\omega(gh) = g(\omega(h))\omega(g)$. It is a cocycle. It is a character if and only if G acts trivially on $k^\times$, if and only if $K_0 = F$. Let ω_i be the composite of ω with τ_i. Then we have $\omega_i = \omega_{i+1}^p$. For any $g \in G$, we write $[g] : \mathcal{S} \to \mathcal{S}$ to be the k-semilinear, E-linear endomorphism of $\mathcal{S}$ as $k \otimes E$-algebra such that $[g](u) = (\omega(g) \otimes 1)u$. Let $\phi : \mathcal{S} \to \mathcal{S}$ be the E-linear, k-Frobenius-semilinear endomorphism of $\mathcal{S}$ such that $\phi(u) = u^p$.

Definition 1.1. Let $\kappa \in [2, p-1]$ be an integer. The category $BrMod^{\kappa-1}_{dd,K/F}$ consists of quintuples $(\mathcal{M}, Fil^{\kappa-1}\mathcal{M}, \phi_{\kappa-1}, \{[g]\}, N)$ where:

(1) $\mathcal{M}$ is a finitely generated $\mathcal{S}$ module, free over $k[u]/u^{ep}$.

(2) $Fil^{\kappa-1}\mathcal{M}$ is an $\mathcal{S}$-submodule of $\mathcal{M}$ containing $u^{e(\kappa-1)}\mathcal{M}$.

(3) $\phi_{\kappa-1} : Fil^{\kappa-1}\mathcal{M} \to \mathcal{M}$ is an E-linear and ϕ-semilinear map with image generating $\mathcal{M}$ as an $\mathcal{S}$-module.

(4) $N : \mathcal{M} \to u\mathcal{M}$ is a $k \otimes E$-linear map such that

$$N(ux) = uN(x) - ux \quad \forall x \in \mathcal{M},$$
$$u^e N(Fil^{\kappa-1}\mathcal{M}) \subset Fil^{\kappa-1}\mathcal{M},$$
$$\phi_{\kappa-1}(u^e N(x)) = (-\pi^e/p)^- N(\phi_{\kappa-1}(x)) \quad \forall x \in Fil^{\kappa-1}\mathcal{M}.$$

Here $(-\pi^e/p)^-$ is the image of $(-\pi^e/p)$ in the residue field k.

(5) $[g] : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in G$, preserving $Fil^{\kappa-1}\mathcal{M}$, commuting with the $\phi_{\kappa-1}$-, N-, and E-actions, and satisfying $[g_1] \circ [g_2] = [g_1 g_2]$ for all $g_1, g_2 \in G$, and $[1]$ is the identity map. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} E$, $m \in \mathcal{M}$, then

$$[g](au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i [g](m).$$

Remark 1.2. (1) If $\kappa = 2$, the category $BrMod^1_{dd,K/F}$ is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ together with an E-action and descent data on the generic fiber from K to F (this equivalence depends on π). In this case it follows from other axioms that there is always a unique N which satisfies the required properties. See for example Proposition 5.1.3 of [1].

(2) If $\kappa \le \kappa'$, then there is a fully faithful functor $L : BrMod^{\kappa-1}_{dd,K/F} \to BrMod^{\kappa'-1}_{dd,K/F}$ which identifies $BrMod^{\kappa-1}_{dd,K/F}$ as a full subcategory of $BrMod^{\kappa'-1}_{dd,K/F}$. More precisely, if $\mathcal{M} = (\mathcal{M}, Fil^{\kappa-1}\mathcal{M}, \phi_{\kappa-1}, \{[g]\}, N)$ is an object in $BrMod^{\kappa-1}_{dd,K/F}$, then $L(\mathcal{M}) = (L(\mathcal{M}), Fil^{\kappa'-1}L(\mathcal{M}), \phi_{\kappa'-1}, \{[g]\}, N)$ where $L(\mathcal{M}) = \mathcal{M}$, $Fil^{\kappa'-1}L(\mathcal{M}) = u^{e(\kappa'-\kappa)}Fil^{\kappa-1}\mathcal{M}$, $\phi_{\kappa'-1}(u^{e(\kappa'-\kappa)}x) = \phi_{\kappa-1}(x)$, and N, $[g]$ remain the same.

(3) Let $Rep_E(G_F)$ be the category of representations of $G_F := Gal(\bar{F}/F)$ over E-vector spaces. In this paper, we use the contravariant functor $T_{st} : BrMod^{\kappa-1}_{dd,,K/F} \to Rep_E(G_F)$ defined in section 2.1 of [3].

In this paper, we assume that $K = K_0((-p)^{1/p^r-1})$ and $F = K_0$. Note that in this case we have $e = p^r - 1$ and K is Galois over K_0 with $Gal(K/K_0) \cong \mathbb{Z}/(p^r-1)\mathbb{Z}$.

2. Rank one Breuil modules

In this section, we classify rank one Breuil modules, determine when we have nontrivial morphisms between two rank one Breuil modules, and prove some other properties. Most of these results are in [3], we sketch the proofs here and refer to [3] for details.

Recall that $S = Hom(k, \bar{\mathbb{F}}_p) \cong \mathbb{Z}/r\mathbb{Z}$ and E contains the image of $\tau_i \in S$, so we have a ring isomorphism $k \otimes_{\mathbb{F}_p} E \simeq E^S$ where the action of $x \otimes 1$ on the τ-component coincides with the action of $1 \otimes \tau(x)$ for $\tau \in S$. Therefore we may write $\mathcal{S} = \oplus_S E[u]/u^{ep}$. We also denote ϕ to be the map $\phi : E[u]/u^{ep} \to E[u]/u^{ep}$ which sends u to u^p and acts trivially on E.

If $\mathcal{M}$ is an object of $BrMod^{\kappa-1}_{dd,K/K_0}$, then

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_r,$$

where $\mathcal{M}_i = \mathcal{M} \otimes_{\mathcal{S},\tau_i} E[u]/u^{ep}$ is a free $E[u]/u^{ep}$-module, which is characterized by the fact that the action of $x \otimes 1$ on $\mathcal{M}_i$ coincides with the action of $1 \otimes \tau_i(x)$ for $\tau_i \in S$. Throughout the paper if $\mathcal{M}$ is a Breuil module over $\mathcal{S}$, then $\mathcal{M}_i$ will always denote the τ_i-component of $\mathcal{M}$. By convention, the subscripts i are always taken modulo r. Similarly, $Fil^{\kappa-1}\mathcal{M}$ has a decomposition

$$Fil^{\kappa-1}\mathcal{M} = Fil^{\kappa-1}\mathcal{M}_1 \oplus Fil^{\kappa-1}\mathcal{M}_2 \oplus \cdots \oplus Fil^{\kappa-1}\mathcal{M}_r,$$

with $u^{e(\kappa-1)}\mathcal{M}_i \subset Fil^{\kappa-1}\mathcal{M}_i \subset \mathcal{M}_i$. The Frobenius action of $k \otimes_{\mathbb{F}_p} E$ maps E_{τ_i} to $E_{\tau_{i+1}}$, $\phi_{\kappa-1}$ induces $\phi_{\kappa-1} : Fil^{\kappa-1}\mathcal{M}_i \to \mathcal{M}_{i+1}$ for $i \in \mathbb{Z}/r\mathbb{Z}$ and the image generates $\mathcal{M}_{i+1}$, and $N(\mathcal{M}_i) \subset \mathcal{M}_i$.

If $\mathcal{M}$ is of rank one as an $\mathcal{S}$-module, then $\mathcal{M}_i$ is free of rank one over $E[u]/u^{ep}$. Therefore, there exists an integer $m_i \in [0, e(\kappa-1)]$ such that $Fil^{\kappa-1}\mathcal{M}_i = u^{m_i}\mathcal{M}_i$.

Let e_1 be a basis of $\mathcal{M}_1$. Define $e_2 = \phi_{\kappa-1}(u^{m_1}e_1)$. Since $\phi_{\kappa-1}(Fil^{\kappa-1}\mathcal{M}_1)$ generates $\mathcal{M}_2$, e_2 is a basis of $\mathcal{M}_2$. Inductively, define $e_{i+1} = \phi_{\kappa-1}(u^{m_i}e_i)$ for $i < r$. Then e_i is a basis of $\mathcal{M}_i$. Finally, we have $\phi_{\kappa-1}(u^{m_r}e_r) = ae_1$ for some $a \in (E[u]/u^{ep})^\times$.

Remark 2.1. Assume that $\lambda \in E[u]/u^{ep}$ is invertible. Replacing e_1 by λe_1 changes a to $a \cdot \phi^r(\lambda)/\lambda$. By the following lemma, we may assume that $a \in E^\times$.

Lemma 2.2. *If $x \equiv 1 \pmod u$, then there exists $y \in \mathcal{S}$, such that $y/\phi^r(y) = x$.*

Proof. Since $x \equiv 1 \pmod u$, $\phi^{r^n}(x) = 1$ for sufficiently large n. Thus we can choose $y = \prod_{n=0}^{\infty} \phi^{r^n}(x)$. □

Note that $\phi_{\kappa-1}(u^e N(u^{m_i}e_i)) = N(\phi_{\kappa-1}(u^{m_i}e_i)) = N(e_{i+1})$. On the other hand, we know that $\phi_{\kappa-1}(u^e N(u^{m_i}e_i)) = u^{ep}\phi_{\kappa-1}(N(u^{m_i}e_i)) = 0$. Therefore $N(e_{i+1}) = 0$ for any i.

We then consider the descent data. By the definition of Breuil modules, the Galois action commutes with other actions, so $[g]$ maps $\mathcal{M}_i$ to $\mathcal{M}_i$. On the i-th piece, the action of G on $E[u]/u^{ep}$ is given by $[g]u = \omega_i(g)u$. Assume that $[g] \cdot e_i = \alpha_i(g)e_i$, where α_i is a function $\alpha_i : G \to (E[u]/u^{ep})^\times$. By definition, $[g][h] = [gh]$, so α_i is a character. Since $\alpha_i(g)$ is an e-th root of unit in $(E[u]/u^{ep})^\times$, $\alpha_i(g) \in E^\times$. We may assume that $\alpha_i = \omega_i^{\mu_i}$ for some $\mu_i \pmod e$. Also, $[g] \circ \phi_{\kappa-1} = \phi_{\kappa-1} \circ [g]$, we have $\mu_{i+1} \equiv p(\mu_i + m_i) (\mathrm{mod}\ e)$. From this, we have

$$p^r m_i + p^{r-1}m_{i+1} + \cdots + p^2 m_{i+r-2} + pm_{i+r-1} \equiv 0 (\mathrm{mod}\ e).$$

We write $(a)_i = \begin{cases} a \text{ if } i = 1, \\ 1 \text{ otherwise.} \end{cases}$

Proposition 2.3. *If $\mathcal{M}$ is a rank one object of $BrMod^{\kappa-1}_{dd,K/K_0}$, then there exist integers $m_i \in [0, e(\kappa-1)]$, $\mu_i \in [0, e-1]$, and $a \in E^\times$, such that we can choose basis e_i for $\mathcal{M}_i$, and*

(1) $Fil^{\kappa-1}\mathcal{M}_i = \langle u^{m_i}e_i \rangle$,

(2) $\phi_{\kappa-1}(u^{m_i}e_i) = (a)_{i+1}e_{i+1}$,

(3) $\mu_{i+1} \equiv p(\mu_i + m_i)(mod\ e)$,
(4) $[g] \cdot e_i = \omega_i^{\mu_i}(g)e_i$,
(5) $N(e_i) = 0$.
We will write the Breuil module with these invariants $\mathcal{M}(m_i, \mu_i, a)$.

We attach to $\mathcal{M}(m_i, \mu_i, a)$ another invariant

$$\mu_{fil,i} = \frac{p^r m_i + p^{r-1} m_{i+1} + \cdots + p^2 m_{i+r-2} + p m_{i+r-1}}{e}.$$

Note that $\mu_{fil,i}$ is an integer divisible by p and $pm_i = p\mu_{fil,i} - \mu_{fil,i+1}$.

For $a \in E^\times$, let $unr(a) : G_{K_0} \to E^\times$ be the unramified character of G_{K_0} sending the geometric Frobenius to a.

Proposition 2.4. *Let* $\mathcal{M} = \mathcal{M}(m_i, \mu_i, a)$ *be as in above, then the character* $T_{st}(\mathcal{M})$ *of* G_{K_0} *is*

$$unr(a) \cdot \omega_i^{(\kappa-1)(1+p+\cdots+p^{r-1})-(\mu_i+\mu_{fil,i})}.$$

Proof. This is Proposition 2.3 of [3]. Note that $\omega_i^{(\mu_i+\mu_{fil,i})}$ is independent of i since $p(\mu_i + \mu_{fil,i}) \equiv (\mu_{i+1} + \mu_{fil,i+1})(\text{mod } e)$. □

The following two propositions are from section 5 of [3]. Let $\mathcal{A}$ and $\mathcal{B}$ be two rank one objects of $BrMod^{\kappa-1}_{dd,K/K_0}$. Assume that $\mathcal{A} = \mathcal{M}(a_i, \alpha_i, a)$ and $\mathcal{B} = \mathcal{M}(b_i, \beta_i, b)$. Write

$$\alpha_{fil,i} = \frac{p^r a_i + p^{r-1} a_{i+1} + \cdots + p^2 a_{i+r-2} + p a_{i+r-1}}{e}$$

and

$$\beta_{fil,i} = \frac{p^r b_i + p^{r-1} b_{i+1} + \cdots + p^2 b_{i+r-2} + p b_{i+r-1}}{e}.$$

Proposition 2.5. *Assume that there is an isomorphism* $f : T_{st}(\mathcal{B}) \to T_{st}(\mathcal{A})$. *Then there exists a non-zero morphism (in the category* $BrMod^{\kappa-1}_{dd,K/K_0}$*)* $\mathcal{A} \to \mathcal{B}$ *if and only if* $\alpha_{fil,i} \leq \beta_{fil,i}$ *for all* i. *In this case, the morphism* $f' : \mathcal{A} \to \mathcal{B}$ *defined by* $A_i \mapsto u^{\beta_{fil,i}-\alpha_{fil,i}} B_i$ *induces an isomorphism* $T_{st}(f')$, *where* A_i *and* B_i *are basis of* $\mathcal{A}_i$ *and* $\mathcal{B}_i$ *respectively.*

Proposition 2.6. *Assume that there is an isomorphism* $f : T_{st}(\mathcal{B}) \to T_{st}(\mathcal{A})$. *Then there exists a third object* $\mathcal{C}$ *in* $BrMod^{\kappa-1}_{dd,K/K_0}$ *of rank one, with morphisms* $f'_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ *and* $f'_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$, *such that* $T_{st}(f'_{\mathcal{A}}) \circ T_{st}(f'_{\mathcal{B}})^{-1}$ *is an isomorphism.*

Proof. If $\alpha_{fil,i} \leq \beta_{fil,i}$ for all i, we may choose $\mathcal{C} = \mathcal{B}$. If $\alpha_{fil,i} \geq \beta_{fil,i}$ for all i, we may choose $\mathcal{C} = \mathcal{A}$.

In general, we construct $\mathcal{C}$ directly as follows. Let $\gamma_{fil,i} = \max(\alpha_{fil,i}, \beta_{fil,i})$, $n_i = \frac{1}{p}\max(0, \beta_{fil,i} - \alpha_{fil,i})$, $c_i = a_i + pn_i - n_{i+1}$, and $\gamma_i = \alpha_i + \alpha_{fil,i} - \gamma_{fil,i}$. Then we may define $\mathcal{C} = \mathcal{M}(c_i, \gamma_i, a)$. See Proposition 5.6 of [3] for more details. □

We define a special type of Breuil modules.

Definition 2.7. Let $J \subset S$. We say $\mathcal{M}(m_i, \mu_i, a)$ is of *type* J if $m_i = e(\kappa - 1)\delta_J(i + 1)$, where $\delta_J(i) = \begin{cases} 1 \text{ if } \tau_i \in J, \\ 0 \text{ otherwise.} \end{cases}$

Proposition 2.8. *Fix $J \subset S$ and a character $\psi : G_{K_0} \to E^\times$ trivial on I_K. Then there exists a unique rank one Breuil module $\mathcal{M}$ of type J such that $T_{st}(\mathcal{M}) \cong \psi$.*

Proof. Since ψ is trivial on I_K, we may write $\psi = unr(a)\omega_0^n$ for some $a \in E^\times$. Define

$$\mu_i \equiv -p^i n + (\kappa - 1)\sum_{j=0}^{r-1} p^j - (\kappa - 1)\sum_{j=0}^{r-1} p^{r-j}\delta_J(i+j+1) \pmod{e}.$$

It is easy to see that

$$\mu_{i+1} \equiv p(\mu_i + e(\kappa-1)\delta_J(i+1)) (\bmod\ e).$$

Then we may define $\mathcal{M} = \mathcal{M}(e(\kappa-1)\delta_J(i+1), \mu_i, a)$. By Proposition 2.4, $T_{st}(\mathcal{M}) \cong \psi$.

The uniqueness follows from Proposition 2.4 and Definition 2.7. □

Corollary 2.9. *Fix a character $\psi : G_{K_0} \to E^\times$ trivial on I_K. Let J and J' be two subsets of S. By the above proposition, we know that there exist two rank one Breuil modules $\mathcal{M}_J$ and $\mathcal{M}_{J'}$ of type J and J' respectively, such that $T_{st}(\mathcal{M}_J) \cong T_{st}(\mathcal{M}_{J'}) \cong \psi$. Then there exists a non-zero morphism $f' : \mathcal{M}_J \to \mathcal{M}_{J'}$ if and only if $J \subset J'$.*

Proof. By definition, $\mathcal{M}_J$ has $m_{J,i} = e(\kappa-1)\delta_J(i+1)$ and $\mathcal{M}_{J'}$ has $m_{J',i} = e(\kappa-1)\delta_{J'}(i+1)$. If $J \subset J'$, it is obvious that $m_{J,i} \le m_{J',i}$ and therefore $\mu_{J,fil,i} \le \mu_{J',fil,i}$. There exists a nonzero morphism $f' : \mathcal{M}_J \to \mathcal{M}_{J'}$ by Proposition 2.5.

If $J \not\subset J'$, we choose $j \in J\backslash J'$, then $\mu_{J,fil,j} > \mu_{J',fil,j}$ and there is no nonzero morphism $f' : \mathcal{M}_J \to \mathcal{M}_{J'}$. □

3. Reducible rank two Breuil modules

In this section, we consider rank two Breuil modules which are extensions of a rank one Breuil module with descent data by another rank one Breuil module with descent data. We will follow the method of section 7 of [6] for the remainder of this section.

First, we forget about the descent data and the monodromy operator N. Let $\mathcal{M}(m_i, a)$ and $\mathcal{M}(n_i, b)$ be two Breuil modules of rank one. Let $\mathcal{M} \in Ext^1(\mathcal{M}(m_i, a), \mathcal{M}(n_i, b))$. Assume that

$$\mathcal{M}(m_i, a) = \oplus_{i\in S}\langle \bar{f}_i\rangle,$$
$$\mathcal{M}(n_i, b) = \oplus_{i\in S}\langle \bar{e}_i\rangle.$$

We may write that

$$\mathcal{M} = \oplus_{i\in S}\mathcal{M}_i = \oplus_{i\in S}\langle e_i, f_i\rangle,$$

and

$$Fil^{\kappa-1}\mathcal{M} = \oplus_{i\in S} Fil^{\kappa-1}\mathcal{M}_i = \oplus_{i\in S}\langle u^{n_i}e_i, u^{m_i}f_i + h_ie_i\rangle,$$

where e_i is the image of $\bar{e}_i$, f_i is a lift of $\bar{f}_i$, and $h_i \in E[u]/u^{ep}$. We will simplify the structure of $\mathcal{M}$ in three steps.

(Step 1) Since $u^{e(\kappa-1)}\mathcal{M}_i \subset Fil^{\kappa-1}\mathcal{M}_i$, we have $u^{e(\kappa-1)}f_i \in \langle u^{n_i}e_i, u^{m_i}f_i + h_ie_i\rangle$. This tells us that $h_i \in u^{max(0, m_i+n_i-e(\kappa-1))}E[u]/u^{ep}$.

(Step 2) We may assume that $\phi_{\kappa-1}(u^{m_i}f_i + h_ie_i) = (a)_{i+1}f_{i+1}$. First, we choose f_1 such that $Fil^{\kappa-1}\mathcal{M}_1 = \langle u^{n_1}e_1, u^{m_1}f_1 + h_1e_1\rangle$ for some $h_1 \in E[u]/u^{ep}$. Then we define f_{i+1} inductively by $\phi_{\kappa-1}(u^{m_i}f_i + h_ie_i) = (a)_{i+1}f_{i+1}$ for $i < r$. Suppose that $\phi_{\kappa-1}(u^{m_r}f_r + h_re_r) = (a)_1(f_1 + Xe_1)$, then we define $f_1' = f_1 + Xe_1$ and $h_1' = h_1 - u^{m_1}X$. Hence $u^{m_1}f_1' + h_1'e_1 = u^{m_1}f_1 + h_1e_1$. From this construction, we see that f_i if $i \neq 1$ or f_1' if

$i=1$ is a lift of $\bar{f}_i$. They give a basis of $\mathcal{M}$ since $\phi_{\kappa-1}(\mathcal{M})$ generates $\mathcal{M}$. The relation $\phi_{\kappa-1}(u^{m_i}f_i+h_ie_i)=(a)_{i+1}f_{i+1}$ $(i\neq 1)$ holds from the construction and $\phi_{\kappa-1}(u^{m_1}f_1'+h_1'e_1)=\phi_{\kappa-1}(u^{m_1}f_1+h_1e_1)=(a)_2f_2$.

(Step 3) Now we determine what kind of transformations we can make to keep the form in (Step 2). Assume that replacing f_i by $f_i'=f_i+X_ie_i$ and h_i by H_i keeps the form

$$\phi_{\kappa-1}(u^{m_i}f_i+h_ie_i)=(a)_{i+1}f_{i+1}.$$

We have

$$\phi_{\kappa-1}(u^{m_i}f_i'+H_ie_i)=(a)_{i+1}f_{i+1}'.$$

The left hand side of the above equation is

$$\begin{aligned}\phi_{\kappa-1}(u^{m_i}f_i'+H_ie_i)&=\phi_{\kappa-1}(u^{m_i}f_i+h_ie_i+(u^{m_i}X_i+H_i-h_i)e_i)\\&=(a)_{i+1}f_{i+1}+\phi_{\kappa-1}((u^{m_i}X_i+H_i-h_i)e_i)\end{aligned}$$

We must have

$$(a)_{i+1}X_{i+1}e_{i+1}=\phi_{\kappa-1}((u^{m_i}X_i+H_i-h_i)e_i).$$

Assume that $u^{m_i}X_i+H_i-h_i=t_{i+1}u^{n_i}$ for some $t_{i+1}\in E[u]/u^{ep}$, then

$$\begin{aligned}&(a)_{i+1}X_{i+1}=(b)_{i+1}\phi(t_{i+1})\\&H_i=h_i+t_{i+1}u^{n_i}-(b/a)_iu^{m_i}\phi(t_i).\end{aligned}\tag{3.1}$$

From the above analysis, we have the following proposition, which generalizes Lemma 5.2.4 of [2].

Proposition 3.1. *If we forget about the descent data and the monodromy operator N, then*

$$Ext^1(\mathcal{M}(m_i,a),\mathcal{M}(n_i,b))\cong\frac{\{(h_i)_{i\in S}|h_i\in u^{max(0,m_i+n_i-e(\kappa-1))}E[u]/u^{ep}\}}{\{(t_{i+1}u^{n_i}-(b/a)_iu^{m_i}\phi(t_i))_{i\in S}\}},$$

where t_i's run through all elements in $E[u]/u^{ep}$.

We add the descent data to our consideration. Let $\mathcal{M}\in Ext^1(\mathcal{M}(m_i,\alpha_i,a),\mathcal{M}(n_i,\beta_i,b))$, such that

$$\begin{gathered}\mathcal{M}_i=\langle e_i,f_i\rangle,\\Fil^{\kappa-1}\mathcal{M}_i=\langle u^{n_i}e_i,u^{m_i}f_i+h_ie_i\rangle,\\\phi_{\kappa-1}(u^{n_i}e_i)=(b)_{i+1}e_{i+1},\quad\phi_{\kappa-1}(u^{m_i}f_i+h_ie_i)=(a)_{i+1}f_{i+1}.\end{gathered}$$

Assume that

$$\begin{gathered}[g]e_i=\omega_i^{\beta_i}(g)e_i,\\ [g]f_i=\omega_i^{\alpha_i}(g)f_i+A_{i,g}e_i.\end{gathered}$$

We show that we may make $A_{i,g}=0$ without changing the forms of $Fil^{\kappa-1}$ and $\phi_{\kappa-1}$.

Lemma 3.2. *$H^q(G,E[u]/u^{ep})=0$ for all $q>0$. Here the Galois action is given by $g(\sum a_ju^j)=\sum a_j\omega_i(g)^ju^j$.*

Proof. Write $E(j)=E$ with G-action given by $g\cdot a=a\omega_i(g)^j$. Then $H^q(G,E(j))=0$ for $q>0$ since $\sharp G=e=p^r-1$ and $\sharp E(j)=p^N$ for some integer N. Therefore, if $q>0$,

$$H^q(G,E[u]/u^{ep})=\oplus_jH^q(G,E(j))=0.$$

□

Lemma 3.3. *All nonzero terms of $A_{i,g}$ have degree divisible by p.*

Proof. We use the relation $\phi_{\kappa-1} \circ [g] = [g] \circ \phi_{\kappa-1}$ to prove this lemma. On one hand,

$$[g] \circ \phi_{\kappa-1}(u^{m_i} f_i + h_i e_i) = (a)_{i+1}(\omega_{i+1}^{\alpha_{i+1}}(g) f_{i+1} + A_{i+1,g} e_{i+1}).$$

On the other hand,

$$\begin{aligned}[g](u^{m_i} f_i + h_i e_i) &= \omega_i^{m_i}(g) u^{m_i}(\omega_i^{\alpha_i}(g) f_i + A_{i,g} e_i) + g(h_i)\omega_i^{\beta_i}(g) e_i \\ &= \omega_i^{m_i+\alpha_i}(g)(u^{m_i} f_i + h_i e_i) + H e_i,\end{aligned}$$

where

$$H = \omega_i^{m_i}(g) u^{m_i} A_{i,g} + g(h_i)\omega_i^{\beta_i}(g) - \omega_i^{m_i+\alpha_i}(g) h_i.$$

Since $[g]$ preserves $Fil^{\kappa-1}$, we have $He_i \in Fil^{\kappa-1}\mathcal{M}$. Therefore, $u^{n_i}|H$. Let H/u^{n_i} be an element in $E[u]/u^{ep}$ such that $u^{n_i}(H/u^{n_i}) = H$. Note that H/u^{n_i} is not unique. (Assume that $H/u^{n_i} = \sum c_i u^i$, then $\sum c_i u^{i+n_i} = H$. Therefore c_i is uniquely determined only for those i with $i + n_i < ep$.) But $\phi(H/u^{n_i}) \in E[u]/u^{ep}$ is unique, because if $i + n_i \geq ep$, then $pi \geq p(ep - n_i) \geq ep$. Then

$$\phi_{\kappa-1} \circ [g](u^{m_1} f_i + h_i e_i) = \omega_i^{(m_i+\alpha_i)}(g)(a)_{i+1} f_{i+1} + \phi(H/u^{n_i})(b)_{i+1} e_{i+1}.$$

Remember that $\alpha_{i+1} \equiv p(m_i + \alpha_i) (\text{mod } e)$. Comparing the above two equations, we see that $A_{i+1,g} = (b/a)_{i+1}\phi(H/u^{n_i})$. All nonzero terms of $A_{i+1,g}$ have degree divisible by p. □

Lemma 3.4. *We may assume that $A_{i,g} = 0$.*

Proof. Recall that $[hg] = [h][g]$. Applying both sides to f_i, we have

$$\begin{aligned}&\omega_i^{\alpha_i}(hg) f_i + A_{i,hg} e_i = \omega_i^{\alpha_i}(g)(\omega_i^{\alpha_i}(h) f_i + A_{i,h} e_i) + h(A_{i,g})\omega_i^{\beta_i}(h) e_i \\ \Rightarrow &\frac{A_{i,hg}}{\omega_i^{\alpha_i}(hg)} = \frac{A_{i,h}}{\omega_i^{\alpha_i}(h)} + \omega_i^{\beta_i-\alpha_i}(h) h(\frac{A_{i,g}}{\omega_i^{\alpha_i}(g)}).\end{aligned}$$

So $(g \mapsto \frac{A_{i,g}}{\omega_i^{\alpha_i}(g)})$ is a cocycle in $H^1(G, E[u]/u^{ep})$, where the Galois action is given by $g \cdot (\sum a_j u^j) = \omega_i^{\beta_i-\alpha_i}(g) \sum a_j \omega_i(g)^j u^j$. By the same argument of Lemma 3.2, this cohomology group is trivial. The cocycle is actually a coboundary.

Let $f_i' = f_i + (b/a)_i\phi(t_i)e_i$, for $t_i \in E[u]/u^{ep}$. Note that by equation (3.1), this change will keep the form of $\mathcal{M}$ as stated after Proposition 3.1. (However, we may get new h_i's.) Then

$$\begin{aligned}[g] f_i' &= [g] f_i + g((b/a)_i\phi(t_i))[g] e_i \\ &= \omega_i^{\alpha_i}(g) f_i' + (A_{i,g} + (g \cdot ((b/a)_i\phi(t_i)) - (b/a)_i\phi(t_i))\omega_i^{\alpha_i}(g)) e_i,\end{aligned}$$

$\frac{A_{i,g}}{\omega_i^{\alpha_i}(g)}$ is changed by the coboundary of $(b/a)_i\phi(t_i)$. We then may assume that $A_{i,g} = 0$ by the above lemma. □

Remark 3.5. (1) By the above lemma, we assume that $A_{i,g} = 0$. Then in the proof of Lemma 3.3, $\phi(H/u^{n_i}) = 0$. So $u^e|(H/u^{n_i})$, i.e., $u^{e+n_i}|H$. If we write $h_i = \sum_j a_j u^j$, since $H = g(h_i)\omega_i^{\beta_i}(g) - \omega_i^{m_i+\alpha_i}(g) h_i$, then

$$H = \sum_j a_j(\omega_i^{j+\beta_i}(g) - \omega_i^{m_i+\alpha_i}(g)) u^j.$$

If $j < e + n_i$ and $a_j \neq 0$, then $j \equiv m_i + \alpha_i - \beta_i (\text{mod } e)$.

(2) If $\{h_i\}$ and $\{h'_i\}$ give isomorphic Breuil modules with $A_{i,g} = 0$. Then we know that

$$h'_i = h_i + t_{i+1}u^{n_i} - (b/a)_i u^{m_i}\phi(t_i)$$

for some $\{t_i\}$. By the proof of Lemma 3.4, we know that $g \cdot ((b/a)_i\phi(t_i)) - (b/a)_i\phi(t_i) = 0$. So if $t_i = \sum_j a_{i,j}u^j$, $a_{i,j} \neq 0$ and $j < e$, then $\beta_i - \alpha_i + jp \equiv 0(\text{mod } e)$.

Therefore, in equation (3.1), if all nonzero terms with degree less than n_i of h_i have degree congruent to $(m_i + \alpha_i - \beta_i)$ (mod e), then all nonzero terms with degree less than n_i of H_i also have degree congruent to $(m_i + \alpha_i - \beta_i)$ (mod e).

Next, we study the group $Ext^1(\mathcal{M}(m_i, \alpha_i, a), \mathcal{M}(n_i, \beta_i, b))$. Let $\mathcal{M} \in Ext^1(\mathcal{M}(m_i, \alpha_i, a), \mathcal{M}(n_i, \beta_i, b))$. Assume that $\mathcal{M} = \oplus_i \mathcal{M}_i$ has the following form.

$$\begin{gathered}\mathcal{M}_i = \langle e_i, f_i \rangle,\\ Fil^{\kappa-1}\mathcal{M}_i = \langle u^{n_i}e_i, u^{m_i}f_i + h_ie_i \rangle,\\ \phi_{\kappa-1}(u^{n_i}e_i) = (b)_{i+1}e_{i+1}, \quad \phi_{\kappa-1}(u^{m_i}f_i + h_ie_i) = (a)_{i+1}f_{i+1},\\ [g]e_i = \omega_i^{\beta_i}(g)e_i, \quad [g]f_i = \omega_i^{\alpha_i}(g)f_i,\\ N(e_i) = 0, \quad N(f_i) = C_ie_i.\end{gathered}$$

From Proposition 3.1, two different sets $\{h_i\}$ and $\{h'_i\}$ give isomorphic Breuil modules only if there exist $t_i \in E[u]/u^{ep}$ such that $h_i - h'_i = t_{i+1}u^{n_i} - (b/a)_iu^{m_i}\phi(t_i)$ for all $i \in S$. We would like to solve $(T_i)_{i\in S}$ from the following equation system:

$$H_i = u^{n_i}T_{i+1} - (b/a)_iu^{m_i}\phi(T_i) \quad i \in S \tag{3.2}$$

where $H_i \in E[u]/u^{ep}$.

Assume that $H_i = \sum_{j_i} H_{i,j_i}u^{j_i}$, $T_i = \sum_{j_i} T_{i,j_i}u^{j_i}$, then the equation system is the same as

$$H_{i,j_i} = T_{i+1,j_i-n_i} - (b/a)_iT_{i,\frac{j_i-m_i}{p}} \quad i \in S\ \forall j_i. \tag{3.3}$$

T_{i,j_i} is required to be zero unless j_i is a nonnegative integer. Set $X_i = \frac{1}{p}(\beta_{fil,i} - \alpha_{fil,i})$, $J_i = X_{i+1} + n_i$. Then $X_{i+1} = pX_i + m_i - n_i$. Note that X_i and J_i are integers.

We will attempt to solve the equation system (3.3) by induction on the least integer greater than or equal to $max_{i\in S}\{|j_i - J_i|\}$. The condition $|j_i - J_i| = 0$ for all $i \in S$ is an empty set unless J_i is a nonnegative integer and $j_i = J_i$, in which case, we have the following base case equation system:

$$H_{i,J_i} = T_{i+1,X_{i+1}} - (b/a)_iT_{i,X_i} \quad i \in S. \tag{3.4}$$

If this is solvable for (T_{i,X_i}), we have our base case. We assume the following inductive hypothesis:

(a) the equation system (3.3) can be solved for all H_{i,j_i} with $|j_i - J_i| \leq N$;

(b) in doing so, all and only the T_{i,j_i} with $|j_i - X_i| \leq N$ or with $j_i \notin \mathbb{Z}_{\geq 0}$ have been determined.

Assume that $N < |j_i - J_i| \leq N + 1$, then

$$N < |(j_i - n_i) - X_{i+1}| \leq N + 1.$$

If $|j_i - J_i| < N + 1$, then j_i is not an integer, we have $T_{i+1,j_i-n_i} = 0$. If $|j_i - J_i| = N + 1$ and $N = 0$, then

$$|\frac{j_i - m_i}{p} - X_i| = |j_i - J_i|/p = 1/p,$$

$\frac{j_i-m_i}{p}$ is not an integer and we set $T_{i+1,j_i-n_i}=H_{i,j_i}$. If $|j_i-J_i|=N+1$ and $N\geq 1$, then

$$|\frac{j_i-m_i}{p}-X_i|=|j_i-J_i|/p\leq N,$$

$T_{i,\frac{j_i-m_i}{p}}$ has been determined. We may take

$$T_{i+1,j_i-n_i}=H_{i,j_i}+(b/a)_iT_{i,\frac{j_i-m_i}{p}}.$$

Note that if $j_i<n_i$, then there is a solution if and only if T_{i+1,j_i-n_i} so obtained is 0.

From the above analysis, we have the following lemma.

Lemma 3.6. *The equation system (3.2) has a solution if and only if*

(a) the base case (3.4) is either vacuous or is non vacuous and has a solution;

(b) whenever $j_i<n_i$, we have

$$T_{i+1,j_i-n_i}=H_{i,j_i}+(b/a)_iT_{i,\frac{j_i-m_i}{p}}=0.$$

Lemma 3.7. *Suppose that the equation system (3.2) has a solution and* $\deg H_i<n_i\ \forall\ i$, *then $H_i=0\ \forall\ i$.*

Proof. First, we prove that $H_{i,J_i}=0\ \forall i$. If $X_i\geq 0\ \forall i$, then by definition, $J_i=X_{i+1}+n_i\geq n_i$, so $H_{i,J_i}=0$. If $X_i<0$ for some i, choose any $l\in S$, we analyze case by case.

(1) $X_l<0$. If $X_{l+1}<0$, then $H_{l,J_l}=T_{l+1,X_{l+1}}-(b/a)_lT_{l,X_l}=0$. If $X_{l+1}\geq 0$, then $J_l=X_{l+1}+n_l\geq n_l$, $H_{l,J_l}=0$.

(2) $X_l\geq 0$. If $X_{l+1}\geq 0$, then $J_l=X_{l+1}+n_l\geq n_l$, $H_{l,J_l}=0$. If $X_{l+1}<0$, then $H_{l,J_l}=c\cdot T_{l,X_l}$. Also $H_{l-1,J_{l-1}}=0$ since $J_{l-1}=X_l+n_{l-1}\geq n_{l-1}$, $H_{l,J_l}=c\cdot T_{l-1,X_{l-1}}$. We continue this step, there exists a minimal a such that $X_{l-a+1}\geq 0$ but $X_{l-a}<0$. Then $H_{l,J_l}=c\cdot T_{l-a,X_{l-a}}=0$.

Now we know that $H_{i,J_i}=0\ \forall i$. The base case has a solution $T_{i,J_i}=0\ \forall i$. Then by the induction procedure, it is easy to see that $T_{i,j}=0\ \forall\ i,j$. So $H_i=0\ \forall\ i$. $\square$

Lemma 3.8. *(1) If the base case for $(H_i)_{i\in S}$ can be solved, or cannot be solved but $X_i<0$ for some i, then there exists a unique $(H_i')_{i\in S}$ such that the equation system for $(H_i')_{i\in S}$ can be solved and such that* $\deg(H_i'-H_i)<n_i$ *for all $i\in S$.*

(2) If the base case cannot be solved and $X_i\geq 0$ for all i, then there exists a unique $(H_i')_{i\in S}$ such that the equation system for $(H_i')_{i\in S}$ can be solved, $\deg(H_i'-H_i)<n_i$ *for $i\neq r$, and the only nonzero term of $H_r'-H_r$ of degree at least n_r if any is of degree $J_r=X_1+n_r$.*

Proof. First, we make a suitable choice of coefficients H'_{i,J_i} for all i: namely, we would like the base case (3.4) to be solvable, and we would like $H'_{i,J_i}=H_{i,J_i}$ whenever $J_i\geq n_i$, except that in case (2) we omit the latter condition when $i=r$.

If the base case is solvable for H_i, we just take $H_i'=H_i$. If the base case is not solvable, we distinguish the two cases (1) and (2). Note that the base case gives us the following equation system:

$$\begin{cases} H_{1,J_1} = T_{2,X_2} - (b/a)T_{1,X_1} \\ H_{2,J_2} = T_{3,X_3} - T_{2,X_2} \\ \quad\cdots\cdots \\ H_{r-1,J_{r-1}} = T_{r,X_r} - T_{r-1,X_{r-1}} \\ H_{r,J_r} = T_{1,X_1} - T_{r,X_r} \end{cases}$$

In case (1), if $X_i < 0$, let $s > 0$ be minimal such that $X_{i+s} < 0$ (we might have $s = r$). If $s = 1$, set $H'_{i,J_i} = 0$. If $s > 1$ and $1 \notin \{i+1, \cdots, i+s-1\}$, summing the equations from i to $i+s-1$ gives $H_{i,J_i} + H_{i+1,J_{i+1}} + \cdots + H_{i+s-1,J_{i+s-1}} = T_{i+s,X_{i+s}} - (b/a)_i T_{i,X_i} = 0$. In this case, we take $H'_{j,J_j} = H_{j,J_j}$ for $i < j < i+s$ and set $H'_{i+s-1,J_{i+s-1}} = -(H_{i,J_i} + \cdots + H_{i+s-2,J_{i+s-2}})$. If $s > 1$ and $1 \in \{i+1, \cdots, i+s-1\}$, summing the equations from i to $i+s-1$ gives $H_{i,J_i} + H_{i+1,J_{i+1}} + \cdots + H_{i+s-1,J_{i+s-1}} = T_{i+s,X_{i+s}} - (b/a)_i T_{i,X_i} + (1-b/a)T_{1,X_1} = (1-b/a)T_{1,X_1}$. In the case $a \neq b$, we take $H'_{j,J_j} = H_{j,J_j}$ for $i < j \leq i+s-1$; in the case $a = b$, we take $H'_{j,J_j} = H_{j,J_j}$ for $i < j < i+s-1$ and set $H'_{i+s-1,J_{i+s-1}} = -(H_{i,J_i} + \cdots + H_{i+s-2,J_{i+s-2}})$. Now the base case is solvable, and $H'_{i,J_i} \neq H_{i,J_i}$ only for some $i \in S$ with $X_{i+1} < 0$, so that $J_i < n_i$.

In case (2), since all $X_i \geq 0$, none of the T_{i,X_i} are forced to be 0 by virtue of having negative degree. Now the insolvability of the base case is equivalent to the insolvability of $\sum_{i=1}^r H_{i,J_i} = (1-b/a)T_{1,X_1}$; this occurs if and only if $a = b$ and $\sum_{i=1}^r H_{i,J_i} \neq 0$. In this case we must take $H'_{i,J_i} = H_{i,J_i}$ for $i \neq r$ and $H'_{r,J_r} = -\sum_{i=1}^{r-1} H_{i,J_i}$.

Having made a suitable choice of the coefficients H'_{i,J_i}, we extend this to a full choice of H'_i's. The only obstruction is that Lemma 3.6(b) must be satisfied. In particular we can certainly set $H'_{i,j_i} = H_{i,j_i}$ whenever $j_i \geq n_i$ (and $j_i \neq J_i$). Recall that in the inductive process for solving the system of the equations that is described prior to Lemma 3.6, if $j_i \neq J_i$ then the coefficient $T_{i,(j_i-m_i)/p}$ has been determined before the coefficient H_{i,j_i} has ever been used in the process. We carry out the inductive process on the H'_i's, except that we initially treat H'_{i,j_i} as an indeterminate whenever $j_i < n_i$ and $j_i \neq J_i$. When we arrive at the determination of T_{i+1,j_i-n_i} in that inductive process (with $j_i < n_i$ and $j_i \neq J_i$), we simply set $H'_{i,j_i} = -(b/a)_i T_{i,(j_i-m_i)/p}$ and carry on.

Finally, the uniqueness in case (1) follows from Lemma 3.7. The uniqueness in case (2) follows from Lemma 3.7 and the fact that $H'_{r,J_r} = -\sum_{i=1}^{r-1} H_{i,J_i}$. □

Now we can state the following theorem which corresponds to Theorem 7.5 of [6].

Theorem 3.9. *Let $\mathcal{M} \in Ext^1(\mathcal{M}(m_i, \alpha_i, a), \mathcal{M}(n_i, \beta_i, b))$. Let $X_i = (\beta_{fil,i} - \alpha_{fil,i})/p$ and $J_i = X_{i+1} + n_i$. Write $\mathcal{M} = \oplus_{i\in S}\mathcal{M}_i$, then there exist e_i and f_i such that*

$$\mathcal{M}_i = \langle e_i, f_i \rangle,$$
$$Fil^{\kappa-1}\mathcal{M}_i = \langle u^{n_i}e_i, u^{m_i}f_i + h_ie_i \rangle,$$
$$\phi_{\kappa-1}(u^{n_i}e_i) = (b)_{i+1}e_{i+1}, \quad \phi_{\kappa-1}(u^{m_i}f_i + h_ie_i) = (a)_{i+1}f_{i+1},$$
$$N(e_i) = 0, \quad N(f_i) = C_ie_i,$$
$$[g]e_i = \omega_i^{\beta_i}(g)e_i, \quad [g]f_i = \omega_i^{\alpha_i}(g)f_i.$$

Here C_i is a polynomial with all nonzero terms have degree congruent to $(\beta_i - \alpha_i) \pmod e$, $h_i \in u^{max(0,m_i+n_i-e(\kappa-1))}E[u]/u^{ep}$ satisfies

(1) all nonzero terms have degree congruent to $m_i + \alpha_i - \beta_i$ (mod e);

(2) if $X_i \geq 0$ for all $i \in S$, $a = b$, and $J_i \equiv m_i + \alpha_i - \beta_i \pmod e$ for some $i \in S$, then $\deg(h_i) < n_i$ if $i \neq r$, h_r may have one nonzero term of degree $J_r \geq n_r$ and every other nonzero term of H_r has degree less than n_r; otherwise, all nonzero terms of h_i have degree less than n_i.

Furthermore, if $\kappa = 2$, each set $(h_i)_{i\in S}$ with the properties as above will give us a well defined rank two Breuil module.

Proof. We first prove the statement about $N(f_i)$. Assume that $N(f_i) = C_i e_i + D_i f_i$ with $u|C_i, D_i$. Because $N([g](f_i)) = [g](N(f_i))$, we have

$$\omega_i^{\alpha_i}(g)(C_i e_i + D_i f_i) = g(C_i)\omega_i^{\beta_i}(g)e_i + g(D_i)\omega_i^{\alpha_i}(g)f_i.$$

So C_i is a polynomial with all nonzero terms have degree congruent to $\beta_i - \alpha_i \pmod e$. $D_i = 0$ follows from the fact that $N(\bar{f}_i) = 0$.

From (Step 1), we have $h_i \in u^{max(0, m_i+n_i-e(\kappa-1))}E[u]/u^{ep}$. Because $u^{e+n_i}e_i \in Fil^{\kappa-1}\mathcal{M}_i$ and $\phi_{\kappa-1}(u^{e+n_i}e_i) = u^{ep}\phi_{\kappa-1}(u^{n_i}e_i) = 0$, we may assume that $\deg(h_i) < (e+n_i)$. Then condition (1) follows from Remark 3.5(1).

From the definition of J_i, we see that $J_i \equiv m_i + \alpha_i - \beta_i (\text{mod } e)$ if and only if $p\beta_i + pn_i + \beta_{fil,i+1} \equiv p\alpha_i + pm_i + \alpha_{fil,i+1} \pmod e$, if and only if $\beta_{i+1} + \beta_{fil,i+1} \equiv \alpha_{i+1} + \alpha_{fil,i+1} \pmod e$. If $J_i \equiv m_i + \alpha_i - \beta_i (\text{mod } e)$ for one $i \in S$, then $J_i \equiv m_i + \alpha_i - \beta_i (\text{mod } e)$ for all $i \in S$. Condition (2) follows from Lemma 3.8.

The last statement follows from Remark 1.2(1). □

Remark 3.10. (1) In the case $\kappa = 2$, we have an equivalence between $BrMod^1_{dd,K/K_0}$ and the category of certain finite flat group schemes over $\mathcal{O}_K$. When m_i's get larger, the corresponding group scheme gets more multiplicative. When m_i's get smaller, the corresponding group scheme gets more etale. The theorem is compatible with the fact that there are many extensions of etale group schemes by multiplicative ones, but none in the other direction. See the remark following Theorem 7.5 of [6] for a similar statement.

(2) In the case $\kappa = 2$, if $\beta_i + \beta_{fil,i} \not\equiv \alpha_i + \alpha_{fil,i} \pmod e$ or $a \neq b$, then all the h_i's can be taken to be monomials. This fact may help us simplify computations.

(3) In general, if $\kappa \neq 2$, a set $(C_i, h_i)_{i\in S}$ where C_i, h_i satisfy the properties stated in the theorem may not give us a well defined Breuil module. The problem is that the monodromy operator N satisfies some equations by the definition of Breuil modules, and these equations give us some equations that C_i and h_i should satisfy.

As mentioned before, in the case $\kappa = 2$, we know that there exists a unique N which satisfies all the conditions in Definition 1.1. We have the following corollary.

Corollary 3.11. *If $\kappa = 2$, assume that $\beta_i + \beta_{fil,i} \not\equiv \alpha_i + \alpha_{fil,i} \pmod e$ or $a \neq b$. Let $B = \sharp\{i \in S \mid \exists x \in \mathbb{Z}$ such that $max(0, m_i + n_i - e) \leq x \leq (n_i - 1)$ and $x \equiv m_i + \alpha_i - \beta_i \pmod e\}$, then*

$$\dim_E Ext^1(\mathcal{M}(m_i, \alpha_i, a), \mathcal{M}(n_i, \beta_i, b)) = B.$$

4. Rank two Breuil modules of type J

Definition 4.1. Let J be a subset of S. We say a reducible rank two Breuil module $\mathcal{M}$ is of type J if it is an extension of a rank one Breuil module of type J by a rank one Breuil module of type J^c. Here $J^c = S\backslash J$.

In this section, we assume that $\kappa = 2$. The main goal is to compute $Ext^1(\mathcal{M},\mathcal{M})$, where $\mathcal{M}$ is a reducible rank two Breuil module of type J such that $\mathcal{M} = \oplus_{i\in S}\mathcal{M}_i$ has the following form.

$$\mathcal{M}_i = E[u]/u^{ep}\langle e_i, f_i\rangle,$$
$$Fil^1\mathcal{M}_i = E[u]/u^{ep}\langle u^{j_i}e_i, u^{e-j_i}f_i + \lambda_i u^{h_i}e_i\rangle,$$
$$\phi_1(u^{j_i}e_i) = (b)_{i+1}e_{i+1}, \quad \phi_1(u^{e-j_i}f_i + \lambda_i u^{h_i}e_i) = (a)_{i+1}f_{i+1},$$
$$[g]e_i = \omega_i^{\beta_i}(g)e_i, \quad [g]f_i = \omega_i^{\alpha_i}(g)f_i,$$

where $\lambda_i \in E$ with $\lambda_i = 0$ if $i+1 \notin J$, $j_i = \begin{cases} e & i+1 \in J \\ 0 & i+1 \notin J \end{cases}$, and $h_i \in [0, e-1]$ with $h_i \equiv \alpha_i - \beta_i (\bmod e)$. Note that $\mathcal{M}$ is split if and only if all the λ_i's are 0. Let $S_0 = \{i \in S \mid \lambda_i = 0\}$ and $S_1 = \{i \in S \mid \lambda_i \neq 0\}$. The main result of this section is the following theorem.

Theorem 4.2. *(1) If $S_0 = S$ ($\mathcal{M}$ is split), then* $\dim_E Ext^1(\mathcal{M},\mathcal{M}) \leq 2 + r$.
(2) If $S_0 \neq S$, then $\dim_E Ext^1(\mathcal{M},\mathcal{M}) \leq 1 + r$.

Let $\mathcal{N} \in Ext^1(\mathcal{M},\mathcal{M})$. Write $\mathcal{N} = \oplus_{i\in S}\mathcal{N}_i$. Assume that

$$\mathcal{N}_i = E[u]/u^{ep}\langle e_i, f_i, e_i', f_i'\rangle,$$

where $e_i' \in \mathcal{N}_i$ (resp. $f_i' \in \mathcal{N}_i$) is a lift of $e_i \in \mathcal{M}_i$ (resp. $f_i \in \mathcal{M}_i$).

Lemma 4.3. *We may assume that*

$$Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle u^{j_i}e_i, u^{e-j_i}f_i + \lambda_i u^{h_i}e_i, u^{j_i}e_i', u^{e-j_i}f_i' + \lambda_i u^{h_i}e_i'\rangle.$$

Proof. If $i+1 \in J$, assume that $Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle u^e e_i, f_i + \lambda_i u^{h_i}e_i, u^e e_i' + A_i e_i + B_i f_i, f_i' + \lambda_i u^{h_i}e_i' + C_i e_i + D_i f_i\rangle$, where $A_i, B_i, C_i, D_i \in E[u]/u^{ep}$. We may assume that $B_i = D_i = 0$. Since $u^e e_i' \in Fil^1\mathcal{N}_i$, we see $u^e | A_i$. So we may assume that $A_i = 0$. Now let $f_i'' = f_i' + C_i e_i$, we may assume that $C_i = 0$.

If $i+1 \notin J$, assume that $Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle e_i, u^e f_i, e_i' + A_i e_i + B_i f_i, u^e f_i' + C_i e_i + D_i f_i\rangle$, where $A_i, B_i, C_i, D_i \in E[u]/u^{ep}$. We may assume that $A_i = C_i = 0$. Let $e_i'' = e_i' + B_i f_i$, we may assume that $B_i = 0$. Since $u^e f_i' \in Fil^1\mathcal{N}_i$, we see $u^e | D_i$ and may assume that $D_i = 0$. □

Assume that $\mathcal{N}$ has the following form.

$$\mathcal{N}_i = E[u]/u^{ep}\langle e_i, f_i, e_i', f_i'\rangle,$$
$$Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle u^{j_i}e_i, u^{e-j_i}f_i + \lambda_i u^{h_i}e_i, u^{j_i}e_i', u^{e-j_i}f_i' + \lambda_i u^{h_i}e_i'\rangle,$$
$$\phi_1(u^{j_i}e_i') = (b)_{i+1}e_{i+1}' + X_{i+1}e_{i+1} + Y_{i+1}f_{i+1},$$
$$\phi_1(u^{e-j_i}f_i' + \lambda_i u^{h_i}e_i') = (a)_{i+1}f_{i+1}' + Z_{i+1}e_{i+1} + W_{i+1}f_{i+1},$$
$$[g](e_i') = \omega_i^{\beta_i}(g)e_i' + A_{i,g}e_i + B_{i,g}f_i,$$
$$[g](f_i') = \omega_i^{\alpha_i}(g)f_i' + C_{i,g}e_i + D_{i,g}f_i,$$

where the X, Y, Z, W and A, B, C, D are in $E[u]/u^{ep}$.

Lemma 4.4. *We may assume that* $A_{i,g} = B_{i,g} = C_{i,g} = D_{i,g} = 0$.

Proof. $[hg]e'_i = \omega_i^{\beta_i}(hg)e'_i + A_{i,hg}e_i + B_{i,hg}f_i$. On the other hand,

$$\begin{aligned}[hg]e'_i &= [h](\omega_i^{\beta_i}(g)e'_i + A_{i,g}e_i + B_{i,g}f_i) \\ &= \omega_i^{\beta_i}(g)(\omega_i^{\beta_i}(h)e'_i + A_{i,h}e_i + B_{i,h}f_i) + h(A_{i,g})\omega_i^{\beta_i}(h)e_i + h(B_{i,g})\omega_i^{\alpha_i}(h)f_i.\end{aligned}$$

Comparing the coefficients, we get the following equations.

$$\frac{A_{i,hg}}{\omega_i^{\beta_i}(hg)} = \frac{A_{i,h}}{\omega_i^{\beta_i}(h)} + h(\frac{A_{i,g}}{\omega_i^{\beta_i}(g)}),$$

and

$$\frac{B_{i,hg}}{\omega_i^{\beta_i}(hg)} = \frac{B_{i,h}}{\omega_i^{\beta_i}(h)} + h(\frac{B_{i,g}}{\omega_i^{\beta_i}(g)})\omega_i^{\alpha_i-\beta_i}(h).$$

Similarly, we have

$$\frac{D_{i,hg}}{\omega_i^{\alpha_i}(hg)} = \frac{D_{i,h}}{\omega_i^{\alpha_i}(h)} + h(\frac{D_{i,g}}{\omega_i^{\alpha_i}(g)}),$$

and

$$\frac{C_{i,hg}}{\omega_i^{\alpha_i}(hg)} = \frac{C_{i,h}}{\omega_i^{\alpha_i}(h)} + h(\frac{C_{i,g}}{\omega_i^{\alpha_i}(g)})\omega_i^{\beta_i-\alpha_i}(h).$$

If we replace e'_i and f'_i by $e''_i = e'_i + P_ie_i + Q_if_i$, $f''_i = f'_i + R_ie_i + S_if_i$, then

$$\begin{aligned}[g](e''_i) &= \omega_i^{\beta_i}(g)e'_i + A_{i,g}e_i + B_{i,g}f_i + g(P_i)\omega_i^{\beta_i}(g)e_i + g(Q_i)\omega_i^{\alpha_i}(g)f_i \\ &= \omega_i^{\beta_i}(g)e''_i + [A_{i,g} + \omega_i^{\beta_i}(g)(g(P_i) - P_i)]e_i \\ &\quad + [B_{i,g} + \omega_i^{\alpha_i}(g)g(Q_i) - \omega_i^{\beta_i}(g)Q_i]f_i,\end{aligned}$$

$$\begin{aligned}[g](f''_i) &= \omega_i^{\alpha_i}(g)f'_i + C_{i,g}e_i + D_{i,g}f_i + g(R_i)\omega_i^{\beta_i}(g)e_i + g(S_i)\omega_i^{\alpha_i}(g)f_i \\ &= \omega_i^{\alpha_i}(g)f''_i + [C_{i,g} + \omega_i^{\beta_i}(g)g(R_i) - \omega_i^{\alpha_i}(g)R_i]e_i \\ &\quad + [D_{i,g} + \omega_i^{\alpha_i}(g)(g(S_i) - S_i)]f_i.\end{aligned}$$

We prove that, in the case $i+1 \in J$, we can choose P_i, Q_i, R_i, S_i to make $A_{i,g} = B_{i,g} = C_{i,g} = D_{i,g} = 0$ without changing the form of Fil^1. (The case $i+1 \notin J$ is similar and the computation is easier.) In this case, $Fil^1\mathcal{N}_i = \langle u^e e_i, f_i + \lambda_i u^{h_i}e_i, u^e e'_i, f'_i + \lambda_i u^{h_i}e'_i\rangle$. In order to keep the form of $Fil^1\mathcal{N}_i$, the equations $e''_i = e'_i + P_ie_i + Q_if_i$, $f''_i = f'_i + R_ie_i + S_if_i$ should give us

$$\langle u^e e_i, f_i + \lambda_i u^{h_i}e_i, u^e e'_i, f'_i + \lambda_i u^{h_i}e'_i\rangle = \langle u^e e_i, f_i + \lambda_i u^{h_i}e_i, u^e e''_i, f''_i + \lambda_i u^{h_i}e''_i\rangle.$$

Note that

$$\begin{aligned}f''_i + \lambda_i u^{h_i}e''_i &= f'_i + R_ie_i + S_if_i + \lambda_i u^{h_i}(e'_i + P_ie_i + Q_if_i) \\ &= f'_i + \lambda_i u^{h_i}e'_i + (S_i + \lambda_i u^{h_i}Q_i)(f_i + \lambda_i u^{h_i}e_i) + \\ &\quad (R_i + \lambda_i u^{h_i}P_i - \lambda_i u^{h_i}S_i - (\lambda_i u^{h_i})^2Q_i)e_i.\end{aligned}$$

To keep the form of $Fil^1\mathcal{N}_i$, we should choose P_i, Q_i, R_i, S_i such that

$$u^e \mid (R_i + \lambda_i u^{h_i}P_i - \lambda_i u^{h_i}S_i - (\lambda_i u^{h_i})^2Q_i).$$

Also note that $[g]$ preserves $Fil^1\mathcal{N}_i$. $[g](u^e e_i') \in Fil^1\mathcal{N}_i$ since $u^e\mathcal{N}_i \subset Fil^1\mathcal{N}_i$.

$$\begin{aligned}
[g](f_i' + \lambda_i u^{h_i} e_i') &= \omega_i^{\alpha_i}(g) f_i' + C_{i,g} e_i + D_{i,g} f_i + \lambda_i u^{h_i} \omega_i^{h_i}(g)(\omega_i^{\beta_i}(g) e_i' + A_{i,g} e_i + B_{i,g} f_i) \\
&= \omega^{\alpha_i}(g)(f_i' + \lambda_i u^{h_i} e_i') + (D_{i,g} + \lambda_i u^{h_i} \omega_i^{h_i}(g) B_{i,g})(f_i + \lambda_i u^{h_i} e_i) \\
&\quad + (C_{i,g} + \lambda_i u^{h_i} \omega_i^{h_i}(g) A_{i,g} - \lambda_i u^{h_i} D_{i,g} - \lambda_i^2 u^{2h_i} \omega_i^{h_i}(g) B_{i,g}) e_i.
\end{aligned}$$

Therefore,

$$u^e \mid (C_{i,g} + \lambda_i u^{h_i} \omega_i^{h_i}(g) A_{i,g} - \lambda_i u^{h_i} D_{i,g} - \lambda_i^2 u^{2h_i} \omega_i^{h_i}(g) B_{i,g}).$$

First, from the above computation and Lemma 3.2, we may choose $P_i, Q_i, R_i, S_i \in E[u]/u^{ep}$ such that

$$\begin{cases} A_{i,g} + \omega_i^{\beta_i}(g)(g(P_i) - P_i) = 0 \\ B_{i,g} + \omega_i^{\alpha_i}(g) g(Q_i) - \omega_i^{\beta_i}(g) Q_i = 0 \\ D_{i,g} + \omega_i^{\alpha_i}(g)(g(S_i) - S_i) = 0 \end{cases}$$

and

$$u^e \mid (R_i + \lambda_i u^{h_i} P_i - \lambda_i u^{h_i} S_i - (\lambda_i u^{h_i})^2 Q_i).$$

Therefore, we may assume that $A_{i,g} = B_{i,g} = D_{i,g} = 0$ and $u^e \mid C_{i,g}$. Then we choose $\Sigma_i \in E[u]/u^{ep}$ such that $u^e \mid \Sigma_i$ and $C_{i,g} + \omega_i^{\beta_i}(g) g(\Sigma_i) - \omega_i^{\alpha_i}(g)\Sigma_i = 0$. Replacing f_i'' by $f_i'' + \Sigma_i e_i$, we may assume that $C_{i,g} = 0$.

If $i+1 \notin J$, the argument is similar. Indeed, by taking $\lambda_i = 0$ and reversing the roles of e_i and f_i in the proof for the case $i+1 \in J$, we get the argument for the case $i+1 \notin J$. □

Lemma 4.5. *We may assume that $X_{i+1}, W_{i+1} \in E$, Y_{i+1} is either 0 or a monomial of degree $e - h_{i+1}$, and Z_{i+1} is either 0 or a monomial of degree h_{i+1}.*

Proof. If $i+1 \in J$, we know that $\mathcal{N}_i$ has the following form.

$$\begin{gathered}
\mathcal{N}_i = E[u]/u^{ep}\langle e_i, f_i, e_i', f_i'\rangle, \\
Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle u^e e_i, f_i + \lambda_i u^{h_i} e_i, u^e e_i', f_i' + \lambda_i u^{h_i} e_i'\rangle, \\
\phi_1(u^e e_i') = (b)_{i+1} e_{i+1}' + X_{i+1} e_{i+1} + Y_{i+1} f_{i+1}, \\
\phi_1(f_i' + \lambda_i u^{h_i} e_i') = (a)_{i+1} f_{i+1}' + Z_{i+1} e_{i+1} + W_{i+1} f_{i+1}, \\
[g](e_i') = \omega_i^{\beta_i}(g) e_i', \quad [g](f_i') = \omega_i^{\alpha_i}(g) f_i'.
\end{gathered}$$

By definition, we have the relation $[g] \circ \phi_1 = \phi_1 \circ [g]$. On one hand,

$$\begin{aligned}
[g](\phi_1(u^e e_i')) &= [g]((b)_{i+1} e_{i+1}' + X_{i+1} e_{i+1} + Y_{i+1} f_{i+1}) \\
&= \omega_{i+1}^{\beta_{i+1}}(g)(b)_{i+1} e_{i+1}' + g(X_{i+1}) \omega_{i+1}^{\beta_{i+1}}(g) e_{i+1} + g(Y_{i+1}) \omega_{i+1}^{\alpha_{i+1}}(g) f_{i+1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\phi_1([g](u^e e_i')) &= \phi_1(\omega_i^{\beta_i}(g) u^e e_i') \\
&= \omega_i^{\beta_i}(g)((b)_{i+1} e_{i+1}' + X_{i+1} e_{i+1} + Y_{i+1} f_{i+1}).
\end{aligned}$$

Comparing the two equations and using the relation $\beta_{i+1} \equiv p\beta_i (\bmod e)$, we see that

$$g(X_{i+1}) = X_{i+1}, \quad g(Y_{i+1}) \omega_{i+1}^{\alpha_{i+1}}(g) = \omega_{i+1}^{p\beta_i}(g) Y_{i+1}.$$

Therefore every nonzero term of X_{i+1} has degree congruent to $0 \pmod e$ and every nonzero term of Y_{i+1} has degree congruent to $(p\beta_i - \alpha_{i+1}) \pmod e$. Note that the action of $[g]$

preserves the degree of a monomial, every single term of X_{i+1} or Y_{i+1} also satisfies the above relation.

We may assume that X_{i+1} and Y_{i+1} are of degree less than e. Because absorbing all the terms with degree $\geq e$ to e'_{i+1} does not change the form of $Fil^1\mathcal{N}_{i+1}$ and $[g]$. (It does not change the form of $Fil^1\mathcal{N}_{i+1}$ because $u^e\mathcal{N} \subset \mathcal{N}$; it does not change the form of $[g]$ because the degrees of nonzero terms satisfy the above congruence equations modulo e.)

Therefore we may assume X_{i+1} is of degree 0 and Y_{i+1} is either 0 or a monomial of degree $\equiv (p\beta_i - \alpha_{i+1}) \pmod e$. Note that $h_{i+1} \equiv \alpha_{i+1} - \beta_{i+1} \equiv \alpha_{i+1} - p\beta_i \pmod e$, we see that Y_{i+1} is either 0 or a monomial of degree $e - h_{i+1}$,

Similarly, we have

$$\begin{aligned}[g](\phi_1(f'_i + \lambda_i u^{h_i} e'_i)) &= [g]((a)_{i+1} f'_{i+1} + Z_{i+1}e_{i+1} + W_{i+1}f_{i+1}) \\ &= \omega_{i+1}^{\alpha_{i+1}}(g)(a)_{i+1}f'_{i+1} + g(Z_{i+1})\omega_{i+1}^{\beta_{i+1}}(g)e_{i+1} \\ &\quad + g(W_{i+1})\omega_{i+1}^{\alpha_{i+1}}(g)f_{i+1},\end{aligned}$$

and

$$\begin{aligned}\phi_1([g](f'_i + \lambda_i u^{h_i} e'_i)) &= \phi_1(\omega_i^{\alpha_i}(g)f'_i + \lambda_i\omega_i^{h_i+\beta_i}(g)u^{h_i}e'_i) \\ &= \omega_i^{\alpha_i}(g)((a)_{i+1}f'_{i+1} + Z_{i+1}e_{i+1} + W_{i+1}f_{i+1}),\end{aligned}$$

where the last equality follows from the congruence $h_i \equiv \alpha_i - \beta_i \pmod e$. Comparing the two equations and using the relation $\alpha_{i+1} \equiv p\alpha_i (\bmod\ e)$, we have

$$g(W_{i+1}) = W_{i+1}, \quad g(Z_{i+1})\omega_{i+1}^{\beta_{i+1}}(g) = \omega_{i+1}^{p\alpha_i}(g)Z_{i+1}.$$

Therefore, by the same argument as before, we may assume that $W_{i+1} \in E$ and Z_{i+1} is either 0 or a monomial of degree h_{i+1}.

The argument for $i+1 \notin J$ is similar. $\square$

Now we prove the theorem. We separate the proof to three cases. (1) $S_0 = S$; (2) $S_1 = S$; (3) $S_0 \neq S$ and $S_1 \neq S$.

Proof of theorem 4.2. In the following argument, we make some change-of-variables for one $i \in S$ at a time to simplify the form of $\mathcal{N}$.

(1) $\mathcal{M}$ is split, all the λ_i's are 0. Fix a single $i \in S$. If $Fil^1\mathcal{N}_i = \langle u^e e_i, f_i, u^e e'_i, f'_i\rangle$, we may assume that $X_i = Y_i = W_i = 0$ because making the following change of variables

$$\begin{cases} e''_i = e'_i + X_i e_i + Y_i f_i \\ f''_i = f'_i + W_i f_i \end{cases}$$

and leaving e'_j and f'_j with $j \neq i$ unchanged do not change the forms of $Fil^1\mathcal{N}_i$ and $[g]$. (It does not change the form of Fil^1 because $\langle u^e e_i, f_i, u^e e'_i, f'_i\rangle = \langle u^e e_i, f_i, u^e e''_i, f''_i\rangle$ by the construction; it does not change the form of $[g]$ since we know that X_i and W_i are elements in E and Y_i is either 0 or a monomial of degree $(e - h_i)$.)

If $Fil^1\mathcal{N}_i = E[u]/u^{ep}\langle e_i, u^e f_i, e'_i, u^e f'_i\rangle$, we may assume that $X_i = Z_i = W_i = 0$ because making the following change of variables

$$\begin{cases} e''_i = e'_i + X_i e_i \\ f''_i = f'_i + Z_i e_i + W_i f_i \end{cases}$$

and leaving e'_j and f'_j with $j \neq i$ unchanged do not change the forms of $Fil^1\mathcal{N}_i$ and $[g]$.

Note that by the change of variables as above, although we may change the values of X_{i+1} and W_{i+1}, the assumptions about the degrees of nonzero terms in $X_{i+1}, Y_{i+1}, Z_{i+1}, W_{i+1}$ are preserved. Now suppose that we have done the above steps for $i = 2, 3, ..., r$. If $Fil^1\mathcal{N}_1 = E[u]/u^{ep}\langle u^e e_1, f_1, u^e e'_1, f'_1\rangle$, we may assume that $Y_1 = 0$ since letting $e''_1 = e'_1 + Y_1 f_1$ does not change anything. Indeed, it does not change the forms of Fil^1 and $[g]$ by the same argument as before. Furthermore, $\phi_1(u^e e''_1) = \phi_1(u^e e'_1) + \phi_1(u^e Y_i f_1) = \phi_1(u^e e'_1)$, it does not change X_2, Y_2, Z_2, or W_2.

If $Fil^1\mathcal{N}_1 = E[u]/u^{ep}\langle e_1, u^e f_1, e'_1, u^e f'_1\rangle$, similarly, we may assume that $Z_1 = 0$. By counting the number of variables, we have our result.

(2) All the λ_i's are nonzero. In this case $J = S$. Fix a single $i \in S$, we may assume that $X_i = Y_i = Z_i = 0$ because we can make the following change of variables

$$\begin{cases} e''_i = e'_i + (b)_i^{-1} X_i e_i + (b)_i^{-1} Y_i f_i \\ f''_i = f'_i - \lambda_i u^{h_i} (b)_i^{-1} X_i e_i + w_i (f_i + \lambda_i u^{h_i} e_i), \end{cases}$$

where $((a)_i w_i - (a/b)_i X_i)\lambda_i u^{h_i} = Z_i$. (This is possible because we know that Z_i is either 0 or a monomial of degree h_i.) It does not change the form of Fil^1 since $f''_i + \lambda_i u^{h_i} e''_i = f'_i + \lambda_i u^{h_i} e'_i + w_i(f_i + \lambda_i u^{h_i} e_i) + \lambda_i (b)_i^{-1} u^{h_i} Y_i f_i$ and $u^e \mid u^{h_i} Y_i$. It does not change the form of $[g]$ because $X_i, W_i \in E$, Y_i is either 0 or a monomial of degree $e - h_i$, and Z_i is either 0 or a monomial of degree h_i. Also, the new ϕ_1 has the following form on $\mathcal{N}_i$:

$$\phi_1((b)_i^{-1} Y_i u^e f_i) = \phi_1((b)_i^{-1} Y_i u^e (f_i + \lambda_i u^{h_i} e_i)) - \phi_1((b)_i^{-1} \lambda_i u^e Y_i u^{h_i} e_i) = 0$$

$$\begin{aligned} \phi_1(u^e e''_i) &= \phi_1(u^e e'_i) + \phi_1((b)_i^{-1} X_i u^e e_i) + \phi_1((b)_i^{-1} Y_i u^e f_i) \\ &= \phi_1(u^e e'_i) + \phi_1((b)_i^{-1} X_i u^e e_i) \end{aligned}$$

$$\begin{aligned} \phi_1(f''_i + \lambda_i u^{h_i} e''_i) &= \phi_1(f'_i + \lambda_i u^{h_i} e'_i) + \phi_1(w_i(f_i + \lambda_i u^{h_i} e_i)) + \phi_1((b)_i^{-1} \lambda_i Y_i u^{h_i} f_i) \\ &= \phi_1(f'_i + \lambda_i u^{h_i} e'_i) + \phi_1(\tilde{Y}_i u^e e_i) + \phi_1(w_i(f_i + \lambda_i u^{h_i} e_i)). \end{aligned}$$

Here $\tilde{Y}_i$ is an element in $E[u]/u^{ep}$ such that $\tilde{Y}_i u^e = -(b)_i^{-1} \lambda_i^2 u^{2h_i} Y_i$. If we make the change of variables for i, then it may change the values of X_{i+1}, W_{i+1}, and Z_{i+1}. We may still assume that $X_{i+1}, W_{i+1} \in E$ and Z_{i+1} is a monomial of degree h_{i+1} by absorbing the terms with degree greater or equal to e to e'_{i+1} or f'_{i+1} as in the proof of Lemma 4.5. We reduce the X, Y, Z, and W as follows. The original ϕ_1 is given by the set of matrices

$$\{\begin{pmatrix} X_2 & Y_2 \\ Z_2 & W_2 \end{pmatrix}, \cdots, \begin{pmatrix} X_r & Y_r \\ Z_r & W_r \end{pmatrix}, \begin{pmatrix} X_1 & Y_1 \\ Z_1 & W_1 \end{pmatrix}\}.$$

We make the change for $i = 1, 2, \cdots, r-1$, the new ϕ_1 is given by the set of matrices

$$\{\begin{pmatrix} 0 & 0 \\ 0 & W'_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 \\ 0 & W'_{r-1} \end{pmatrix}, \begin{pmatrix} X'_r & Y'_r \\ Z'_r & W'_r \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & W'_1 \end{pmatrix}\}.$$

We then make the change for $i = r$. By the above equations, this may change X_1, Z_1, and W_1, the new ϕ_1 is given by the set of matrices

$$\{\begin{pmatrix} 0 & 0 \\ 0 & W'_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 \\ 0 & W'_{r-1} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & W'_r \end{pmatrix}, \begin{pmatrix} X'_1 & 0 \\ Z'_1 & W''_1 \end{pmatrix}\}.$$

Finally, we make the change for $i=1$ again. Note now $Y_1'=0$, this does not change Z_2'. Thus ϕ_1 is given by the following matrices

$$\{\begin{pmatrix} X_2'' & 0 \\ 0 & W_2'' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & W_3' \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 \\ 0 & W_r' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & W_1''' \end{pmatrix}\}.$$

By counting the potentially nonzero variables, we get the right bound of the dimension.

(3) In this case, without loss of generality, we assume that $\lambda_r=0$, $\lambda_1\neq 0$. Fix an $i\in S$. As in (1), if $Fil^1\mathcal{N}_i=E[u]/u^{ep}\langle u^e e_i, f_i, u^e e_i', f_i'\rangle$, we may assume that $X_i=Y_i=W_i=0$. If $Fil^1\mathcal{N}_i=E[u]/u^{ep}\langle e_i, u^e f_i, e_i', u^e f_i'\rangle$, we may assume that $X_i=Z_i=W_i=0$.

As in (2), if $Fil^1\mathcal{N}_i=E[u]/u^{ep}\langle u^e e_i, f_i+\lambda_i u^{h_i}e_i, u^e e_i', f_i'+\lambda_i u^{h_i}e_i'\rangle$ with $\lambda_i\neq 0$, we may assume that $X_i=Y_i=Z_i=0$.

Now suppose that we have done the above steps for $i=1,2,...,r-1$. We do this for $i=r$ and this will only change the value of X_1 and W_1 since $\lambda_r=0$. The potentially nonzero terms are X_1, W_1 and one of X_i, Y_i, Z_i, W_i for each $i\neq 1$. $\square$

Acknowledgement I would like to thank the referee for his constructive comments and suggestions, which substantially helped improving the quality of the paper. Some of the arguments (for example Lemma 3.8) are due to the referee. I also want to thank Matthew Emerton for his support and encouragement.

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