DEFORMATIONS OF FORMAL π-DIVISIBLE $\mathcal{O}$-MODULES VIA $\mathcal{O}$-DISPLAYS

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Abstract. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π and R be an $\mathcal{O}$-algebra with π nilpotent in R. In this paper, we study deformations of $\mathcal{O}$-displays over R by explicit computation. Since the category of nilpotent $\mathcal{O}$-displays over R is equivalent to the category of formal π-divisible $\mathcal{O}$-modules over R, we obtain results on deformations of formal π-divisible $\mathcal{O}$-modules, which generalize the corresponding results on formal p-divisible groups.

1. Introduction

The theory of displays, which was developed by Zink and Lau in a series of papers ([13, 14, 8, 9, 10] etc.), is a powerful tool in the study of p-divisible groups. One of the main results of this theory is a classification result, which says that, for any ring R with p nilpotent in it, the category of formal p-divisible groups over R and the category of nilpotent displays over R are equivalent. Moreover, if R is a Noetherian local ring with perfect residue field of characteristic p, the category of p-divisible groups over R and the category of Dieudonné displays over R are equivalent.

The above classification result was generalized in [1, 2]. In particular, we have the following result, which is the starting point of this paper. Let $p > 2$ be a prime. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π. Let R be an $\mathcal{O}$-algebra with π nilpotent in it. Denote by $\mathrm{ndisp}_{\mathcal{O}}/R$ the category of nilpotent $\mathcal{O}$-displays over R. From [2, Theorem 1.1], there exists a covariant functor $\mathrm{BT}_{\mathcal{O}}$

$$\mathrm{BT}_{\mathcal{O}} : \mathrm{ndisp}_{\mathcal{O}}/R \to (\pi\text{-divisible formal } \mathcal{O}\text{-modules}/R),$$

which is an equivalence of categories.

The classification results in [13, 14, 8, 9, 10] have many applications in the study of p-divisible groups. In [2, 3], the authors generalized the classification results and obtained several applications in the study of π-divisible $\mathcal{O}$-modules. A simple idea is that, a π-divisible $\mathcal{O}$-module X is a p-divisible group with a special $\mathcal{O}$-action and this special action includes extra information of the structure of X. Hence if we confine our study in the category of π-divisible $\mathcal{O}$-modules, we should obtain stronger results than those regarding general p-divisible groups.

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In this paper, following the idea in [13, Sections 2.2, 2.5], we study deformations of $\mathcal{O}$-displays by explicit computation. Then by [2, Theorem 1.1], we translate the properties of $\mathcal{O}$-displays to properties of π-divisible $\mathcal{O}$-modules. To state the main results, we first fix some notation.

Let $p > 2$ be a prime. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F} = \mathbb{F}_q$. The category of $\mathcal{O}$-algebras is denoted by $\mathrm{Alg}_{\mathcal{O}}$. For $A \in \mathrm{Alg}_{\mathcal{O}}$, $W_{\mathcal{O}}(A)$ is the ring of ramified Witt vectors. The Frobenius and Verschiebung morphisms on $W_{\mathcal{O}}(A)$ are denoted by ^{F} and ^{V}. The Teichmüller lift of $a \in A$ is denoted by $[a] \in W_{\mathcal{O}}(A)$. Denote by $I_{\mathcal{O}}(A)$ the image of the Verschiebung, i.e., $I_{\mathcal{O}}(A) = {}^V W_{\mathcal{O}}(A)$. See [2, Section 1.2.1] for more details.

For a π-divisible $\mathcal{O}$-module X, $X[\pi^n]$ denotes the π^n-torsion of X. If X is of height h and dimension d, we say that X is of *type* (h, d).

For $\mathcal{O}$-displays and $\mathcal{O}$-windows, we will use without comment the notation of [2, 3]. For an $\mathcal{O}$-display $\mathcal{P} = (P, Q, F, F_1)$ over $R \in \mathrm{Alg}_{\mathcal{O}}$, we say that $\mathcal{P}$ is of *type* (h, d) if P is free of rank h over $W_{\mathcal{O}}(R)$ and P/Q is free of rank d over R.

We prove the following results, which are well-known for p-divisible groups (cf. [7, 4]).

Theorem 1.1. *Let $R \in \mathrm{Alg}_{\mathcal{O}}$ such that π is nilpotent in R.*

(1) *Let X be a formal π-divisible $\mathcal{O}$-module over R with type (h, d). The deformation functor $\mathbb{D}_X$ (cf. Section 3.1) is pro-representable by a formal π-divisible $\mathcal{O}$-module over $R[[t_1, \ldots, t_{d(h-d)}]]$.*

(2) *Let X and Y be two formal π-divisible $\mathcal{O}$-modules over R with $X[\pi^n] = Y[\pi^n]$ for a positive integer n. Let $\tilde{X}$ be a deformation of X over $S \in \mathrm{Aug}_R$ (cf. Section 2.2). Then there exists a deformation $\tilde{Y}$ of Y over S such that $\tilde{Y}[\pi^n] \cong \tilde{X}[\pi^n]$.*

Remark 1.2. If $R = k \in \mathrm{Alg}_{\mathcal{O}}$ is a perfect field of characteristic p, then using [2, Theorem 1.5] and the theory of Dieudonné $\mathcal{O}$-displays, the same argument in this paper proves the following claims in equal-characteristic case.

(1) Let X be a π-divisible $\mathcal{O}$-module over k with type (h, d). The deformation functor $\mathbb{D}_X$ is pro-representable by a π-divisible $\mathcal{O}$-module over $k[[t_1, \ldots, t_{d(h-d)}]]$.

(2) Let X and Y be two π-divisible $\mathcal{O}$-modules over k with $X[\pi^n] = Y[\pi^n]$ for a positive integer n. Let $\tilde{X}$ be a deformation of X over $S \in \mathrm{Aug}_k$. Then there exists a deformation $\tilde{Y}$ of Y over S such that $\tilde{Y}[\pi^n] \cong \tilde{X}[\pi^n]$.

Let $(\mathcal{O}', \pi')$ be a totally ramified extension of $(\mathcal{O}, \pi)$ with degree e. Let $\tilde{X}$ over $\mathcal{O}^u = W_{\mathcal{O}'}(\bar{\mathbb{F}})$ be the π'-divisible Lubin-Tate group associated with the $\mathcal{O}'$-display

$$(W_{\mathcal{O}'}(W_{\mathcal{O}'}(\bar{\mathbb{F}})),\ I_{\mathcal{O}'}(W_{\mathcal{O}'}(\bar{\mathbb{F}})),\ {}^F,\ {}^{V^{-1}}).$$

Let X over $\bar{\mathbb{F}}$ be the π'-divisible Lubin-Tate group associated with the $\mathcal{O}'$-display

$$(W_{\mathcal{O}'}(\bar{\mathbb{F}}),\ I_{\mathcal{O}'}(\bar{\mathbb{F}}),\ {}^F,\ {}^{V^{-1}}).$$

Then $X = \tilde{X} \otimes \bar{\mathbb{F}}$ and is a formal π-divisible $\mathcal{O}$-module over $\bar{\mathbb{F}}$ with a special $\mathcal{O}'$-action (cf. [2, Section 1.2.3]). As a formal π-divisible $\mathcal{O}$-module, the endomorphism ring $\mathrm{End}(X) = \mathcal{O}_D$, where D is the central simple $\mathrm{Frac}(\mathcal{O})$-algebra with invariant $1/e$ and $\mathcal{O}_D$ is the maximal order of D. Let X_m be the base change $\tilde{X} \otimes_{\mathcal{O}^u} \mathcal{O}^u/(\pi')^{m+1}$. Then we have the following result, which may be considered as a relative version of a result of Gross (cf. [6] and [13, Proposition 79]).

Theorem 1.3. *With the notation as above, we have*

$$\mathrm{End}(X_m) = \mathcal{O}' + (\pi')^m \mathcal{O}_D,$$

for all $m \in \mathbb{Z}_{\geq 0}$.

2. Deformations of $\mathcal{O}$-displays

In this section, we study deformations of $\mathcal{O}$-displays and obstructions of lifting homomorphisms. In particular, we show that the deformation functor is pro-representable and describe the universal object explicitly. Since we are interested in nilpotent objects, the $\mathcal{O}$-displays in the rest of this paper are all assumed to be nilpotent without further comment.

2.1. **Liftings of an $\mathcal{O}$-display.** Let R be an $\mathcal{O}$-algebra. Let $\mathcal{P}$ be an $\mathcal{O}$-display over R. Let $S \to R$ be a surjection of $\mathcal{O}$-algebras. A *lifting* of $\mathcal{P}$ to S is an $\mathcal{O}$-display $\mathcal{P}'$ over S such that the base change of $\mathcal{P}'$ with respect to $S \to R$ is isomorphic to $\mathcal{P}$. It is known that to lift $\mathcal{P}$ to S is equivalent to lifting the Hodge filtration (cf. [3, Lemma 2.18])

$$\mathrm{Fil}^1_{\mathcal{P}}(R)(:= Q/I_{\mathcal{O}}(R)P) \subset \mathrm{Fil}_{\mathcal{P}}(R)(:= P/I_{\mathcal{O}}(R)P).$$

Note that this is denoted by $\mathcal{D}^1_{\mathcal{P}}(R) \subset \mathcal{D}_{\mathcal{P}}(R)$ in [13].

Let us consider the special case, where $S \to R$ is a surjection with kernel $\mathfrak{a}$, such that $\mathfrak{a}^2 = 0$. Define an abelian group $\mathcal{G}$ by

$$\mathcal{G} := \mathrm{Hom}(\mathrm{Fil}^1_{\mathcal{P}}(R), \mathfrak{a} \otimes_R (\mathrm{Fil}_{\mathcal{P}}(R)/\mathrm{Fil}^1_{\mathcal{P}}(R))). \tag{2.1}$$

We define an action of $\mathcal{G}$ on the set of liftings of $\mathcal{P}$ to S as follows. Two liftings of $\mathcal{P}$ to S correspond to two liftings E_1 and E_2 of the Hodge filtration, i.e., E_1 and E_2 are both direct summand of $\mathrm{Fil}_{\mathcal{P}}(S)$ that lifts $\mathrm{Fil}^1_{\mathcal{P}}(R)$. Consider the natural homomorphism

$$E_1 \subset \mathrm{Fil}_{\mathcal{P}}(S) \to \mathrm{Fil}_{\mathcal{P}}(S)/E_2. \tag{2.2}$$

Since $E_1 \equiv E_2 \pmod{\mathfrak{a}}$, the homomorphism (2.2) factors as

$$E_1 \to \mathfrak{a}(\mathrm{Fil}_{\mathcal{P}}(S)/E_2) \subset \mathrm{Fil}_{\mathcal{P}}(S)/E_2. \tag{2.3}$$

Moreover, since $\mathfrak{a}^2 = 0$, we have an isomorphism $\mathfrak{a}(\mathrm{Fil}_{\mathcal{P}}(S)/E_2) \cong \mathfrak{a} \otimes_R (\mathrm{Fil}_{\mathcal{P}}(R)/\mathrm{Fil}^1_{\mathcal{P}}(R))$. Hence we obtain a homomorphism

$$u : \mathrm{Fil}^1_{\mathcal{P}}(R) \to \mathfrak{a} \otimes_R (\mathrm{Fil}_{\mathcal{P}}(R)/\mathrm{Fil}^1_{\mathcal{P}}(R)).$$

Define $E_1 - E_2 = u$. It is easy to check from the construction that

$$E_2 = \{e - \widetilde{u(e)} \mid e \in E_1\}, \tag{2.4}$$

where $\widetilde{u(e)} \in \mathfrak{a}\,\mathrm{Fil}_{\mathcal{P}}(S)$ denotes any lifting of $u(e)$. We have the following result (cf. [13, Corollary 49]).

Proposition 2.1. *Let $\mathcal{P}$ be an $\mathcal{O}$-display over R. Let $S \to R$ be a surjection with kernel $\mathfrak{a}$ such that $\mathfrak{a}^2 = 0$. The action of $\mathcal{G}$ on the set of liftings of $\mathcal{P}$ to S constructed as above is simply transitive. If $\mathcal{P}_0$ is a lifting of $\mathcal{P}$ and $u \in \mathcal{G}$, we denote the action by $\mathcal{P}_0 + u$.*

Proof. The transitivity follows from the construction. Moreover, if $E_1 = E_2$, then the object u constructed above is trivial. Hence the action is simple. The proposition follows. □

Remark 2.2. The above action could be described more explicitly. Consider $\mathfrak{a}$ as an ideal of $W_{\mathcal{O}}(\mathfrak{a})$ and we equip $\mathfrak{a}$ with the trivial divided $\mathcal{O}$-pd-structure (cf. [3, Section 2.8]). Let $\mathcal{P}_0 = (P_0, Q_0, F, F_1)$ be a lifting of $\mathcal{P}$ to S. Let $\alpha : P_0 \to \mathfrak{a}P_0 \subset W_{\mathcal{O}}(\mathfrak{a})P_0$ be a homomorphism. For the pair (P_0, Q_0), we define a new $\mathcal{O}$-display structure by setting

$$\begin{aligned} F_\alpha x &= Fx - \alpha(Fx) \text{ for } x \in P_0, \\ F_{1\alpha} y &= F_1 y - \alpha(F_1 y) \text{ for } y \in Q_0. \end{aligned} \tag{2.5}$$

By Proposition 2.1, there is an element $u \in \mathcal{G}$ such that $\mathcal{P}_\alpha = \mathcal{P}_0 + u$. This u could be described as follows. We have a natural isomorphism $\mathfrak{a}P_0 \cong \mathfrak{a} \otimes_R P/I_{\mathcal{O}}(R)P$. Hence the homomorphism α factors uniquely through a morphism

$$\tilde{\alpha} : P/I_{\mathcal{O}}(R)P \to \mathfrak{a} \otimes_R P/I_{\mathcal{O}}(R)P.$$

Conversely, any such R-module homomorphism $\tilde{\alpha}$ determines a unique α. Let $u \in \mathcal{G}$ be the composite of

$$Q/I_{\mathcal{O}}(R)P \subset P/I_{\mathcal{O}}(R)P \xrightarrow{\tilde{\alpha}} \mathfrak{a} \otimes_R P/I_{\mathcal{O}}(R)P \to \mathfrak{a} \otimes_R P/Q.$$

Then it is easy to check that $\mathcal{P}_\alpha = \mathcal{P}_0 + u$.

2.2. **Deformations of an $\mathcal{O}$-display.** Let Λ be a topological $\mathcal{O}$-algebra of the following type. The topology on Λ is given by a filtration of $\mathcal{O}$-ideals

$$\Lambda = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n \supset \ldots, \tag{2.6}$$

such that $\mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$. We assume that π is nilpotent in $\Lambda/\mathfrak{a}_1$ and hence in any quotient $\Lambda/\mathfrak{a}_i$. Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with the discrete topology. Suppose we are given a continuous surjective homomorphism $\varphi : \Lambda \to R$.

Let $\mathrm{Aug}_{\Lambda \to R}$ be the category of morphisms of discrete Λ-algebras $\psi_S : S \to R$, such that ψ_S is surjective and has a nilpotent kernel. If $\Lambda = R$, we denote this category simply by Aug_R.

Let Nil_R be the category of nilpotent R-algebras. Let $\mathcal{N} \in \mathrm{Nil}_R$. We associated with $\mathcal{N}$ an augmented R-algebra $R|\mathcal{N}|$ as follows. As an R-module, $R|\mathcal{N}| = R \oplus \mathcal{N}$. The multiplication is given by

$$(r_1 \oplus n_1)(r_2 \oplus n_2) = (r_1 r_2) \oplus (r_1 n_2 + r_2 n_1 + n_1 n_2) \text{ for all } r_1,\ r_2 \in R \text{ and } n_1,\ n_2 \in \mathcal{N}.$$

Let M be an R-module. We regard M as an object in Nil_R by setting $M^2 = 0$. Hence we obtain fully faithful functors $\mathrm{Mod}_R \subset \mathrm{Nil}_R \subset \mathrm{Aug}_{\Lambda \to R}$.

Definition 2.3. Let F be a set-valued functor on $\mathrm{Aug}_{\Lambda \to R}$. The restriction of this functor to the category of R-modules is denoted by t_F and is called the *tangent functor* of F.

Definition 2.4. Let $\mathcal{P}$ be an $\mathcal{O}$-display over R. Let $S \to R$ be a surjection of $\mathcal{O}$-algebras such that the kernel is nilpotent. A *deformation* of $\mathcal{P}$ to S is an isomorphism class of pairs $(\mathcal{P}', \iota)$, where $\mathcal{P}'$ is an $\mathcal{O}$-display over S and $\iota : \mathcal{P} \to \mathcal{P}'_R$ is an isomorphism. Here $\mathcal{P}'_R$ is the base change of $\mathcal{P}'$ with respect to $S \to R$ (cf. [2, Section 2.2]).

The *deformation functor* of $\mathcal{P}$ is defined by

$$\begin{aligned} \mathbb{D}_{\mathcal{P}} : \mathrm{Aug}_{\Lambda \to R} &\to \mathrm{Sets} \\ S &\mapsto \{\text{deformations of } \mathcal{P} \text{ to } S\}. \end{aligned} \tag{2.7}$$

We show that the functor $\mathbb{D}_{\mathcal{P}}$ is pro-representable and construct the universal object. First we compute the tangent functor of $\mathbb{D}_{\mathcal{P}}$. Let M be an R-module. We study the liftings of $\mathcal{P}$ to $R|M|$ with respect to the canonical map $R|M| \to R$. In this case, the kernel of $R|M| \to R$ is square-zero, we may apply Proposition 2.1 to this situation. In particular, we have an isomorphism:

$$\mathrm{Hom}_R(Q/I_{\mathcal{O}}(R)P, M \otimes_R P/Q) \to \mathbb{D}_{\mathcal{P}}(R|M|).$$

Note that in this case, we have a canonical choice for $\mathcal{P}_0 = \mathcal{P}_{R|M|}$ (cf. Remark 2.2). The tangent space of the functor $\mathbb{D}_{\mathcal{P}}$ is isomorphic to the finitely generated projective R-module $\mathrm{Hom}_R(Q/I_{\mathcal{O}}(R)P, P/Q)$. Define $\omega = \mathrm{Hom}_R(P/Q, Q/I_{\mathcal{O}}(R)P)$. Then we have an isomorphism

$$\mathrm{Hom}_R(\omega, M) \to \mathbb{D}_{\mathcal{P}}(R|M|).$$

The identical endomorphism of ω defines a morphism of functors

$$\mathrm{Spf}\, R|\omega| \to \mathbb{D}_{\mathcal{P}}. \tag{2.8}$$

Let $\tilde{\omega}$ be a finitely generated projective Λ-module with $\tilde{\omega} \otimes_\Lambda R \cong \omega$. Let $S_\Lambda(\tilde{\omega})$ be the symmetric algebra. Let A be the completion of the augmented algebra $S_\Lambda(\tilde{\omega})$ with respect to the augmentation ideal. The morphism (2.8) may be lifted to a morphism

$$\mathrm{Spf}\, A \to \mathbb{D}_{\mathcal{P}}. \tag{2.9}$$

By our construction, the morphism (2.9) induces an isomorphism on the tangent spaces. Hence it is an isomorphism. Now we could describe the universal $\mathcal{O}$-display $\mathcal{P}^{\mathrm{univ}}$ as follows. Let $u : Q/I_{\mathcal{O}}(R)P \to \omega \otimes_R P/Q$ be the map induced by the identical endomorphism of ω. Let $\alpha : P \to \omega \otimes_R P/Q$ be any map that induces u (cf. Remark 2.2). Then we obtain an $\mathcal{O}$-display $\mathcal{P}_\alpha$ over $R|\omega|$. Lifting $\mathcal{P}_\alpha$ to A, we obtain $\mathcal{P}^{\mathrm{univ}}$.

Remark 2.5. We may write down the universal object explicitly in terms of structure equation as follows (cf. [12, Section (1.12)] and [13, Equation (87)]). Assume that $\mathcal{P} = (P, Q, F, F_1)$ and $P = L \oplus T$ is a normal decomposition of $\mathcal{P}$. Then $\mathcal{P}$ is determined by its structure equation

$$\Phi := F_1 \oplus F : L \oplus T \to P.$$

Here $F_1 \oplus F$ is an ^{F}-linear isomorphism. Assume further that L and T are finitely generated free $W_{\mathcal{O}}(R)$-modules, which is automatic if $W_{\mathcal{O}}(R)$ is local. Assume that the rank of L is c and the rank of T is d. Fix a basis of L and T, hence a basis of P, $F_1 \oplus F$ is given by a matrix $M_{\mathcal{P}} \in \mathrm{GL}_h(W_{\mathcal{O}}(R))$. Here $h = c + d$. We choose indeterminates $\{t_{ij} \mid 1 \leq i \leq c,\ 1 \leq j \leq d\}$ and set $A = \Lambda[[t_{ij}]]$. Define an invertible matrix in $\mathrm{GL}_h(W_{\mathcal{O}}(A))$ by

$$\begin{pmatrix} \mathrm{id}_c & [t_{ij}] \\ 0 & \mathrm{id}_d \end{pmatrix} \tilde{M}_{\mathcal{P}}.$$

Here $\tilde{M}_{\mathcal{P}}$ is a lifting of $M_{\mathcal{P}}$ in $\mathrm{GL}_h(W_{\mathcal{O}}(A))$ and $[t_{ij}]$ is the Teichmüller representative of t_{ij}. This matrix defines an $\mathcal{O}$-display $\mathcal{P}^{\mathrm{univ}}$ over the topological ring A. Then the pair $(A, \mathcal{P}^{univ})$ pro-represents the functor $\mathbb{D}_{\mathcal{P}}$ on the category $\mathrm{Aug}_{\Lambda \to R}$.

We could also see the meaning of $t_1, \cdots, t_{dc}$ in Remark 2.5 explicitly when we consider the infinitesimal deformations, i.e., deformations over the dual numbers $R[\epsilon] = R[x]/(x^2)$.

Lemma 2.6. *Let $\mathcal{P} = (P, Q, F, F_1)$ and $\mathcal{P}' = (P', Q', F, F_1)$ be two $\mathcal{O}$-displays over R. Then we have an exact sequence*

$$0 \to \mathrm{Hom}_{F,\mathrm{Fil}}(P, P') \to \mathrm{Hom}_F(P, P') \to \mathrm{Ext}^1(\mathcal{P}, \mathcal{P}') \to 0. \tag{2.10}$$

Here $\mathrm{Hom}_F(P, P')$ means ^{F}-linear maps $P \to P'$, $\mathrm{Hom}_{F,\mathrm{Fil}}(P, P')$ means ^{F}-linear maps $P \to P'$ that send Q to Q', and the second arrow is given by $\beta \mapsto (\beta\Phi^{\mathcal{P}} - \Phi^{\mathcal{P}'}\beta)$.

Proof. The proof is standard. Assume that we have a short exact sequence of $\mathcal{O}$-displays

$$0 \to \mathcal{P}' \to \mathcal{P}'' = (P'', Q'', F, F_1) \to \mathcal{P} \to 0.$$

We may write $P'' = P \oplus P'$ and $Q'' = Q \oplus Q'$. Choose normal decompositions of $\mathcal{P}$ and $\mathcal{P}'$, say $P = L \oplus T$ and $P' = L' \oplus T'$. Then $\mathcal{P}''$ is determined by the structure equation $F_1 \oplus F : (L \oplus L') \oplus (T \oplus T') \to (P \oplus P')$, which may be written as

$$F_1 \oplus F = \begin{pmatrix} F_1 \oplus F & \alpha \\ 0 & F_1 \oplus F \end{pmatrix},$$

where $\alpha \in \mathrm{Hom}_F(P, P')$. Conversely, any element $\alpha \in \mathrm{Hom}_F(P, P')$ gives rise to an extension of $\mathcal{O}$-displays. Moreover, two elements α and α' give rise to isomorphic extensions if there exists an element $\beta \in \mathrm{Hom}_{F,\mathrm{Fil}}(P, P')$ such that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_1 \oplus F & \alpha \\ 0 & F_1 \oplus F \end{pmatrix} = \begin{pmatrix} F_1 \oplus F & \alpha' \\ 0 & F_1 \oplus F \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Hence the lemma follows. □

In the situation as in the lemma, assume further that $\mathcal{P}' = \mathcal{P}$ are $\mathcal{O}$-displays over R of type (h, d), then we have the following result.

Corollary 2.7. *Let $\mathcal{P}$ be an $\mathcal{O}$-display over R of type (h, d). Then*

$$\mathrm{Rank}_{W_{\mathcal{O}}(R)} \mathrm{Ext}^1(\mathcal{P}, \mathcal{P}) = \mathrm{Rank}_{W_{\mathcal{O}}(R)} \mathbb{D}_{\mathcal{P}}(R[\epsilon]) = d(h - d).$$

2.3. **Lifting homomorphisms: part one.** Let $\bar{\mathcal{P}} = (\bar{P}, \bar{Q}, F, F_1)$ and $\bar{\mathcal{P}}' = (\bar{P}', \bar{Q}', F, F_1)$ be two $\mathcal{O}$-displays over R. Let $S \to R$ be a surjection with nilpotent kernel $\mathfrak{a}$. Let $\mathcal{P} = (P, Q, F, F_1)$ be a lifting of $\bar{\mathcal{P}}$ to S. Assume that there exists a homomorphism of $\mathcal{O}$-displays

$$\bar{f} : (\bar{P}, \bar{Q}, F, F_1) \to (\bar{P}', \bar{Q}', F, F_1).$$

Then we have the following result.

Proposition 2.8. *With the notation as above. There exists a lifting $\mathcal{P}' = (P', Q', F, F_1)$ of $\bar{\mathcal{P}}'$ to S and a homomorphism*

$$f : (P, Q, F, F_1) \to (P', Q', F, F_1),$$

such that f lifts $\bar{f}$.

Proof. Since a homomorphism $\alpha : X \to Y$ could be encoded by the automorphism $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ on $X \oplus Y$, to prove the proposition, we may assume that $\bar{f}$ is an automorphism. Moreover, every nilpotent $\mathcal{N} \in \mathrm{Alg}_R$ admits a filtration

$$\mathcal{N} = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_m \supset \mathcal{N}_{m+1} = 0,$$

such that $\mathcal{N}_i^2 \subset \mathcal{N}_{i+1}$ $(0 \le i \le m)$. Hence we may assume that $\mathfrak{a}^2 = 0$. Therefore, the proposition follows from the following lemma. □

Lemma 2.9. *Let $\mathcal{P} = (P, Q, F, F_1)$ be a lifting of $\bar{\mathcal{P}} = (\bar{P}, \bar{Q}, F, F_1)$ from R to $S = R|\mathcal{N}|$ with $\mathcal{N}^2 = 0$. Let $\bar{f}$ be an automorphism of $\bar{\mathcal{P}}$. Then there exists another lifting $\mathcal{P}' = (P', Q', F', F_1')$ of $\bar{\mathcal{P}}$ to S and an isomorphism*

$$f : (P, Q, F, F_1) \to (P', Q', F, F_1),$$

such that f lifts $\bar{f}$.

Proof. Assume that $\mathcal{P}$ is of type (h, d). We fix a normal decomposition $\bar{P} = \bar{L} \oplus \bar{T}$ of $\bar{\mathcal{P}}$ and a basis for both $\bar{L}$ and $\bar{T}$. The structure of $\bar{\mathcal{P}}$ is determined by a matrix $\Phi \in \mathrm{GL}_h(W_{\mathcal{O}}(R))$, which corresponds to the ^{F}-linear isomorphism $F_1 \oplus F : \bar{L} \oplus \bar{T} \to \bar{P}$. The automorphism $\bar{f}$ corresponds to a matrix $X \in \mathrm{GL}_h(W_{\mathcal{O}}(R))$, such that X sends $\bar{L} \oplus I_{\mathcal{O}}(R)\bar{T}$ into $\bar{L} \oplus I_{\mathcal{O}}(R)\bar{T}$. The structure of $\mathcal{P}$ corresponds to a matrix $\Phi + \Phi_{\mathcal{N}} \in \mathrm{GL}_h(W_{\mathcal{O}}(S))$. Here we consider Φ as a matrix in $\mathrm{GL}_h(W_{\mathcal{O}}(S))$ via the natural embedding $W_{\mathcal{O}}(R) \hookrightarrow W_{\mathcal{O}}(S)$, $\Phi_{\mathcal{N}}$ is a matrix in $M_h(W_{\mathcal{O}}(\mathcal{N}))$.

Finding the pair $(\mathcal{P}',\ f)$ is equivalent to finding matrices $\Phi'_{\mathcal{N}} \in M_h(W_{\mathcal{O}}(\mathcal{N}))$ and $X_{\mathcal{N}} \in M_h(W_{\mathcal{O}}(\mathcal{N}))$ with the property

$$(\Phi + \Phi'_{\mathcal{N}})(X + X_{\mathcal{N}}) = (X + X_{\mathcal{N}})(\Phi + \Phi_{\mathcal{N}}), \tag{2.11}$$

because then we may take $\mathcal{P}'$ to be the $\mathcal{O}$-display with structure equation given by $\Phi + \Phi'_{\mathcal{N}}$, f to be the homomorphism given by $X + X_{\mathcal{N}}$.

Note that $\Phi X = X\Phi$ since X induces a homomorphism of $\mathcal{O}$-displays. Define

$$\begin{cases} \Phi'_{\mathcal{N}} = \Phi X \Phi_{\mathcal{N}} \Phi^{-1} X^{-1}, \\ X_{\mathcal{N}} = -X \Phi_{\mathcal{N}} \Phi^{-1} = -\Phi^{-1}\Phi'_{\mathcal{N}} X. \end{cases} \tag{2.12}$$

Since $\mathcal{N}^2 = 0$, we have $\Phi'_{\mathcal{N}} X_{\mathcal{N}} = X_{\mathcal{N}} \Phi'_{\mathcal{N}} = 0$. It is easy to check that

$$\Phi X_{\mathcal{N}} - X_{\mathcal{N}} \Phi = -\Phi'_{\mathcal{N}} X + X \Phi_{\mathcal{N}}.$$

The pair $(\Phi'_{\mathcal{N}}, X_{\mathcal{N}})$ defined by equation (2.12) satisfies equation (2.11). The lemma follows. □

By the same discussion as above, we have the following result.

Proposition 2.10. *Let $\bar{\mathcal{P}} = (\bar{P}, \bar{Q}, F, F_1)$ and $\bar{\mathcal{P}}' = (\bar{P}', \bar{Q}', F, F_1)$ be two $\mathcal{O}$-displays over R. Let $S \to R$ be a surjection with nilpotent kernel. Let $\mathcal{P} = (P, Q, F, F_1)$ be a lifting of $\bar{\mathcal{P}}$ to S. Assume that there exists a homomorphism between quadruples*

$$\bar{f} : (\bar{P}/\pi^n, \bar{Q}/\pi^n, F, F_1) \to (\bar{P}'/\pi^n, \bar{Q}'/\pi^n, F, F_1)$$

for some $n \in \mathbb{Z}_{\ge 0}$. Then there exists a lifting $\mathcal{P}' = (P', Q', F, F_1)$ of $\bar{\mathcal{P}}'$ to S and a homomorphism

$$f : (P/\pi^n, Q/\pi^n, F, F_1) \to (P'/\pi^n, Q'/\pi^n, F, F_1),$$

such that f lifts $\bar{f}$.

2.4. **Lifting homomorphisms: part two.** In Section 2.3, we saw that liftings of a homomorphism $\bar{f} : \bar{\mathcal{P}}_1 \to \bar{\mathcal{P}}_2$ always exist if we are allowed to change the liftings of the $\mathcal{O}$-displays. The situation changes completely if we fix the liftings of the $\mathcal{O}$-displays, as we shall see in this section.

Let $S \to R$ be an $\mathcal{O}$-pd-thickening with kernel $\mathfrak{a}$. Assume that π is nilpotent in S. Let $\mathcal{P}_i = (P_i, Q_i, F, F_1)$ $(i = 1, 2)$ be two $\mathcal{O}$-displays over S. Denote by $\bar{\mathcal{P}}_i = (\bar{P}_i, \bar{Q}_i, F, F_1)$ the base change of $\mathcal{P}_i$ to R. Let $\bar{\varphi} : \bar{\mathcal{P}}_1 \to \bar{\mathcal{P}}_2$ be a morphism of $\mathcal{O}$-displays. It lifts to a morphism of $\mathcal{O}$-windows over $\mathcal{W}_{S/R}$ (cf. [3, Section 2.8])

$$\varphi : (P_1, \hat{Q}_1, F, F_1) \to (P_2, \hat{Q}_2, F, F_1). \tag{2.13}$$

Note that in [13, Section 2.5], Zink used $\mathcal{P}$-triples, which are the same as $\mathcal{O}$-windows over $\mathcal{W}_{S/R}$. The morphism φ does not induce a morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$ in general. We may describe the obstruction as follows. Consider the composition

$$\operatorname{Obst} \bar{\varphi} : Q_1/I_{\mathcal{O}}(S)P_1 \hookrightarrow P_1/I_{\mathcal{O}}(S)P_1 \xrightarrow{\varphi} P_2/I_{\mathcal{O}}(S)P_2 \to P_2/Q_2. \tag{2.14}$$

Since $\bar{\varphi}(\bar{Q}_1) \subset \bar{Q}_2$, $\operatorname{Obst} \bar{\varphi}$ is trivial modulo $\mathfrak{a}$. Hence we obtain a map

$$\operatorname{Obst} \bar{\varphi} : Q_1/I_{\mathcal{O}}(S)P_1 \to \mathfrak{a} \otimes_S P_2/Q_2, \tag{2.15}$$

which is zero if and only if $\bar{\varphi}$ lifts to a morphism of $\mathcal{O}$-displays $\mathcal{P}_1 \to \mathcal{P}_2$ (i.e., φ sends Q_1 into Q_2). We call it the *obstruction* to lift $\bar{\varphi}$ to S.

Remark 2.11. The obstruction has functorial property. Assume that we have a morphism $\alpha : \mathcal{P}_2 \to \mathcal{P}_3$ of $\mathcal{O}$-displays over S. Let $\bar{\alpha} : \bar{\mathcal{P}}_2 \to \bar{\mathcal{P}}_3$ be its reduction over R. Then $\operatorname{Obst} \bar{\alpha} \circ \bar{\varphi}$ is the composite of the following maps

$$Q_1/I_{\mathcal{O}}(S)P_1 \xrightarrow{\operatorname{Obst} \bar{\varphi}} \mathfrak{a} \otimes_S P_2/Q_2 \xrightarrow{1 \otimes \alpha} \mathfrak{a} \otimes_S P_3/Q_3.$$

We denote this fact by

$$\operatorname{Obst} \bar{\alpha}\bar{\varphi} = \alpha \operatorname{Obst} \bar{\varphi}. \tag{2.16}$$

Remark 2.12. In the case $\mathfrak{a}^2 = 0$, we have $\mathfrak{a} \otimes_S P_2/Q_2 \cong \mathfrak{a} \otimes_R \bar{P}_2/\bar{Q}_2$. In this case, the obstruction may be considered as a map

$$\operatorname{Obst} \bar{\varphi} : \bar{Q}_1/I_{\mathcal{O}}(R)\bar{P}_1 \to \mathfrak{a} \otimes_R \bar{P}_2/\bar{Q}_2.$$

This is compatible with Proposition 2.1. Equation (2.16) may be written as

$$\operatorname{Obst} \bar{\alpha}\bar{\varphi} = \bar{\alpha} \operatorname{Obst} \bar{\varphi}. \tag{2.17}$$

Let S and $\tilde{S}$ be $\mathcal{O}$-algebras such that $\pi S = \pi \tilde{S} = 0$. Let $S \to R$ be a surjection with kernel $\mathfrak{a}$ such that $\mathfrak{a}^q = 0$. Let $\tilde{S} \to S$ be a surjection with kernel $\mathfrak{b}$ such that $\mathfrak{b}^q = 0$. We equip $\mathfrak{a}$ and $\mathfrak{b}$ with the trivial $\mathcal{O}$-pd-structure, hence $S \to R$ and $\tilde{S} \to S$ are both $\mathcal{O}$-pd-thickening.

Assume that $\mathcal{P}_i$ is the base change of an $\mathcal{O}$-display $\tilde{\mathcal{P}}_i$ over $\tilde{S}$ with respect to $\tilde{S} \to S$ $(i = 1, 2)$. Consider $\pi\bar{\varphi} : \bar{\mathcal{P}}_1 \to \bar{\mathcal{P}}_2$, a morphism of $\mathcal{O}$-displays over R. It lifts to a morphism

$$\pi\varphi : (P_1, \hat{Q}_1, F, F_1) \to (P_2, \hat{Q}_2, F, F_1).$$

This morphism induces a morphism $\pi\varphi : \mathcal{P}_1 \to \mathcal{P}_2$, as $\operatorname{Obst} \pi\bar{\varphi}$ is trivial.

Remark 2.13. The morphism $\bar{\varphi} : \bar{\mathcal{P}}_1 \to \bar{\mathcal{P}}_2$ also lifts to a morphism

$$\varphi : (P_1, \hat{Q}_1, F, F_1) \to (P_2, \hat{Q}_2, F, F_1).$$

But φ does not induce a morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$ in general. On the other hand, $\pi \cdot \varphi$ does as $\pi\mathfrak{a} = 0$ and the obstruction vanishes.

In the following, we study the obstruction to lift $\pi\varphi$ to a homomorphism of $\mathcal{O}$-displays $\tilde{\mathcal{P}}_1 \to \tilde{\mathcal{P}}_2$, i.e., the map

$$\operatorname{Obst} \pi\varphi : \tilde{Q}_1/I_{\mathcal{O}}(\tilde{S})\tilde{P}_1 \to \mathfrak{b} \otimes \tilde{P}_2/\tilde{Q}_2. \tag{2.18}$$

The obstruction $\operatorname{Obst} \pi\varphi$ may be computed in terms of $\operatorname{Obst} \bar{\varphi}$. In order to do so, we need to define two other maps.

The map $V^\sharp$: The image of $F_1 : \tilde{Q}_1 \to \tilde{P}_1$ generates $\tilde{P}_1$, hence it induces a surjection

$$F_1^\sharp : \tilde{S} \otimes_{\tilde{S},\mathrm{Frob}} \tilde{Q}_1/I_{\mathcal{O}}(\tilde{S})\tilde{P}_1 \to \tilde{P}_1/(I_{\mathcal{O}}(\tilde{S})\tilde{P}_1 + W_{\mathcal{O}}(\tilde{S})F\tilde{P}_1).$$

Using the normal decomposition of $\tilde{P}_1$, one sees that the left hand side and the right hand side are projective $\tilde{S}$-modules of the same rank. Hence $F_1^\sharp$ is an isomorphism. Let $V^\sharp$ be the inverse of $F_1^\sharp$. Note that $\mathfrak{b}$ is in the kernel of the Frobenius morphism, we have an isomorphism

$$\tilde{S} \otimes_{\tilde{S},\mathrm{Frob}} \tilde{Q}_1/I_{\mathcal{O}}(\tilde{S})\tilde{P}_1 \cong \tilde{S} \otimes_{S,\mathrm{Frob}} Q_1/I_{\mathcal{O}}(S)P_1.$$

It induces the following map, which we still denote by $V^\sharp$

$$V^\sharp : \tilde{P}_1 \to \tilde{S} \otimes_{S,\mathrm{Frob}} Q_1/I_{\mathcal{O}}(S)P_1.$$

The map $F^\sharp$: We have assumed that $\mathfrak{b}^q = 0$, so the operator F on $\tilde{P}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2$ factors as

$$\begin{array}{ccc} \tilde{P}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2 & \xrightarrow{\quad F \quad} & \tilde{P}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2 \\ & \searrow \quad \nearrow_{F^b} & \\ & P_2/I_{\mathcal{O}}(S)P_2 & \end{array} \tag{2.19}$$

Moreover, from the definition of $\mathcal{O}$-displays, $F(x) = \pi F_1(x)$ if $x \in \tilde{Q}_2$. Hence $\tilde{Q}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2 \in \operatorname{Ker}(F)$ and we obtain a Frobenius-linear map

$$F^b : P_2/Q_2 \to \tilde{P}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2.$$

Restricting F^b to $\mathfrak{a}(P_2/Q_2)$, we obtain

$$F^b : \mathfrak{a}(P_2/Q_2) \to \mathfrak{b}(\tilde{P}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2).$$

Note that we may view $\mathfrak{b}$ as an ideal of $W_{\mathcal{O}}(\mathfrak{b})$ (cf. [3, Section 2.8]). Hence we may and do identify $\mathfrak{b}(\tilde{P}_2/I_{\mathcal{O}}(\tilde{S})\tilde{P}_2)$ with $\mathfrak{b}\tilde{P}_2$. Denote by $F^\sharp$ the linearization of F^b

$$F^\sharp : \tilde{S} \otimes_{S,\mathrm{Frob}} \mathfrak{a}(P_2/Q_2) \to \mathfrak{b}\tilde{P}_2. \tag{2.20}$$

Proposition 2.14. *The following diagram is commutative*

$$\begin{array}{ccccc} \tilde{Q}_1/I_{\mathcal{O}}(\tilde{S})\tilde{P}_1 & \xrightarrow{V^\sharp} & \tilde{S}\otimes_{S,\mathrm{Frob}} Q_1/I_{\mathcal{O}}(S)P_1 & \xrightarrow{\tilde{S}\otimes \mathrm{Obst}\,\bar{\varphi}} & \tilde{S}\otimes_{S,\mathrm{Frob}} \mathfrak{a}(P_2/Q_2) \\ & \searrow^{\mathrm{Obst}(\pi\varphi)} & & & \downarrow F^\sharp \\ & & & & \mathfrak{b}(\tilde{P}_2/\tilde{Q}_2). \end{array} \tag{2.21}$$

Sketch of the proof. The morphism of $\mathcal{O}$-displays $\pi\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ lifts to a uniquely determined morphism of $\mathcal{O}$-windows over $\mathcal{W}_{\tilde{S}/S}$

$$\tilde{\psi} : (\tilde{P}_1, \hat{\tilde{Q}}_1, F, F_1) \to (\tilde{P}_2, \hat{\tilde{Q}}_2, F, F_1).$$

Let $\tilde{\varphi} : \tilde{P}_1 \to \tilde{P}_2$ be any $W_{\mathcal{O}}(\tilde{S})$-linear map that lifts $\varphi : P_1 \to P_2$ (cf. Remark 2.13). It does not induce a morphism $\tilde{\mathcal{P}}_1 \to \tilde{\mathcal{P}}_2$ of $\mathcal{O}$-windows over $\mathcal{W}_{\tilde{S}}$ since it does not commutes with F_1 in general. On the other hand, we have

$$\tilde{\psi} = \pi\tilde{\varphi} + \omega, \tag{2.22}$$

where $\omega : \tilde{P}_1 \to \mathfrak{b}\tilde{P}_2 \subset W_{\mathcal{O}}(\mathfrak{b})\tilde{P}_2$ is the composite of the following maps

$$\tilde{P}_1 \xrightarrow{V^\sharp} \tilde{S}\otimes_{S,\mathrm{Frob}} Q_1/I_{\mathcal{O}}(S)P_1 \xrightarrow{\tilde{S}\otimes \mathrm{Obst}\,\bar{\varphi}} \tilde{S}\otimes_{S,\mathrm{Frob}} \mathfrak{a}(P_2/Q_2) \xrightarrow{F^\sharp} \mathfrak{b}\tilde{P}_2.$$

Equation (2.22) could be proven by the same argument of [13, Corollary 74], which is closely related to [13, Theorem 44] and [3, Theorem 2.12]. Then the proposition follows easily. □

3. Deformations of formal π-divisible $\mathcal{O}$-modules

In this section, we translate the results in Section 2 via [2, Theorem 1.1]. In particular, we obtain Theorem 1.1.

3.1. **The universal deformation.** Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with π nilpotent in it. Let X be a formal π-divisible $\mathcal{O}$-module over R. Let $S \to R$ be a surjection with nilpotent kernel. A *deformation* of X to S is an isomorphism class of pairs (X', ι), where X' is a formal π-divisible $\mathcal{O}$-module over S and $\iota : X' \times_S R \cong X$ is an isomorphism of formal π-divisible $\mathcal{O}$-modules. The *deformation functor* of X is defined by

$$\begin{aligned} \mathbb{D}_X : \mathrm{Aug}_{\Lambda\to R} &\to \mathrm{Sets} \\ S &\mapsto \{\text{deformations of } X \text{ to } S\}. \end{aligned} \tag{3.1}$$

Theorem 3.1. *With the notation as above, if $X = \mathrm{BT}_{\mathcal{O}}(\mathcal{P})$, then the two functors $\mathbb{D}_X$ and $\mathbb{D}_{\mathcal{P}}$ are equivalent. Therefore, there exists a formal π-divisible $\mathcal{O}$-module $\mathcal{X} \to \mathrm{Spf}(\Lambda[[t_1, \cdots, t_{dc}]])$ which is universal for the functor $\mathbb{D}_X$, i.e.,*

$$\mathbb{D}_X(S) = \mathrm{Hom}(\Lambda[[t_1, \cdots, t_{dc}]], S) \tag{3.2}$$

and every deformation of X over S is a base change induced by a morphism in the left hand side of equation (3.2). Here $c = h - d$ and X is of type (h, d).

3.2. **On the truncations.** Let $R \in \mathrm{Alg}_{\mathcal{O}}$ with π nilpotent in it. Let X_1 and X_2 be formal π-divisible $\mathcal{O}$-modules over R.

Theorem 3.2. *If $X_1[\pi^n] \cong X_2[\pi^n]$, then for any deformation $\tilde{X}_1$ of X_1 over S, there exists a deformation $\tilde{X}_2$ of X_2 over S, such that $\tilde{X}_1[\pi^n] \cong \tilde{X}_2[\pi^n]$.*

Proof. Let $\mathrm{BT}_{\mathcal{O},n}$ be the category of special truncated formal π-divisible $\mathcal{O}$-modules with level n. (Here *special* means that the truncated $\mathcal{O}$-modules are kernels of isogenies of formal π-divisible $\mathcal{O}$-modules.) Then $\mathrm{BT}_{\mathcal{O},n}$ is a smooth Artin algebraic stack with affine diagonal. The truncation morphism $\mathrm{BT}_{\mathcal{O},n+1} \to \mathrm{BT}_{\mathcal{O},n}$ is smooth and surjective by the same argument of [11, Proposition 3.15]. (See also [2, Lemma 4.4].) The theorem then follows from Proposition 2.8. □

Remark 3.3. This result was also indicated in [5, Section 8].

Remark 3.4. Let X be a formal π-divisible $\mathcal{O}$-module over R. Let $\mathcal{P} = (P, Q, F, F_1)$ be the corresponding $\mathcal{O}$-display. Then by [2, Theorem 2.12], X is determined by the following exact sequence

$$0 \to \widehat{Q}_{\mathcal{N}} \xrightarrow{\mathrm{id} - F_1} \widehat{P}_{\mathcal{N}} \to X(\mathcal{N}) \to 0.$$

By Snake Lemma, $X[\pi^n]$ lies in the exact sequence

$$X[\pi^n](\mathcal{N}) \to \widehat{Q}_{\mathcal{N}}/\pi^n \xrightarrow{\mathrm{id} - F_1} \widehat{P}_{\mathcal{N}}/\pi^n.$$

If the first arrow is an injection, then $X[\pi^n]$ is determined by the quadruple $(P/\pi^n, Q/\pi^n, F, F_1)$ and the theorem follows from Proposition 2.10. In general, the first arrow has non-trivial kernel and we need to adapt to stacks $\mathrm{BT}_{\mathcal{O},n}$ to prove our claim.

For formal p-divisible groups, Theorem 3.2 follows from [7, Théorème 4.4], which is proved by a different method.

3.3. **A result of Keating.** Let $k \in \mathrm{Alg}_{\mathcal{O}}$ be an algebraically closed field of characteristic p. Let X_0 be a π-divisible $\mathcal{O}$-module of height 2 and dimension 1. Then $\mathrm{End}(X_0)$ is the ring of integers in a quaternion algebra D with center $\mathrm{Frac}(\mathcal{O})$. Let $O_D = \mathrm{End}(X_0)$. Let $\alpha \mapsto \alpha^*$ be the main involution on O_D. Fix $\alpha \in O_D$ such that $\alpha \notin \mathcal{O}$ and set $\iota = \mathrm{ord}_{O_D}(\alpha - \alpha^*)$. Define $c(\alpha) \in \mathbb{N}$ by

$$c(\alpha) = \begin{cases} q^{\iota/2} + 2\sum_{j=1}^{\iota/2} q^{\iota/2-j} & \text{if } 2 \mid \iota, \\ 2\sum_{j=0}^{\frac{\iota-1}{2}} q^{\frac{\iota-1}{2}-j} & \text{if } 2 \nmid \iota. \end{cases}$$

Let $\mathcal{X}$ over $k[[t]]$ be the universal deformation of X_0 in equal characteristic.

Theorem 3.5. *With the notation as above, α lifts to an endomorphism of $\mathcal{X} \otimes_{k[[t]]} k[[t]]/t^{c(\alpha)}$ but does not lift to an endomorphism of $\mathcal{X} \otimes_{k[[t]]} k[[t]]/t^{c(\alpha)+1}$.*

Proof. If we translate the above statement on π-divisible $\mathcal{O}$-modules to a statement on $\mathcal{O}$-displays and use Proposition 2.14, the proof then goes entirely similar as the proof of [13, Proposition 75]. □

4. On Lubin-Tate groups

In this section, we study Lubin-Tate groups and prove Theorem 1.3. The main idea is to use the relation between $\mathcal{O}'$-displays and $(\mathcal{O}, \mathcal{O}')$-displays, which is an essential ingredient in the proof of [2, Theorem 1.1].

4.1. **The general set-up.** Let A be an $\mathcal{O}$-algebra and S be an A-algebra. An $(\mathcal{O}, A)$-display over S is a pair $(\mathcal{P}, \iota)$, where $\mathcal{P}$ is an $\mathcal{O}$-display over S and $\iota : A \to \mathrm{End}(\mathcal{P})$ is a ring homomorphism, such that the action of A on P/Q induced from ι coincides with action from the structure morphism $A \to S$.

Let $a \in A$ be a fixed element. Set $R = S/a$ and $R_i = S/a^{i+1}$. Then we have a sequence of surjections

$$S \to \cdots \to R_i \to R_{i-1} \to \cdots \to R_0 = R.$$

Let $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$ be $\mathcal{O}$-displays over S. By base change, we have $\mathcal{O}$-displays $\mathcal{P}_1^{(i)}$ and $\mathcal{P}_2^{(i)}$ over R_i for each $i \in \mathbb{Z}_{\geq 0}$. Set $\mathcal{P}_1 = \mathcal{P}_1^{(0)}$ and $\mathcal{P}_2 = \mathcal{P}_2^{(0)}$. Let $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of $\mathcal{O}$-displays over R. Assume that φ lifts to a morphism $\varphi^{(i-1)} : \mathcal{P}_1^{(i-1)} \to \mathcal{P}_2^{(i-1)}$. To lift $\varphi^{(i-1)}$ to a morphism $\mathcal{P}_1^{(i)} \to \mathcal{P}_2^{(i)}$ gives us the following obstruction morphism

$$\mathrm{Obst}\,\varphi^{(i-1)} : Q_1^{(i)}/I_{\mathcal{O}}(R_i)P_1^{(i)} \to (a^i)/(a^{i+1}) \otimes_{R_i} P_2^{(i)}/Q_2^{(i)},$$

which factors through (cf. Remark 2.12)

$$\mathrm{Obst}_i\,\varphi : Q_1/I_{\mathcal{O}}(R)P_1 \to (a^i)/(a^{i+1}) \otimes_R P_2/Q_2.$$

Moreover, the obstruction to lift $\iota(a)\varphi^{(i-1)}$ to a morphism $\mathcal{P}_1^{(i+1)} \to \mathcal{P}_2^{(i+1)}$ is given by

$$\mathrm{Obst}(\iota(a)\varphi^{(i-1)}) : Q_1^{(i+1)}/I_{\mathcal{O}}(R_{i+1})P_1^{(i+1)} \to (a^i)/(a^{i+2}) \otimes_{R_{i+1}} P_2^{(i+1)}/Q_2^{(i+1)},$$

which factors through

$$\mathrm{Obst}_{i+1}(\iota(a)\varphi) : Q_1/I_{\mathcal{O}}(R)P_1 \to (a^i)/(a^{i+2}) \otimes_R P_2/Q_2.$$

Since $\iota(a)$ acts on $P^{(i+1)}/Q^{(i+1)}$ by multiplication by a, we have the following commutative diagram

$$\begin{array}{ccc} Q_1/I_{\mathcal{O}}(R)P_1 & \xrightarrow{\mathrm{Obst}_i(\varphi)} & (a^i)/(a^{i+1}) \otimes_R P_2/Q_2 \\ & \searrow^{\mathrm{Obst}_{i+1}(\iota(a)\varphi)} & \downarrow{\scriptstyle a\otimes \mathrm{id}} \\ & & (a^i)/(a^{i+2}) \otimes_R P_2/Q_2. \end{array} \tag{4.1}$$

Therefore, we have the following result.

Lemma 4.1. *Let $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$ be $\mathcal{O}$-displays over S. By base change, we have $\mathcal{O}$-displays $\mathcal{P}_1^{(i)}$ and $\mathcal{P}_2^{(i)}$ over R_i for each $i \in \mathbb{Z}_{\geq 0}$. Set $\mathcal{P}_1 = \mathcal{P}_1^{(0)}$ and $\mathcal{P}_2 = \mathcal{P}_2^{(0)}$. Let $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of $\mathcal{O}$-displays over R. Assume that φ lifts to a morphism $\varphi^{(i-1)} : \mathcal{P}_1^{(i-1)} \to \mathcal{P}_2^{(i-1)}$. Then $\iota(a)\varphi$ lifts to a morphism $\varphi^{(i)} : \mathcal{P}_1^{(i)} \to \mathcal{P}_2^{(i)}$.*

4.2. **The Lubin-Tate $\mathcal{O}$-display.** Let E' be a totally ramified extension of $E = \mathrm{Frac}(\mathcal{O})$ with degree $e \geq 2$. Let $\mathcal{O}'$ be the ring of integers of E' and π' be a uniformizer of $\mathcal{O}'$. In the following, we study a particular $(\mathcal{O}, \mathcal{O}')$-display. Let S be the $\mathcal{O}'$-algebra $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(\bar{\mathbb{F}})$. Denote by a the image of π' in S.

Let $P = \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(S)$. It is a free $W_{\mathcal{O}}(S)$-module with basis $\{(\pi')^i \otimes 1 \mid 0 \leq i \leq e-1\}$. Hence it is a free $W_{\mathcal{O}}(S)$-module with basis

$$1 \otimes 1,\ (\pi')^i \otimes 1 - 1 \otimes [a^i] \text{ for } 1 \leq i \leq e-1.$$

Define

$$T = W_{\mathcal{O}}(S)\langle 1 \otimes 1\rangle, \quad L = W_{\mathcal{O}}(S)\langle (\pi')^i \otimes 1 - 1 \otimes [a^i] \mid 1 \le i \le e-1\rangle.$$

Then $P = L \oplus T$. Define $Q = L \oplus I_{\mathcal{O}}(S)T$. We define an $\mathcal{O}$-display structure on the pair (P, Q) by writing down the structure equation explicitly. More precisely (cf. [13, Pages 24-25]),

$$(4.2) \qquad \begin{cases} F_1((\pi')^i \otimes 1 - 1 \otimes [a^i]) = \frac{\pi' \otimes 1 - 1 \otimes [a^{iq}]}{\pi' \otimes 1 - 1 \otimes [a^q]} = \sum_{\substack{0 \le k, l \le i-1 \\ k+l=i-1}} (\pi')^k \otimes [a^{lq}], \\ F(1 \otimes 1) = \tau^{-1} \frac{(\pi')^e \otimes 1 - 1 \otimes [a^{eq}]}{\pi' \otimes 1 - \pi' \otimes [a^q]}. \end{cases}$$

Here $\tau = \pi^{-1}((\pi')^e \otimes 1 - 1 \otimes [a^{eq}])$. It is a unit in $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(S)$ by [2, Lemma 2.24].

Let $\tilde{P} = (\tilde{P}, \tilde{Q}, F, F_1)$ be the $\mathcal{O}$-display over S defined as above. Let $\mathcal{P} = (P, Q, F, F_1)$ be the $\mathcal{O}$-display over $R = S/aS = \bar{\mathbb{F}}$ defined by base change. Then $Q = \pi' P$ and

$$F_1((\pi')^i) = (\pi')^{i-1} \text{ for } i \ge 1,$$

where $\pi' = \pi' \otimes 1 \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(\bar{\mathbb{F}})$.

Let $\varphi : \mathcal{P} \to \mathcal{P}$ be an endomorphism of $\mathcal{P}$. The obstruction to lift φ to $R_1 = S/a^2S$ is

$$\mathrm{Obst}_1(\varphi) : Q/I_{\mathcal{O}}(R)P \to (a)/(a^2) \otimes_R P/Q.$$

The endomorphism φ induces an endomorphism on $P/Q \cong \bar{\mathbb{F}}$, which is the multiplication of some element in $\bar{\mathbb{F}}$. Denote this element by $\mathrm{Lie}\,\varphi$. Let σ be the Frobenius endomorphism of $\bar{\mathbb{F}}$ given by $x \mapsto x^q$.

Lemma 4.2. *With the notation as above, we have the following commutative diagram*

$$(4.3) \qquad \begin{array}{ccccc} Q/I_{\mathcal{O}}(R)P = Q/\pi P & \xrightarrow{(\pi')^{-1}} & P/\pi' P = P/Q & \xrightarrow{\sigma^{-1}(\mathrm{Lie}\,\varphi) - \mathrm{Lie}\,\varphi} & P/Q \\ & \searrow^{\mathrm{Obst}_1(\varphi)} & & & \downarrow a \\ & & & & (a)/(a^2) \otimes_R P/Q. \end{array}$$

Proof. For simplicity, if $x \in \mathcal{O}'$, we still denote by x for the image $x \otimes 1 \in \mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(\bar{\mathbb{F}}) = P$. Write

$$(4.4) \qquad \varphi(1) = \xi_0 + \xi_1 \pi' + \cdots + \xi_{e-1}(\pi')^{e-1}, \ \xi_i \in W_{\mathcal{O}}(\bar{\mathbb{F}}).$$

Since $\varphi((\pi')^{i-1}) = \varphi(F_1((\pi')^i)) = F_1(\varphi((\pi')^i))$, we have

$$\varphi((\pi')^i) = {}^{F^{-i}}\xi_0(\pi')^i + {}^{F^{-i}}\xi_1(\pi')^{i+1} + \cdots + {}^{F^{-i}}\xi_{e-1}(\pi')^{e-1+i},$$

for all $i \in \mathbb{Z}_{\ge 0}$. Consider $R_1 \to R$ as an $\mathcal{O}$-pd-thickening by equipping aR_1 with the trivial $\mathcal{O}$-pd-structure. Then the category of $\mathcal{O}$-windows over $\mathcal{W}_{R_1/R}$ is equivalent to the category of $\mathcal{O}$-windows over $\mathcal{W}_R$, hence equivalent to the category of $\mathcal{O}$-displays over R (cf. [3, Proposition 2.21]). Let $(P^{(1)}, \widehat{Q}^{(1)}, F, F_1)$ be the $\mathcal{O}$-window over the frame $\mathcal{W}_{R_1/R}$ corresponding to $\mathcal{P}$ via the above equivalence. Then $P^{(1)} = \mathcal{O}' \otimes W_{\mathcal{O}}(R_1)$ and

$$F_1(\pi')^i = (\pi')^{i-1} \text{ for } i \ge 2, \quad F1 = \frac{\pi}{\pi'}.$$

The lifting $\tilde{\varphi} \in \mathrm{End}(P^{(1)}, \widehat{Q}^{(1)}, F, F_1)$ of $\varphi \in \mathrm{End}(\mathcal{P})$ is defined by the same formula (4.4), i.e., we have

$$(4.5) \qquad \tilde{\varphi} = \varphi \otimes_{W_{\mathcal{O}}(\bar{\mathbb{F}})} W_{\mathcal{O}}(R_1).$$

We need to understand the obstruction to lift $\tilde{\varphi}$ to a morphism of $\mathcal{W}_{R_1}$-windows. The map $\tilde{\varphi}$ induces an $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R_1)$-module homomorphism

$$Q^{(1)}/I_{\mathcal{O}}(R_1)P^{(1)} \to P^{(1)}/Q^{(1)}. \tag{4.6}$$

As an R_1-module, $Q^{(1)}/I_{\mathcal{O}}(R_1)P^{(1)}$ is free with basis $\{(\pi')^i - a^i \mid 1 \leq i \leq e-1\}$. Here we write π' for $\pi' \otimes 1 \in \mathcal{O}' \otimes_{\mathcal{O}} R_1$ and a for $1 \otimes a$. Since $a^2 = 0$ in R_1, it is easy to see that $(\pi')^i \in Q^{(1)}$ if $i \geq 2$. Because $\tilde{\varphi}$ is an $\mathcal{O}' \otimes_{\mathcal{O}} W_{\mathcal{O}}(R_1)$-module homomorphism, $\tilde{\varphi}((\pi')^i) \in Q^{(1)}$ for $i \geq 2$. To understand the obstruction, it suffices to understand $\tilde{\varphi}(\pi' - a)$. Since $\tilde{\varphi}$ is defined by the formula (4.4), we have

$$\begin{aligned} \tilde{\varphi}(\pi - a) &= ({}^{F^{-1}}\xi_0(\pi') + {}^{F^{-1}}\xi_1(\pi')^2 + \cdots + {}^{F^{-1}}\xi_{e-1}(\pi')^e) - \tilde{\varphi}(a) \\ &\equiv {}^{F^{-1}}\xi_0\pi' - \xi_0 a \pmod{Q^{(1)}} \\ &\equiv ({}^{F^{-1}}\xi_0 - \xi_0)a \pmod{Q^{(1)}}. \end{aligned} \tag{4.7}$$

The lemma follows since $\mathrm{Lie}(\varphi) = \xi_0 \pmod{\pi}$. □

Proposition 4.3. *With the notation as above. Let $\mathcal{O}_D = \mathrm{End}(\mathcal{P})$ be the endomorphism ring, which is isomorphic to the maximal order of the central simple E-algebra with invariant $1/e$. Let $\mathcal{O}^u$ be the ring of integers of the maximal unramified extension of E' with residue field $\bar{\mathbb{F}}$. Then*

$$\mathrm{End}(\tilde{\mathcal{P}}_{\mathcal{O}^u/(\pi')^{m+1}}) = \mathcal{O}' + (\pi')^m \mathcal{O}_D, \quad m \geq 0.$$

Proof. Let $\varphi \in \mathcal{O}_D$. Then $(\pi')^m\varphi$ lifts to an endomorphism of $\tilde{\mathcal{P}}$ over $\mathcal{O}^u/(\pi')^{m+1}$ by Lemma 4.1. Moreover, we have

$$\mathrm{Obst}_{m+1}(\pi')^m\varphi = (\pi')^m \,\mathrm{Obst}_1 \varphi,$$

where $(\pi')^m$ on the right hand side denotes the map

$$(\pi')^m : (a)/(a^2) \otimes_R P/Q \to (a^{(m+1)})/(a^{(m+2)})P/Q.$$

Let $\psi \in (\mathcal{O}' + (\pi')^m\mathcal{O}_D) - (\mathcal{O}' + (\pi')^{m+1}\mathcal{O}_D)$. We claim that ψ does not lift to an endomorphism of $\tilde{\mathcal{P}}_{\mathcal{O}^u/(\pi')^{m+2}}$.

Indeed, since π' is a uniformizer of $\mathcal{O}_D$, we may write

$$\psi = [a_0] + [a_1]\pi' + \cdots + [a_m](\pi')^m + \cdots,$$

where $a_i \in \mathbb{F}'$. Here $\mathbb{F}'$ is the degree e extension of $\mathbb{F}$. By our assumption on ψ, we have $a_i \in \mathbb{F}$ for $i < m$ and $a_m \notin \mathbb{F}$. Then

$$\mathrm{Obst}_{m+1}\psi = \mathrm{Obst}_{m+1}([a_m](\pi')^m + \cdots) = (\pi')^m\,\mathrm{Obst}_1([a_m] + \pi'[a_{m+1}] + \cdots).$$

By Lemma 4.2, $\mathrm{Obst}_{m+1}\psi$ does not vanish since $\sigma(a_m) \neq a_m$. The claim follows. The proposition then follows from the following lemma. □

Lemma 4.4. *Let S be an $\mathcal{O}$-algebra such that π is nilpotent in S. Let $\mathfrak{a} \subset S$ be an ideal with $\mathcal{O}$-pd-structure. Let $R = S/\mathfrak{a}$. Let $\mathcal{P} = (P, Q, F, F_1)$ and $\mathcal{P}' = (P', Q', F, F_1)$ be two $\mathcal{O}$-displays over S. Then the natural map*

$$\mathrm{Hom}(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}(\mathcal{P}_R, \mathcal{P}'_R)$$

is injective.

Proof. Let $u : \mathcal{P} \to \mathcal{P}'$ be a morphism of $\mathcal{O}$-displays that is zero modulo $\mathfrak{a}$. Hence $u(P) \subset W_{\mathcal{O}}(\mathfrak{a})P$. Since $S \to R$ is an $\mathcal{O}$-pd-thickening, the map $F_1 : Q' \to P'$ extends to the map $F_1 : W_{\mathcal{O}}(\mathfrak{a})P' + Q' \to P'$ which maps $W_{\mathcal{O}}(\mathfrak{a})P'$ to $W_{\mathcal{O}}(\mathfrak{a})P'$. We claim that the following diagram is commutative

$$\begin{array}{ccc} P & \xrightarrow{u} & W_{\mathcal{O}}(\mathfrak{a})P' \\ {\scriptstyle V^\sharp}\downarrow & & \uparrow{\scriptstyle F_1^\sharp} \\ W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(S),F} P & \xrightarrow{1\otimes u} & W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(S),F} W_{\mathcal{O}}(\mathfrak{a})P'. \end{array} \tag{4.8}$$

Here $F_1^\sharp$ is the linearization of F_1, $V^\sharp : P \to W_{\mathcal{O}}(S) \otimes_{W_{\mathcal{O}}(S),F} P$ is the unique $W_{\mathcal{O}}(S)$-linear map satisfies, for all $w \in W_{\mathcal{O}}(S), x \in P$ and $y \in Q$, (cf. [2, Lemma 2.2] and [13, Lemma 10])

$$\begin{aligned} V^\sharp(wFx) &= \pi \cdot w \otimes x, \\ V^\sharp(wF_1 y) &= w \otimes y. \end{aligned}$$

Indeed, since $P = W_{\mathcal{O}}(S)\langle F_1 Q\rangle$, it suffices to show the commutativity for elements of the form $wF_1 l$, where $w \in W_{\mathcal{O}}(S)$ and $l \in Q$. But in this case the commutativity is obvious, hence the claim holds.

Iterating the diagram, for any $N \in \mathbb{Z}_{\geq 1}$, we have

$$(F_1^\sharp)^N (1 \otimes_{F^N} u)(V^{N\sharp}) = u.$$

Therefore $u = 0$ since $\mathcal{P}$ is nilpotent. The lemma follows. □

Finally, Theorem 1.3 follows from Proposition 4.3 and the fact that $\Gamma_2(\mathcal{O}, \mathcal{O}')$ in [2, Proposition 2.29] is an equivalence on nilpotent objects.

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References

[1] T. Ahsendorf, *$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules.* Thesis 2012.
[2] T. Ahsendorf; C. Cheng; T. Zink, *$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules.* Journal of Algebra Volume 457, July 2016, 129-193.
[3] C. Cheng, *Breuil $\mathcal{O}$-windows and π-divisible $\mathcal{O}$-modules.* Trans. Amer. Math. Soc. 370 (2018), no. 1, 695-726
[4] J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry.* Inst. Hautes Études Sci. Publ. Math., Vol. 82 (1995), pp. 5-96.
[5] G. Faltings, *Group schemes with strict $\mathcal{O}$-action.* MOSCOW MATHEMATICAL JOURNAL, Volume 2, Number 2, April-June 2002, 249-279.
[6] B. Gross, *On canonical and quasicanonical liftings.* Invent. math. 84, 321-326, (1986).
[7] L. Illusie, *Déformations des groupes de Barsotti-Tate(d'après A. Grothendieck).* Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84), pp. 151-198, J. Astérisque **127**, Soc. Math. de France, Paris, 1985.
[8] E. Lau, *A duality theorem for Dieduonne displays.* Ann. Sci. Ec. Norm. Sup. (4) 42 (2009) 241-259.
[9] E. Lau, *Displays and formal p-divisible groups.* Invent. Math. 171 (2008) 617–628.
[10] E. Lau, *Frames and Finite Group Schemes over Complete Regular Local Rings.* Doc. Math. 15 (2010) 545–569.

[11] E. Lau, *Smoothness of the truncated display functor*. J. Amer. Math. Soc. 26 (2013), 129-165
[12] F. Oort, *Newton polygons and formal groups: conjectures by Manin and Grothendieck.* Ann. of Math. (2) 152 (2000), no. 1, pp. 183-206.
[13] T. Zink, *The display of a formal p-divisible group.* Astrisque No. 278 (2002), 127-248.
[14] T. Zink, *A Dieudonné theory for p-divisible groups.* In Class Field Theory, Its Centenary and Prospect, 139-160, Advanced Studies in Pure Mathematics 30, Math. Soc. Japan 2001.