EVERY IRREDUCIBLE REPRESENTATION OF $\mathrm{GL}_2(\mathbb{F}_q)$ DOES PHASE RETRIEVAL

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Abstract. Let $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of a finite group G. In this paper, we show that if each irreducible component of $\pi \otimes \pi^* : G \to \mathbf{U}(V \otimes V)$ has multiplicity less than or equal to two, then the representation π admits maximal spanning vectors and hence does phase retrieval. In particular, for the finite field $\mathbb{F}_q$, we show that every irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ does phase retrieval.

1. Introduction

In [13], Li-Han-etc. conjectured that every irreducible representation of a finite group does phase retrieval and confirmed the conjecture for projective representations of abelian groups. Further progress has been made since [13]. In [7, 8, 9], the authors generalized the conjecture to locally compact groups and verified the conjecture for central extensions of several types of locally compact abelian groups via Fourier analysis. In [1] Bartusel-Führ-Oussa confirmed the conjecture for the affine group $\mathrm{GA}(1, p)$, where p is a prime number. In [11] Führ-Oussa showed that irreducible representations of nilpotent Lie groups do phase retrieval via tools from Lie algebras and studied the phase retrieval property of representations of p-groups. In [6], Cheng generalized the result of [1] and proved the phase retrievability for all affine groups $\mathrm{GA}(1, q)$. Let $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of a finite group G. Then [6, Theorem 1.1] states that π does phase retrieval if π is unramified (i.e. each irreducible component of $\pi\otimes\pi^* : G \to \mathbf{U}(V\otimes V)$ has multiplicity one). In this paper, we adapt the idea of [6] and prove the following result.

Theorem 1.1. *Let $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of G. If each irreducible component of $\pi \otimes \pi^* : G \to \mathbf{U}(V \otimes V)$ has multiplicity less than or equal to two, then (π, G, V) admits maximal spanning vectors and the set of maximal spanning vectors is open dense in V. In particular, π does phase retrieval.*

Although we just relaxed the condition on the multiplicity from one to less than or equal to two, Theorem 1.1 is much stronger than [6, Theorem 1.1]. Moreover, the proof of the theorem provides a description of all the maximal spanning vectors once the representation π is given explicitly (cf. Proposition 4.2). The key ingredient in the proof is the basic algebraic fact that the polynomial ring over a field is a unique factorization domain (cf. Lemma 3.1). As an application of Theorem 1.1, we show that principal series representations of $\mathrm{GL}_2(\mathbb{F}_q)$ admit maximal spanning vectors. It is shown in [6] that the cuspidal representations and the special representations of $\mathrm{GL}_2(\mathbb{F}_q)$ admit maximal spanning vectors, we then obtain the following result, which provides another evidence for [13, Conjecture] since

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$\mathrm{GL}_2(\mathbb{F}_q)$ is not nilpotent when q is large. See Section 5 and the references there for more information on representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

Theorem 1.2. *Let $\mathbb{F}_q$ be the finite field with q elements. Every irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ admits maximal spanning vectors, hence does phase retrieval.*

Notation and conventions. In this paper, $\mathbb{F}_q$ denotes the finite field with q elements, $\mathbf{i}$ denotes a fixed square root of -1. In this paper G is a finite group. Denote by $L^2(G)$ the space of functions on G with inner product given by

$$\langle f, f' \rangle = \sum_{g \in G} f(g)\overline{f'(g)}.$$

All vector spaces are finite dimensional $\mathbb{C}$-spaces. For a Hilbert space V, $\mathbf{U}(V)$ denotes the set of unitary operators on V, $\mathrm{HS}(V)$ denotes the set of Hilbert-Schmidt operators on V. For a matrix M, we denote its transpose by M' and conjugate transpose by M^*. Denote by $M_d(\mathbb{C})$ the space of $d \times d$ complex matrices with inner product defined by

$$\langle A, B \rangle = \mathrm{Tr}(AB^*), \quad A,\ B \in M_d(\mathbb{C}).$$

If $\pi : G \to \mathbf{U}(V)$ is a representation of G on V, we denote it by (π, V) or (π, G, V). Denote the character of (π, V) by χ_V or χ_π. Denote the dual representation of π by π^*. A representation π is *multiplicity free* if each irreducible component of π has multiplicity one. A representation π is *unramified* if the tensor product representation $\pi \otimes \pi^* : G \to \mathbf{U}(V \otimes V)$ is multiplicity free.

2. The phase retrieval problem of group frames

We review the phase retrieval problem of group frames. We refer to [4, 15] for basics of frames and group frames. Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional irreducible representation. Let $v \in V$ be a nontrivial vector. Then $\Phi_v := \{\pi(g)v \mid g \in G\}$ is a tight frame for V. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. If the frame Φ_v is phase retrievable, i.e. the map

$$\begin{aligned} t_v : V/\mathbb{T} &\to \mathbb{R}^{|G|} \\ x &\mapsto (|\langle x, \pi(g)v \rangle|)_{g \in G} \end{aligned}$$

is injective, then we call the vector v *phase retrievable.* If (π, G, V) admits a phase retrievable vector, we say that the representation π *does phase retrieval.* In the above situation, $v \in V$ has *maximal span* (or is *maximal spanning*) if $\mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = \mathrm{HS}(V)$. Here for any $x,\ y \in V$, $x \otimes y$ is the projection

$$\begin{aligned} V &\to V \\ u &\mapsto \langle u, y \rangle x. \end{aligned}$$

In other words, $v \in V$ has maximal span if and only if $\dim \mathrm{Span}\{\pi(g)v \otimes \pi(g)v \mid g \in G\} = (\dim V)^2$. It is well known and easy to see that if v has maximal span, then $v \in V$ is phase retrievable (cf. [2]). For $u, v \in V$, let $\mathfrak{c}_{u,v}$ be the matrix coefficient

$$\begin{aligned} \mathfrak{c}_{u,v} : G &\to \mathbb{C} \\ g &\mapsto \langle \pi(g)u, v \rangle. \end{aligned}$$

Denote by $C_\pi \subset L^2(G)$ the subspace spanned by the matrix coefficients $\mathfrak{c}_{u,v}$ $(u, v \in V)$. The following three conditions are equivalent (cf. [6, Section 1]).

(1) $v \in V$ is a maximal spanning vector for (π, G, V);
(2) $v \otimes v \in V \otimes V$ is a cyclic vector for the representation $(\pi \otimes \pi^*, G, V \otimes V)$;
(3) $\mathfrak{c}_{v,v} \in C_\pi$ is a cyclic vector for the representation (ad, G, C_π).

Here π^* is the dual representation of π, $\mathrm{ad} : G \to \mathbf{U}(C_\pi)$ is the adjoint representation defined by $\mathrm{ad}(g)(f)(x) = f(g^{-1}xg)$ for $g \in G$ and $f \in C_\pi$.

Using the above equivalences, it is shown in [6] that unramified irreducible representations admit maximal spanning vectors and do phase retrieval. Moreover, for ramified representations, it provides a method to find maximal spanning vectors once we have an explicit description of (ad, G, C_π). Then one proves that the special representations and the cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$ admit maximal spanning vectors and do phase retrieval.

The method in [6] does not apply to the principal series representations of $\mathrm{GL}_2(\mathbb{F}_q)$, since these representations are ramified and we do not have an explicit decomposition of C_π (cf. [6, Remark 4.8]). On the other hand, the principal series representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are *ramified with small degree* in the sense that each irreducible component of $\pi \otimes \pi^*$ has multiplicity less than or equal to two. We may then circumvent the difficulties by a property on the independence of sesquilinear forms.

3. Quadratic forms and sesquilinear forms

We study the independence of quadratic forms and sesquilinear forms in this section. Let K be a field and $x_1, \ldots, x_n$ be variables. A *linear form* (resp. *quadratic form*) with respect to $x_1, \ldots, x_n$ is a homogeneous polynomial in $K[x_1, \ldots, x_n]$ of degree one (resp. degree two).

Lemma 3.1. *Let f_{11}, f_{12}, f_{21}, $f_{22} \in K[x_1, \ldots, x_n]$ be nontrivial quadratic forms. Suppose that*

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = 0.$$

Then at least one of the following conditions holds

(1) *there exists $k \in K^\times$ such that $f_{11} = kf_{21}$, $f_{12} = kf_{22}$;*
(2) *there exists $\ell \in K^\times$ such that $f_{11} = \ell f_{12}$, $f_{21} = \ell f_{22}$;*
(3) *there exist linear forms L_1, L_2, F_1, F_2 such that $f_{11} = F_1L_1$, $f_{12} = F_1L_2$, $f_{21} = F_2L_1$, $f_{22} = F_2L_2$.*

Proof. If one of f_{ij} $(i, j \in \{1, 2\})$ is irreducible, then condition (1) or condition (2) holds because $K[x_1, \ldots, x_n]$ is an unique factorization domain. If all the f_{ij} $(i, j \in \{1, 2\})$ are reducible and conditions (1) and (2) do not hold, then condition (3) holds. $\square$

Let V be an n-dimensional complex vector space, $\mathcal{S} : V \times V \to \mathbb{C}$ be a sesquilinear form on V. Fixing a basis $\{e_1, \ldots, e_n\}$ of V, we may identify V with $\mathbb{C}^n$ and then $\mathcal{S}$ corresponds to a matrix $S = (s_{ij})_{n\times n} \in M_n(\mathbb{C})$, where $s_{ij} = \mathcal{S}(e_i, e_j)$. Let $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$ be the coordinates of an element $v \in V$. Set

$$Q(v) := \mathcal{S}(v, v) = z^* S z = \sum_{i,j=1}^{n} s_{ij} z_i \bar{z}_j.$$

Note that we have the polarization identity

$$4\mathcal{S}(u,v) = Q(u+v) - Q(u-v) + \mathbf{i}Q(u+\mathbf{i}v) - \mathbf{i}Q(u-\mathbf{i}v)$$

for all $u, v \in V$, hence Q also determines $\mathcal{S}$.

Write $z_k = x_k + \mathbf{i}y_k$, where x_k is the real part of z_k and y_k is the imaginary part of z_k. Write $S = A + \mathbf{i}B$, where $A \in M_n(\mathbb{R})$ is the real part of S and $B \in M_n(\mathbb{R})$ is the imaginary part of S. Then

$$\begin{aligned} z^*Sz &= (x' - \mathbf{i}y')(A+\mathbf{i}B)(x+\mathbf{i}y) \\ &= x'Ax - x'By + y'Ay + y'Bx + \mathbf{i}(x'Ay + x'Bx - y'Ax + y'By) \\ &= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} A+\mathbf{i}B & -B+\mathbf{i}A \\ B-\mathbf{i}A & A+\mathbf{i}B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3.1) \\ &= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} S & \mathbf{i}S \\ -\mathbf{i}S & S \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

If we view the x_k and y_k as indeterminates, then z^*Sz is a quadratic form in $\mathbb{C}[x_i, y_i : 1 \le i \le n]$. We call it the quadratic form associated with S and denote it by q_S.

Lemma 3.2. *The quadratic form q_S is nontrivial and reducible if and only if S has rank one.*

Proof. If S has rank one, say $S = (s_1 \ \cdots \ s_n)'(t_1 \ \cdots \ t_n)$, where $(s_1 \ \cdots \ s_n)'$, $(t_1 \ \cdots \ t_n)' \in \mathbb{C}^n$ are nonzero vectors, then

$$\begin{aligned} z^*Sz &= z^*(s_1 \ \cdots \ s_n)'(t_1 \ \cdots \ t_n)z \\ &= \left(\sum_{1\le k\le n} s_k\overline{z}_k\right)\left(\sum_{1\le k\le n} t_k z_k\right) \\ &= \left(\sum_{1\le k\le n} s_k x_k - \sum_{1\le k\le n} \mathbf{i}s_k y_k\right)\left(\sum_{1\le k\le n} t_k x_k + \sum_{1\le k\le n} \mathbf{i}t_k y_k\right). \end{aligned}$$

The quadratic form q_S is nontrivial and reducible. On the other hand, assume that q_S is nontrivial and reducible, say

$$\begin{aligned} q_S &= \left(\sum_{1\le k\le n} a_k x_k + \sum_{1\le k\le n} b_k y_k\right)\left(\sum_{1\le k\le n} c_k x_k + \sum_{1\le k\le n} d_k y_k\right) \\ &= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \gamma' & \delta' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

where $\alpha = (a_1 \ \cdots \ a_n)'$, $\beta = (b_1 \ \cdots \ b_n)'$, $\gamma = (c_1 \ \cdots \ c_n)'$, $\delta = (d_1 \ \cdots \ d_n)' \in \mathbb{C}^n$ and $x = (x_1 \ \cdots \ x_n)'$, $y = (y_1 \ \cdots \ y_n)'$. Then

$$\begin{pmatrix} S & \mathbf{i}S \\ -\mathbf{i}S & S \end{pmatrix} + \begin{pmatrix} S & \mathbf{i}S \\ -\mathbf{i}S & S \end{pmatrix}' = \begin{pmatrix} \alpha\gamma' & \alpha\delta' \\ \beta\gamma' & \beta\delta' \end{pmatrix} + \begin{pmatrix} \alpha\gamma' & \alpha\delta' \\ \beta\gamma' & \beta\delta' \end{pmatrix}'.$$

Comparing the entries, we have identities

$$\begin{cases} S+S' &= \alpha\gamma' + \gamma\alpha', \\ \mathbf{i}(S'-S) &= \beta\gamma' + \delta\alpha', \\ \mathbf{i}(S-S') &= \alpha\delta' + \gamma\beta', \\ S+S' &= \beta\delta' + \delta\beta'. \end{cases}$$

From the first two identities, we have

$$2S = \alpha\gamma' + \gamma\alpha' + \mathbf{i}(\beta\gamma' + \delta\alpha'),\ 2S' = \alpha\gamma' + \gamma\alpha' - \mathbf{i}(\beta\gamma' + \delta\alpha').$$

From the last two identities, we have

$$2S = \beta\delta' + \delta\beta' - \mathbf{i}(\alpha\delta' + \gamma\beta'),\ 2S' = \beta\delta' + \delta\beta' + \mathbf{i}(\alpha\delta' + \gamma\beta').$$

Therefore,

$$\begin{aligned} 0 = 2S - 2S &= \alpha\gamma' + \gamma\alpha' + \mathbf{i}(\beta\gamma' + \delta\alpha') - (\beta\delta' + \delta\beta' - \mathbf{i}(\alpha\delta' + \gamma\beta')) \\ &= (\alpha + \mathbf{i}\beta)(\gamma + \mathbf{i}\delta)' + (\gamma + \mathbf{i}\delta)(\alpha + \mathbf{i}\beta)' \end{aligned}$$

and

$$\begin{aligned} 0 = 2S' - 2S' &= \alpha\gamma' + \gamma\alpha' - \mathbf{i}(\beta\gamma' + \delta\alpha') - (\beta\delta' + \delta\beta' + \mathbf{i}(\alpha\delta' + \gamma\beta')) \\ &= (\alpha - \mathbf{i}\beta)(\gamma - \mathbf{i}\delta)' + (\gamma - \mathbf{i}\delta)(\alpha - \mathbf{i}\beta)'. \end{aligned}$$

Note that if x and $y \in \mathbb{C}^n$ are column vectors and $0 = xy' + yx'$, then at least one of x and y is the zero vector. Since q_S is nontrivial, at least one of $\alpha + \mathbf{i}\beta$ and $\alpha - \mathbf{i}\beta$ is nonzero and at least one of $\gamma + \mathbf{i}\delta$ and $\gamma - \mathbf{i}\delta$ is nonzero. Without loss of generality, we may then assume that

$$\alpha - \mathbf{i}\beta = \gamma + \mathbf{i}\delta = 0,\ \alpha + \mathbf{i}\beta \neq 0,\ \gamma - \mathbf{i}\delta \neq 0.$$

Hence

$$\begin{aligned} &\sum_{1\le k\le n} a_k x_k + \sum_{1\le k\le n} b_k y_k \\ =& \sum_{1\le k\le n} a_k \frac{1}{2}(z_k + \overline{z}_k) + \sum_{1\le k\le n} b_k \frac{\mathbf{i}}{2}(-z_k + \overline{z}_k) \\ =& \sum_{1\le k\le n} (\frac{1}{2}a_k + \frac{\mathbf{i}}{2}b_k)\overline{z}_k. \end{aligned}$$

Similarly,

$$\sum_{1\le k\le n} c_k x_k + \sum_{1\le k\le n} d_k y_k = \sum_{1\le k\le n} (\frac{1}{2}c_k - \frac{\mathbf{i}}{2}d_k) z_k.$$

Therefore, $S = (\frac{1}{2}\alpha + \frac{\mathbf{i}}{2}\beta)(\frac{1}{2}\gamma - \frac{\mathbf{i}}{2}\delta)'$ has rank one. This completes the proof. □

Lemma 3.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be sesquilinear forms with associated matrices S and T respectively. If there exists a constant $c \in \mathbb{C}^\times$ with $q_S = cq_T$, then $S = cT$ and $\mathcal{S} = c\mathcal{T}$.*

Proof. The assumption $q_S = cq_T$ is equivalent to

$$\begin{pmatrix} S & \mathbf{i}S \\ -\mathbf{i}S & S \end{pmatrix} + \begin{pmatrix} S & \mathbf{i}S \\ -\mathbf{i}S & S \end{pmatrix}' = c\begin{pmatrix} T & \mathbf{i}T \\ -\mathbf{i}T & T \end{pmatrix} + c\begin{pmatrix} T & \mathbf{i}T \\ -\mathbf{i}T & T \end{pmatrix}'.$$

Therefore

$$S + S' = c(T + T'), \quad S - S' = c(T - T').$$

The lemma follows. □

Proposition 3.4. *Let $f_{ij}(z) = z^* S_{ij} z$ ($i,\ j \in \{1,\ 2\}$) be sesquilinear forms on $\mathbb{C}^n$. Suppose that for any $z \in \mathbb{C}^n$,*

$$\begin{vmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z) \end{vmatrix} = 0.$$

Then at least one of the following conditions holds

1. *there exists $k \in \mathbb{C}^\times$ such that $f_{11} = k f_{21}$, $f_{12} = k f_{22}$;*
2. *there exists $\ell \in \mathbb{C}^\times$ such that $f_{11} = \ell f_{12}$, $f_{21} = \ell f_{22}$;*
3. *there exist linear forms $L_1,\ L_2,\ F_1,\ F_2$ such that $f_{11} = F_1 \overline{L}_1$, $f_{12} = F_1 \overline{L}_2$, $f_{21} = F_2 \overline{L}_1$, $f_{22} = F_2 \overline{L}_2$. Here $\overline{L}_i(z) = \overline{L_i(z)}$.*

Proof. From equation (3.1), for $i,\ j \in \{1,\ 2\}$, we may write

$$f_{ij}(z) = r_{ij}(x, y) + \mathbf{i} t_{ij}(x, y),$$

where r_{ij} and t_{ij} are real quadratic forms with respect to the real and imaginary parts of z. From the assumption, for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{vmatrix} r_{11}(x, y) + \mathbf{i} t_{11}(x, y) & r_{12}(x, y) + \mathbf{i} t_{12}(x, y) \\ r_{21}(x, y) + \mathbf{i} t_{21}(x, y) & r_{22}(x, y) + \mathbf{i} t_{22}(x, y) \end{vmatrix} = 0.$$

Therefore, $r_{11}r_{22} - t_{11}t_{22} - r_{12}r_{21} + t_{12}t_{21}$ and $r_{11}t_{22} + t_{11}r_{22} - r_{12}t_{21} - r_{21}t_{12}$ are zero polynomials with respect to x and y. Hence, as complex quadratic forms, the polynomials $q_{S_{11}}(x, y)$, $q_{S_{12}}(x, y)$, $q_{S_{21}}(x, y)$, $q_{S_{22}}(x, y)$ satisfy

$$\begin{vmatrix} q_{S_{11}}(x, y) & q_{S_{12}}(x, y) \\ q_{S_{21}}(x, y) & q_{S_{22}}(x, y) \end{vmatrix} = 0.$$

The proposition then follows form Lemmas 3.1, 3.2, 3.3. □

4. The maximal spanning vectors

4.1. **Explicit decomposition of a representation.** Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. Let W_i ($1 \le i \le h$) be the irreducible representations of G with $\mathrm{Hom}_G(W_i, V)$ nontrivial. Denote by V_i the direct sum of irreducible components of V which are isomorphic to W_i ($1 \le i \le h$). Then we have the canonical decomposition

$$V = V_1 \oplus \cdots \oplus V_h.$$

Following [14, Section 2.7], we describe an explicit decomposition of V_i into a direct sum of subrepresentations isomorphic to W_i. Let W_i be given in matrix form $(r_{\alpha\beta}(s))$ with respect to a basis of W_i. Let $n = \dim W_i$. For each pair of integers $1 \le \alpha,\ \beta \le n$, let $p_{\alpha\beta}$ denote the linear map of V into V defined by

$$p_{\alpha\beta} = \frac{n}{|G|} \sum_{s \in G} r_{\beta\alpha}(s^{-1}) \rho(s).$$

The following result is [14, Section 2.7, Proposition 8]. We state it here for the convenience of the readers.

Proposition 4.1. *With the notation as above, the following statements hold.*

(1) *The map $p_{\alpha\alpha}$ is a projection; it is zero on the V_j, $j \neq i$. Its image $V_{i,\alpha}$ is contained in V_i, and V_i is the direct sum of the $V_{i,\alpha}$ for $1 \leq \alpha \leq n$. The map $p_i = \sum_\alpha p_{\alpha\alpha}$ is the projection of V onto V_i.*

(2) *The linear map $p_{\alpha\beta}$ is zero on V_j, $j \neq i$, as well as on the $V_{i,\gamma}$ for $\gamma \neq \beta$; it defines an isomorphism from $V_{i,\beta}$ onto $V_{i,\alpha}$.*

(3) *Let x_1 be a nonzero element of $V_{i,1}$ and let $x_\alpha = p_{\alpha 1}(x_1) \in V_{i,\alpha}$. The x_α are linearly independent and generate a vector space $W(x_1)$ stable under G and of dimension n. For each $s \in G$, we have*

$$\rho(s)(x_\alpha) = \sum_\beta r_{\beta\alpha}(s) x_\beta.$$

In particular, $W(x_1)$ is isomorphic to W_i.

(4) *If $(x_1^{(1)}, \ldots, x_1^{(m)})$ is a basis of $V_{i,1}$, the representation V_i is the direct sum of the subrepresentations $W(x_1^{(1)}),\ \ldots,\ W(x_1^{(m)})$.*

4.2. **The cyclic vectors of $\pi \otimes \pi^*$.** Let $\pi : G \to \mathbf{U}(\mathbb{C}^d)$ be an irreducible representation of G with degree $d \geq 2$. Let $\rho := \pi \otimes \pi^* : G \to \mathbf{U}(\mathbb{C}^d \otimes \mathbb{C}^d)$. We identify $\mathbb{C}^d \otimes \mathbb{C}^d$ with $V := M_d(\mathbb{C})$, then $\rho : G \to \mathbf{U}(M_d(\mathbb{C}))$ is given by

$$\rho(g)M = \pi(g) M \pi(g)^*,$$

for $M \in M_d(\mathbb{C})$. Let $V = V_1 \oplus \cdots \oplus V_h$ be the canonical decomposition of ρ as in Section 4.1. Assume that $V_i \cong W_i^{\oplus m_i}$ for $1 \leq i \leq h$. Moreover, assume that $m_i \geq 2$ for $1 \leq i \leq \ell$, $m_i = 1$ for $\ell + 1 \leq i \leq h$. Let $d_i = \dim W_i$. Then $m_i \leq d_i$ for all $1 \leq i \leq h$ and $d^2 = \sum_i m_i d_i$.

We apply Proposition 4.1 to $(\rho = \pi \otimes \pi^*, G, V = M_d(\mathbb{C}))$ to construct a decomposition of V_i ($1 \leq i \leq \ell$). For each V_i, fix a basis $(A_{i,1}^{(1)}, \ldots, A_{i,1}^{(m_i)})$ of $V_{i,1}$. Let $A_{i,\alpha}^{(j)} = p_{\alpha 1}(A_{i,1}^{(j)})$ ($1 \leq \alpha \leq d_i$, $1 \leq j \leq m_i$). We remark that the projections $p_{\alpha 1}$ depend on i, we omit the subscript i to ease the notation. Let

$$W_i(A_{i,1}^{(j)}) = \mathrm{Span}\{A_{i,\alpha}^{(j)} :\ 1 \leq \alpha \leq d_i\},\ 1 \leq j \leq m_i.$$

Then $W_i(A_{i,1}^{(j)})$ is stable under G and isomorphic to W_i. We have a decomposition

$$V_i = W_i(A_{i,1}^{(1)}) \oplus \cdots \oplus W_i(A_{i,1}^{(m_i)}).$$

Moreover, by Proposition 4.1(3), the map

$$\begin{aligned} W_i(A_{i,1}^{(j)}) &\to W_i(A_{i,1}^{(1)}) \\ A_{i,\alpha}^{(j)} &\mapsto A_{i,\alpha}^{(1)}, \quad 1 \leq \alpha \leq d_i \end{aligned} \tag{4.1}$$

is an isomorphism of G-representations. Applying [6, Propositions 2.3, 2.5], we have the following proposition.

Proposition 4.2. *Let $A \in M_d(\mathbb{C})$ be a nontrivial matrix. Then A is a cyclic vector for the representation $(\rho = \pi \otimes \pi^*, G, V = M_d(\mathbb{C}))$ if and only if the following two conditions are satisfied.*

(1) *For $\ell + 1 \leq i \leq h$, the projection of A in V_i is nontrivial.*

(2) *For $1 \leq i \leq \ell$, the matrix*

$$\mathfrak{P}_i = \begin{pmatrix} \langle A, A_{i,1}^{(1)} \rangle & \langle A, A_{i,2}^{(1)} \rangle & \cdots & \langle A, A_{i,d_i}^{(1)} \rangle \\ \langle A, A_{i,1}^{(2)} \rangle & \langle A, A_{i,2}^{(2)} \rangle & \cdots & \langle A, A_{i,d_i}^{(2)} \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle A, A_{i,1}^{(m_i)} \rangle & \langle A, A_{i,2}^{(m_i)} \rangle & \cdots & \langle A, A_{i,d_i}^{(m_i)} \rangle \end{pmatrix}$$

has rank m_i.

In particular, a column vector $v \in \mathbb{C}^d$ is maximal spanning for $(\pi, G, \mathbb{C}^d)$ if and only if the following two conditions are satisfied.

(1) *For $\ell + 1 \leq i \leq h$, the projection of vv^* in V_i is nontrivial.*
(2) *For $1 \leq i \leq \ell$, the matrix*

$$\mathfrak{P}_i = \begin{pmatrix} v^* B_{i,1}^{(1)} v & v^* B_{i,2}^{(1)} v & \cdots & v^* B_{i,d_i}^{(1)} v \\ v^* B_{i,1}^{(2)} v & v^* B_{i,2}^{(2)} v & \cdots & v^* B_{i,d_i}^{(2)} v \\ \cdots & \cdots & \cdots & \cdots \\ v^* B_{i,1}^{(m_i)} v & v^* B_{i,2}^{(m_i)} v & \cdots & v^* B_{i,d_i}^{(m_i)} v \end{pmatrix}$$

has rank m_i. Here $B_{i,\alpha}^{(j)}$ is the conjugate transpose of $A_{i,\alpha}^{(j)}$.

Proof. From [6, Propositions 2.3], A is a cyclic vector for the representation $(\rho = \pi \otimes \pi^*, G, V = M_d(\mathbb{C}))$ if and only if the projection of A into V_i is a cyclic vector for (ρ, G, V_i). The first statement then follows from [6, Propositions 2.5] and the fact that equation (4.1) is an isomorphism of G-representations.

From the equivalent conditions in Section 2, a column vector $v \in \mathbb{C}^d$ is maximal spanning for $(\pi, G, \mathbb{C}^d)$ if and only if vv^* is a cyclic vector for $(\rho = \pi \otimes \pi^*, G, V = M_d(\mathbb{C}))$. Then the second statement holds as

$$\langle vv^*, M \rangle = \mathrm{Tr}(M^* vv^*) = \mathrm{Tr}(v^* M^* v) = \langle v, Mv \rangle \tag{4.2}$$

for $v \in \mathbb{C}^d$ and $M \in M_d(\mathbb{C})$. $\square$

For $\ell + 1 \leq i \leq h$, let C_i be the set of vectors $v \in \mathbb{C}^d$ such that the projection of vv^* in V_i is nontrivial. It is easy to see that C_i $(\ell + 1 \leq i \leq h)$ is open dense in $\mathbb{C}^d$. Indeed, the complement of C_i is a finite union of the zero set of nontrivial polynomials on the real and imaginary parts of the coordinates of v (cf. equations (3.1), (4.2), and proof of [6, Theorem 1.1]).

For $1 \leq i \leq \ell$, let C_i be the set of vectors $v \in \mathbb{C}^d$ such that $\mathrm{rank}(\mathfrak{P}_i) = m_i$. Note that $\mathrm{rank}(\mathfrak{P}_i) < m_i$ is a closed condition given by polynomials on the real and imaginary parts of the coordinates of v, C_i $(1 \leq i \leq \ell)$ is open dense in $\mathbb{C}^d$ if it is nonempty. Because the set of maximal spanning vectors for $(\pi, G, \mathbb{C}^d)$ is the intersection $\cap_{1 \leq i \leq h} C_i$, it is open dense in $\mathbb{C}^d$ if it is nonempty.

In general it is not easy to check the condition $\mathrm{rank}(\mathfrak{P}_i) = m_i$ because we do not have enough information on $A_{i,\alpha}^{(j)}$. In [6], for $G = \mathrm{GA}(1, q)$, this is done by constructing an explicit set of matrices $A_{i,\alpha}^{(j)}$ (cf. the matrix $\mathfrak{P}$ in [6, Section 4]). In the following, applying Proposition 3.4, we check the condition $\mathrm{rank}(\mathfrak{P}_i) = m_i$ under the assumption that $m_i = 2$ for $1 \leq i \leq \ell$.

Lemma 4.3. *Fix $1 \le i \le \ell$. If $m_i = 2$, then C_i is open dense in $\mathbb{C}^d$.*

Proof. As explained above, to show that C_i is open dense in $\mathbb{C}^d$, it suffices to show that there exists one column vector $v \in \mathbb{C}^d$ such that $\mathrm{rank}(\mathfrak{P}_i) = 2$. Suppose otherwise, for all $v \in \mathbb{C}^d$, we have $\mathrm{rank}(\mathfrak{P}_i) \le 1$. Since the matrices $A^{(j)}_{i,\alpha}$ are linearly independent, by Proposition 3.4, there exist linear forms $F_1,\ F_2,\ L_1,\ \ldots,\ L_{d_i}$ with

$$v^* B^{(j)}_{i,\alpha} v = F_\alpha(v)\overline{L_j(v)},\ \alpha = 1,\ 2,\ 1 \le j \le d_i.$$

In other words, there exist column vectors $u_1,\ u_2,\ w_1,\ \ldots,\ w_{d_i} \in \mathbb{C}^d$ with

$$A^{(j)}_{i,\alpha} = u_\alpha w_j^*,\ \alpha = 1,\ 2,\ 1 \le j \le d_i.$$

Since for each α, $\mathrm{Span}\{A^{(j)}_{i,\alpha} :\ 1 \le j \le d_i\}$ is G-stable under the action $\rho(g)M = \pi(g)M\pi(g)^*$, therefore

$$(\pi(g)u_\alpha)(\pi(g)w_j)^* = \rho(g)(u_\alpha w_j^*) \in \mathrm{Span}\{A^{(j)}_{i,\alpha} = u_\alpha w_j^* :\ 1 \le j \le d_i\}.$$

Hence u_α is a common eigenvector for $\pi(g)$ $(g \in G)$ and $\mathbb{C}\langle u_\alpha\rangle \subset \mathbb{C}^d$ is a subrepresentation of π, which contradicts to the irreducibility of π. The lemma follows. □

Combining the above discussion, we obtain the following result, which generalizes [6, Theorem 1.1].

Theorem 4.4. *Let $\pi : G \to \mathbf{U}(V)$ be an irreducible representation of G. If each irreducible component of $(\pi\otimes\pi^*, G, V\otimes V)$ has multiplicity one or two, then (π, G, V) admits maximal spanning vectors and the set of maximal spanning vectors is open dense in V.*

If (π, G, V) is an irreducible representation of G with relatively large (resp. small) dimension among the irreducible representations of G, then (π, G, V) is likely to be more ramified (resp. unramified). In particular, for the symmetric group S_n with $n \le 8$, one could see this from the table in [12, Appendix I.A and I.I]. One could also make a list of representations for S_n with $n \le 8$ that satisfy the condition in Theorem 4.4.

Corollary 4.5. *Let $G = S_n$ be the symmetric group with $n \le 5$. Every irreducible representation of S_n admits maximal spanning vectors, hence does phase retrieval.*

5. Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

We check that the principal series representations of $\mathrm{GL}_2(\mathbb{F}_q)$ satisfy the conditions in Theorem 4.4 and then obtain Theorem 1.2. We first review the basic properties of the group $\mathrm{GL}_2(\mathbb{F}_q)$ and its representations. The readers may find more details in the literatures, for example [3, Section 6] and [10, Section 5.2].

Since $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$, this case follows from Corollary 4.5. See also [7, Section 4.3.2] for explicit computation. We assume that $q \ge 3$ in the following. We have $|\,\mathrm{GL}_2(\mathbb{F}_q)| = (q-1)^2 q(q+1)$. Let $\epsilon \in \mathbb{F}_q^\times$ be a non-square element. There are four types of conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$. We collect the information in the following table (cf. [10, Section 5.2]).

There are four types of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

(1) The one-dimensional representations U_α, induced by $\det : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{F}_q^\times$ and characters $\alpha : \mathbb{F}_q^\times \to \mathbb{C}^\times$. There are $(q-1)$ one-dimensional representations.

TABLE 1. Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$

Representative	No. of elements in each class	No. of classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1	$q-1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	q^2-1	$q-1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $x \neq y$	q^2+q	$(q-1)(q-2)/2$
$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$, $y \neq 0$	q^2-q	$q(q-1)/2$

(2) The q-dimensional representations V_α (the **special representations**). Consider the permutation representation of $\mathrm{GL}_2(\mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$, which has dimension $q+1$. It contains the trivial representation, let V be the complementary q-dimensional representation. Then V is irreducible. Define $V_\alpha = V \otimes U_\alpha$. There are $(q-1)$ special representations.

(3) The $(q+1)$-dimensional representations $W_{\alpha,\beta}$ (the **principal series representations**). Let $B \subset \mathrm{GL}_2(\mathbb{F}_q)$ be the subset of upper triangular matrices. For each pair α, β of characters of $\mathbb{F}_q^\times$, let $\alpha \otimes \beta : B \to \mathbb{C}^\times$ be the character that takes $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to $\alpha(a)\beta(d)$. Let $W_{\alpha,\beta}$ be the induced representation $\mathrm{Ind}_B^G \alpha \otimes \beta$. It is irreducible if $\alpha \neq \beta$ and $W_{\alpha,\beta} \cong W_{\beta,\alpha}$. There are $\frac{1}{2}(q-1)(q-2)$ principal series representations. If $\alpha = \beta$, then $W_{\alpha,\alpha} \cong U_\alpha \oplus V_\alpha$.

(4) The $(q-1)$-dimensional representations X_φ (the **cuspidal representations**). Let $\varphi : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ be a character of the multiplicative group of the quadratic extension of $\mathbb{F}_q$ and $\varphi \neq \varphi^q$. There exists an irreducible $(q-1)$-dimensional representation X_φ inside $\mathrm{Ind}_{\mathbb{F}_{q^2}^\times}^G \varphi$ and $X_\varphi = X_{\varphi^q}$. There are $\frac{1}{2}q(q-1)$ cuspidal representations.

For the construction of cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$, we identify $\mathbb{F}_{q^2}^\times$ with a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ by mapping $x + y\sqrt{\epsilon}$ to $d_{x,y} \in \mathrm{GL}_2(\mathbb{F}_q)$. The character table of $\mathrm{GL}_2(\mathbb{F}_q)$ is as follows (cf. [10, Section 5.2]).

TABLE 2. Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

$\mathrm{GL}_2(\mathbb{F}_q)$	a_x	b_x	$c_{x,y}$	$d_{x,y} (= \zeta \in \mathbb{F}_{q^2}^\times)$
Size of class	1	q^2-1	q^2+q	q^2-q
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\zeta^{q+1})$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\zeta^{q+1})$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta)+\varphi(\zeta^q))$

Theorem 5.1. *Let $\mathbb{F}_q$ be the finite field with q elements. Every irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ admits maximal spanning vectors, hence does phase retrieval.*

Proof. By [6, Corollary 1.3], cuspidal representations and special representations of $\mathrm{GL}_2(\mathbb{F}_q)$ admit maximal spanning vectors and we need to show that every principal series representation of $\mathrm{GL}_2(\mathbb{F}_q)$ admits maximal spanning vectors, hence does phase retrieval. Let $\pi = W_{\alpha,\beta}$ be a principal series representation of $\mathrm{GL}_2(\mathbb{F}_q)$. Since $W_{\alpha,\beta} \cong U_\alpha \otimes W_{1,\alpha^{-1}\beta}$ and $W_{\alpha,\beta} \otimes W^*_{\alpha,\beta} \cong W_{1,\alpha^{-1}\beta} \otimes W^*_{1,\alpha^{-1}\beta}$, we may assume that $\pi = W_{1,\beta}$ and β is not trivial. By Theorem 4.4, it suffices to show that each irreducible component of $\rho := W_{1,\beta} \otimes W^*_{1,\beta}$ has multiplicity less than or equal to two. This is a straightforward verification from the character table. In the following, we give details for the case q odd.

On the four types of conjugacy classes a_x, b_x, $c_{x,y}$, $d_{x,y}$, the character χ_ρ of ρ takes value

$$(q+1)^2,\ 1,\ 2+\beta(xy^{-1})+\beta(x^{-1}y),\ 0,\ \text{respectively.}$$

We compute the inner product $\langle \chi_X, \chi_\rho\rangle$ for each irreducible character χ_X and then

$$\dim \mathrm{Hom}(X,\rho) = \frac{1}{|\mathrm{GL}_2(\mathbb{F}_q)|}\langle \chi_X, \chi_\rho\rangle = \frac{1}{(q-1)^2 q(q+1)}\langle \chi_X, \chi_\rho\rangle.$$

Assume first that $\beta^2 \neq 1$.

(1) For U_α, we have

$$\begin{aligned}
&\langle \chi_{U_\alpha}, \chi_\rho\rangle \\
&= \sum_{x\in\mathbb{F}_q^\times} \alpha(x^2)(q+1)^2 + \sum_{x\in\mathbb{F}_q^\times} \alpha(x^2)(q^2-1) \\
&\quad + \frac{1}{2}\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}} \alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y))(q^2+q) + 0 \\
&=(q+1)^2 \sum_{x\in\mathbb{F}_q^\times}\alpha(x^2) + (q^2-1)\sum_{x\in\mathbb{F}_q^\times}\alpha(x^2) \\
&\quad + \frac{1}{2}(q^2+q)\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}} \alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y)) \\
&= \begin{cases} (q-1)^2 q(q+1) & \text{if } \alpha = 1, \\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{5.1}$$

Here the last "=" follows from

(5.2)
$$
\begin{aligned}
&\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}} \alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y))\\
=&\sum_{x,y\in\mathbb{F}_q^\times} \alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y)) - \sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x=y}} \alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y))\\
=&\sum_{x,y\in\mathbb{F}_q^\times} (2\alpha(xy)+\alpha\beta(x)\alpha\beta^{-1}(y)+\alpha\beta^{-1}(x)\alpha\beta(y)) - 4\sum_{x\in\mathbb{F}_q^\times}\alpha(x^2)\\
=&2(\sum_{x\in\mathbb{F}_q^\times}\alpha(x))^2+\sum_{x\in\mathbb{F}_q^\times}\alpha\beta(x)\sum_{y\in\mathbb{F}_q^\times}\alpha\beta^{-1}(y)+\sum_{x\in\mathbb{F}_q^\times}\alpha\beta^{-1}(x)\sum_{y\in\mathbb{F}_q^\times}\alpha\beta(y)-4\sum_{x\in\mathbb{F}_q^\times}\alpha(x^2)\\
=&2(\sum_{x\in\mathbb{F}_q^\times}\alpha(x))^2-4\sum_{x\in\mathbb{F}_q^\times}\alpha(x^2).
\end{aligned}
$$

One may also obtain equation (5.1) from

$$\langle\chi_{U_\alpha},\chi_\rho\rangle=\sum_{g\in\mathrm{GL}_2(\mathbb{F}_q)}\chi_{U_\alpha}\chi_{W_{1,\beta}}\overline{\chi_{W_{1,\beta}}}(g)=|\,\mathrm{GL}_2(\mathbb{F}_q)|\dim\mathrm{Hom}(U_\alpha\otimes W_{1,\beta},W_{1,\beta}).$$

We compute equation (5.2) as it appears several times in the following.

(2) For V_α, we have

$$
\begin{aligned}
&\langle\chi_{V_\alpha},\chi_\rho\rangle\\
=&\sum_{x\in\mathbb{F}_q^\times} q\alpha(x^2)(q+1)^2+0+\frac{1}{2}\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}}\alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y))(q^2+q)+0\\
=&\begin{cases} 2(q-1)^2q(q+1) & \text{if } \alpha=1,\\ (q-1)^2q(q+1) & \text{if } \alpha\neq1,\ \text{but } \alpha^2=1,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

(3) For W_{α_1,α_2}, we have

$$\begin{aligned}
&\langle \chi_{W_{\alpha_1,\alpha_2}}, \chi_\rho\rangle \\
=&\sum_{x\in\mathbb{F}_q^\times}(q+1)\alpha_1(x)\alpha_2(x)(q+1)^2+\sum_{x\in\mathbb{F}_q^\times}\alpha_1(x)\alpha_2(x)(q^2-1)\\
&+\frac{1}{2}\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}}(\alpha_1(x)\alpha_2(y)+\alpha_1(y)\alpha_2(x))(2+\beta(xy^{-1})+\beta(x^{-1}y))(q^2+q)+0\\
=&((q+1)^3+q^2-1)\sum_{x\in\mathbb{F}_q^\times}\alpha_1\alpha_2(x)\\
&+\frac{1}{2}\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}}(\alpha_1\alpha_2(x)\alpha_2(x^{-1}y)+\alpha_1\alpha_2(x)\alpha_1(x^{-1}y))(2+\beta(xy^{-1})+\beta(x^{-1}y))\\
=&\begin{cases}2(q-1)^2q(q+1) & \text{if } \alpha_1\alpha_2=1,\ \alpha_1=\beta \text{ or } \alpha_1=\beta^{-1},\\ (q-1)^2q(q+1) & \text{if } \alpha_1\alpha_2=1,\ \alpha_1\neq\beta \text{ and } \alpha_1\neq\beta^{-1},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

(4) For X_φ, we have

$$\begin{aligned}
&\langle \chi_{X_\varphi}, \chi_\rho\rangle \\
=&\sum_{x\in\mathbb{F}_q^\times}(q-1)\varphi(x)(q+1)^2+\sum_{x\in\mathbb{F}_q^\times}(-\varphi(x))(q^2-1)+0+0\\
=&\begin{cases}(q-1)^2q(q+1) & \text{if } \varphi|_{\mathbb{F}_q^\times}=1,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

From the above computation, we see that each irreducible component of $W_{1,\beta}\otimes W_{1,\beta}^*$ ($\beta^2\neq 1$) has multiplicity less than or equal to two. Moreover, V and $W_{\beta,\beta^{-1}}$ are the two irreducible components with multiplicity two. The verification for $W_{1,\beta}\otimes W_{1,\beta}^*$ ($\beta^2=1$) is similar.

(1) For U_α, we have

$$\begin{aligned}
&\langle \chi_{U_\alpha}, \chi_\rho\rangle \\
=&\sum_{x\in\mathbb{F}_q^\times}\alpha(x^2)(q+1)^2+\sum_{x\in\mathbb{F}_q^\times}\alpha(x^2)(q^2-1)\\
&+\frac{1}{2}\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}}\alpha(xy)(2+\beta(xy^{-1})+\beta(x^{-1}y))(q^2+q)+0\\
=&\begin{cases}(q-1)^2q(q+1) & \text{if } \alpha=1, \text{ or } \alpha=\beta,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

(2) For V_α, we have

$$\begin{aligned}
&\langle \chi_{V_\alpha}, \chi_\rho \rangle \\
=& \sum_{x \in \mathbb{F}_q^\times} q\alpha(x^2)(q+1)^2 + 0 + \frac{1}{2} \sum_{\substack{x,y \in \mathbb{F}_q^\times \\ x \neq y}} \alpha(xy)(2 + \beta(xy^{-1}) + \beta(x^{-1}y))(q^2+q) + 0 \\
=& \begin{cases} 2(q-1)^2 q(q+1) & \text{if } \alpha = 1, \\ 2(q-1)^2 q(q+1) & \text{if } \alpha \neq 1, \text{ but } \alpha^2 = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(3) For W_{α_1,α_2}, we have

$$\begin{aligned}
&\langle \chi_{W_{\alpha_1,\alpha_2}}, \chi_\rho \rangle \\
=& \sum_{x \in \mathbb{F}_q^\times} (q+1)\alpha_1(x)\alpha_2(x)(q+1)^2 + \sum_{x \in \mathbb{F}_q^\times} (q+1)\alpha_1(x)\alpha_2(x)(q^2-1) \\
&+ \frac{1}{2} \sum_{\substack{x,y \in \mathbb{F}_q^\times \\ x \neq y}} (\alpha_1(x)\alpha_2(y) + \alpha_1(y)\alpha_2(x))(2 + \beta(xy^{-1}) + \beta(x^{-1}y))(q^2+q) + 0 \\
=& \begin{cases} (q-1)^2 q(q+1) & \text{if } \alpha_1\alpha_2 = 1, \ \alpha_1 \neq \beta, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(4) For X_φ, we have

$$\begin{aligned}
&\langle \chi_{X_\varphi}, \chi_\rho \rangle \\
=& \sum_{x \in \mathbb{F}_q^\times} (q-1)\varphi(x)(q+1)^2 + \sum_{x \in \mathbb{F}_q^\times} (-\varphi(x))(q^2-1) + 0 + 0 \\
=& \begin{cases} (q-1)^2 q(q+1) & \text{if } \varphi|_{\mathbb{F}_q^\times} = 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For q even, the computation is almost the same, except that there are no characters of $\mathbb{F}_q^\times$ with order two. This completes the proof of the theorem. □

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