

SOME RESULTS ON GALOIS REPRESENTATIONS ATTACHED TO MOTIVES OF GL_2 -TYPE

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Abstract: In this paper, we study the structure of $End(M)$ where M is a motive of GL_2 -type over a totally real field. Under some standard conjectures for motives, we then prove some results about Galois representations attached to such motives.

Keywords: Galois representation, motives of GL_2 -type.

1. Introduction

In this paper, a motive always means a motive for absolute Hodge cycles. All the terminology about motives used in this paper will be introduced in Section 2. (See Section 2 for some basic properties and [6] for details.)

Let K be a number field. An abelian variety A over K is of GL_2 -type if the \mathbb{Q} -algebra $End_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a number field of degree $\dim(A)$ over \mathbb{Q} . In [7], the author investigates abelian varieties of GL_2 -type and proves that all abelian varieties of GL_2 -type over \mathbb{Q} are modular under Serre's conjecture on mod p Galois representations of $G_{\mathbb{Q}}$. In [12], the author proves a similar modularity result for motives of GL_2 -type over \mathbb{Q} . In this paper, we continue the work in [12] and prove several results on Galois representations attached to motives of GL_2 -type over totally real fields. As in [12], some of the arguments in this paper are identical to those in [7] and [8].

Let M be a motive over a number field K . We say that M is of GL_2 -type if (1) M is pure of weight w with Hodge type $(p_{\sigma}, q_{\sigma}) + (q_{\sigma}, p_{\sigma})$ ($p_{\sigma} > q_{\sigma}$) with respect to $\sigma : K \rightarrow \mathbb{C}$, and w is equal to the dimension of the underlying variety of M if w is even; (2) there exists a number field E such that $E \hookrightarrow End_K(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $[E : \mathbb{Q}] = \frac{1}{2} \text{rank}(M)$. By functoriality, E acts on $H_B(M)$ which is a \mathbb{Q} -vector space of dimension $\text{rank}(M)$.

In this paper, we always assume that K is a totally real field. We have the following theorem.

Theorem 1.1. *Let M be a motive over K of GL_2 -type. Then $M_{/K}$ is simple if and only if $E = \text{End}_K(M) \otimes \mathbb{Q}$.*

Let N be a motive over $\bar{\mathbb{Q}}$ which is pure of weight. We say that N has complex multiplication if it is the product of motives N_i such that $\text{End}_{\bar{\mathbb{Q}}}(N_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a number field of degree $\text{rank}(N_i)$ over \mathbb{Q} .

We have the following theorem. The case when $K = \mathbb{Q}$ is proved in Theorem 4.3 of [12].

Theorem 1.2. *Let M be a simple motive over K of GL_2 -type such that $M_{\bar{\mathbb{Q}}} = M \times_K \bar{\mathbb{Q}}$ does not contain submotives with complex multiplication. Let $E \hookrightarrow \text{End}_K(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $[E : \mathbb{Q}] = \frac{1}{2} \text{rank}(M)$. Then $\mathfrak{D} := \text{End}_{\bar{\mathbb{Q}}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a central simple algebra over a totally real field $L \subset E$, the Schur index of \mathfrak{D} is 1 or 2, and \mathfrak{D} contains E as a maximal subfield.*

Let M be a simple motive of GL_2 -type over K which is pure of weight w . Assume that $E = \text{End}_K(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since M has a polarization (see Section 6 of [4]), we see that E is a totally real or a CM field. Let $V_l = H_l(M)$ be the l -adic realization of M for each prime number l and E_λ be the completion of E at a finite place $\lambda \mid l$ of E . By functoriality, E acts on V_l which is a \mathbb{Q}_l -vector space of dimension $\text{rank}(M)$. Then V_l is free of rank 2 over $E \otimes \mathbb{Q}_l$, $V_\lambda := V_l \otimes_{E \otimes \mathbb{Q}_l} E_\lambda$ is a vector space of dimension 2 over E_λ . Therefore, we get a family of rank 2 Galois representations:

$$\rho_\lambda : G_K \rightarrow GL_{E \otimes \mathbb{Q}_l}(V_l) \cong GL_2(E \otimes \mathbb{Q}_l) \rightarrow GL_2(E_\lambda),$$

where $G_K = \text{Gal}(\bar{\mathbb{Q}}/K)$ is the absolute Galois group of K .

For each finite place λ of E , let $\delta_\lambda = \det(\rho_\lambda) : G_K \rightarrow E_\lambda^\times$. Let χ_l be the l -adic cyclotomic character. Fix embeddings $E \hookrightarrow \mathbb{C}$ and $E_\lambda \hookrightarrow \mathbb{C}$. Then we have the following theorem. The corresponding results for motives over \mathbb{Q} are proved in Section 5 of [12].

Theorem 1.3.

- (1) *Let M be a simple motive of GL_2 -type over K which is pure of weight w . Then there is a character of finite order $\epsilon : G_K \rightarrow E_\lambda^\times$ such that $\delta_\lambda = \epsilon \chi_l^w$ for each λ . Furthermore, this character is unramified at each prime which is a prime of good reduction for M .*
- (2) *For each λ , δ_λ is totally odd in the sense that $\delta_\lambda(c) = -1$ for any complex conjugation $c \in G_K$.*

We prove the following theorem. These results in the case $K = \mathbb{Q}$ are mentioned in Section 5 of [12]. The abelian varieties analogue is proved in [7]. Note that if $K = \mathbb{Q}$, Serre’s conjecture is now a theorem, thus ρ_λ is modular and the following results follow immediately from the properties of Galois representations attached to modular forms.

Theorem 1.4. *Under Tate conjecture (see Section 2 for the statement), we have the following results.*

- (1) *For each prime \mathfrak{p} of good reduction of M , $a_{\mathfrak{p}} = \epsilon(\text{Frob}_{\mathfrak{p}})\bar{a}_{\mathfrak{p}}$, where $\text{Frob}_{\mathfrak{p}}$ is arithmetic Frobenius, $a_{\mathfrak{p}} = \text{Trace}(\rho_{\lambda}(\text{Frob}_{\mathfrak{p}}))$, and $\bar{}$ means the complex conjugation.*
- (2) *Let S be a finite set of finite primes at which M has bad reduction. Then $E = \mathbb{Q}(a_{\mathfrak{p}} \mid \mathfrak{p} \notin S)$.*
- (3) *Let F be the subfield of E generated by the numbers $a_{\mathfrak{p}}^2/\epsilon(\text{Frob}_{\mathfrak{p}})$ with $\mathfrak{p} \notin S$, then F is totally real and the extension E/F is abelian.*
- (4) *The field L in Theorem 1.2 is the same as the field F in (3). In particular, E is an abelian extension of L .*

2. Motives for absolute Hodge cycles

In this section, we review the construction of the category of motives for absolute Hodge cycles. Let X be a smooth projective variety over a subfield K of \mathbb{C} . We consider three Weil cohomologies H_{dR} , $H_{B,\sigma}$, and H_l . (They are de Rham cohomology, Betti cohomology with respect to $\sigma : K \hookrightarrow \mathbb{C}$, and l -adic etale cohomology.) These cohomologies are furnished with cycle maps:

$$cl^r : CH^r(X) \rightarrow H_{?}^{2r}(X)(r),$$

where $CH^r(X)$ is the abelian group of algebraic cycles of codimension r on X modulo rational equivalence, and $? = dR, (B, \sigma),$ or l . Define

$$H_{total}^{2r}(X)(r) := H_{dR}^{2r}(X)(r) \times \prod_l H_l^{2r}(X)(r) \times \prod_{\sigma} H_{B,\sigma}^{2r}(X, \mathbb{Z}) \otimes \mathbb{Z}(r).$$

Let $I_{\sigma} : H_{B,\sigma}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{dR}(X) \otimes_{K,\sigma} \mathbb{C}$ and $I_{l,\sigma} : H_{B,\sigma}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} H_l(X)$ be the comparison isomorphisms. Then write $I_{\sigma,\mathbb{Z}}$ (resp. $I_{l,\sigma,\mathbb{Z}}$) for the composition of the natural inclusion $H_{B,\sigma}^{2r}(X, \mathbb{Z}) \otimes \mathbb{Z}(r) \hookrightarrow H_{B,\sigma}(X)(r)$ with I_{σ} (resp. $I_{l,\sigma}$).

Definition 2.1. *An absolute Hodge cycle x of codimension r on X is an element of the finitely generated \mathbb{Z} -module:*

$$C_{AH}^r(X) := \{(x_{dR}, x_l, x_{\sigma})_{l,\sigma} \in H_{total}^{2r}(X)(r) \mid I_{\sigma,\mathbb{Z}}(x_{\sigma}) = x_{dR}, I_{l,\sigma,\mathbb{Z}}(x_{\sigma}) = x_l\}.$$

The cup product of each cohomology theory induces the ring structure on $C_{AH}^{2\dim(X)}(X \times X)$ for a smooth projective variety X .

The category \mathcal{M}_{AH} of motives for absolute Hodge cycles is defined as follows. The objects of \mathcal{M}_{AH} are triples (X, p, m) , where X is a smooth projective variety, $p \in C_{AH}^{2\dim(X)}(X \times X) \otimes \mathbb{Q}$ is a projector, and m is an integer. For two objects $M = (X, p, m)$ and $N = (Y, q, n)$ of \mathcal{M}_{AH} , the morphism set is defined by

$$\text{Hom}(M, N) := p \circ \text{Corr}_{AH}^{-m+n}(X, Y) \circ q,$$

where $\text{Corr}_{AH}^r(X, Y) = \prod_{i=1}^n C_{AH}^{r+\dim(X_i)}(X_i \times Y) \otimes \mathbb{Q}$ if X has irreducible components X_1, \dots, X_n .

Remark 2.2. There are several advantages to work with motives for absolute Hodge cycles. First, the category of motives for absolute Hodge cycles by extending the coefficients to \mathbb{Q} is a semi-simple \mathbb{Q} -linear abelian category. (See for example [6] and Section 6 of [4].) Then we can control the decomposition of motives using their endomorphism algebras and consider the image of a morphism between motives. See [12] for applications of this fact. Second, there is a well defined theory of polarizations for motives for absolute Hodge cycles. (See Section 6 of [4].) Therefore, we may generalize certain arguments in the case of abelian varieties to the case of motives.

In the rest of this section, we recall some definitions related to motives for absolute Hodge cycles.

Dual motives. Let $M = (X, p, m)$ be a pure motive, we define the dual motive $M^\vee = (X, p^t, \dim(X) - m)$, where p^t means the transpose of p .

The motive $\mathbb{L} = (\mathbb{P}^1, \pi_2, 0)$, $\pi_2 = \mathbb{P}^1 \times \{pt\}$, is the so-called Lefschetz motive and $\mathbb{Q}(1) := \mathbb{L}^\vee = (\mathbb{P}^1, \pi_0, 1)$, $\pi_0 = \{pt\} \times \mathbb{P}^1$, is the so-called Tate motive. For $M = (X, p, m)$, we have isomorphism $M \cong (X, p, 0) \otimes \mathbb{Q}(m)$.

Realizations of motives. For each Weil cohomology theory $H_?$, we define the realizations of a motive $M = (X, p, 0)$ by

$$H_?^*(M) = \bigoplus_{i=0}^{2 \dim(X)} H_?^i(M),$$

where $H_?^i(M) = \text{Im}(p_* : H_?^i(X) \rightarrow H_?^i(X))$. Here p_* is the composite map of the following maps

$$H_?^i(X) \xrightarrow{p_1^*} H_?^i(X \times X) \xrightarrow{\cup \gamma(p)} H_?^{i+2 \dim(X)}(X \times X) \xrightarrow{p_{2*}} H_?^i(X),$$

where $p_j : X \times X \rightarrow X$ is the j -th projection, $\gamma : \text{Corr}^0(X, X) \rightarrow H_?^{2 \dim(X)}(X)$ is the cycle map, and p_{2*} is the Gysin homomorphism.

Definition 2.3. Let $M = (X, p, m)$ be a motive. If $H_?^*(M) \subset H_?^i(X)(m)$ for each Weil cohomology $H_? = H_{B,\sigma}, H_{dR}, H_{et}$, then we say that M is pure of weight $w = i - 2m$. Note that if M is pure of weight w , then $M^\vee = M(w)$.

We have Hodge decomposition $H_{B,\sigma}^i(X)(m) = \bigoplus_{p+q=i} H_{B,\sigma}^{p,q}(X)(m)$. We say M is of Hodge type $(p_\sigma, q_\sigma) + (q_\sigma, p_\sigma)$ if $H_{B,\sigma}^*(M) \subset H_{B,\sigma}^{p_\sigma, q_\sigma}(X)(m) \oplus H_{B,\sigma}^{q_\sigma, p_\sigma}(X)(m)$.

If $M = (X, p, m)$ is pure of weight $i - 2m$ and M is defined over a number field K (i.e., X and p are both defined over K), then the realizations of M can be explicitly described. See Section 2 of [12] and Section 4 of [6] for details.

The rank of a motive. We have comparison isomorphisms between realizations of motives:

$$I_\sigma : H_{B,\sigma}(M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{dR}(M) \otimes_{K,\sigma} \mathbb{C}$$

and

$$I_{l,\sigma} : H_{B,\sigma}(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} H_l(M).$$

We define the *rank* of M to be $\dim_{\mathbb{Q}} H_{B,\sigma}(M)$. By the comparison theorem, $\text{rank}(M) = \dim_{\mathbb{Q}_l}(H_l(M)) = \dim_K(H_{dR}(M))$.

Field of definition of an endomorphism. As in [12], we define the field of definition of absolute Hodge cycles as follows. For a motive M over K , we say $\phi \in \text{End}(M)$ is defined over a field extension L of K if $(\phi)_l$ is member of $H_l(M)^{\text{Gal}(K/L)}$ for all prime l . We write $\text{End}_L(M)$ for the ring consisting of endomorphisms of M defined over L .

Tate conjecture. We recall the Tate conjecture as stated in Section 5 of [12]. For more details, see the references cited in [12].

Conjecture 2.4 (Tate conjecture). *Let M and N be motives over a number field K for absolute Hodge cycles which have same weight. Let B be a positive number such that $H_{\text{et}}(M, \mathbb{Z}_l)$ and $H_{\text{et}}(N, \mathbb{Z}_l)$ are free \mathbb{Z}_l -modules for any prime number $l \geq B$. Let $\rho_l : G_K \rightarrow GL_{\mathbb{Z}_l}(H_{\text{et}}(M, \mathbb{Z}_l))$ be the l -adic representation attached to M . Then we have the following.*

- (1) $H_{\text{et}}(M, \mathbb{Z}_l)$ is a semi-simple G_K -module.
- (2) The natural map

$$\alpha_K : \text{Hom}_K(N, M) \otimes_{\mathbb{Z}} \mathbb{Z}_l \rightarrow \text{Hom}_{G_K}(H_{\text{et}}(N, \mathbb{Z}_l), H_{\text{et}}(M, \mathbb{Z}_l))$$

is an isomorphism.

- (3) For almost all $l \geq B$, the subalgebra $\mathbb{Z}_l[\rho_l(G_K)]$ of $\text{End}_{\mathbb{Z}_l}(H_{\text{et}}(M, \mathbb{Z}_l))$ is the full commutator of the image of the natural map $\beta_K : \text{End}_K(M) \rightarrow \text{End}_{\mathbb{Z}_l}(H_{\text{et}}(M, \mathbb{Z}_l))$.
- (4) Let $\bar{\rho}_l$ be the reduction of $\rho_l \pmod{l}$ for $l \geq B$. Then for almost all $l \geq B$, the subalgebra $\mathbb{F}_l[\bar{\rho}_l(G_K)]$ of $\text{End}_{\mathbb{F}_l}(H_{\text{et}}(M, \mathbb{Z}_l)/lH_{\text{et}}(M, \mathbb{Z}_l))$ is the full commutator of the image of the natural map $\beta_K : \text{End}_K(M) \otimes \mathbb{Z}/l\mathbb{Z} \rightarrow \text{End}_{\mathbb{F}_l}(H_{\text{et}}(M, \mathbb{Z}_l)/lH_{\text{et}}(M, \mathbb{Z}_l))$.
- (5) $\text{End}_{\mathbb{F}_l}(H_{\text{et}}(M, \mathbb{Z}_l)/lH_{\text{et}}(M, \mathbb{Z}_l))$ is a semi-simple G_K -module.

3. Endomorphisms of M

3.1. Proof of Theorem 1.1 and Theorem 1.2

We prove Theorem 1.1 and 1.2 in this subsection. The strategy is essentially the same as in the case $K = \mathbb{Q}$.

Proof of theorem 1.1. Let $\mathfrak{X} = \text{End}_K(M) \otimes \mathbb{Q}$. By assumption, we have embedding $E \hookrightarrow \mathfrak{X}$. Let D be the commutant of E in \mathfrak{X} . First, we show that D is a field.

Let $\phi \in D \setminus \{0\}$ and let $N = \phi(M)$. Certainly, E acts on N . ϕ induces an endomorphism of $H_{dR}(M)$ which preserves the filtration of $H_{dR}(M)$. Since K is totally real, by Lemma 5.1.1 of [5], $H_{dR}(M)$ (resp. $H_{dR}(N)$) has a filtration $F(M) = \text{Fil}^{[w/2]}H_{dR}(M)$ (resp. $F(N)$) which is a free $E \otimes K$ -module of rank one (resp. of rank $\frac{\text{rank}(N)}{2}$). Since

$$[E : \mathbb{Q}] \leq \frac{\text{rank}_K(H_{dR}(N))}{2} \leq \frac{\text{rank}_K(H_{dR}(M))}{2} = [E : \mathbb{Q}],$$

we have $\text{rank}(M) = \text{rank}(N)$ and ϕ is an isogeny. D is a division algebra over \mathbb{Q} such that $D \supset E$.

Because endomorphisms of M preserve the filtration of $H_{dR}(M)$, $F(M)$ can be viewed as a D -vector space. Then $\dim_K D \otimes K \mid \text{rank}_K(F(M)) = [E : \mathbb{Q}][K : \mathbb{Q}]$. Therefore $[D : \mathbb{Q}] \mid [E : \mathbb{Q}]$. So $E = D$. In particular, the center F of \mathfrak{X} is a subfield of $D = E$.

Assume that $M \sim_K \prod_i N_i^{n_i}$, where N_i is a simple motive over K . Then $\mathfrak{X} = \prod_i M_{n_i}(D_i)$, where $D_i = \text{End}_K(N_i) \otimes \mathbb{Q}$ is a division algebra over F . Since $E \hookrightarrow \mathfrak{X}$, and the commutant of E in \mathfrak{X} is E itself, we may assume that $M \sim_K N_i^{n_i}$ for one i . Let $t_i = \sqrt{\dim_{\mathbb{Q}} D_i}$, then $[E : F] = n_i t_i$. As before, $F(N_i)$ is a vector space over D_i , we have $\dim_{\mathbb{Q}} D_i \mid \text{rank}_K(F(N_i))$. Therefore $n_i \cdot \dim_{\mathbb{Q}} D_i \mid n_i \cdot \text{rank}_K(F(N_i))$. Note that $n_i \cdot \dim_{\mathbb{Q}} D_i = n_i t_i^2 [F : \mathbb{Q}]$, $n_i \cdot \text{rank}_K(F(N_i)) = \text{rank}_K(F(M)) = [E : \mathbb{Q}] = t_i n_i [F : \mathbb{Q}]$, so $t_i = 1$, $F = D_i$. The theorem follows. ■

Proof of theorem 1.2. Since M/K is simple, we have $E = \text{End}_K(M) \otimes \mathbb{Q}$. First, we assume that M is $\bar{\mathbb{Q}}$ simple. Let D be the commutant of E in $\mathfrak{D} = \text{End}_{\bar{\mathbb{Q}}}(M) \otimes \mathbb{Q}$. By the same argument as in the proof of Theorem 1.1, $\dim_{\mathbb{Q}} D \mid [E : \mathbb{Q}]$ and $D = E$. Let $F = Z(\mathfrak{D})$ be the center of \mathfrak{D} . Assume that \mathfrak{D} has Schur index t , then we have $[E : F] = t$, $[\mathfrak{D} : F] = t^2$. Note that $\text{rank}(M) = 2[E : \mathbb{Q}] = 2t[F : \mathbb{Q}]$. Since $H_B(M)$ is vector space over \mathfrak{D} , we have $t^2[F : \mathbb{Q}] = \text{rank}_{\mathbb{Q}}(\mathfrak{D}) \mid \text{rank}_{\mathbb{Q}}(H_B(M)) = 2t[F : \mathbb{Q}]$. Therefore $t = 1$ or 2 .

We still have to prove that F is totally real. This is a statement about motives over $\bar{\mathbb{Q}}$ (it has nothing to do with the field of definition of M). It follows from the argument in the proof of Theorem 4.3 of [12].

Now assume that $M \sim_{\bar{\mathbb{Q}}} \prod_i N_i^{n_i}$, where N_i are $\bar{\mathbb{Q}}$ simple. Then

$$\text{End}_{\bar{\mathbb{Q}}}(M) = \prod_i M_{n_i}(\text{End}_{\bar{\mathbb{Q}}}(N_i)).$$

Because E is a \mathbb{Q} subalgebra of $\text{End}_{\bar{\mathbb{Q}}}(M) \otimes \mathbb{Q}$, we must have $E \hookrightarrow M_{n_i}(\text{End}_{\bar{\mathbb{Q}}}(N_i))$ for each i . Then

$$\frac{\text{rank}(M)}{2} = [E : \mathbb{Q}] \mid \frac{n_i}{2} \text{rank}(N_i) \leq \sum_i \frac{n_i}{2} \text{rank}(N_i) = \frac{\text{rank}(M)}{2}.$$

We must have $M \sim_{\bar{\mathbb{Q}}} N^n$. Then $\text{End}_{\bar{\mathbb{Q}}}(M) = M_n(\text{End}_{\bar{\mathbb{Q}}}(N))$, the statements hold. ■

3.2. K -motives

Let M be a motive over $\bar{\mathbb{Q}}$, K be a number field. We say M is a K -motive if (1) M is pure of weight w with Hodge type $(p_\sigma, q_\sigma) + (q_\sigma, p_\sigma)$ ($p_\sigma > q_\sigma$), and w is equal to the dimension of the underlying variety of M if w is even; (2) $\text{End}_{\bar{\mathbb{Q}}}(M) \otimes \mathbb{Q}$ is a division algebra over a totally real field F with Schur index $t = 1$ or 2 , and $2t[F : \mathbb{Q}] = \text{rank}(N)$; (3) for each $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/K)$, there exists a non-zero isogeny

$$\mu_\sigma : M^\sigma \rightarrow M,$$

such that $\mu_\sigma \circ \sigma(\phi) = \phi \circ \mu_\sigma$ for all $\phi \in \text{End}_{\bar{\mathbb{Q}}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. We then have the following proposition.

Proposition 3.1. *Let N be a $\bar{\mathbb{Q}}$ simple factor of a motive M of GL_2 -type over K which does not contain submotives with complex multiplication. Then N is a K -motive.*

Proof. From the proof of Theorem 1.2, we may assume that $M \sim_{\bar{\mathbb{Q}}} N^n$, where N is a $\bar{\mathbb{Q}}$ simple motive. We have proved that the endomorphism algebra $End_{\bar{\mathbb{Q}}}(N) \otimes \mathbb{Q}$ is a division algebra over a totally real field F with Schur index $t = 1$, or 2, and the rank of N is $\text{rank}(N) = t[F : \mathbb{Q}]$. Then the result follows exactly by the same argument as in the proof of Proposition 6.3 of [12]. ■

4. Galois representations attached to M

4.1. Proof of Theorem 1.3 and Theorem 1.4

Proof of theorem 1.3. The proof is the same as the proof of Lemma 5.1 of [12]. The only difference is that, for statement (2), we have a complex conjugation for each embedding $K \hookrightarrow \bar{\mathbb{C}}$. But the argument in [12] still holds since the complex conjugation induced from $\sigma : K \rightarrow \mathbb{C}$ sends $H^{p_\sigma, q_\sigma}(M)$ to $H^{q_\sigma, p_\sigma}(M)$ and $p_\sigma \neq q_\sigma$. ■

Proof of theorem 1.4. (1) Let $\sigma : E \hookrightarrow \bar{\mathbb{Q}}_l$ be an embedding of fields. Let $\bar{\sigma}$ be the complex conjugation of σ defined by $\bar{\sigma}(x) = \sigma(\bar{x})$. (Recall that we have fixed an embedding $E \hookrightarrow \mathbb{C}$.) Let $V_\sigma = V_l \otimes_{E \otimes \mathbb{Q}_l} \bar{\mathbb{Q}}_l$, where the tensor product is taken relative to the map $E \otimes \mathbb{Q}_l \hookrightarrow \bar{\mathbb{Q}}_l$ induced by σ . Define $V_{\bar{\sigma}}$ similarly. Fix a polarization of M defined over K . It gives us a bilinear map

$$\phi : V_l \times V_l \rightarrow \mathbb{Q}_l(w)$$

with the following properties:

$$\begin{aligned} \phi(ex, y) &= \phi(x, \bar{e}y) & \forall e \in E, x, y \in V_l, \\ \phi(gx, gy) &= g\phi(x, y) & \forall g \in G_K. \end{aligned}$$

Here $\bar{e} = e$ if E is totally real, \bar{e} is the complex conjugation of e if E is CM. After extending scalars from \mathbb{Q}_l to $\bar{\mathbb{Q}}_l$, we find an isomorphism of $\bar{\mathbb{Q}}_l[G_K]$ -modules

$$V_{\bar{\sigma}} \cong Hom(V_\sigma, \bar{\mathbb{Q}}_l(w)).$$

By Theorem 1.3, the determinant of V_σ is a one-dimensional vector space over $\bar{\mathbb{Q}}_l$ on which G_K acts by the character $\epsilon^\sigma \chi_l^w$. Since V_σ is of dimension two over $\bar{\mathbb{Q}}_l$, this gives us another isomorphism

$$Hom(V_\sigma, \bar{\mathbb{Q}}_l(\epsilon^\sigma \chi_l^w)) \cong V_\sigma.$$

By the above two isomorphisms, we have $V_\sigma \cong V_{\bar{\sigma}}(\epsilon^\sigma)$. For $\mathfrak{p} \nmid l$ a prime of good reduction, the trace of $\text{Frob}_\mathfrak{p}$ acting on V_σ is $\sigma(a_\mathfrak{p})$, the trace of $\text{Frob}_\mathfrak{p}$ acting on $V_{\bar{\sigma}}(\epsilon^\sigma)$ is $\epsilon^\sigma(\text{Frob}_\mathfrak{p})\bar{\sigma}(a_\mathfrak{p}) = \epsilon^\sigma(\text{Frob}_\mathfrak{p})\sigma(\bar{a}_\mathfrak{p})$. The result follows.

(2) By Tate conjecture, we have $E \otimes \mathbb{Q}_l = (End_K M) \otimes \mathbb{Q}_l = End_{G_K} V_l$. Let $\bar{V}_l = V_l \otimes \mathbb{Q}_l$. Then we have $End_{\bar{\mathbb{Q}}_l[G_K]} \bar{V}_l = E \otimes \mathbb{Q}_l$. Also, V_l is a semi-simple G_K -module by Tate conjecture, so \bar{V}_l is a semi-simple $\bar{\mathbb{Q}}_l[G_K]$ -module. Define V_σ as in (1). Then V_σ are semi-simple $\bar{\mathbb{Q}}_l[G_K]$ -modules and they are pairwise non-isomorphic. (In fact, they are semi-simple since \bar{V}_l is. By the same argument in the proof of Proposition 4.2, the commutator of their product is $\prod_\sigma \bar{\mathbb{Q}}_l$.) It follows that their traces are pairwise distinct. Let $E' = \mathbb{Q}(a_{\mathfrak{p}} \mid \mathfrak{p} \notin S) \subset E$, we have $\sigma|_{E'} \neq \tau|_{E'}$ for any pair $\sigma, \tau : E \hookrightarrow \bar{\mathbb{Q}}_l$. So we must have $E = E' = \mathbb{Q}(a_{\mathfrak{p}} \mid \mathfrak{p} \notin S)$.

(3) For $\mathfrak{p} \notin S$, by (1), we have

$$\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})} = \bar{a}_{\mathfrak{p}}^2 \epsilon(\text{Frob}_{\mathfrak{p}}) = \frac{\bar{a}_{\mathfrak{p}}^2}{\bar{\epsilon}(\text{Frob}_{\mathfrak{p}})},$$

where $\bar{}$ is complex conjugation. So F is totally real. Then E is contained in the extension of F obtained by adjoining to F the square roots of all $\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})}$ and all roots of unity. Therefore E is an abelian extension of F .

(4) Choose a prime number l which splits completely in E , so that every embedding $E \hookrightarrow \bar{\mathbb{Q}}_l$ factors as $E \hookrightarrow \mathbb{Q}_l \hookrightarrow \bar{\mathbb{Q}}_l$. Choose a finite extension K' of \mathbb{Q} which is sufficiently large such that: (1) all endomorphisms of M are defined over K' ; (2) $H := Gal(\bar{\mathbb{Q}}/K')$ is contained in the kernel of ϵ . By Tate conjecture, we have $\mathfrak{D} \otimes \mathbb{Q}_l = End_{\mathbb{Q}_l[H]} V_l$. The center of $\mathfrak{D} \otimes \mathbb{Q}_l$ is $L \otimes \mathbb{Q}_l$. The space $V_l = H_l(M)$ which is free of rank two over $E \otimes \mathbb{Q}_l$ decomposes as a product $\prod_\sigma V_\sigma$, where σ runs over the set Ω of injections $\sigma : E \rightarrow \mathbb{Q}_l$, $V_\sigma = V_l \otimes_{E \otimes \mathbb{Q}_l} \mathbb{Q}_l$ with respect to $\sigma : E \hookrightarrow \mathbb{Q}_l$. By Tate conjecture, the action of H on V_l is semi-simple. So V_σ is a simple $\mathbb{Q}_l[H]$ -module because M has no sub motives with complex multiplication. Hence $End_H V_\sigma = \mathbb{Q}_l$ for all σ .

Now consider M as a motive defined over K' . Certainly, we have a Galois representation $r : Gal(\bar{\mathbb{Q}}/K') \rightarrow GL_E(H_B(M)) \cong GL_2(E)$. For each prime v of K' which is not over l and at which M has good reduction, let $t_v \in E$ be the number $\text{Trace}(r(\text{Frob}_v))$. Write r_l for the Galois representation

$$r_l : Gal(\bar{\mathbb{Q}}/K') \rightarrow GL_{E \otimes \mathbb{Q}_l}(H_l(M)) \cong GL_2(E \otimes \mathbb{Q}_l) \cong \prod_{\sigma} GL(V_{\sigma}),$$

we have $\text{Trace}(\text{Frob}_v|_{V_{\sigma}}) = \sigma(t_v)$ for each σ . For two different $\sigma, \tau \in \Omega$, V_{σ} and V_{τ} are isomorphic as $\mathbb{Q}_l[H]$ -modules if and only if they have the same trace, i.e., $\sigma(t_v) = \tau(t_v)$ for all v , if and only if $\sigma|_J = \tau|_J$, where J is the field $\mathbb{Q}(t_v)$. This implies that the center of $\mathfrak{D} \otimes \mathbb{Q}_l$ is $J \otimes \mathbb{Q}_l$. Thus $L \otimes \mathbb{Q}_l = J \otimes \mathbb{Q}_l$, which implies that $L = J$.

Assume $\sigma|_J = \tau|_J$, so that $V_{\sigma} \cong V_{\tau}$ as representations of H . Therefore $Ind_H^{G_K}(V_{\sigma}|_H) \cong Ind_H^{G_K}(V_{\tau}|_H)$ as representations of G_K . Since V_{σ} is a simple factor of $Ind_H^{G_K}(V_{\sigma}|_H)$ and V_{τ} is a simple factor of $Ind_H^{G_K}(V_{\tau}|_H)$, we see that there exists a character $\psi : G_K \rightarrow \mathbb{Q}_l^{\times}$ such that $V_{\sigma} \cong V_{\tau} \otimes \psi$ as representations of G_K . (In fact, we see that ψ factors as $G_K \rightarrow Gal(K'/K) \rightarrow \mathbb{Q}_l^{\times}$.) Since both V_{σ} and V_{τ} are unramified at all primes $\mathfrak{p} \nmid l$ and at which M has good reduction, so is ψ . Taking the traces of the two representations $V_{\sigma} \cong V_{\tau} \otimes \psi$, we

have $\sigma(a_{\mathfrak{p}}) = \psi(\text{Frob}_{\mathfrak{p}})\tau(a_{\mathfrak{p}})$ for all such primes. Taking the determinant, we have $\epsilon^\sigma = \psi^2 \cdot \epsilon^\tau$. Therefore, $\sigma(\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})}) = \tau(\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})})$ for all $\mathfrak{p} \nmid l$ and at which M has good reduction. By Galois theory, we have $\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})} \in J$ for all $\mathfrak{p} \nmid l$ at which M has good reduction. By varying l , we see that $\mathbb{Q}(\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})} \mid \mathfrak{p} \notin S) \subset J$.

We have to prove $J \subset \mathbb{Q}(\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})} \mid \mathfrak{p} \notin S)$. If $\sigma(\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})}) = \tau(\frac{a_{\mathfrak{p}}^2}{\epsilon(\text{Frob}_{\mathfrak{p}})})$ for all $\mathfrak{p} \notin S$, by Cebotarev density theorem, we have

$$\frac{(\text{Trace } V_\sigma)^2}{\det(V_\sigma)} = \frac{(\text{Trace } V_\tau)^2}{\det(V_\tau)}.$$

Restricting V_σ and V_τ to H , we have $\text{Trace}(V_\sigma|_H) = \pm \text{Trace}(V_\tau|_H)$. We may choose K' even larger, so H smaller, and get an equality $\text{Trace}(V_\sigma|_H) = \text{Trace}(V_\tau|_H)$. (See page 324 of [11].) Therefore $V_\sigma \cong V_\tau$ as $\mathbb{Q}_l[H]$ -modules. ■

We construct a map $\alpha : G_K \rightarrow E^\times/F^\times$ as follows. For any element $g \in G_K$ and $\phi \in \text{End}_{\overline{\mathbb{Q}}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$, ϕ and $g(\phi)$ are inner by Skolem-Noether Theorem. So there exists an element $\alpha(g) \in \text{End}_{\overline{\mathbb{Q}}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$, such that $g(\phi) = \alpha(g) \circ \phi \circ \alpha(g)^{-1}$. Since $E = \text{End}_K(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ is its own centralizer in $\text{End}_{\overline{\mathbb{Q}}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$, we must have $\alpha(g) \in E$ and $\alpha(g)$ is determined up to a scalar multiple of F^\times . We have the following proposition.

Proposition 4.1. *With the notation as above, we have $\alpha^2 \equiv \epsilon \pmod{F^\times}$. Moreover, suppose that \mathfrak{p} is a prime of good reduction for M such that $a_{\mathfrak{p}} \neq 0$, then $\alpha(\text{Frob}_{\mathfrak{p}}) \equiv a_{\mathfrak{p}} \pmod{F^\times}$.*

Proof. Fix a prime number l which splits completely in E . To prove that $\alpha^2 \equiv \epsilon \pmod{F^\times}$, it suffices to prove that $\sigma(\frac{\alpha^2(g)}{\epsilon(g)}) = \tau(\frac{\alpha^2(g)}{\epsilon(g)})$ for any σ and τ are embeddings $E \hookrightarrow \mathbb{Q}_l$ which agree on F . Let σ and τ be embeddings $E \hookrightarrow \mathbb{Q}_l$ which agree on F , then from the proof of the above theorem, we have $V_\sigma \cong V_\tau \otimes \psi$ for some character $\psi : G_K \rightarrow \mathbb{Q}_l^\times$. Consider the $\text{Hom}_{\mathbb{Q}_l}(V_\sigma, V_\tau)$ as a Galois module. On one hand, G_K acts on it by multiplication $\psi(g)^{-1}$. On the other hand, G_K acts on it by conjugation by $\alpha(g)$. Note that $\alpha(g)$ acts on V_σ and V_τ by $\sigma(\alpha(g))$ and $\tau(\alpha(g))$ respectively, $\alpha(g)$ acts on $\text{Hom}_{\mathbb{Q}_l}(V_\sigma, V_\tau)$ as $\tau(\alpha(g))/\sigma(\alpha(g))$. Therefore, $\psi(g)^{-1} = \tau(\alpha(g))/\sigma(\alpha(g))$ as elements in \mathbb{Q}_l . From the proof of the above theorem, we have $\epsilon^\sigma = \psi^2 \cdot \epsilon^\tau$. It is easy to see that $\sigma(\frac{\alpha^2(g)}{\epsilon(g)}) = \tau(\frac{\alpha^2(g)}{\epsilon(g)})$.

From the above argument, we have the following two equations: (1) $\sigma(a_{\mathfrak{p}}) = \tau(a_{\mathfrak{p}})\psi(\text{Frob}_{\mathfrak{p}})$ because $V_\sigma \cong V_\tau \otimes \psi$; (2) $\psi(g)^{-1} = \tau(\alpha(g))/\sigma(\alpha(g))$ proved above. Therefore, we have $\sigma(\frac{\alpha(\text{Frob}_{\mathfrak{p}})}{a_{\mathfrak{p}}}) = \tau(\frac{\alpha(\text{Frob}_{\mathfrak{p}})}{a_{\mathfrak{p}}})$. The result follows. ■

4.2. Modularity

The following two propositions generalize Proposition 5.4 and 7.1 of [12], where the statements are about the case $K = \mathbb{Q}$. The proof is the similar. We give details here for completeness.

Proposition 4.2. *Under Tate conjecture, the Galois representation $\rho_\lambda : G_K \rightarrow GL_2(E_\lambda)$ is absolutely irreducible for all λ . In particular, we have $End_{G_K}(\rho_\lambda) = E_\lambda$.*

Proof. By Tate conjecture, we have the following equality

$$\prod_{\lambda|l} E_\lambda = E \otimes_{\mathbb{Q}} \mathbb{Q}_l = End_{\mathbb{Q}_l[G_K]}(V_l) = \prod_{\lambda|l} End_{\mathbb{Q}_l[G_K]}(V_\lambda).$$

We must have $E_\lambda = End_{\mathbb{Q}_l[G_K]}(V_\lambda)$. It is obvious that

$$End_{E_\lambda[G_K]}(V_\lambda) \subset End_{\mathbb{Q}_l[G_K]}(V_\lambda),$$

therefore $End_{E_\lambda[G_K]}(V_\lambda) = E_\lambda$. ■

Proposition 4.3. *Under Tate conjecture, $\bar{\rho}_\lambda$ is absolutely irreducible for all but finitely many λ .*

Proof. Since $E = End_K(M) \otimes \mathbb{Q}$, we may assume that $End_K(M) = \mathcal{O}_E$, the ring of integers of E . By Tate conjecture, the subalgebra $\mathbb{F}_l[\bar{\rho}_l(G_K)]$ of $End_{\mathbb{F}_l}(H_{et}(M, \mathbb{Z}_l)/lH_{et}(M, \mathbb{Z}_l))$ is a semi-simple \mathbb{F}_l -algebra. Let $\Phi : End_K(M) \otimes \mathbb{Z}/\mathbb{Z}_l \rightarrow End_{\mathbb{F}_l}(H_{et}(M, \mathbb{Z}_l)/lH_{et}(M, \mathbb{Z}_l))$ be the natural map. Then by bicommutation theorem, we have

$$End_{\mathbb{F}_l[G_K]}(H_{et}(M, \mathbb{Z}_l)/lH_{et}(M, \mathbb{Z}_l)) = (\mathbb{F}_l[\bar{\rho}_l(G_K)])^\circ Im(\Phi),$$

where \circ means commutator. Therefore, we have the following surjection map

$$End_K(M) \otimes \mathbb{Z}/\mathbb{Z}_l = \mathcal{O}_E \otimes \mathbb{Z}/\mathbb{Z}_l \rightarrow End_{\mathbb{F}_l[G_K]}(H_{et}(M, \mathbb{Z}_l)/lH_{et}(M, \mathbb{Z}_l)).$$

By Tate conjecture again, the source and target have the same rank over \mathbb{F}_l . Therefore, the map is actually bijective. Let k_λ be the residue field of \mathcal{O}_{E_λ} , then we have

$$k_\lambda = End_{k_\lambda[G_K]}(H_{et}(M, \mathbb{Z}_l)/lH_{et}(M, \mathbb{Z}_l)).$$

The statement follows. ■

In [1], the authors made the following conjecture.

Conjecture 4.4 (Serre’s conjecture). *Let K be a totally real field. If $\bar{\rho} : G_K \rightarrow GL_2(\bar{\mathbb{F}}_p)$ is a continuous, irreducible, and totally odd Galois representation, then $\bar{\rho}$ is modular in the sense that $\bar{\rho} \cong \bar{\rho}_f$, where $\bar{\rho}_f$ is the modular Galois representation attached to a Hilbert modular eigenform.*

We immediately have that the following corollary.

Corollary 4.5. *Under Tate conjecture and Serre’s conjecture, the Galois representation $\bar{\rho}_\lambda$ attached to a motive M of GL_2 -type is modular.*

References

- [1] K. Buzzard, F. Diamond, F. Jarvis, *On Serre's conjecture for mod ℓ Galois representations over totally real fields*, Duke Math Journal **55** (2010), no. 1, 105–161.
- [2] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proceedings of Symposia in Pure Mathematics **33** (1979), no. 2, 313–346.
- [3] P. Deligne, J. Milne, *Hodge cycles on abelian varieties*, Hodge cycles, Motives, and Shimura varieties, Lecture Notes in Math. Vol. **900**, 1982.
- [4] P. Deligne, J. Milne, *Tanakian categories*, Hodge cycles, Motives, and Shimura varieties, Lecture Notes in Math. Vol **900**, 1982.
- [5] H. Hida, *On the search of genuine p -adic modular L -functions for $GL(n)$* , Monograph, Memoires SMF **67**, 1996.
- [6] A. Panchishkin, *Motives for absolute Hodge cycles*, in Motives vol 1, Amer. Math. Soc., Providence, RI, 1994, 461–483.
- [7] K. Ribet, *Abelian varieties over \mathbb{Q} and modular forms*, Algebra and topology 1992, 53–79, Korea Adv. Inst. Sci. Tech., Taejŏn, 1992.
- [8] K. Ribet, *Fields of definition of abelian varieties with real multiplication*, Arithmetic geometry (Tempe, AZ, 1993), 107–118, Contemp. Math. **174**, Amer. Math. Soc., Providence, RI, 1994.
- [9] A.J. Scholl, *Motives for modular forms*, Inventiones Mathematicae **100**, no. 1, 419–430.
- [10] J.-P. Serre, *Modular forms of weight one and Galois representations*, in Algebraic number fields: L-functions and Galois properties. Proc. Sympos., Univ. Durham, Durham, 1975, Academic Press, London, 1977, 193–268.
- [11] J.-P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259–331
- [12] T. Yamauchi, *\mathbb{Q} -motives and modular forms*, Journal of Number Theory **128** (2008), 1485–1505.

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